0 as both coordinates.

If we were looking at $T_{(1,1)}\mathbb{C}^2$ instead (with the same tangent vector $(2,1)$), then we would have obtained $x \mapsto 1+2\varepsilon$ and $y \mapsto 1+\varepsilon$ instead. Why didn't I take $T_{(1,1)}Z_3$? While $(1,1) \in Z_3$, the tangent vector $(2,1)$ does not belong to $T_{(1,1)}Z_3$. The latter can be calculated as follows: Substitute $a := x-1$ and $b := y-1$, then

$$\mathbb{C}[x,y]/(y^2-x^3) = \mathbb{C}[a,b]/(b^2+2b-a^3-3a^2-3a),$$

and we look at the point $(a,b)=(0,0)$. The cotangent space is $(a,b)/(a^2,ab,b^2,2b-3a)$, and the vector $(2,1)$ does not define a correct linear map

$$(a,b)/(a^2,ab,b^2,2b-3a) \to \mathbb{C}, \quad a \mapsto 2,\ b \mapsto 1.$$

On the other hand, it does not make sense to compare the tangent spaces of some X within different points, anyway. We have no parallel transport.

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12. Aufgabenblatt zum 10.7.2023

Problem 101. Let F be a locally free sheaf on an integral, i.e. irreducible and reduced scheme X. Show that, for open subsets $U \subseteq X$, the restriction map $\Gamma(X, F) \to \Gamma(U, F)$ is injective.
Give counter examples for the cases when one of the assumptions is violated.

Solution: W.l.o.g. $X = \operatorname{Spec} A$ with A being a domain and $F = \mathcal{O}_X$. Now, all localizations are injective.

Examples for violated assumptions: (i) $X = \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[x]$ and $F = \widetilde{M}$ with $M = \mathbb{C}[x]/(x) = \mathbb{C}$ yields the 0-map to $U := X \setminus \{0\}$, and (ii) $X = \operatorname{Spec}(\mathbb{C} \times \mathbb{C})$ is the disjoint union of two points. Even $F = \mathcal{O}_X$ violates the claim now.

Problem 102. a) Show directly that the diagonal $\Delta : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ is a closed embedding. What is the homogeneous ideal of $\Delta(\mathbb{P}^1_{\mathbb{C}}) \subseteq \mathbb{P}^3_{\mathcal{C}}$ after additionally using the Segre embedding? Do you see the Veronese embedding within this picture?
b) Let $X := \mathbb{A}^1_{\mathbb{C}} \cup \mathbb{A}^1_{\mathbb{C}}$ glued along the common $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$. Show directly that there are affine open $U_1, U_2 \subseteq X$ such that either $U_1 \cap U_2$ is not affine or that $U_1 \cap U_2 = U$ is affine with $U_i = \operatorname{Spec} A_i$ and $U = \operatorname{Spec} B$ such that $A_1 \otimes_{\mathbb{C}} A_2 \to B$ is not surjective.
c) In the situation of (b) show that $\Delta(X) \subseteq X \times_{\operatorname{Spec} \mathbb{C}} X$ is not a closed subset.

Solution: (a) The Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$, $(x_0 : x_1), (y_0 : y_1) \mapsto (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1)$ corresponds to the homogeneous coordinate rings $\mathbb{C}[z_0, \ldots, z_3] \to$ $\to \mathbb{C}[z_0, \ldots, z_3]/(z_0 z_3 - z_1 z_2)$. The diagonal $\Delta : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$ maps $(w_0 : w_1) \mapsto (w_0 : w_1), (w_0 : w_1) \mapsto (w_0^2 : w_0 w_1 : w_0 w_1 : w_1^2)$. In particular, It is obtained from the homogeneous ring homomorphism

$$\mathbb{C}[z_0, \ldots, z_3] \to \mathbb{C}[w_0, w_1], \quad z_0 \mapsto w_0^2,\ z_1, z_2 \mapsto w_0 w_1,\ z_3 \mapsto w_1^2$$

factoring through $\mathbb{C}[z_0, \ldots, z_3]/(z_0 z_3 - z_1 z_2)$. It is a surjection not onto $\mathbb{C}[w_0, w_1]$, but onto its even part. In the projective situation, this is sufficient for providing a closed embedding. The kernel is generated by $(z_1 - z_2,\ z_0 z_3 - z_1 z_2)$. Thus, the embedding $\Delta : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ factors via the second Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$, followed by a linear embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^3$.

(b) In $X := U_1 \cup U_2$ (with $U_i = \mathbb{A}^1$ for $i = 1, 2$) the intersection of U_1 and U_2 is, via definition, $U_{12} = \mathbb{A} \setminus \{0\}$. The restriction maps ϱ_i on the coordinate rings are both the localization maps $\varrho_i : \mathbb{C}[x] \to \mathbb{C}[x]_x$. The two copies of $\mathbb{C}[x]$ do not generate the larger ring $\mathbb{C}[x]_x$.

(c) Just looking at the closed points, X consists of $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ and two points 0_A and 0_B. In $X \times X$, all 4 pairs $(0_A, 0_A)$, $(0_A, 0_B)$, $(0_B, 0_A)$, $(0_B, 0_B)$ belong to the closure of $\Delta(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}) = \{(t, t) \mid t \neq 0\}$. However, $\Delta(\mathbb{A}^1_{\mathbb{C}})$ contains only two of them, namely $(0_A, 0_A)$ and $(0_B, 0_B)$.

Problem 103. a) Show that d-dimensional k-varieties (with a perfect field k) are birational equivalent to hypersurfaces in $\mathbb{P}^{d+1}$.
(*Hint:* Use the theorem of the primitive element.)

b) Let $f, g \in k[x]$ be two different polynomials with simple roots. Construct a hypersurface of $\mathbb{C}^2$ that is birational equivalent to $V(y^2 - f(x),\, z^2 - g(x)) \subseteq \mathbb{C}^3$.

Solution: (a) k = perfect $\Rightarrow$ for each field extension $K = k(\alpha_1, \ldots, \alpha_m) \supseteq k$ there is an $e \subseteq \{\alpha_1, \ldots, \alpha_m\}$ with $K \supseteq k(e) \supseteq k$ (separable|transzendent), cf. [ZS, ch. II, Th 30+31, S.104]. "Satz vom primitiven Element" $\Rightarrow$ d-dimensional k-varieties are birational equivalent to hypersurfaces in $\mathbb{P}^{d+1}$.

(b) Let $\pm y$ and $\pm z$ be the respective roots of the minimal polynomials $m_y(t) = t^2 - f(x)$ and $m_z(t) = t^2 - g(x)$ over $k(x)$. Theorem of the primitive element (actually, its proof) $\rightsquigarrow$ every $\gamma := y + cz$ with $c \in k(x)$ such that $y + cz \neq (-y) + c(-z)$, i.e., $c \neq -y/z$ generates the extension field $K := k(x)(y, z)$ over $k(x)$.
With $c := 1$, i.e., $\gamma := y + z$, we obtain $\big(\gamma^2 - (f+g)\big)^2 = 4fg$. This leads to the hypersurface equation $\gamma^4 - 2(f+g)\gamma^2 + (f-g)^2 = 0$.

Problem 104. Assume that the ring A is factorial. Show that this implies $\operatorname{Pic}(\operatorname{Spec} A) = 0$, i.e. every invertible sheaf on $\operatorname{Spec} A$ is isomorphic to $\mathcal{O}_{\operatorname{Spec} A}$.

(*Hint:* For invertible sheaves $\mathcal{L}$ one is supposed to use the cocycle description on an open covering $\{D(g_i)\}$ with $\mathcal{L}|_{D(g_i)} \cong \mathcal{O}_{D(g_i)}$, cf. Problem 95. Via induction by the overall number of prime factors of the g_i, one can reduce the claim to the special case that all elements g_i are prime. Now, using again Problem 95, one can attain that $h_{ij} \in A^*$ for all i, j.)

Solution: Let p be a prime divisor of $g_1 \cdots g_N$ – via induction by the number of prime divisors of $g_1 \cdots g_N$ we may assume that $\mathcal{L}$ is trivial on $D(p) = \operatorname{Spec} A_p$. On the other hand, (prime divisors of $g_1 \cdots g_N$) $\supseteq (g_1, \ldots, g_N) = (1)$. Thus, we may suppose that all g_i are prime.
Now, every $h_{ij} \in A^*_{g_i g_j}$ (using the notation of Problem 95) can be expressed as $h_{ij} = u_{ij} \cdot g_i^{e_i}/g_j^{e_j}$ with $u_{ij} \in A^*$. The elements u_{ij}'s do still satisfy the cocycle condition. Hence, we can represent them as $u_{ij} = u_{i0}/u_{j0}$.

Problem 105. Show (by using the toric language via polytopes in $M_{\mathbb{R}}$) that the blowing up of $\mathbb{P}^2$ in two points is isomorphic to the blowing up of $\mathbb{P}^1 \times \mathbb{P}^1$ in one single point.

Solution: