# COMMUTATIVE ALGEBRA/ ALGEBRAIC GEOMETRY (BMS-LECTURE WS 2022/23)

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## 1. Rings and ideals

week 1 (1)

1.1. **Rings and ideals.** Units, zerodivisors, nilpotent elements, prim- and maximal ideals in rings (Example: in $\mathbb{Z}/n\mathbb{Z}$ und $k[X,Y]/(X^2-Y^3) = k[t^2,t^3]$ with $k$ being a field). Operations with Ideals: $+, \cap, \cdot, \sqrt{\ }$; moving ideals along ring homomorphisms.

week 2 (3)

1.2. **Algebraic sets.** $k = \overline{k}$ field $\rightsquigarrow k[\mathbf{x}] := k[x_1,\ldots,x_n]$ is the ring of "regular functions" $A(k^n)$; "*closed algebraic subsets*" of $k^n$ are the vanishing loci $V(J) \subseteq k^n$ for subsets or (radical) ideals $J \subseteq k[\mathbf{x}] \rightsquigarrow$ $\boxed{\text{Zariski } \textit{topology}}$ on $k^n$: $\bigcap_i V(J_i) = V(\bigcup_i J_i) = V(\sum_i J_i)$ and $V(J_1) \cup V(J_2) \subseteq V(J_1 \cap J_2) \subseteq V(J_1 J_2) \subseteq V(J_1) \cup V(J_2)$.
*Examples:* $V(y^2 - x^3)$, $V\Big(\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{pmatrix} \leq 1\Big)$, $\mathrm{SL}(n,k) \subseteq \mathbb{M}(n,k) = k^{n^2}$.

Subset $Z \subseteq k^n \rightsquigarrow$ radical ideal $I(Z) := \{f \in k[\mathbf{x}] \mid f|_Z = 0\} \subseteq k[\mathbf{x}]$. Properties: $I(\subseteq) = \supseteq$ and $I(Z_1 \cup Z_2) = I(Z_1) \cap I(Z_2)$. Moreover, $Z \subseteq V(I(Z)) =$ "algebraic closure" and $I(V(J)) \supseteq \sqrt{J}$ (even "=" by HNS (7.3)). In particular, for $Z = V(J)$ algebraic: $Z \subseteq V\big(I(V(J))_{\supseteq J}\big) \subseteq V(J) = Z$. Thus, HNS (7.3) $\rightsquigarrow$ order reversing bijection

$$\{\text{closed, algebraic} \subseteq k^n\} \underset{I(\bullet)}{\overset{V(\bullet)}{\rightleftarrows}} \{\text{radical ideals in } k[x_1,\ldots,x_n]\}.$$

*Properties:* $I(\bigcap_i Z_i) = \sqrt{\sum_i I(Z_i)}$; $Z$ is irreducible $\Leftrightarrow I(Z)$ is a prime ideal.

"Regular functions" on closed algebraic $Z = V(J)$: Reduced "coordinate ring" $A(Z) := k[\mathbf{x}]/I(Z)$ (integral for irreducible $Z =$ "*affine varieties*"); same bijection as above for $Z$ and $A(Z)$; the smallest example is $Z = \{p\}$ with $A(\{p\}) = k[\mathbf{x}]/\mathfrak{m}_p = k$.
week 3 (5) Open subsets $D(g \in A(Z)) := [g \neq 0] = Z \setminus V(g)$ yields a basis of the open subsets; $D(g_i)(i \in I)$ cover $Z \Leftrightarrow V(g_i \mid i \in I) = \emptyset \Leftrightarrow (g_i)_{i \in I} = (1)$ in $A(Z)$ by HNS.

1.3. **Functoriality of algebraic sets.** Regular algebraic maps $f : k^m \to k^n$ are, by definition, $n$-tuples $f = (f_1,\ldots,f_n)$ with $f_i \in k[\mathbf{x}] = k[x_1,\ldots,x_m]$. This is equivalent to $k$-algebra homomorphisms $f^* : k[\mathbf{y}] := k[y_1,\ldots,y_n] \to k[\mathbf{x}]$ sending $y_i \mapsto f_i(\mathbf{x})$. This map coincides with the pull-back of regular functions, i.e. $f^*(g \in$

$k[\mathbf{y}]) = g \circ f$. If $J \subseteq k[\mathbf{y}]$, then $f^{-1}V(J) = V(f^*(J)k[\mathbf{x}])$, i.e. regular fuctions are continous.

More generalized: If $X \subseteq k^m$ and $Y \subseteq k^n$ are (Zariski-) closed algebraic subsets, then regular maps $f : X \to Y$ are, by definition, given by asking for extendability to commutative diagrams

$$\begin{array}{ccc} k^m & \xrightarrow{F} & k^n \\ \uparrow & & \uparrow \\ X & \xrightarrow{f} & Y \end{array}$$

where $F$ is regular as before. An equivalent condition for such a diagram is a ring homomorphism $F^* : k[\mathbf{y}] \to k[\mathbf{x}]$ with $F^*(I(Y)) \subseteq I(X)$. In particular, regular maps $f : X \to Y$ are provided by $k$-algebra homomorphisms $f^* : A(Y) \to A(X)$. This category is equivalent to the opposite of the category of reduced, finitely generated $k$-algebras.

Thus, (Zariski-) closed algebraic subsets form a category; their isomorphism classes (i.e. neglecting the embedding into an ambient space $k^n$) are called *affine sets*. This category is equivalent to the opposite of the category of reduced, finitely generated $k$-algebras.

A special case: If $f \in k[\mathbf{x}]$, then we obtain $g(\mathbf{x}, t) := f(\mathbf{x}) \cdot t - 1 \in k[\mathbf{x}, t]$ and $Z_f := V(g)$ is a closed subset of $k^{m+1}$, i.e.

$$\begin{array}{ccc} k^{m+1} & \xrightarrow{\mathrm{pr}} & k^m \\ \uparrow & & \uparrow \\ Z_f & \xrightarrow{p} & D(f) \end{array}$$

where $p$ denotes the restriction of the projection map $\mathrm{pr} : (\mathbf{x}, t) \mapsto \mathbf{x}$. It is bijective; the inverse map is $\mathbf{x} \mapsto (\mathbf{x}, 1/f(\mathbf{x}))$. While all maps are continous with respect to the Zariski topology, $p$ does even become a homeomorphism. Moreover, despite it is not a closed subset in $k^m$, this construction provides $k[\mathbf{x}, t]/(ft-1) = k[\mathbf{x}, 1/f(\mathbf{x})] \subseteq k(\mathbf{x})$ as the associated ring of regular functions.

1.4. **Prime avoiding and two radicals.** A ring is local $\Leftrightarrow$ the non-units form an ideal. "*Nil radical*": $\sqrt{(0)} = \bigcap\{\text{prime ideals}\}$ (*Proof*: $f \notin \sqrt{(0)} \Rightarrow 0 \notin \{f^{\mathbb{N}}\} =: S$, and use Zorn's lemma with ideals disjoint to $S$). "*Jacobson radical*": $\bigcap\{\text{maximal ideals}\} = \{a \in R \mid 1 + aR \subseteq R^*\}$.

**Lemma 1.** 1) *Prime ideal* $P \supseteq IJ \Leftrightarrow P \supseteq I \cap J \Leftrightarrow P \supseteq I$ *or* $P \supseteq J$.
2) $J \subseteq \bigcup_{i=1}^k P_i$ *(prime ideales with at most 2 exceptions)* $\Rightarrow \exists\, i \colon J \subseteq P_i$.

*Proof.* $J \not\subseteq P_i \Rightarrow$ induction yields $x_i \in J \setminus \bigcup_{j \neq i} P_j \rightsquigarrow x_i \in P_i$. For $k = 2$ consider $y := x_1 + x_2 \in J \setminus \bigcup_i P_i$; for $k \geq 3$ consider $y := x_1 + x_2 \cdot \ldots \cdot x_k$ if $P_1 =$ prime. □

1.5. **Chinese Remainders.** The generalization of $\mathbb{Z}/(mn)\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ is:

**Proposition 2** (Chinese Remainder Theorem)**.** $I_1, \ldots, I_k \subseteq R$ *with* $I_i + I_j = (1)$ *for all* $i \neq j$. *Then,* $\prod_i I_i = \bigcap_i I_i$, *and* $\pi : R/\prod_i I_i \xrightarrow{\sim} \prod_i R/I_i$ *is an isomorphism.*

*Proof.* $k=2$: $x_1+x_2=1$ ($x_i\in I_i$) and $y\in I_1\cap I_2$ yields $y=x_1y+x_2y\in I_1I_2$. Moreover, $\pi(x_1)=(1,0)$; $\pi(x_2)=(0,1)$ imply the surjectivity of $\pi$.

week 4 (7) Induction: Since $x_i+x_k^{(i)}=1$ (with $x_i\in I_i$, $x_k^{(i)}\in I_k$) yields $\prod_i x_i=\prod_i(1-x_k^{(i)})\in\big(\prod_{i=1}^{k-1}I_i\big)\cap(1+I_k)$, we have $\big(\prod_{i=1}^{k-1}I_i\big)+I_k=(1)$. □

1.6. **The spectrum of a ring.** $\operatorname{Spec}R:=\{P\subseteq R\,|\,\text{primie ideales}\}\supseteq\operatorname{MaxSpec}R$ is a topological space ("ZARISKI-*Topology*"): $V(J):=\{P\supseteq J\}$ are the closed subsets; $V(I)\cup V(J)=V(I\cap J)=V(IJ)$ and $\bigcap_{i\in I}V(J_i)=V(\sum_{i\in I}J_i)$.

$\operatorname{Spec}R$ is quasicompact: $\bigcap_{i\in I}V(J_i)=\emptyset\Leftrightarrow\sum_{i\in I}J_i\ni 1$. Basis of the open subsets via $D(f):=(\operatorname{Spec}R)\backslash V(f)=\{P\in\operatorname{Spec}R\,|\,f\notin P\}$; one has $D(f)\cap D(g)=D(fg)$.

*Examples:* $\mathbb{A}^n:=\operatorname{Spec}\mathbb{Z}[x_1,\dots,x_n]$ and $\boxed{\operatorname{Spec}A\supseteq V(J)=\operatorname{Spec}A/J}$.

Subset $Z\subseteq\operatorname{Spec}R\rightsquigarrow$ reduced ideal $I(Z):=\bigcap_{P\in Z}P\subseteq R$. Properties: $I(\subseteq)=\supseteq$ and $I(Z_1\cup Z_2)=I(Z_1)\cap I(Z_2)$. Moreover, $Z\subseteq V(I(Z))=$ "algebraic closure" and $I(V(J))=\bigcap_{P\supseteq J}P=\sqrt{J}$ (no HNS needed!). In particular, for $Z=V(J)$ algebraic: $Z=V(I(Z))$ as in (1.2).

1.7. **Affine schemes.** $k=\overline{k}$ as in (1.2) $\rightsquigarrow$ another form of HNS (7.3): Every maximal ideal of $k[\mathbf{x}]$ is of the form $\mathfrak{m}_p=(x_1-p_1,\dots,x_n-p_n)$ for some $p\in k^n$. Thus, $k^n\overset{\sim}{\to}\operatorname{MaxSpec}k[\mathbf{x}]$, $p\mapsto\mathfrak{m}_p$ is a homeomorphism. Moreover, $\operatorname{MaxSpec}k[\mathbf{x}]\subseteq\mathbb{A}^n_k$ is exactly the set of closed points.

Hence, in $X=\operatorname{Spec}R$, the ring $R$ is considered the ring of regular functions on $X$: The value of $r\in R$ in $P\in X$ is $\overline{r}\in K(P):=\operatorname{Quot}R/P$ (Example: $K(\mathfrak{m}_p)=k[\mathbf{x}]/\mathfrak{m}_p=k$). In particular, $r\in R$ vanishes on $P\in X\Leftrightarrow r\in P$, and $r\in R$ vanishes on $Z\subseteq X\Leftrightarrow r\in P$ for all $P\in Z\Leftrightarrow r\in I(Z)$.

Regular maps in (1.2): Continuous $f:(Z\subseteq k^n)\to(Z'\subseteq k^{n'})$ such that $f^*:g\mapsto g\circ f$ induces a ring homomorphism $f^*:A(Z')\to A(Z)$ (equivalent: $f=(f_1,\dots,f_{n'})$ with $k[\mathbf{x}]\twoheadrightarrow A(Z)\ni f_i$). The embedding $Z\hookrightarrow k^n$ corresponds to $k[\mathbf{x}]\twoheadrightarrow A(Z)$.

Ring homomorphisms $\varphi:R\to S\rightsquigarrow$ continuous $\varphi^\#:\operatorname{Spec}S\to\operatorname{Spec}R$; example: $\mathbb{C}[x,y]/(y^2-x^3)=\mathbb{C}[t^2,t^3]\subseteq\mathbb{C}[t]$. "*Affine scheme*" $\operatorname{Spec}R:=(\operatorname{Spec}R,R)$ with morphisms $\operatorname{Hom}_{\mathit{affSch}}(\operatorname{Spec}S,\operatorname{Spec}R):=\operatorname{Hom}_{\mathcal{R}ings}(R,S)$, cf. (19.3), (19.1), and Proposition 55.

## 2. $R$-modules, localization/factorization

2.1. **Basics of $R$-modules.** Operations $\oplus$, $\sum$, $\cap$, Hom, $\otimes$ of $R$-modules – the latter

week 5 (9) is defined via $\operatorname{Hom}_R(M\otimes_R N,P)=\operatorname{Hom}_R(M,N;P):=\{\text{bilinear maps }M\times N\to P\}$. If $M,N\subseteq L$ (e.g. $M,N=$ ideals), then $(M:N):=\{r\in R\,|\,rN\subseteq M\}$. This includes $(0:N)=\operatorname{Ann}_R N$. Exact sequences; the 5-lemma.

2.2. **Testing exactness by applying the** Hom **functor.**

**Lemma 3.** *$M_\bullet = [0 \to M_1 \to M_2 \to M_3]$ is exact $\Leftrightarrow$ $\forall K$: $\mathrm{Hom}_R(K, M_\bullet)$ is exact. Similarily for $[M_1 \to M_2 \to M_3 \to 0]$ und $\mathrm{Hom}_R(\bullet, N)$. In particular, both* Hom *functors are left exact.* week 6 (11)

*Proof.* Choose $K := R$ for the first claim and $N := \mathrm{coker}(M_2 \to M_3)$ and $N := \mathrm{coker}(M_1 \to M_2)$ for the second. □

For $R$-modules $M, N, P$ we have $\mathrm{Hom}_R(M \otimes_R N,\, P) = \mathrm{Hom}_R(M,\, \mathrm{Hom}_R(N, P))$, i.e. $(\otimes_R N) \dashv \mathrm{Hom}(N, \bullet)$ ("*adjoint*"). The functor $(\otimes_R N)$ admits a right adjoint $\Rightarrow$ $(\otimes_R N)$ is right exact.

2.3. **Localization.** $S \subseteq R$ is called *multiplicative closed* $:\Leftrightarrow 1 \in S$ and $S \cdot S \subseteq S$; *Localization* $S^{-1}M := \{m/s \mid m \in M, s \in S\}$ (with $m/s = m'/s' :\Leftrightarrow \exists t \in S : t(ms' - m's) = 0$) is $(S^{-1}R)$-module; $M \to S^{-1}M$ $(m \mapsto m/1)$ is injective $\Leftrightarrow$ $S$ does not contain $M$-zero divisors.

Examples: $f \in R$, $S := \{f^{\mathbb{N}}\} \rightsquigarrow M_f$. Prime ideal $P \in \mathrm{Spec}\, R$, $S := R \setminus P \rightsquigarrow M_P$; this turns $\boxed{R_P}$ into a $\boxed{\text{local ring}}$ (via 2.5). "Total quotient ring": $S := \{$Non-zero divisors of $R\}$.

2.4. **Comparison with factorization.** (LocFac1) $I \subseteq R$ ideal; $S \subseteq R$ multiplicative closed $\Rightarrow$ $R \to R/I$ is universal with $I \to 0$; $R \to S^{-1}R$ is universal with $S \to \{\text{units}\}$.

(LocFac2) $(R/I)$-modules are $R$-modules with $IM = 0$; $(S^{-1}R)$-modules are $R$-modules with $[S \to \mathrm{Aut}_R(M)] \subseteq [R \to \mathrm{End}_R(M)]$.

(LocFac3) $M \mapsto M/IM = M \otimes_R R/I$ is right exact; $M \mapsto S^{-1}M = M \otimes_R S^{-1}R$ is exact ($R \to S^{-1}R$ is $\boxed{\text{flat}}$).

2.5. **Behavior of ideals via** $[R \to S^{-1}R]$**.** Let $I \subseteq R$, $J \subseteq S^{-1}R$ be ideals $\Rightarrow$ $I \cdot S^{-1}R = S^{-1}I$ with $S^{-1}I = R \Leftrightarrow I \cap S \neq \emptyset$. Moreover, $S^{-1}(J \cap R) = J$; $I \subseteq (S^{-1}I) \cap R$, but only for *prime* ideals $P \subseteq R \setminus S$ it holds true that $[a/s \in S^{-1}P \Rightarrow a \in P]$, hence $P = (S^{-1}P) \cap R$. This implies week 7 (13)

(LocFac4) $\mathrm{Spec}\, S^{-1}R = \{P \in \mathrm{Spec}\, R \mid P \cap S = \emptyset\}$, in particular, $\mathrm{Spec}\, R_f = D(f) := \mathrm{Spec}\, R \setminus V(f) \subseteq \mathrm{Spec}\, R$ is an open subset. The set $\mathrm{Spec}\, R/I = V(I) \subseteq \mathrm{Spec}\, R$ is closed.

(LocFac4') For $P \in \mathrm{Spec}\, R$ we have: In $R/P$ ideals $\boxed{\text{above } P}$ survive; in $R_P$ ideals $\boxed{\text{below } P}$ survive.

(LocFac5) $S^{-1}(R/I) = S^{-1}R \otimes_R R/I = (S^{-1}R)/(S^{-1}I)$.

2.6. **Local tests.** Many properties of $R$-modules can be tested locally:

**Proposition 4.** *An $R$-linear map $f : M \to N$ is zero/surjective/injective/an isomorphism $\Leftrightarrow$ the same holds true for all $f_\mathfrak{m} : M_\mathfrak{m} \to N_\mathfrak{m}$ with $\mathfrak{m} \in \operatorname{MaxSpec} R$.*

*Proof.* $a \in M$ with $a/1 = 0$ in all $M_\mathfrak{m} \Rightarrow \forall \mathfrak{m}$: $\operatorname{Ann} a \not\subseteq \mathfrak{m} \Rightarrow \operatorname{Ann} a = R$, i.e. $a = 0$. In particular, $[\forall \mathfrak{m}$: $M_\mathfrak{m} = 0]$ implies $M = 0$. □

**Corollary 5.** *Exactness is a local property. $M$ is $R$-flat $\Leftrightarrow \forall \mathfrak{m}$: $M_\mathfrak{m}$ is $R_\mathfrak{m}$-flat.*

2.7. **The Nakayama lemma.** Let $M$ be a $\boxed{\text{finitely generated}}$ $R$-module.

**Proposition 6** (Cayley-Hamilton)**.** *$I \subseteq R$ ideal, $\varphi : M \to IM \Rightarrow \exists p = \sum_j p_j x^{n-j} \in R[x]$: $p_0 = 1$, $p_j \in I^j$ and $p(\varphi) = 0$ in $\operatorname{End}_R(M)$.*

*Proof.* $m_1, \ldots, m_k \in M$ generators; $\varphi(m_i) = \sum_j a_{ij} m_j \Rightarrow (xI_k - A) \cdot \underline{m} = 0 \in M^k$ ($M$ turns, via $\varphi$, into an $R[x]$-module). Multiplication with $\operatorname{adj}(xI_k - A) \rightsquigarrow p(x) := \det(xI_k - A)$ kills all $m_i$, thus $M$. □

**Corollary 7.** 1) *$M = IM \Rightarrow \exists\, p \in 1 + I \subseteq R$: $pM = 0$ $(1 + I \subseteq R^* \Rightarrow M = 0)$.*
2) *$f : M \to M$ surjective $\Rightarrow f$ is an isomorphism.*
3) ("Nakayama-Lemma") *$(R, \mathfrak{m})$ local, $m_i \in M$ $\boxed{\text{generate } M/\mathfrak{m}M \Rightarrow \text{generate } M}$.*

*Proof.* (1) $\varphi := \mathrm{id}_M$; (2) $I := (x) \subseteq R[x] =: R$ with $x$ acting as $f \Rightarrow p(x) = 1 + xq(x)$ kills $M$ since (1), thus $f^{-1} = -q(f)$; (3) $N := \operatorname{span}_R\{m_i\} \Rightarrow$ apply (1) to $M/N$. □

*Application:* Minimal sets of generators, minimal resolutions for modules over local rings $(R, \mathfrak{m})$. If $F = R^s$, then $p : F \twoheadrightarrow M$ induces an isomorphism $\overline{p} : F/\mathfrak{m}F \xrightarrow{\sim} M/\mathfrak{m}M \Leftrightarrow \ker p \subseteq \mathfrak{m}F$.

week 8 (15)

2.8. **Support of modules.** $M$=$R$-module $\rightsquigarrow \operatorname{supp} M := \{P \in \operatorname{Spec} R \mid M_P \neq 0\}$ and, by abuse of notation, $\operatorname{supp} I := \operatorname{supp} R/I$.

- $0 \to M' \to M \to M'' \to 0$ exact $\Rightarrow \operatorname{supp} M = (\operatorname{supp} M') \cup (\operatorname{supp} M'')$;
- $M$ finitely generated $\Rightarrow (S^{-1}N : S^{-1}M) = S^{-1}(N : M) \Rightarrow \operatorname{supp} M = V(\operatorname{Ann} M)$ (via $(0 : M)_P \neq (1) \Leftrightarrow P \supseteq \operatorname{Ann} M$).

2.9. Hom **commutes with flat base change.** $R \to S$ algebra $\rightsquigarrow$ canonical $S$-linear map $\alpha_M : \operatorname{Hom}_R(M, N) \otimes_R S \to \operatorname{Hom}_S(M \otimes_R S, N \otimes_R S)$.

**Proposition 8.** *$R \to S$ flat, $M$ finitely presented $\Rightarrow \alpha_M$ is an isomorphism.* (Example: Localisations $R \to S^{-1}R$.)

*Proof.* $R^a \to R^b \to M \to 0 \Rightarrow$ w.l.o.g.: $M = R^n$. □

## 3. Noetherian rings

3.1. **Chain conditions.** $(\Sigma, \leq)$ poset $\rightsquigarrow$ [strongly ascending chains do always terminate $\Leftrightarrow$ each subset of $\Sigma$ has maximal elements].
(Examples: open subsets of topological spaces with $\subseteq$, submodules with $\subseteq / \supseteq$).

**Definition 9.** $M$ is a *noetherian* $R$-module $:\Leftrightarrow$ each submodule is finitely generated $\Leftrightarrow \Sigma := \{\text{submodules}\}$ satisfies the $\boxed{\textit{ascending}\text{ chain condition}}$ (ACC).

**Lemma 10.** $0 \to M' \to M \xrightarrow{\pi} M'' \to 0$ *exact* $\Rightarrow$ [$M$ *noetherian* $\Leftrightarrow M'$, $M''$ *noetherian*]. (Special case: $M = M' \oplus M''$, thus finite direct sums.)

*Proof.* For ($\Leftarrow$) consider intersections with $M'$ and images in $M''$; afterwards one uses: $N_1 \subseteq N_2 \subseteq M$ with $N_1 \cap M' = N_2 \cap M'$ and $\pi(N_1) = \pi(N_2) \Rightarrow N_1 = N_2$. (This follows from $0 \to N_i \cap M' \to N_i \to \pi(N_i) \to 0$ by using the 5-lemma.) □

$R$ = "*noetherian ring*" $:\Leftrightarrow$ all ideals are finitely generated $\Leftrightarrow$ $R$ is a noetherian $R$-module. If $R$ is noetherian, then all finitely generated $R$-modules are noetherian, i.e. "f.g." is bequeathed to the submodules and implies "finitely presented".

3.2. **Hilbert's basis theorem.** The property "noetherian ring" is bequeathed as follows:

**Proposition 11.** 1) *$R$ noetherian $\Rightarrow R/I$ and $S^{-1}R$ are noetherian.*
2) *$R$ noetherian $\Rightarrow$ finitely generated $R$-algebras* (*as* $R[x]$) *are noetherian.*

*Proof.* $S^{-1}R$: For $J_i \subseteq S^{-1}R$ use $J_i = S^{-1}(J_i \cap R)$.
"Hilbert's basis theorem": $R$ noetherian; $I \subseteq R[x]$ ideal $\rightsquigarrow$ let $I_0 \subseteq R$ be the ideal of the highest coefficients of polynomials from $I \Rightarrow I_0 = (a^1, \ldots, a^k)$. Choose $f_i \in I$ with highest coefficient $a^i \rightsquigarrow I' := (f_1, \ldots, f_k) \subseteq R[x]$. Defining $N := \max_i(\deg f_i)$ we conclude $I = I' + (\langle 1, x, \ldots, x^{N-1}\rangle \cap I)$, and the second summand is a submodule of a finitely generated $R$-module. Thus, $I$ is finitely generated. □

In particular, localizations of $\boxed{\text{finitely generated } \mathbb{Z}\text{- or } k\text{-algebras}}$ are noetherian.

3.3. **An important filtration.** Let $R$ be a noetherian ring and $M$ a finitely generated $R$-module (*Example*: $M = k[\mathbf{x}]/[\text{monomial ideal}]$).

**Proposition 12.** *There is a finite* ("nice") *filtration $M = M_0 \supseteq \ldots \supseteq M_m = 0$ with factors $M_{k-1}/M_k \cong R/P_k$ for suitable (possibly equal) prime ideals $P_k \subseteq R$.*

*Proof.* Induction by #(generators of $M$) $\rightsquigarrow$ w.l.o.g. $M = R/I$. If $I$ is not a prime ideal $\rightsquigarrow x, y \in R \setminus I$ with $xy \in I$. We obtain $I + (x) \supsetneq I$ and $I : (x) \supseteq I + (y) \supsetneq I$ and $0 \to R/(I : x) \xrightarrow{\cdot x} R/I \to R/[I + (x)] \to 0$. Because of "noetherian", these enlargements of $I$ terminate. □

week 9 (17)

3.4. **Associated primes.** Let $R$ and $M$ be as in (3.3).
$\mathrm{Ass}(M) := \{P \in \mathrm{Spec}\, R \,|\, \exists\, R/P \hookrightarrow M\} = \{\mathrm{Ann}(m) \in \mathrm{Spec}\, R\}_{m\in M} \subseteq V(\mathrm{Ann}\, M)$.
In particular, using the notation of Proposition 12, $P_m \in \mathrm{Ass}(M) \rightsquigarrow \mathrm{Ass}(M) \neq \emptyset$.

**Proposition 13.** $0 \to M' \to M \to M'' \to 0$ *exact* $\Rightarrow \mathrm{Ass}(M') \subseteq \mathrm{Ass}(M) \subseteq \mathrm{Ass}(M') \cup \mathrm{Ass}(M'')$. *In particular* (cf. Prop. 12), $\mathrm{Ass}(M) \subseteq \{P_1, \ldots, P_m\}$ *is finite.*

*Proof.* Let $P \in \mathrm{Ass}(M)\backslash\mathrm{Ass}(M') \Rightarrow R/P \hookrightarrow M \twoheadrightarrow M''$ with kernel $K := M' \cap R/P$. Since each $0 \neq a \in K$ would yield an $R/P \overset{a}{\hookrightarrow} K \subseteq M'$, we obtain $K = 0$. □

3.5. **Minimal primes.** Denote $\mathrm{Min}(M) := \{\text{minimal primes above } \mathrm{Ann}(M)\}$.

**Lemma 14.** *For each ideal $I$ there exists a finite representation $\sqrt{I} = P_1 \cap \ldots \cap P_k$.*

*Proof.* If $\sqrt{I}$ is not prime, then choose $x, y \notin \sqrt{I} \ni xy \rightsquigarrow \sqrt{I} = \sqrt{I+(x)} \cap \sqrt{I+(y)}$: Assume $\sqrt{I} = 0$ ($R$ is now reduced) and $a \in \sqrt{(x)} \cap \sqrt{(y)}$. Then $a^m \in (x)$ and $a^n \in (y)$, hence $a^{m+n} \in (xy) = 0$. Now do noetherian induction. □

Lemma 1 implies that unshortenable representations fulfill $\{P_1, \ldots, P_k\} = \mathrm{Min}(R/I)$ and, moreover, that each $P \in V(I) \subseteq \mathrm{Spec}\, R$ contains an element of $\mathrm{Min}(R/I)$.

**Proposition 15.** *Let $R$ be a noetherian ring and $M$ a finitely generated $R$-module.*
1) *For multiplicative closed $S \subseteq R$ we have* $\mathrm{Ass}(S^{-1}M) = \mathrm{Ass}(M) \cap \mathrm{Spec}(S^{-1}R)$.
2) $P \supseteq \mathrm{Ann}\, M$ *minimal prime above* $\mathrm{Ann}\, M \Rightarrow P \in \mathrm{Ass}(M)$.

*Proof.* (1) Let $F : S^{-1}R/S^{-1}P \hookrightarrow S^{-1}M$ be given by $1 \mapsto m/s \Rightarrow \exists t \in S$: $P \cdot tm = 0$. Then, $f : R/P \to M$, $1 \mapsto tm$ is well-defined, and $S^{-1}f \sim F$ is injective. Eventually, the injectivity of $R/P \hookrightarrow S^{-1}(R/P)$ implies this of $f$.
2) $P = \mathfrak{m}$ in a local ring $(R, \mathfrak{m}) \Rightarrow \emptyset \neq \mathrm{Ass}(M) \subseteq V(\mathrm{Ann}\, M) = \{\mathfrak{m}\}$. □

$$\boxed{\mathrm{Min}(M) \subseteq \mathrm{Ass}(M) \subseteq \{P_1, \ldots, P_m \text{ of Proposition 12}\} \subseteq \mathrm{supp}(M) = \overline{\mathrm{Min}(M)}.}$$

week 10 (19) 3.6. **Zero divisors.** Let $R$ be noetherian and $M$ a finitely generated $R$-module $\Rightarrow$ $\bigcup \mathrm{Ass}(M) = \{\text{zero divisors of } M\} \cup \{0\}$:
Let $r \in R$ be a zero divisor, i.e. $r \in \mathrm{Ann}(m) \neq (1)$ for some $m \in M$. If $\mathrm{Ann}(m)$ is not prime, then there are $x, y \in R$ with $xy \in \mathrm{Ann}(m)$, but $x, y \notin \mathrm{Ann}(m)$. Thus $\mathrm{Ann}(m) \subsetneq \mathrm{Ann}(xm) \neq (1) \rightsquigarrow$ Noether induction.

## 4. Modules of finite length and Artin rings

4.1. **Composition series.** $R$ = ring, $M$ = finitely generated $R$-module $\rightsquigarrow$ "*composition series*" (the factors are simple, i.e. isomorphic to $R/\mathfrak{m}$); $\ell(M) :=$ "*length* of (the shortest composition series of) $M$" $\leq \infty$.
*Examples*: 1) $(R, \mathfrak{m})$ local $k$-algebra with field extension $k \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$ of degree $d \Rightarrow d \cdot \ell(M) = \dim_k M$.
2) $(R, \mathfrak{m})$ local with $\sqrt{0} = \mathfrak{m}$ ($\Leftrightarrow \mathrm{Spec}\, R = \{\mathfrak{m}\}$) $\Rightarrow \ell(M) < \infty$ (Proposition 12).

**Proposition 16.** $\boxed{\ell(\bullet) \text{ is additive}}$ *(in particular, strictly monotonic increasing), each filtration of an $R$-module $M$ has length $\leq \ell(M)$ and (in case of $\ell(M) < \infty$) can be refined toward a composition series of $M$. The latter are characterized by* [$\ell$(factors) $= 1$] *or by* [length $= \ell(M)$].

*Proof.* $\ell(\bullet)$ is strictly monotonic increasing: $N \subsetneq M \Rightarrow$ each minimal composition series $\{M_j\}$ of $M$ yields the $N$-filtration $\{N_j := M_j \cap N\}$ with $N_j/N_{j+1} \subseteq M_j/M_{j+1}$. Thus, for an *arbitrary filtration* $\{M_j\}$ of $M$ on has $\ell(M_j) > \ell(M_{j+1})$, i.e. $\ell(M) \geq$ [length of the filtration]. □

4.2. **Artinian $R$-modules.** :⇔ {submodules} satisfies the $\boxed{\textit{descending}}$ chain condition (DCC); similarily: "*Artinian ring*"; Lemma 10 does still apply.

*Examples*: (0) $k[\varepsilon]/\varepsilon^2$. (1) $\mathbb{Z}$ is noetherian, but not artinan. (2) $A := \mathbb{Z}_p/\mathbb{Z}$ is an artinian, but not noetherian $\mathbb{Z}$-Modul: $\gcd(a,p) = 1 \Rightarrow a/p^n \sim 1/p^n$ ($ab + p^n c = 1$ implies $1/p^n = b \cdot a/p^n$); hence $A_n := 1/p^n \cdot \mathbb{Z} \subseteq A$ are the only submodules at all. (3) $\mathbb{Z}_p$ satisfies neither (ACC)/(DCC).

4.3. **Artinian rings.** Despite (2) in (4.2), *rings $R$* satisfy:

**Proposition 17.** *$R$ is* $\boxed{\textit{artinian} \Leftrightarrow \ell_R(R) < \infty}$ *⇔ $R$ is noetherian with* $\operatorname{MaxSpec} R = \operatorname{Spec} R$, *i.e. every prime ideal is maximal. If so, then* $\operatorname{Spec} R$ *is a finite set.*

*Proof.* (i) "$\ell_R(R) < \infty$" implies "artinian" and "noetherian" via Proposition 16.
(ii) Let $R$ be noetherian with $\ell_R(R) = \infty$; let $I \subseteq R$ be maximal with "$\ell_R(R/I) = \infty$" $\Rightarrow I$ is prime: ↗ proof of Proposition 12. On the other hand, since $\ell_R(R/I) = \infty$, the domain $R/I$ is not a field.
(iii) Let $R$ be artinian; let $J \subseteq R$ be the *smallest* ideal being the product of finitely many maximal ideals $\Rightarrow J^2 = J$ and $J = J\mathfrak{m} \subseteq \mathfrak{m}$ ($\forall \mathfrak{m} \in \operatorname{MaxSpec} R$) $\Rightarrow J = 0$ (Nakayama – if $J$ is finitely generated).
*WorkAround* (if $J$ is not finitely generated): Let $I$ be the smallest ideal with $IJ \neq 0$ $\Rightarrow IJ = I$ (since $(IJ)J = IJ^2 = IJ \neq 0$), and there is an $f \in I$: $fJ \neq 0 \rightsquigarrow I = (f)$. Thus, $I$ is finitely generated, hence Nakayama applies, hence $I = 0$ (↯).
(iv) $(0) = \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_k$ provides a filtration of $R$ with its factors being the finite-dimensional (because of "artinian") $R/\mathfrak{m}_i$-vector spaces $\mathfrak{m}_1 \ldots \mathfrak{m}_{i-1}/\mathfrak{m}_1 \ldots \mathfrak{m}_i$.
(v) $P \in \operatorname{Spec} R \Rightarrow P \supseteq \mathfrak{m}_1 \cdot \ldots \cdot \mathfrak{m}_k \Rightarrow \exists i$: $P \supseteq \mathfrak{m}_i$. □

4.4. **Multiplicities.** $M$ = finitely generated module over a noetherian ring $R$; let $P \supseteq \operatorname{Ann}(M)$ be minimal above $\operatorname{Ann}(M)$, i.e. $P \in \operatorname{Min}(M)$.

**Proposition 18.** *In each "nice" filtration of $M$* (according to Proposition 12) *the factor* $\boxed{R/P \textit{ appears exactly } \ell_{R_P}(M_P)\textit{-times.}}$ *In particular, this multiplicity* ($< \infty$) *does not depend on the special choice of the filtration.*

*Proof.* [filtration] $\otimes_R R_P \rightsquigarrow$ factors $R/Q$ with $Q \not\subseteq P$ disappear, and $R/P$ becomes the field $R_P/PR_P = \operatorname{Quot}(R/P)$. □

## 5. Primary decomposition

week 11 (21)

5.1. ***P*-primary ideals.** $R$ = noetherian. $Q \subseteq R$ *primary* $:\Leftrightarrow$ in $R/Q$ the zero divisors are nilpotent.
$Q \subseteq R$ primary $\Rightarrow P := \sqrt{Q}$ is prime ("$Q$ is $P$-primary") $\Rightarrow \exists\, n : P^n \subseteq Q \subseteq P$.
An ideal $Q$ with prime $P := \sqrt{Q}$ is ($P$-) primary $\Leftrightarrow \forall x, y \in R$: $[xy \in Q,\, x \notin Q \Rightarrow y \in P]$. Thus, intersections of $P$-primary ideales are $P$-primary.
*Examples:* 1) $Q = (x, y^2) \subseteq k[x, y]$ is $P$-primary with $P = (x, y)$; $P^2 \subseteq Q \subseteq P$.
2) $P := (x, z) \subseteq k[x, y, z]/(xy - z^2) \Rightarrow P^2$ is not primary(!)
$Q \subseteq R$ with $\boxed{\mathfrak{m} := \sqrt{Q} \text{ maximal ideal} \Rightarrow Q \text{ is } \mathfrak{m}\text{-primary}}$ ($\sqrt{Q} = \sqrt{0} \subseteq R/Q$ is then the only prime ideal, hence $\{R/Q - \text{zero divisors}\} = \bigcup \operatorname{Ass}(R/Q) = \sqrt{(0)}$).

5.2. **Existence.** $R$= noetherian $\rightsquigarrow$ every ideal $I \subseteq R$ is a finite intersection of $\cap$-irreducible ideals.

**Lemma 19.** *In noetherian rings, all $\cap$-irreducible ideals are primary.*

*Proof.* $\forall y \in R\ \exists k$: $\operatorname{Ann}(y^k) = \operatorname{Ann}(y^{k+1}) \Rightarrow \operatorname{Ann}(y) \cap (y^k) = (0)$. Hence, if $(0)$ is irreducible, then $\operatorname{Ann}(y) \neq 0$ (i.e. $y$ is a zero divisor) implies $y^k = 0$. □

In particular, all $I \subseteq R$ admit a primary decomposition $I = \bigcap_{i=1}^k Q_i$ which is minimal, i.e. unshortenable with mutually different radicals $P_i = \sqrt{Q_i}$. Example in [Eis, 3.8, S.103-105]: $(x) \cap (x^2, xy, y^2) = (x^2, xy) = (x) \cap (x^2, y)$.

5.3. **First uniqueness.** Let $Q$ be $P$-primary; $x \in R \Rightarrow (Q : x) = (1)$ if $x \in Q$, and $(Q : x) = P$-primary otherwise (from $Q \subseteq (Q : x) \subseteq P$ one derives $\sqrt{(Q : x)} = P$).

**Theorem 20.** *$I = \bigcap_i Q_i$ minimal primary decomposition $\Rightarrow \{P_i := \sqrt{Q_i}\} = \operatorname{Ass}(R/I)$. In particular, we obtain $\sqrt{I} = \bigcap \operatorname{Ass}(R/I) = \bigcap \operatorname{Min}(R/I)$ again.*

*Proof.* $I = 0$. $x \in R \Rightarrow \sqrt{\operatorname{Ann} x} = \bigcap_i \sqrt{(Q_i : x)} = \bigcap_{x \notin Q_i} P_i$. If $\operatorname{Ann} x$ is prime, then so is $\sqrt{\operatorname{Ann} x}$, hence $\operatorname{Ann} x = \sqrt{\operatorname{Ann} x} = P_i$ for some $i$.
Conversely, if $0 \neq x \in I_i := \bigcap_{j \neq i} Q_j$, then $x \notin Q_i$ and $\sqrt{\operatorname{Ann} x} = P_i$. If $0 \neq x \in P_i^m I_i$ with $P_i^{m+1} I_i = 0$ (exists because of $P_i^{\gg 0} \subseteq Q_i$), then $P_i x = 0$, hence $P_i \subseteq \operatorname{Ann} x \subseteq \sqrt{\operatorname{Ann} x} = P_i$. □

In particular, primary ideals $Q$ are alternatively characterized by $\# \operatorname{Ass}(R/Q) = 1$.

5.4. **Second uniqueness.** The primary $Q_i$ partners of the associated $P_i \in \operatorname{Ass}(R/I)$ are not all uniquely determined, but:

**Theorem 21.** *For $\boxed{\text{minimal}}$ $P_i \in \operatorname{Min}(R/I)$, the $Q_i$ are uniquely determined by $I$.*

*Proof.* $\otimes_R R_{P_i}$ respects intersections (exact) and kills all $Q_j$ with $P_j \not\subseteq P_i \Rightarrow I R_{P_i} = Q_i R_{P_i}$. On the other hand, for primary ideals, $Q_i R_{P_i} = Q_i' R_{P_i}$ implies $Q_i = Q_i'$. □

5.5. **Monomial ideals.** Generalizing the example in (5.2), let $I \subseteq k[x,y]$ be a monomial ideal $\rightsquigarrow S := \{a \in \mathbb{N}^2 \mid x^a \notin I\}$ "*standard monomials*" with $[S \ni a \geq b \in \mathbb{N}^2$ (i.e. $a - b \in \mathbb{N}^2) \Rightarrow b \in S]$; assume $S \neq \mathbb{N}^2$.

$S(1) := \{a \in S \mid a + (0 \times \mathbb{N}) \subseteq S\} = [0, \alpha] \times \mathbb{N}$ for some (maximal) $\alpha \in \mathbb{Z}_{\geq -1}$
$S(2) := \{a \in S \mid a + (\mathbb{N} \times 0) \subseteq S\} = \mathbb{N} \times [0, \beta]$ for some (maximal) $\beta \in \mathbb{Z}_{\geq -1}$
$S(12) := \overline{S \setminus (S(1) \cup S(2))}$ (closure with respect to "$\leq$") is finite.

$\Rightarrow S(1), S(2), S(12)$ correspond to ideals being $(x)$-, $(y)$- and, $(x,y)$-primary, and $S = S(1) \cup S(2) \cup S(12)$ yields a decomposition. Here, $S(12)$ could be replaced by each larger, "$\leq$"-closed, but still finite set.

## 6. Integral ring extensions

6.1. **Integral vs. finite.** $A \subseteq B$ rings: $x \in B$ is *integral* over $A :\Leftrightarrow x$ satisfies an equation $x^n + \sum_{v=0}^{n-1} a_v x^v = 0$ with $a_v \in A$; integral closure $=: \overline{A}^{(B)}$.
*Examples:* $R$ factorial $\Rightarrow R$ is integrally closed in $\operatorname{Quot}(R)$ ("normal"); $w := (\sqrt{5}+1)/2$ satisfies $w^2 - w + 1 = 0$ (over $\mathbb{Z}$).

**Proposition 22.** *For $A \subseteq B \ni b$ the following facts are equivalent:*

(1) *$b$ is integral over $A$,*
(2) *$B \supseteq A[b]$ is a finite $A$-algebra, i.e. finitely generated as an $A$-module,*
(3) *$\exists$ a finite $A$-algebra $C$: $A[b] \subseteq C \subseteq B$,*
(4) *$\exists$ $B \supseteq A[b]$-module $M$: $\operatorname{Ann}_{A[b]} M = 0$, and $M$ is finitely generated over $A$.*

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial; (4) $\Rightarrow$ (1) follows from Proposition 6: $\varphi := (\cdot\, b)$; $I = R = A$. □

*Consequences:* $b_i \in B$ are integral over $A \Leftrightarrow A[b_1, \ldots, b_k]$ is a finitely generated $A$-module; the $A$-integral elements of $B$ form a subring; integrality of ring extensions is transitiv.

Integrality (and "integral closure") is a local property, i.e. $b$ is integral over $A \Leftrightarrow$ it is integral over all $A_P$; $A$ is normal $\Leftrightarrow$ all $A_P$ are normal (even the $A_\mathfrak{m}$ suffice): For the first, lift from the $A_P$ to $A_{f_i}$ with $(f_1, \ldots, f_k) = (1)$. The normality statement follows from $A = \bigcap_{\mathfrak{m} \in \operatorname{MaxSpec} A} A_\mathfrak{m}$ (for $b \in \operatorname{Quot} A$ consider $\{a \in A \mid ab \in A\}$).

6.2. **Integrality over ideals.** $I \subseteq A$ ideal $\rightsquigarrow$ analogous notion "$b \in B \supseteq A$ is integral over $I$" via $b^n + \sum_{v=0}^{n-1} a_v b^v = 0$ with $a_v \in I$. We have $\overline{I}^{(B)} = \sqrt{I\,\overline{A}^B}$: If $b \in I\,\overline{A}^{(B)}$, thus $b = \sum_v a_v c_v$ with $a_v \in I$ and $c_v \in \overline{A}^{(B)}$, then $M := A[c_\bullet]$ is a finitely generated $A$-module. Now, one uses Proposition 6 with $\varphi := (\cdot\, b)$ and $I$.

**Proposition 23.** *$A \subseteq B$ domains with normal $A$. Then, $b \in B$ is integral over $I \subseteq A \Leftrightarrow b$ is algebraic over* $\operatorname{Quot} A$ *with minimal polynomial from $x^n + \sqrt{I}[x]_{<n}$.*

*Proof.* The coefficients of the minimal polynomial are from $\operatorname{Quot} A$. On the other hand, as symmetric functions in the roots ($\in \overline{\operatorname{Quot} A}$, integral over $I$) they are also integral over $I$. □

6.3. **Going up and down.** Let $A \subseteq B$ be an integral extension; denote $\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$, $Q \mapsto Q \cap A$.

**Proposition 24.** (1) *If $A, B =$ domains, then [$A$ is a field $\Leftrightarrow$ $B$ is a field].*

(2) *$Q \in \operatorname{Spec} B$ is maximal $\Leftrightarrow$ $Q \cap A$ is maximal in $A$.*

(3) *$\varphi$ is injective $\boxed{\text{on chains}}$ of prime ideals of $B$, i.e. $Q_2 \subseteq Q_1$ together with $\varphi(Q_2) = \varphi(Q_1)$ implies $Q_2 = Q_1$.*

(4) *$\varphi$ is surjective (on chains) – a successively increasing lifting is possible.*

(5) *$A$, $B$ integral domains, $A =$ normal $\Rightarrow$ successively decreasing liftings are possible, too.*

*Proof.* (1) $\Rightarrow$ (2) via factorisation; (2) $\Rightarrow$ (3) via localization by $P := Q_i \cap A$.
(4) If $(A, \mathfrak{m})$ is local, then by (2) every maximal ideal in $B$ is a preimage of $\mathfrak{m}$; localization $\rightsquigarrow$ general case.
(5) Let $P_2 \subseteq P_1 \subseteq A$ and $Q_1 \subseteq B$ with $P_1 = Q_1 \cap A$; we show that $P_2$ is the restriction of a prime ideal via $A_{P_1} \hookrightarrow B_{Q_1}$. *Problem*: This inclusion is not integral anymore – thus one has to check directly that $\boxed{P_2 B_{Q_1} \cap A \subseteq P_2}$ (and can, afterwards, choose a maximal ideal in $(A \setminus P_2)^{-1} B_{Q_1}$ over $P_2$): Let $A \ni x = y/s$ with $y \in P_2 B$ and $s \in B \setminus Q_1 \Rightarrow y$ is integral over $P_2$, i.e. it has over $\operatorname{Quot} A$ a minimal polynomial $y^n + a_1 y^{n-1} + \ldots + a_n = 0$ with $a_v \in P_2$. For $s$ the minimal polynomial becomes $s^n + (a_1/x) s^{n-1} + \ldots + (a_n/x^n) = 0$; integrality $\Rightarrow a_v/x^v \in A$ with $x^v \cdot (a_v/x^v) \in P_2$. Finally, if $x \notin P_2$, then we would obtain $s^n \in P_2 B \subseteq Q_1$. □

week 12 (23)

6.4. **Finiteness of the normalization.** Integral closures of domains in fields are, under sufficiently good assumptions, finitely generated modules:

**Proposition 25.** *Let $A$ be a domain and $L \supseteq \operatorname{Quot} A$ a finite field extension. If*

(i) *$A$ is a finitely generated $k$-algebra* (with e.g. $L = \operatorname{Quot} A$), *or*

(ii) *$A$ is noetherian, normal, and $L | \operatorname{Quot} A$ is separable,*

*then $B := \overline{A}^{(L)}$ is a finitely generated $A$-module.*

*Proof.* $A =$ finitely generated $k$-Algebra: See [ZS, ch. V, Th 9, S.267].
*Normal/Separable:* Let $K := \operatorname{Quot} A$ and $b_1, \ldots, b_m \in B$ a $K$-basis of $L = \operatorname{Quot} B = B \otimes_A K$ (the equality follows from $s \in L \Rightarrow \exists a \in A$: The minimal polynomial of $s$ turns into an integrality relation of $as$). With $d := \det \operatorname{Tr}_{L|K}(b_i b_j) \in A \setminus \{0\}$ (separable!), the $\operatorname{Tr}_{L|K}(\bullet, \bullet)$-dual basis is some $b'_1, \ldots, b'_m \in \frac{1}{d} B$. For $b \in L$ it follows that $b = \sum_i \operatorname{Tr}_{L|K}(b b_i) b'_i$, and for $b \in B$, the coefficients stem from $A$. Hence, $B \subseteq \sum_i A b'_i$. □

## 7. The Hilbert Nullstellensatz

7.1. **The Weierstrass Preparation Theorem.** (Trivial form for polynomials) $\#k = \infty$, $f \in k[x_1, \ldots, x_n] \Rightarrow$ there is a linear change of coordinates $\psi : x_i \mapsto x_i + a_i x_n$ ($\mathbf{a} \in k^n$; $i = n$: $x_n \mapsto x_n$, but $a_n := 1$) with

$$\psi(f) = (\text{const} \neq 0) \cdot x_n^N + \sum_{i=0}^{N-1} c_i(x_1, \ldots, x_{n-1}) \cdot x_n^i \quad (\text{and } \deg c_i \leq N - i).$$

($N := \deg f \Rightarrow f \mapsto \psi(f)$ produces $x_n^N$ with coefficients $\mathbf{a}^r$ for every monomial $\mathbf{x}^r$ of degree $N$. The entire coefficient of $x_n^N$ in $\psi(f)$ is then $f_{[\deg=N]}(\mathbf{a})$; hence choose an $\mathbf{a} = (a_1, \ldots, a_{n-1}, 1) \in k^n$ with $f_{[\deg=N]}(\mathbf{a}) \neq 0$.)

**Proposition 26** ("Noether-Normalization")**.** *$\#k = \infty$, $k[x_1, \ldots, x_n] \twoheadrightarrow A$ finitely generated $k$-algebra $\Rightarrow \exists y_1, \ldots, y_d \in \operatorname{span}_k(x_1, \ldots, x_n)$: $k[y_1, \ldots, y_d] \hookrightarrow A$ is integral.*

*Proof.* $k[x_1, \ldots, x_n] \to A$ finite, not injective $\Rightarrow f \in \ker$ has w.l.o.g. the above shape $\Rightarrow k[x_1, \ldots, x_{n-1}] \to k[x_1, \ldots, x_n]/f$ is finite. □

7.2. **The cool version of the HNS.**

**Corollary 27** (HNS1)**.** *Let $k$ be a field and $A$ a finitely generated $k$-algebra being a field, too. Then, $A|k$ is a finite field extension, i.e. $[A : k] < \infty$. In particular, if $k = \overline{k}$, then this implies $A = k$.*

*Proof.* a) [$\#k = \infty$]: Proposition 26 $\Rightarrow k[y_1, \ldots, y_d] \hookrightarrow A$ is integral; then Proposition 24 implies that $k[y_1, \ldots, y_d]$ is a field $\Rightarrow d = 0$.
b) [Without Proposition 26]: Let $a_1, \ldots, a_n \in A$ be algebra generators. If $n = 0$, then we are done. We proceed by induction on $n$:
$k \hookrightarrow k[a_1] \hookrightarrow k(a_1) \hookrightarrow A \Rightarrow [A : k(a_1)] < \infty$. Let $f \in k[a_1]$ be a common denomiator of the integrality relations of the remaining $a_2, \ldots, a_n \Rightarrow A$ is integral over $k[a_1]_f \Rightarrow k[a_1]_f$ is a field, i.e. $a_1$ is not transzendental over $k$. □

7.3. **The standard version of the HNS.** Let $k = \overline{k}$ be an algebraically closed field.

**Proposition 28** (HNS2)**.** *Let $k = \overline{k}$.*

(1) *Every maximal ideal of $k[\mathbf{x}]$ is of the form $\mathfrak{m}_p = (x_1 - p_1, \ldots, x_n - p_n)$.*
(2) *Let $J \subseteq k[\mathbf{x}]$ be an ideal with $V(J) = \emptyset$ in the sense of* (1.2) $\Rightarrow J = (1)$.
(3) *$J \subseteq k[\mathbf{x}]$ ideal $\Rightarrow I(V(J)) = \sqrt{J}$ in the sense of* (1.2).

*Proof.* Corollary 27 $\Rightarrow$ (1) $\Rightarrow$ (2). (3): $f \in I(V(J)) \Rightarrow V(J, f(\mathbf{x})\,t - 1) = \emptyset \Rightarrow J + (ft - 1) = (1)$. Now, substitute $t \mapsto 1/f$ in the coefficients. □

7.4. **Algebraically not closed fields.** Example for $k \subset \overline{k}$: $J := (x^2 + 1) \subseteq \mathbb{R}[x]$. In (1.7) we have defined $f(P) \in K(P) := \operatorname{Quot}(R/P) = R_P/PR_P$ ("residue field" of $P$). *Example:* $R = k[\mathbf{x}] \Rightarrow x_i \in R$ yields $x_i(P) \in K(P) =$ "$i$-th coordinate". If $\mathfrak{m} := (x^2 + 1) \in \operatorname{Spec} \mathbb{R}[x] \Rightarrow K(\mathfrak{m}) = \mathbb{R}[x]/(x^2 + 1) = \mathbb{C}$, and $x(\mathfrak{m}) = \sqrt{-1}$.

## 8. Projective resolutions

week 13 (25)

8.1. **Projective modules.** $:\Leftrightarrow \operatorname{Hom}_R(P,\bullet)$ is exact $\Leftrightarrow$ all $M \twoheadrightarrow P$ split $\Leftrightarrow P$ is the direct $\boxed{\text{summand of a free } R\text{-module}}$ $R^I := R^{\oplus I}$ ($\operatorname{Hom}(P,\bullet)$ is then a summand of $\operatorname{Hom}(R^I,\bullet)$) $\Rightarrow P$ is flat (for the same reason with $P\otimes$ and $R^I\otimes$).

Base change (e.g. localization) preserves "projective" ($R^I$-summands); for $P$ with finite presentation it holds true: $P$ is projective $\Leftrightarrow \forall \mathfrak{m} \in \operatorname{MaxSpec} R$: $P_\mathfrak{m}$ is a projective $R_\mathfrak{m}$-module (for $M \twoheadrightarrow N$ localize $\operatorname{Hom}(P,M) \to \operatorname{Hom}(P,N)$).

*Example:* $(2, 1+\sqrt{-5}) \subseteq \mathbb{Z}[\sqrt{-5}]$ is projective, but not free; smooth points of an affine, elliptic curve yield those ideals, too.

$(R,\mathfrak{m})$ local, $P$ = projective with finite presentation $\Rightarrow P$ is $\boxed{\text{(locally) free}}$: Let $R^n \twoheadrightarrow P$ be minimal and $R^n = P \oplus P' \Rightarrow P' \otimes R/\mathfrak{m} = 0$, hence $P' = 0$ (Nakayama).

8.2. **Complexes and Qis'.** $\mathcal{A}$ = abelian category (e.g. $\mathcal{M}od_R = \{R\text{-modules}\}$). complexes $M_\bullet$ (with $d_i : M_i \to M_{i-1}$, left shift $M[1]_i := M_{i-1}$, $M^i := M_{-i}$, hence $d^i : M^i \to M^{i+1}$ and $M[1]^i = M^{i+1}$; $d[1] := -d$); (co-)homlogy $\operatorname{H}_i(M_\bullet) := \operatorname{Z}_i(M_\bullet)/\operatorname{B}_i(M_\bullet)$ with $\operatorname{H}_i(M_\bullet) = \operatorname{H}_0(M_\bullet[-i])$; morphisms of complexes $f : M_\bullet \to N_\bullet$; the long exact homology sequence (is functorial); $\boxed{\text{“Qis”:=“quasiisomorphisms”}}$ (*not* stable under the application of functors).

*Example:* $f : 0_\bullet \to M_\bullet := [0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0]$ exact $\Rightarrow f$ is Qis and $\operatorname{H}_\bullet(\mathrm{id}_M) = 0$. However, $N := \mathbb{Z}/2\mathbb{Z}$ yields $f \otimes \mathrm{id}_N \neq$ Qis and $\operatorname{H}_\bullet(\mathrm{id}_M \otimes \mathrm{id}_N) \neq 0$.

Double complexes $M_{\bullet\bullet}$ have differentials $d' : M_{i\bullet} \to M_{(i-1)\bullet}$ and $d'' : M_{\bullet j} \to M_{\bullet(j-1)}$ with $d'd'' + d''d' = 0$; the associated "total complex" is $\operatorname{Tot}_\bullet(M_{\bullet\bullet})$ with $\operatorname{Tot}_n := \oplus_{i+j=n} M_{ij}$ and $d := d' + d''$.

8.3. **Mapping cones.** $f : M_\bullet \to N_\bullet \rightsquigarrow$ *mapping cone* $\operatorname{Cone}(f)_\bullet := \operatorname{Tot}(M_\bullet \to N_\bullet)$ where the complexes $M_\bullet$ and $N_\bullet$ sit in row 1 and 0, respectively, with $d' := f$ and $d'' := (-d_M)/d_N$. Down to earth, this means that $\operatorname{Cone}(f)_\bullet := N_\bullet \oplus M_\bullet[1]$ with differential $d_{\text{Cone}} := \begin{pmatrix} d_N & f \\ 0 & -d_M \end{pmatrix}$; in particular, $0 \to N_\bullet \to \operatorname{Cone}(f)_\bullet \to M_\bullet[1] \to 0$ is an exact sequence of complexes (where each layer separately splits); the connecting homomorphism equals $\operatorname{H}_\bullet(f)$. The complex $\operatorname{Cone}(f)$ is exact $\Leftrightarrow f : M_\bullet \to N_\bullet$ is Qis.

*Note:* $M_\bullet[1] \hookrightarrow \operatorname{Cone}(f)$ and $\operatorname{Cone}(f) \twoheadrightarrow N_\bullet$ are *not* maps of complexes, i.e. they are not compatible with the respective differentials. In particular, the above sequence does not split as a sequence of complexes.

8.4. **Homotopies.** A *homotopy* $H : f \sim 0$ is a $H : M_\bullet \to N_\bullet[-1]$ (not compatible with $d$) with $Hd + dH = f$. Homotopies $H : 0 \sim 0$ are degree one morphisms of complexes.

$K^{(+/-/b)}(\mathcal{A}) :=$ homotopy category of bounded (from below, above, or both) $\mathcal{A}$-complexes with

$$\mathrm{Hom}_K(M,N) := \{\text{maps of complexes}\}/\text{homotopy} = \mathrm{H}_0\,\mathrm{Hom}_\bullet(M,N)$$

$\rightsquigarrow$ *homotopy equivalences* ($f : M_\bullet \to N_\bullet$ and $g : N_\bullet \to M_\bullet$ with $gf \sim \mathrm{id}_M$ and $fg \sim \mathrm{id}_N$) become isomorphisms in $K(\mathcal{A}) \rightsquigarrow$ Qis's:

**Proposition 29.** 1) *$f \sim 0 \Rightarrow \mathrm{H}_\bullet(f) = 0$. Thus, $\mathrm{H}_0 : K(\mathcal{A}) \to \mathcal{A}$ makes sense.*
2) *Let $P_\bullet \in K^-(\mathrm{proj}\,\mathcal{A}) \subseteq K^-(\mathcal{A})$ "projective" (i.e. all $P_i$ are projective) and $C_\bullet \in K(\mathcal{A})$ exact $\Rightarrow$ every $f : P_\bullet \to C_\bullet$ is 0-homotopic.* □ week 14 (27)

8.5. **The** Hom **complex.** Let $M_\bullet, N_\bullet \in K^b(\mathcal{A})$; then we define the double complex $\mathrm{Hom}_{\bullet\bullet}(M_\bullet, N_\bullet)$ via $\mathrm{Hom}_{ij} := \mathrm{Hom}(M_{-i}, N_j)$. The ordinary Hom complex is obtained as $\mathrm{Hom}_\bullet(M_\bullet, N_\bullet) := \mathrm{Tot}\,\mathrm{Hom}_{\bullet\bullet}(M_\bullet, N_\bullet)$, i.e. $\mathrm{Hom}_n(M_\bullet, N_\bullet) = \oplus_j \mathrm{Hom}(M_{j-n}, N_j)$ with $d(\varphi) = d_N\,\varphi - \varphi\, d_M$. In particular, $\mathrm{Z}_n(\mathrm{Hom}_\bullet)$ is the set of degree $n$ homomorphisms of complexes.

For $f : M_\bullet \to N_\bullet$ and $A_\bullet \in K^b(\mathcal{A})$ the functor $\mathrm{Hom}_\bullet(A_\bullet, -)$ and the Cone construction commute; in particular, we obtain an exact sequence of complexes

$$0 \to \mathrm{Hom}_\bullet(A_\bullet, N_\bullet) \to \mathrm{Hom}_\bullet(A_\bullet, \mathrm{Cone}\,(f)_\bullet) \to \mathrm{Hom}_\bullet(A_\bullet, M_\bullet[1]) \to 0.$$

(Note that one has to be more careful with unbounded complexes; direct sums might be to replaced by direct products...)

8.6. **Projective resolutions become canonical.** Assume that the abelian category $\mathcal{A}$ has *enough projectives*, i.e. every object attracts a surjection from a projective one. Then, in $K^-(\mathcal{A})$ there exist unique and functorial projective resolutions (similar with injective resolutions in $K^+(\mathcal{A})$):

**Proposition 30.** 1) *Let $P_\bullet \in K^-(\mathrm{proj}\,\mathcal{A})$ be "projective" and $A_\bullet \stackrel{q}{\to} B_\bullet$ be a Qis in $K(\mathcal{A}) \Rightarrow q$ induces an isomorphism $\mathrm{Hom}_{K(\mathcal{A})}(P_\bullet, A_\bullet) \stackrel{\sim}{\to} \mathrm{Hom}_{K(\mathcal{A})}(P_\bullet, B_\bullet)$.*
2) *Each $M_\bullet \in K^-(\mathcal{A})$ admits a unique projective resolution $P_\bullet \stackrel{\mathrm{qis}}{\to} M_\bullet$. This construction yields a functor $K^-(\mathcal{A}) \to K^-(\mathrm{proj}\,\mathcal{A})$ transforming Qis' into isomorphisms.*

*Proof.* 1) Since $q$ is a Qis, the complex $\mathrm{Cone}\,(q)$ is exact, i.e. for all $n \in \mathbb{Z}$ we have $\mathrm{H}_n(\mathrm{Hom}_\bullet(P_\bullet, \mathrm{Cone}\,(q)) = \mathrm{Hom}_{K(\mathcal{A})}(P_\bullet, \mathrm{Cone}\,(q)[n]) = 0$ by Proposition 29(2). Using the exact sequence of (8.5), this means that $\mathrm{Hom}_\bullet(P_\bullet, A_\bullet) \to \mathrm{Hom}_\bullet(P_\bullet, B_\bullet)$ is a qis.

2) Let $f_\bullet : P_\bullet \stackrel{\mathrm{qis}}{\to} M_\bullet$ for $< i$, and $f_i : P_i \to M_i$ inducing a surjective $\ker(P_i \to P_{i-1}) \twoheadrightarrow \ker(M_i \to M_{i-1})$. Then, one lifts $P'_{i+1} \twoheadrightarrow f_i^{-1}\big(\mathrm{im}(M_{i+1} \to M_i)\big) \cap Z_i(P_\bullet) \to \to \mathrm{im}(M_{i+1} \to M_i)$ toward $M_{i+1}$. Hence, $P_{i+1} := P'_{i+1} \oplus P''_{i+1}$ with surjective $P''_{i+1} \to \to \ker(M_{i+1} \to M_i)$.

$$\begin{array}{ccc} P_\bullet & \xrightarrow{\text{qis}} & M_\bullet \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\ P'_\bullet & \xrightarrow{\text{qis}} & M'_\bullet \end{array}$$

(i) For a given $f$ and for given resolutions $P_\bullet \to M_\bullet$ and $P'_\bullet \to M'_\bullet$, there exists a unique $F$ in $K^-(\operatorname{proj}\mathcal{A})$.

(ii) If $f = \mathrm{id}$ (or $f = \mathrm{qis}$) then $F$ is a qis, too. Its inverse within $K^-(\mathcal{A})$ can be obtained via $\mathrm{Hom}_K(P', P) \xrightarrow{\sim} \mathrm{Hom}_K(P', P') \ni \mathrm{id}$.

*Why is $G \mapsto \mathrm{id}_{P'}$ inverse to $F$?* By definition, we know that $F \circ G = \mathrm{id}_{P'}$. In particular, $G$ is a qis. This yields

$$\mathrm{Hom}_K(P,P) \xrightarrow[\sim]{F} \mathrm{Hom}_K(P,P') \xrightarrow[\sim]{G} \mathrm{Hom}_K(P,P) \xrightarrow[\sim]{F} \mathrm{Hom}_K(P,P').$$

$$\Phi \qquad\qquad \mathrm{id}_{P'}$$

Thus, since we already know that the horizontal maps in the previous line are isomorphisms, $\mathrm{Hom}_K(G) = \mathrm{Hom}_K(F)^{-1}$, hence, for the map $\Phi : \mathrm{Hom}_K(P,P) \to \mathrm{Hom}_K(P,P)$ we get,

$$\mathrm{Hom}_K(GF) = \mathrm{Hom}_K(G) \circ \mathrm{Hom}_K(F) = \mathrm{id}\,.$$

These two incarnations of $\Phi$, however, send $\mathrm{id}_P$ to $GF = \mathrm{id}_P$, respectively. □

## 9. Tor(sion) and Ext(ensions)

Every object $M \in \mathcal{A}$ gives rise to a complex supported on the 0-th spot only. Then, for a complex $P_\bullet = [\dots P_2 \to P_1 \to P_0]$, a quasiisomorphism $P_\bullet \to M$ is equivalent to an exact sequence $\dots P_2 \to P_1 \to P_0 \to M \to 0$.

9.1. **Derived functors.** Let $\mathcal{A}$ a be an abelian category with *enough projectives* and let $F : \mathcal{A} \to \mathcal{B}$ be an (additive) right exact functor, e.g. $F = (\otimes_R N) : \mathcal{M}od_R \to \mathcal{M}od_R$. Then, the *derived functors* $\mathrm{L}_i F : \mathcal{A} \to \mathcal{B}$ $(i \geq 0)$ are charcterized by (i) $\boxed{\mathrm{L}_0 F = F}$, (ii) $\boxed{\mathrm{L}_{\geq 1} F(\text{projective}) = 0}$, and (iii) $[0 \to M' \to M \to M'' \to 0] \mapsto$ [natural transformation $\mathrm{L}_i F(M'') \to \mathrm{L}_{i-1} F(M')$ with $\boxed{\text{long exact homology sequence}}$]. In particular, $\mathrm{L}_{\geq 1}(\text{exact } F) = 0$.

*Construction*: $P_\bullet \to M$ projective resolution $\rightsquigarrow \mathrm{L}_i F(M) := \mathrm{H}_i\big(F(P_\bullet)\big)$.

(Proof of (iii): Projective resolutions $P'_\bullet \overset{\text{qis}}{\to} M'$ and $P_\bullet \overset{\text{qis}}{\to} M \rightsquigarrow f : P'_\bullet \to P_\bullet \rightsquigarrow \operatorname{Cone}(f) \overset{\text{qis}}{\to} \operatorname{Cone}(M' \to M) \overset{\text{qis}}{\to} M''$; now take the long exact homology sequence for $F\big(0 \to P_\bullet \to \operatorname{Cone}(f) \to P'_\bullet[1] \to 0\big)$.

*The overall picture*: $M_\bullet \in K^-(\mathcal{A})$ with projective resolution $K^-(\operatorname{proj}\mathcal{A}) \ni P_\bullet \xrightarrow{\text{qis}} M_\bullet \Rightarrow \mathbb{L}F(M_\bullet) := F(P_\bullet) \in K^-(\mathcal{B})$. There is a natural transformation $\mathbb{L}F \to F$, and $\mathbb{L}_i F M_\bullet := \mathrm{H}_i(\mathbb{L}F M_\bullet)$. If $f : M_\bullet \overset{\text{qis}}{\to} N_\bullet$ is a qis, then, in contrast to $F(f)$, the map $\mathbb{L}F(f)$ preserves this property. However, if $F$ is exact, then $F(P_\bullet) \to F(M_\bullet)$ stays a qis, hence $\mathbb{L}F \to F$ is a qis, too.

9.2. Tor **and** Ext **as derived functors.** $\mathrm{Tor}^R(\bullet, N) := (\otimes^{\mathbb{L}}_R N)$, $\mathrm{Ext}_R(\bullet, N) := \mathbb{R}\,\mathrm{Hom}_R(\bullet, N)$; example: $R = \mathbb{Z}$; compatibility of $\mathrm{Tor}_i^R$ with flat base change $R \to S$ ($P_\bullet \to M$ yields projective $S$-resolution $P_\bullet \otimes_R S \to M \otimes_R S$) – and similarly for $\mathrm{Ext}^i_R$, if the $P_i$ are of finite presentation. Moreover, one can choose the argument to resolve ($\rightsquigarrow$ usage of $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ for Ext):

week 15 (29)

**Proposition 31.** *Let* $P_\bullet \xrightarrow{\mathrm{qis}} M$, $Q_\bullet \xrightarrow{\mathrm{qis}} N$, *and* $N \xrightarrow{\mathrm{qis}} I^\bullet$ *be projective and injective resolutions, respectively. Then,* $\mathrm{Tor}_i^R(M, N) = \mathrm{H}_i(P_\bullet \otimes_R N) = \mathrm{H}_i(M \otimes_R Q_\bullet)$ *and* $\mathrm{Ext}^i_R(M, N) = \mathrm{H}^i\,\mathrm{Hom}(P_\bullet, N) = \mathrm{H}^i\,\mathrm{Hom}(M, I^\bullet)$.

*Proof.* The first equalities are the definitions; for the second check the properties (i)-(iii) from (9.1). ☐

9.3. **Yoneda's Extensions.** $\mathrm{Ex}^1_R(M, N) := \{0 \to N \to \bullet \to M \to 0\}/\mathrm{isom} \rightsquigarrow$ provides a bifunctor on $\mathcal{A}^{\mathrm{opp}} \times \mathcal{A}$ ($m : M' \to M$ induces $0 \to N \to \bullet \times_M M' \to M' \to 0$; similarily for $n : N \to N'$) with $R$-algebra structure (addition via doubling the sequence and additional application of $M \to M \oplus M$ and $N \oplus N \to N$).

**Proposition 32.** $\mathrm{Ext}^1_R(M, N) \xrightarrow{\sim} \mathrm{Ex}^1_R(M, N)$ *as* $R$*-modules.*

*Proof.* $M \twoheadleftarrow P_0$ projective $\rightsquigarrow$ $(*)$ $0 \to K \to P_0 \to M \to 0$. With $\mathrm{Hom}(P_0, N) \to \mathrm{Hom}(K, N) \to \mathrm{Ext}^1_R(M, N) \to 0$ let $\mathrm{Hom}(K, N) \ni p \mapsto p_*(*)$. ☐

## 10. Flatness and Syzygies

**10.1.** [$M$ projective $\Leftrightarrow \mathrm{Ext}^1_R(M, \bullet) = 0$] and [$N$ flat $\Leftrightarrow \mathrm{Tor}_1^R(\bullet, N) = 0$].

**Proposition 33.** *Let* $N$ *be an* $R$*-module of finite presentation. Then,* $N$ *is projective* $\Leftrightarrow$ $N$ *is flat* $\Leftrightarrow \forall \mathfrak{m} \in \mathrm{MaxSpec}\, R$: $\mathrm{Tor}_1^R(R/\mathfrak{m}, N) = 0$.

*Proof.* Projectivity can be checked locally, $\mathrm{Tor}_i^R$ commutes with localization $\rightsquigarrow$ w.l.o.g.. $(R, \mathfrak{m})$ is a local ring. Copy (8.1): $R^n \twoheadrightarrow N$ minimal $\rightsquigarrow$ Nakayama. ☐

If $0 \to K \to P_{n-1} \to \ldots \to P_0 \to N \to 0$ is exact (with projective $P_i$) $\Rightarrow$ $\mathrm{L}_i F(K) = \mathrm{L}_{i+n} F(N)$ for $i \geq 1$. In particular, it follows for finitely generated $N$ over noetherian rings $R$: If $\mathrm{Tor}^R_{n+1}(R/\mathfrak{m}, N) = 0$ (for all $\mathfrak{m}$), then $K$ is projective, i.e. $\mathrm{pd}(N) \leq n$.

**Corollary 34** (Hilbert syzygy theorem)**.** *Every finitely generated* $\mathbb{C}[x_1, \ldots, x_n]$*-module has a projective* <u>*resolution of length* $n$</u>*, i.e. its projective dimension is* $\leq n$.

*Proof.* The *Koszul complex* of (10.2) (e.g. $\mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{x}]^2 \to \mathbb{C}[\mathbf{x}]$ for $n = 2$) provides a free resolution of length $n$ of $\mathbb{C}[\mathbf{x}]/\mathfrak{m} \cong \mathbb{C}$. ☐

10.2. **The Koszul complex.** Over $\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1, \ldots, x_n]$, we construct a free resolution of $\mathbb{C} = \mathbb{C}[\mathbf{x}]/(\mathbf{x})$: For $p \in \mathbb{N}$ let

$$K^p := \Lambda^p\, \mathbb{C}[\mathbf{x}]^n = \oplus_{1\le i_1<\ldots<i_p\le n}\, \mathbb{C}[\mathbf{x}] \cdot e_{i_1} \wedge \ldots \wedge e_{i_p} = \oplus_{\underline{i}}\, \mathbb{C}[\mathbf{x}] \cdot e(\underline{i})$$

and $d : K^p \to K^{p+1}$ be the wedge product $\wedge(\sum_{\nu=1}^n x_\nu e_\nu)$. The complex is $\mathbb{Z}^n$-graded by $\deg(\mathbf{x}^r \in \mathbb{C}[\mathbf{x}]) := r$ and $\deg e_i := -e_i$, i.e. $\deg(e(\underline{i})) = -\sum_{v=1}^p e_{i_v}$. Then, if $r_1, \ldots, r_\ell \ge 0$ and $r_{\ell+1} = \ldots = r_n = -1$, the degree $r$ part of $K^\bullet$ equals $\mathbf{x}^r \cdot \boxed{\Lambda^{\bullet - n+\ell}\, \mathbb{C}^\ell} \otimes_{\mathbb{C}} \Lambda^{n-\ell}\, \mathbb{C}^{n-\ell}$ with $\mathbb{C}^\ell$-basis $f_\nu := x_\nu e_\nu$ and differential $d : \Lambda^p \mathbb{C}^\ell \to \Lambda^{p+1}\mathbb{C}^\ell$ equal to $\wedge(\sum_{\nu=1}^\ell f_\nu)$, for the first factor, and where the second factor $\Lambda^{n-\ell}\,\mathbb{C}^{n-\ell} = \mathbb{C} \cdot e(\ell+1, \ldots, n)$ does not matter at all.
If $\ell \ge 1$, then $h : \Lambda^{p+1}\mathbb{C}^\ell \to \Lambda^p\mathbb{C}^\ell$ with $h(e_1 \wedge e_{i_1} \wedge \ldots \wedge e_{i_p}) := e_{i_1} \wedge \ldots \wedge e_{i_p}$ (and 0 otherwise) provides a homotopy $\mathrm{id} \sim^h 0$. If $\ell = 0$ then $K^\bullet(r)$ is concentrated in $K^n(r) = \mathbb{C}$ and provides an isomorphism from this to $\mathbb{C} = \mathbb{C}[\mathbf{x}]/(\mathbf{x})$.

10.3. **No finite generation.** Flatness encodes "continuity" of families $\operatorname{Spec} S \to \operatorname{Spec} R$. (*Example*: Flat projection $R = \mathbb{C}[t] \hookrightarrow \mathbb{C}[x,t]/(x^2 - t) = S$ of the parabola and the non-flat projection $\mathbb{C}[t] \to \mathbb{C}[x,t]/(tx - t)$; comparison of the fibers in $\pm 1$, 0 (and in the generic point $\eta$) in both cases – also over $\mathbb{R}$.) Higher dimension of the fibers $\rightsquigarrow$ the occuring modules (e.g. $S$ over $R$) are no longer finitely generated! week 1 (31)

**Proposition 35.** *Let $N$ be an $R$-module. Then, $N$ is flat $\Leftrightarrow \operatorname{Tor}_1^R(R/I, N) = 0$ for all finitely generated ideals $I \subseteq R$*

*Proof.* ($\Leftarrow$) $M' \subseteq M \rightsquigarrow$ it suffices to test injectivity of $M' \otimes_R N \to M \otimes_R N$ only for finitely generated $M', M$: $x = \sum_i m_i \otimes n_i \in M' \otimes_R N \Rightarrow$ for $x \mapsto 0$ only finitely many bilinear relations in $M \otimes_R N$ are used. Thus, using filtrations, everything can be reduced to $I \subseteq R$, and again the finitely generated ideals suffice. □

*Applications:* 1) A $k[\varepsilon]/\varepsilon^2$-module $N$ is flat $\Leftrightarrow N/\varepsilon N \xrightarrow{\cdot\varepsilon} \varepsilon N$ is (also) injective. Identifying $k[\varepsilon]/\varepsilon^2$-modules $N$ with pairs $(V, \varphi)$ consisting of a $k$-vector space $V$ and $\varphi \in \operatorname{End}_k(V)$ with $\varphi^2 = 0$, i.e. with $\operatorname{im}\varphi \subseteq \ker\varphi$, then $(V, \varphi)$ is flat iff $\operatorname{im}\varphi = \ker\varphi$.
2) $R$ = domain $\rightsquigarrow$ [flat $\Rightarrow$ torsion free]; $R$ = principal ideal domain $\rightsquigarrow$ "$\Leftrightarrow$".
(Counter) *examples:* $\mathbb{Z}/2\mathbb{Z}$ is a flat (even projective) $\mathbb{Z}/6\mathbb{Z}$-module. The ideal $(x, y) \subset k[x, y]$ is torsion free, but not flat.

## 11. Graded rings and modules

11.1. **Graded rings and modules.** $\mathbb{Z}$ or more general abelian grading groups $A$; example $S = k[\mathbf{x}]$; homogeneous ideals and graded submodules; shifts $M(d)$ or $S(d)$; homogeneous resolutions.
*Example:* $(xz - y^2, wy - z^2, xw - yz)$, using $w(xz - y^2) + y(wy - z^2) + z(xw - yz) = 0$, with respect to the usual $\mathbb{Z}$-grading or to $\deg(x, y, z, w) := (1, i)$ with $i = 1, 2, 3, 4$.

week 2 (33) 11.2. **Homogenization.** $w \in \mathbb{R}^n_{\geq 0} \rightsquigarrow \deg_w x_i := w_i$ defines a grading on $k[\mathbf{x}]$; *homogenization*: $f \in k[\mathbf{x}] \rightsquigarrow k[t,\mathbf{x}] \ni f^h(t,\mathbf{x}) := t^{\deg_w f} f(t^{-w}\mathbf{x}) = \operatorname{in}_w f + t \cdot \text{remainder}$; with $\deg t := 1$ the $f^h$ becomes homogeneous of degree $\deg_w f$; *dehomogenization* $f^h(1,\mathbf{x}) = f(\mathbf{x})$. For $a + \deg f = \deg g$ one has $t^a f^h + g^h = t^\bullet (f+g)^h$. This follows from $F(1,\mathbf{x})^h \cdot t^\bullet = F(t,\mathbf{x})$ for homogeneous $F(t,\mathbf{x})$.

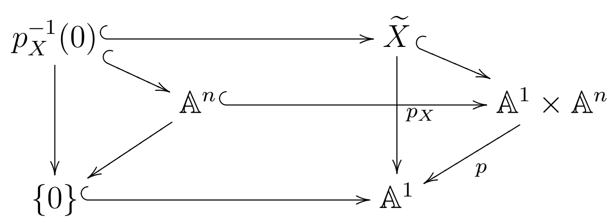
If $I \subseteq k[\mathbf{x}]$ is an ideal and $\leq_w$ is a term order breaking ties for $\deg_w \Rightarrow \operatorname{in}_w I$ is generated by $\operatorname{in}_w\{\leq_w\text{-GB of } I\}$; $I^h := (f^h \mid f \in I)$ is a homogeneous ideal; substituting $t \mapsto 1$ yields $I^h \mapsto I$.

*Example*: $w = \underline{1}$ and $I = (y - x^2, z - x^2)$ (GB for $y, z > x^2$ but not for $x^2 > y, z$; the latter requires $y - z$) yields $I^h = (yt - x^2, zt - x^2, y - z)$.

**Lemma 36.** *Let* $I = (f_1, \ldots, f_k)$. *Then* $I^h = ((f_1^h, \ldots, f_k^h) : t^\infty) = (I^h : t^\infty)$. *If* $\{f_1, \ldots, f_k\} =$[$\leq_w$-Gröbner basis], *then* $(f_1^h, \ldots, f_k^h)$ *is already* $t$*-saturated.*

*Proof.* $g(t,\mathbf{x})$ homogeneous with $t^\ell g \in I^h \Rightarrow g(1,\mathbf{x}) \in I \Rightarrow g(t,\mathbf{x}) = t^\bullet g(1,\mathbf{x})^h \in I^h$. Alternatively, $g(1,\mathbf{x}) = \sum_i \lambda_i(\mathbf{x}) f_i(\mathbf{x}) \Rightarrow \exists\, k, k_i \geq 0:\ t^k g(1,\mathbf{x})^h = \sum_i t^{k_i} \lambda_i^h f_i^h \Rightarrow g \in ((f_1^h, \ldots, f_k^h) : t^\infty)$. If $\{f_i\} = \text{GB}$, then $\operatorname{in}_{\leq_w}(\lambda_i f_i) \leq \operatorname{in}_{\leq_w} g(1,\mathbf{x}) \Rightarrow \deg_w \lambda_i + \deg_w f_i \leq \deg_w g(1,\mathbf{x}) \Rightarrow k = 0$ is possible, i.e. $g \in (f_1^h, \ldots, f_k^h)$. $\square$

11.3. **Gröbner degenerations understood as flat families.** $w \in \mathbb{R}^n_{\geq 0} \rightsquigarrow X := \operatorname{Spec} k[\mathbf{x}]/I \subseteq \mathbb{A}^n$, $\widetilde{X} := \operatorname{Spec} k[t,\mathbf{x}]/I^h \subseteq \mathbb{A}^1 \times \mathbb{A}^n \xrightarrow{p} \mathbb{A}^1$




$\boxed{p_X \text{ is flat}}$ since $k[t,\mathbf{x}]/I^h$ is a flat $k[t]$-module $\Leftrightarrow$ $t$-torsion free $\Leftrightarrow I^h = (I^h : t^\infty)$ and $p_X^{-1}(\mathbb{A}^1 \setminus 0) \cong X \times (\mathbb{A}^1 \setminus 0)$ via the $k[t^{\pm 1}]$-linear $\begin{array}{ccc} k[t^{\pm1},\mathbf{x}]/I & \xrightarrow{\sim} & k[t^{\pm1},\mathbf{x}]/I^h \\ \mathbf{x}, f & \mapsto & t^{-w}\mathbf{x},\ t^{-\deg f} f^h; \end{array}$
$p_X^{-1}(0) = \operatorname{Spec} k[t,\mathbf{x}]/((t) + I^h) = \operatorname{Spec} k[\mathbf{x}]/\operatorname{in}_w(I)$.

11.4. **Limits.** The punctured $p_X^{-1}(\mathbb{A}^1 \setminus 0) \to (\mathbb{A}^1 \setminus 0)$ is a trivial family. Moreover, by Problem 57(c), $\overline{p_X^{-1}(\mathbb{A}^1 \setminus 0)} = \overline{V(I^h) \setminus V(t)} = V(I^h : t^\infty) = V(I^h)$; hence $X_0 := p_X^{-1}(0) = \boxed{\text{“}\lim_{t \to 0}\text{”}}\, p_X^{-1}(t)$.

$I = (f_1, \ldots, f_k)$ Gröbner basis $\Rightarrow X = V(f_i)$, $\widetilde{X} = V(f_i^h)$ and $X_0 = V(\operatorname{in}_w f_i)$. For non-GB we just have $X_0 \subseteq V(\operatorname{in}_w f_i)$.

*Example:* $I = (x - z, y - z)$, $w = (0,0,1) \Rightarrow V(tx - z, ty - z) = k \cdot (1,1,t)$ over $\mathbb{A}^1 \setminus 0$, but has the 0-fiber $V(z)$ which is bigger than the wanted $V(z, x - y)$.

Different term orders yield different degenerations: See [Eis, S.342-347]. This motivates the usage of non-reduced, 0-dimensional schemes.

11.5. **Artin-Rees.** $A$ noetherian; $I \subseteq A$ ideal $\rightsquigarrow \widetilde{A} := \oplus_{\nu\geq 0} I^\nu$ is a finitely generated $A$-algebra $\Rightarrow$ noetherian, too. $M =$ finitely generated $A$-module with "$I$-*filtration*", i.e. $\{M_\nu\}_{\nu\geq 0}$ with $IM_\nu \subseteq M_{\nu+1} \subseteq M_\nu$ (Example: $M_\nu = I^\nu M$) $\rightsquigarrow \widetilde{M} := \oplus_{\nu\geq 0} M_\nu$ is a graded $\widetilde{A}$-module.

**Proposition 37.** *$\widetilde{M}$ is noetherian $\Leftrightarrow$ $M_{\nu+1} = IM_\nu$ for $\nu \gg 0$ ("$I$-stable").*

*Proof.* ($\Rightarrow$) $M^k := (\oplus_{\nu\leq k} M_\nu) \oplus (\oplus_{\nu\geq 1} I^\nu M_k)$ is an ascending chain in $\widetilde{M}$. □

**Corollary 38.** (1) *$M' \subseteq M \Rightarrow I(I^\nu M \cap M') = I^{\nu+1}M \cap M'$ for $\nu \gg 0$, i.e. $\exists c$: $I^k M' \supseteq I^k(I^c M \cap M') = I^{k+c} M \cap M' \supseteq I^{k+c} M'$ for $k \geq 0$ ("Artin-Rees").*
(2) *$1 + I \subseteq A^*$ (e.g. $I = \mathfrak{m}$ in a local ring) $\Rightarrow \bigcap_{k\geq 0} I^k M = 0$.*

*Proof.* (1) $\widetilde{M'} = \oplus_\nu (I^\nu M \cap M') \subseteq \oplus_\nu I^\nu M = \widetilde{M}$ is a noetherian $\widetilde{A}$-module.
(2) follows from (1) with $M' := \bigcap_k I^k M$ and Nakayama. □

↗ §16

11.6. **The local criterion of flatness.** A homomorphism of local rings $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is called *local* :$\Leftrightarrow \varphi(\mathfrak{m}) \subseteq \mathfrak{n} \Leftrightarrow \varphi^\#(\mathfrak{n}) = \mathfrak{m}$. Counter *example:* $\mathbb{C}[x]_{(x)} \hookrightarrow \mathbb{C}(x)$.

**Proposition 39** (Local criterion of flatness)**.** *Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphismus of noetherian rings, let $N$ be a finitely generated $S$-module. Then $N$ is flat over $R$ $\Leftrightarrow$ $\mathrm{Tor}_1^R(R/\mathfrak{m}, N) = 0$.*

*Proof.* Let $I \subseteq R$ be an ideal and $I \otimes_R N \ni u \mapsto 0 \in IN \subseteq N$; we show that $u = 0$: $I \otimes_R N$ is a finite $S$-Modul, and $\mathfrak{m}^a(I \otimes_R N) \subseteq \mathfrak{n}^a(I \otimes_R N) \Rightarrow \bigcap_a \mathfrak{m}^a(I \otimes_R N) = 0$; Artin-Rees $\rightsquigarrow \mathfrak{m}^{a'\gg a} \cap I \subseteq \mathfrak{m}^a I \Rightarrow$ it suffices to show that $u$ is contained, for all $a' \in \mathbb{N}$, in the image of $(\mathfrak{m}^{a'} \cap I) \otimes_R N \to I \otimes_R N$ i.e. that $u$ vanishes in $I/(\mathfrak{m}^{a'} \cap I) \otimes_R N$.

$$\begin{array}{ccccc} u & & I \otimes_R N & \longrightarrow & I/(\mathfrak{m}^{a'} \cap I) \otimes_R N \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & N = R \otimes_R N & \longrightarrow & R/\mathfrak{m}^{a'} \otimes_R N \end{array}$$

On the other hand, the right hand column is injective since $\mathrm{Tor}_1^R(M, N) = 0$ for all $R$-modules $M$ of finite length – this follows via induction from the hypothesis. □

## 12. Hilbert polynomials

week 16 (31)

12.1. **Poincaré series.** $S_0$ noetherian ring; $\lambda$: {finitely generated $S_0$-modules} $\to \mathbb{N}$ additive. $S = \oplus_{\nu\geq 0} S_\nu$ finitely generated, graded $S_0$-algebra: $a_1, \ldots, a_n$ homogeneous generators with $\deg a_i = d_i \geq 1$. If $M =$ finitely generated, graded $S$-module $\Rightarrow$ "*Poincaré series*" $P(M, t) := \sum_{\nu\geq 0} \lambda(M_\nu) \cdot t^\nu \in \mathbb{N}[[t]]$ (cut off the negative part).

**Theorem 40** (Hilbert-Serre)**.** $\prod_{i=1}^n (1 - t^{d_i}) \cdot P(M, t) \in \mathbb{Z}[t]$.

*Proof.* $n = 0 \Rightarrow P(M,t) \in \mathbb{N}[t]$. In general: $K, L :=$ kernel/cokernel of $M \xrightarrow{\cdot a_n} M$ $\Rightarrow \lambda(K_\nu) - \lambda(M_\nu) + \lambda(M_{\nu+d_n}) - \lambda(L_{\nu+d_n}) = 0$, hence

$$t^{d_n} P(K,t) - t^{d_n} P(M,t) + P(M,t) - P(L,t) = \sum_{v=0}^{d_n-1} (\lambda(M_v) - \lambda(L_v)) t^v =: g \in \mathbb{N}[t]$$

$\Rightarrow (1 - t^{d_n}) P(M,t) = P(L,t) - t^{d_n} P(K,t) + g(t)$. And, since $a_n$ annhilates the modules $K$, $L$, they are modules over $S_0[a_1, \dots, a_{n-1}] \subseteq S$. □

12.2. **Pole orders.** $d(M) :=$ [pole order of $P(M,t)$ in $t = 1$] $\leq n$. On the other hand, $d(M) \leq 0$ indicates that $M$ does $\lambda$-live only in finitely many degrees: $P(M,t) \cdot \prod_i (\sum_{v=0}^{d_i-1} t^v) \in (1-t)^{-d(M)} \mathbb{Z}[t] \subseteq \mathbb{Z}[t]$ enforces that $P(M,t) \in \mathbb{N}[t]$. From $d(M) < 0$ it even follows that $P(M,t) = 0$.

*Example:* $P(k[x_1, \dots, x_n], t) = \sum_\nu \binom{\nu+n-1}{n-1} t^\nu = 1/(1-t)^n$ (this easily follows via the $\mathbb{Z}^n$ grading and $\sum_{r \in \mathbb{N}^n} t^r = \prod_i \sum_{k \geq 0} t_i^k) \Rightarrow d(k[x_1, \dots, x_n]) = n$.

**Proposition 41.** *If $a \in S$ is a non-zero divisor of $M$ with $\deg a \geq 1$, then $d(M/aM) = d(M) - 1$.*

*Proof.* $M \xrightarrow{\cdot a} M$ has $K = 0$, hence $(1 - t^{\deg a}) P(M,t) = P(M/aM, t) + g(t)$ with $g \in \mathbb{N}[t]$. In the case $d(M/aM) = 0$ it first follows that $d(M) \leq 1$ and $P(M/aM, t) + g(t) \in \mathbb{N}[t]$. However, with $d(M) = 0$ one would additionally obtain that $P(M/aM, 1) + g(1) = 0$. □

12.3. **Numerical polynomials.** The coefficients $\lambda(M_\nu)$ of $P(M,t)$ themselves behave like polynomials in $\nu$ ("*Hilbert polynomial*"); $f \in \mathbb{R}[t]$ is called a *numerical polynomial* $:\Leftrightarrow f(g) \in \mathbb{Z}$ for sufficiently large $g \in \mathbb{Z} \Leftrightarrow f = \sum_{i=0}^{\deg f} c_i \binom{t}{i}$ with (uniquely determined) $c_i \in \mathbb{Z}$.
(($\Rightarrow$) via induction by $\deg f$: $g(t) := f(t+1) - f(t) = \sum_{i=0}^{\deg f - 1} c_{i+1} \binom{t}{i}$.)

**Proposition 42.** *Let $S$ be generated in degree 1 over $S_0$ ($d_i = 1$) $\Rightarrow$ for $\nu \gg 0$, one has $[\nu \mapsto \lambda(M_\nu)] = H_M(\nu) \in \mathbb{Q}[\nu]$ with $\deg H_M = d(M) - 1$.*

*Proof.* $P(M,t) = f(t)/(1-t)^{d(M)} = f(t) \cdot \sum_{k \geq 0} \binom{d+k-1}{d-1} t^k \Rightarrow$ with $f(t) = \sum_{k=0}^N a_k t^k$ we have $\lambda(M_\nu) = \sum_{k=0}^N a_k \binom{d+\nu-k-1}{d-1}$ for $\nu \geq N$. Since $\sum_k a_k = f(1) \neq 0$, the coefficients of $v^{d-1}$ do not cancel each other. □

*Example*: $H_{k[x_0, \dots, x_n]}(v) = \binom{v+n}{n} = 1/n!\, v^n + \dots$ and, for a homogeneous $f \in k[\mathbf{x}]_d$, $H_{k[\mathbf{x}]/f}(v) = \binom{v+n}{n} - \binom{v+n-d}{n} = d/(n-1)!\, v^{n-1} + \dots$. In particular, for $S = k[\mathbf{x}]/I$, the *degree* $\deg(S) := \deg(H_S)! \cdot$ [leading coefficient of $H_S$] generalizes the degree of a polynomial.

## 13. Dimension of local rings

13.1. **$\mathfrak{m}$-primary ideals.** $(A, \mathfrak{m})$ noetherian local ring, $\mathfrak{m}^r \subseteq Q \subseteq \mathfrak{m}$ ($\mathfrak{m}$-primary) ideal $\Rightarrow S := \mathrm{Gr}_Q(A) := \oplus_{\nu \geq 0} Q^\nu / Q^{\nu+1}$ with $S_0 = A/Q$ (artinian) and $\lambda := \ell =$ length.

**Proposition 43.** *$M$ finitely generated $A$-module $\Rightarrow \nu \mapsto g(\nu) := \ell(M/Q^\nu M) < \infty$ equals a polynomial $\chi_Q^M \in \sum_{i=0}^n \mathbb{Z}\binom{\nu}{i}$ of degree $d(\mathrm{Gr}_Q(M)) \le n := \#\{Q\text{-generators}\}$ for $\nu \gg 0$.*

*Proof.* $\mathrm{Gr}_Q(M) := \oplus_{\nu\ge 0} Q^\nu M/Q^{\nu+1}M$ is a finitely and in deg $=1$ generated $\mathrm{Gr}_Q(A)$-module; Proposition 42 $\rightsquigarrow g(\nu+1) - g(\nu) = \ell(\mathrm{Gr}_Q^\nu(M))$ is a polynomial of degree $< n$. □

$\boxed{d(A) := \deg \chi_Q^A} = d(\mathrm{Gr}_Q(A)) - 1 + 1$ does not depend on $Q$: $\mathfrak{m}^r \subseteq Q \subseteq \mathfrak{m} \Rightarrow \chi_\mathfrak{m}^A(\nu) \le \chi_Q^A(\nu) \le \chi_\mathfrak{m}^A(r\nu)$ for $\nu \gg 0$. Hence $\boxed{d(A) \le \delta(A) := \min_Q \#\{Q\text{-generators}\}}$.

*Example:* $A := k[x_1, \ldots, x_n]_{(\mathbf{x})} \hookrightarrow k[|\mathbf{x}|] =: \hat{A}$ have both $\mathrm{Gr}_{(\mathbf{x})}(A) = \mathrm{Gr}_{(\mathbf{x})}(\hat{A}) = k[\mathbf{x}]$. Hence, $\chi_{(\mathbf{x})}(k) = \binom{k-1+n}{n}$; indeed, $H_{\mathrm{Gr}}(k) = \chi(k+1) - \chi(k) = \binom{k+n}{n} - \binom{k-1+n}{n} = \binom{k+n-1}{n-1}$. Thus, $d(\mathbb{A}^n, 0) = n$ and $\mathrm{mult}(\mathbb{A}^n, 0) = 1$ with $\mathrm{mult}(A) := d(A)! \cdot [\text{leading coefficient of } \chi_\mathfrak{m}^A]$.

13.2. **Hypersurfaces.** Let $a \in (A, \mathfrak{m})$ be a non-zero divisor for $M$. Comparable to Proposition 41 we obtain:

**Proposition 44.** $\deg \chi_\mathfrak{m}^{M/aM} \le \deg \chi_\mathfrak{m}^M - 1$*; in particular $d(A/aA) \le d(A) - 1$ for $M := A$.*

*Proof.* $M/\mathfrak{m}^\nu M \twoheadrightarrow M/(a + \mathfrak{m}^\nu M)$ yields $\chi_\mathfrak{m}^M(\nu) - \chi_\mathfrak{m}^{M/aM}(\nu) = \ell(aM/(aM \cap \mathfrak{m}^\nu M))$, and $\mathfrak{m}^\nu(aM) \subseteq aM \cap \mathfrak{m}^\nu M \overset{\mathrm{Cor38}}{=} \mathfrak{m}^{\nu-\nu_0}(aM \cap \mathfrak{m}^{\nu_0} M) \subseteq \mathfrak{m}^{\nu-\nu_0}(aM)$ implies $\chi_\mathfrak{m}(\nu - \nu_0) \le \ell(\ldots) \le \chi_\mathfrak{m}(\nu)$. Hence $\chi_\mathfrak{m}^M(\nu) - \chi_\mathfrak{m}^{M/aM}(\nu)$ is a polynomial of the same degree and with the same leading coefficient as $\chi_\mathfrak{m}^M$. □

*Example*: $f \in \mathfrak{m}^d \setminus \mathfrak{m}^{d+1}$ in $A$, then $\mathrm{in}(f) := \bar{f} \in \mathfrak{m}^d/\mathfrak{m}^{d+1} = \mathrm{Gr}_\mathfrak{m}^d(A)$; in particular, there is a natural surjection $\Phi : \mathrm{Gr}_\mathfrak{m}(A)/\,\mathrm{in}(f) \twoheadrightarrow \mathrm{Gr}_\mathfrak{m}(A/f)$. If $\mathrm{in}(f)$ is a non-zero divisor in $\mathrm{Gr}_\mathfrak{m}(A)$, then $\Phi$ is an isomorphism.
Hence, for $k[x_1, \ldots, x_n]_{(\mathbf{x})}/(f) \subseteq k[|\mathbf{x}|]/f$ (with $f = f_d + f_{d+1} + \ldots$ in the latter) we obtain $\chi(k) = \chi^{(\mathbb{A}^n,0)}(k) - \chi^{(\mathbb{A}^n,0)}(k-d) = \binom{k-1+n}{n} - \binom{k-1-d+n}{n}$. In particular, $d(k[\mathbf{x}]_{(\mathbf{x})}/(f) = n-1$ and $\mathrm{mult}(k[\mathbf{x}]_{(\mathbf{x})}/(f) = d$.

13.3. **Towers of primes.** "*Height*" of prime ideals $\rightsquigarrow$ "*Krull dimension*" $\dim A := \dim(\operatorname{Spec} A) := \max\{\operatorname{ht} P \mid P \in \operatorname{Spec} A\}$; $P \subseteq A$ is a *minimal* prime ideal $\Leftrightarrow \operatorname{ht} P = 0$.

**Proposition 45.** *$(A, \mathfrak{m})$ noetherian local ring $\Rightarrow$ $\boxed{\operatorname{ht}\mathfrak{m} =: \dim A \le d(A)}$. In particular, the height of prime ideals in noetherian local rings is always finite.*

*Proof.* $d(A) = 0 \Rightarrow \chi_\mathfrak{m}^A(\nu) = \ell(A/\mathfrak{m}^\nu)$ constant $\Rightarrow \mathfrak{m}^\nu = 0$ (Nakayama) for $\nu \gg 0$.
*Induction by $d(A)$:* $P_0 \subset \ldots \subset P_r$ chain of prime ideals, $a \in P_1 \setminus P_0 \Rightarrow \overline{A} := A/P_0 + (a)$ does still contain the chain $\overline{P}_1 \subset \ldots \subset \overline{P}_r$, and $d(\overline{A}) < d(A/P_0) \le d(A)$. □

week 13 (55)

**Theorem 46.** $(A, \mathfrak{m})$ *noetherian local ring* $\Rightarrow$ $\boxed{\dim(A) = d(A) = \delta(A)}$. *For non-zero divisors* $a \in A$ *one has* $\dim A/a = \dim A - 1$.

*Proof.* For $v \le \dim A$ construct inductively $(a_1, \ldots, a_\nu)$ with $[P \supseteq (a_1, \ldots, a_\nu) \Rightarrow \operatorname{ht} P \ge \nu]$: If $P_1, \ldots, P_N$ are the minimal primes over $(a_1, \ldots, a_{\nu-1})$ with $\operatorname{ht} P_i = \nu - 1 < \dim A \Rightarrow P_i \subset \mathfrak{m} \Rightarrow \exists a_\nu \in \mathfrak{m} \setminus \bigcup_i P_i$.
For $\nu = \dim A$ it follows that $Q := (a_1, \ldots, a_{\dim A})$ is $\mathfrak{m}$-primary $\rightsquigarrow \delta(A) \le \dim A$.
"$\ge$" (holding without the non-zero divisor assumption): $(\overline{a_1}, \ldots, \overline{a_d}) = \overline{\mathfrak{m}}$-primary in $A/aA \Rightarrow (a, a_1, \ldots, a_d) = \mathfrak{m}$-primary in $A$. □

## 14. Regular local rings

14.1. **Tangent cones.** $\dim(A, \mathfrak{m}) = d \rightsquigarrow Q = (a_1, \ldots, a_d)$ $\mathfrak{m}$-primary ("*parameter system*") $\rightsquigarrow \Phi : (A/Q)[x_1, \ldots, x_d] \twoheadrightarrow \operatorname{Gr}_Q A$. It holds true: $\Phi(f) = 0 \Rightarrow f \mapsto 0 \in (A/\mathfrak{m})[x_1, \ldots, x_d]$. (Otherwise, by Problem **??**, the (homogeneous) $f$ is a non-zero divisor, hence

$$d = d(\operatorname{Gr}_Q A) \le d((A/Q)[x_1, \ldots, x_d]/f) < d((A/Q)[x_1, \ldots, x_d]) = d.)$$

If $Q = \mathfrak{m}$ is possible, then $\Phi$ becomes an isomorphism!

**Definition 47.** $(A, \mathfrak{m})$ is "*regular*" $:\Leftrightarrow \operatorname{Gr}_{\mathfrak{m}}(A)$ is a polynomial ring $\Leftrightarrow \boxed{\dim \mathfrak{m}/\mathfrak{m}^2} = \dim A \overset{\text{Nakayama}}{\Longleftrightarrow} \mathfrak{m}$ is generated by $(\dim A)$ many elements.
(If $\operatorname{Gr}_{\mathfrak{m}}(A)$ is a polynomial ring, then $\#(\text{variables}) = d(\operatorname{Gr}_{\mathfrak{m}}(A)) = \dim(A)$.) Regular rings are automatically integral domains (is a consequence of Problem 65).

14.2. **Projective dimension of the residue field.** Regularity of rings can be tested homologically:

**Proposition 48.** $(A, \mathfrak{m})$ *is regular* $\Leftrightarrow \operatorname{Tor}^A_{\gg 0}(A/\mathfrak{m}, A/\mathfrak{m}) = 0 \Leftrightarrow$ *every finitely generated* $A$*-module admits a* $\boxed{\text{finite free resolution}}$.

*Proof.* The equivalence of the two right conditions follows from (10.1).
$(\Rightarrow)$ $\mathfrak{m} = (a_1, \ldots, a_d) \Rightarrow$ the *Koszul complex* is a free $A$-resolution of $k = A/\mathfrak{m}$ – this follows via $\operatorname{Gr}_{\mathfrak{m}}(A)$ from the corresponding result for polynomial rings in (10.2): $M' \to M \to M''$ with exact $\operatorname{Gr}_{\mathfrak{m}}(M') \to \operatorname{Gr}_{\mathfrak{m}}(M) \to \operatorname{Gr}_{\mathfrak{m}}(M'')$ (homogeneous maps of degree 1) $\Rightarrow \ker \cap \mathfrak{m}^i M \subseteq \operatorname{im} + \big(\ker \cap \mathfrak{m}^{i+1} M\big) \Rightarrow \exists i_0 : \forall i \ge i_0 : \ker \subseteq \operatorname{im} + (\ker \cap \mathfrak{m}^i M) = \operatorname{im} + \mathfrak{m}^{i-i_0}(\ker \cap \mathfrak{m}^{i_0} M) \subseteq \operatorname{im} + \mathfrak{m}^{i-i_0} \ker$.
$(\Leftarrow)$ $\mathfrak{m} \setminus \mathfrak{m}^2$ contains non-zero divisors $a$: Otherwise, by (3.6) and "prime avoidance" (Lemma 1), $\mathfrak{m} \in \operatorname{Ass}(A)$, i.e. $0 \to A/\mathfrak{m} \overset{\cdot s}{\to} A \to A/s \to 0 \Rightarrow \operatorname{Tor}^A_i(A/s, k) \overset{\sim}{\to} \operatorname{Tor}^A_{i-1}(k, k)$. Along the lines of (10.1), it follows that $0 = \operatorname{Tor}^A_{\operatorname{pd}(k)+1}(A/s, k) = \operatorname{Tor}^A_{\operatorname{pd}(k)}(k, k)$ which cannot be true.
$\dim A/a = \dim A - 1$; $\dim_k \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 = \dim_k \mathfrak{m}/(a + \mathfrak{m}^2) = \dim_k \mathfrak{m}/\mathfrak{m}^2 - 1 \rightsquigarrow$ induction: Let $F_\bullet \overset{\text{qis}}{\to} \mathfrak{m}$ be a finite, free $A$-resolution; since $\operatorname{H}_{\ge 1}(F_\bullet \otimes A/a) = \operatorname{Tor}^A_{\ge 1}(\mathfrak{m}, A/a) =$

$\mathrm{H}_{\geq 1}(\mathfrak{m} \stackrel{\cdot a}{\to} \mathfrak{m}) = 0$, the morphism $F_\bullet \otimes A/a \stackrel{\text{qis}}{\to} \mathfrak{m}/a\mathfrak{m}$ becomes a free $A/a$-resolution. The exact sequence $0 \to A/\mathfrak{m} \stackrel{\cdot a}{\to} \mathfrak{m}/a\mathfrak{m} \to \mathfrak{m}/a \to 0$ splits ($A/\mathfrak{m} \hookrightarrow \mathfrak{m}/a\mathfrak{m} \twoheadrightarrow \mathfrak{m}/\mathfrak{m}^2$ has a section), hence $\mathrm{Tor}^{A/a}_{\gg 0}(k,k) = 0$. □

**Corollary 49.** *Localizations of regular rings in prime ideals are regular.*

*Proof.* $\mathrm{Tor}_i^{A_P}(A_P/PA_P, A_P/PA_P) = \mathrm{Tor}_i^A(A/P, A/P) \otimes_A A_P = 0$ for $i \gg 0$. □

## 15. Global dimension

15.1. **Height vs. codimension.** Let $a_i \in A$ and $P \supseteq (a_1, \ldots, a_r)$ be a minimal prime ideal $\Rightarrow$ $\mathrm{ht}\, P \leq r$ (in $A_P$ the ideal $P$ is the only prime above $(a_1, \ldots, a_r)$; thus, the latter is $P$-primary).

**Proposition 50.** 1) *$a \in A$ non-zero divisor $\Rightarrow$ minimal prime ideals $P$ above $(a)$ have height* $\mathrm{ht}\, P = 1$ ("Krull principal ideal theorem").
2) $A =$ *integral domain* $\rightsquigarrow$ [*factorial* $\Leftrightarrow$ *prime ideals of height 1 are principal*].

*Proof.* 1) $\mathrm{ht}\, P = 0 \rightsquigarrow \dim A_P/a \leq \dim A_P - 1 = -1$.
2) Use "factorial" $\Leftrightarrow$ irreducible $f \in A$ yield prime ideals $(f)$: ($\Leftarrow$) $f \in A \Rightarrow$ a minimal $P \ni f$ has $\mathrm{ht} = 1$; ($\Rightarrow$) $\mathrm{ht}\, P = 1 \Rightarrow$ choose an irreducible $f \in P$. □

In particular, "factorial" implies "regular in codimension (height) one". The reversed implication fails: $\mathbb{C}[x,y,z]/(y^2 - xz)$.

15.2. **Krull dimension.** $\dim A := \max_{P \in \mathrm{Spec}\, A} \mathrm{ht}\, P = \dim A/\sqrt{0}$; if $P_1, \ldots, P_r$ are the minimal primes, then $\dim A = \max_i \dim A/P_i$. Proposition 24 implies [$A \subseteq B$ integral $\Rightarrow \dim A = \dim B$].

*Example:* $A = k[x_1, \ldots, x_n] \rightsquigarrow$ w.l.o.g. $k = \overline{k}$ ($\overline{k}[\mathbf{x}]$ is integral over $k[\mathbf{x}]$) $\stackrel{\text{HNS}}{\Rightarrow}$ $(\mathbf{x})$ is a "typical" maximal ideal $\Rightarrow \dim k[\mathbf{x}] = \dim k[\mathbf{x}]_{(\mathbf{x})} = n$. A chain of primes: $(x_1, \ldots, x_i)$.

15.3. **Transzendental degree.** Let $A$ be a finitely generated $k$-algebra without zero divisors $\rightsquigarrow X := \mathrm{Spec}\, A$ is irreducible with $K[X] := A$ and "*function field*" $K(X) := \mathrm{Quot}\, A$.

**Proposition 51.** 1) $\boxed{\dim A = \text{tr-deg}_k\, \mathrm{Quot}\, A}$.
2) *$P \subseteq A$ prime ideal* $\Rightarrow \dim A = \dim A/P + \mathrm{ht}\, P = \dim A/P + \dim A_P$. *In particular,* $\dim A = \dim A_{\mathfrak{m}}$ *for maximal ideals* $\mathfrak{m}$.

*Proof.* (1) Proposition 26 $\Rightarrow \exists\, k[y_1, \ldots, y_r] \hookrightarrow A$ finite, hence $\text{tr-deg}_k\, \mathrm{Quot}\, A = \text{tr-deg}_k\, k(\mathbf{y}) = r = \dim k[\mathbf{y}] = \dim A$.
(2) w.l.o.g. $\mathrm{ht}\, P = 1$ and $A = k[\mathbf{y}]$: Proposition 24(5) $\Rightarrow \mathrm{ht}\, P = \mathrm{ht}(P \cap k[\mathbf{y}])$ and $\dim A/P = \dim k[\mathbf{y}]/(P \cap k[\mathbf{y}])$. Factoriallity of $k[\mathbf{y}] \rightsquigarrow P = (f)$ with an irreducible $f \in k[\mathbf{y}]$; by (7.1) $k[y_1, \ldots, y_r]/f$ is finite over (w.l.o.g.) $k[y_1, \ldots, y_{r-1}]$, hence it is $(r-1)$-dimensional. □

*Applications:* $\dim A_f = \dim A$, $\dim(X \times Y) = \dim X + \dim Y$. ↗ §21

## 16. Projective varieties

week 3 (35)

**16.1. Recalling affine varieties and spectra.** Equivalences of categories ($k = \overline{k}$):

$$\{\text{closed affine subsets } Z \subseteq \mathbb{A}^n_k\} \leftrightarrow \{\text{radical ideals } I \subseteq k[x_1, \dots, x_n]\}$$
$$\leftrightarrow \{k[x_1, \dots, x_n] \twoheadrightarrow A = \text{ reduced}\}$$

or, forgetting the embedding, {affine algebr $k$-varieties } $\leftrightarrow$ {f.g. red $k$-algebras}. Without $k$, this generalizes to the scheme setup, i.e. to the equivalence of categories

$$\{\text{affine schemes } (\operatorname{Spec} A, A)\} \leftrightarrow \{\text{commutative rings } A\}^{\text{opp}}.$$

$A$ becomes the ring of regular functions on $\operatorname{Spec} A$, we allow nilpotent elements in $A$, and we do not need a field $k$ at this point.

*Examples*: 1) Functor of affine toric varieties $\mathbb{TV}(N, \sigma)$ via $(\sigma \subseteq N_{\mathbb{R}}) \mapsto (\sigma^\vee \subseteq M_{\mathbb{R}})$ and $\mathbb{TV}(N, \sigma) := \operatorname{Spec} k[\sigma^\vee \cap M]$;
2) surjections $A \twoheadrightarrow B$ corresponds to closed embeddings $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} A$;
3) localizations $A \to A_g$ yield $\operatorname{Spec} A_g = D(g) := (\operatorname{Spec} A) \setminus V(g) \subseteq \operatorname{Spec} A$.
4) Faces $\tau \leq \sigma \subseteq N_{\mathbb{R}}$ lead to open embeddings $\mathbb{TV}(N, \tau) \hookrightarrow \mathbb{TV}(N, \sigma)$.

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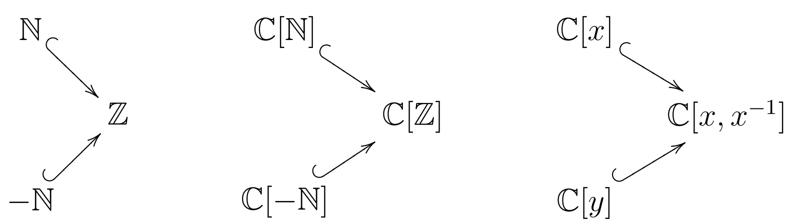
16.1.1. *The toric $\mathbb{P}^1$-construction.* The easiest concrete instance of (4) is the following: Let $\Sigma := \{\sigma^+, \sigma^-, 0\}$ consisting of the 1-dimensional cones $\sigma^\pm := \mathbb{R}_{\geq/\leq 0}$ and their intersection 0. Then the associated semigroups are $\mathbb{N}$, $-\mathbb{N}$, and $\mathbb{Z}$.




where the $y$ from the bottom right corner maps to $x^{-1}$. Geometrically, this means that we glue two copies of $\mathbb{A}^1 = \mathbb{C}^1$ with coordinates $x$ and $y$, respectively, along their open subsets $\mathbb{C}^*$. However, the identification of the two "tori" is done via $y = x^{-1}$.

**16.2. The projective space.** The affine varieties $\mathbb{C}^n$ and, e.g., the quadric $V(x^2 + y^2 - 1) \subseteq \mathbb{C}^2$ and the "elliptic curve" $E := V(y^2 - x^3 + x) \subseteq \mathbb{C}^2$ are not compact when considered in the classical topology (the quasicompactness of the Zariski topology is misleading here). week 4 (37)

$k = \overline{k}$ field $\rightsquigarrow \mathbb{P}^n_k := \mathbb{P}(k^{n+1})$ with $\boxed{\mathbb{P}_k(V) := (V^\vee \setminus \{0\})/k^*}$; the complex $\mathbb{P}^n_{\mathbb{C}} = S^{2n-1}/\mathbb{C}_1$ is compact in the classical topology; "*projective algebraic subsets*":= vanishing loci $V(J) = V_{\mathbb{P}}(J) \subseteq \mathbb{P}^n_k$ for *homogeneous* ideales $J \subseteq k[\mathbf{z}]$ with $\mathbf{z} =$

$(z_0, \ldots, z_n) \rightsquigarrow$ similarily to (1.2): Zariski *topology* on $\mathbb{P}^n_k$; $g \in k[\mathbf{z}]$ homogeneous $\rightsquigarrow$ $D_+(g) := \mathbb{P}^n_k \setminus V(g)$ yield a basis of the open subsets. The special charts $D_+(z_i) \cong k^n$ will be identifyed with the affine schemes $\operatorname{Spec} k[\mathbf{z}]_{(z_i)}$ where

$$k[\mathbf{z}]_{(z_i)} = k[\tfrac{z_0}{z_i}, \ldots, \tfrac{z_n}{z_i}] \subset k[\mathbf{z}, \tfrac{1}{z_i}] = k[\mathbf{z}]_{z_i}$$

denotes the *homogeneous localization* consisting of the degree 0 elements of the latter, ordinary localization. $\mathbb{P}^n = \bigcup_{i=0}^n D_+(z_i)$ is an open, affine covering. And $\mathbb{P}^n_k = D_+(z_0) \sqcup \mathbb{P}^{n-1}_k$ with $D_+(z_0) = \operatorname{Spec} k[\mathbf{x}]$ where $\mathbf{x} = (x_1, \ldots, x_n)$ and $x_i = z_i/z_0$.

16.3. **Projective subsets.** If we start with an ideal $I \subseteq k[\mathbf{x}]$ corresponding to the affine $\operatorname{Spec} k[\mathbf{x}]/I = V(I) \subseteq \mathbb{A}^n_k \rightsquigarrow$ homogenization $I^h \subseteq k[\mathbf{z}]$ ($\subseteq k[t, \mathbf{x}]$ in (11.2) with $w = \underline{1}$), i.e. after substituting $x_i \mapsto z_i/z_0$ one multiplies with the minimal $z_0$-power killing all denominators in the polynomials from $I \rightsquigarrow \boxed{V_{\mathbb{P}}(I^h) = \overline{V_{\mathbb{A}}(I)}}$ inside $\mathbb{P}^n$. *Example:* $\overline{E} = V_{\mathbb{P}}(y^2z - x^3 + xz^2) \subseteq \mathbb{P}^2$ carries a group structure; usually the neutral element is chosen as $(0:1:0)$ which is $\overline{E} \setminus E$, cf. (16.2).

The opposite construction: If $J \subseteq k[\mathbf{z}]$ is a homogeneous ideal, then $J^i := J_{(z_i)} \subseteq k[\frac{\mathbf{z}}{z_i}] = k[\mathbf{x}^{(i)}]$ is obtained from substituting $z_\nu \mapsto x^{(i)}_\nu = z_\nu/z_i$ (thus $z_i \mapsto 1$) in the arguments of the polynomials from $J$. Then, the local structure of $V_{\mathbb{P}}(J) \subseteq \mathbb{P}^n$ in the chart $D_+(z_i)$ is obtained by identifying $V_{\mathbb{P}}(J) \cap D_+(z_i) = \operatorname{Spec} k[\mathbf{x}^{(i)}]/J^i$.

The maximal ideal $(\mathbf{z})=$ is called the *irrelevant ideal.* $V(J)$ and $V(J : \mathbf{z}^\infty)$ have the same local structure, e.g. $V(\mathbf{z})$ and $V(1)$, or $V(z_0^2 - z_0z_1, z_0z_1 - z_1^2)$ and $V(z_0 - z_1)$, and the ideal $(J : \mathbf{z}^\infty)$ is maximal with this property.

*Example:* $\operatorname{Grass}(d, V) \subseteq \mathbb{P}(\Lambda^d V^\vee)$ is given by the Plücker relations.

For $Z = V(J) \subseteq \mathbb{P}^n$ we call $\boxed{S(Z) := k[\mathbf{z}]/(J : \mathbf{z}^\infty)}$ the *homogeneous coordinate ring*; it is $\mathbb{Z}$-graded; the affine coordinate ring of the $i$-th chart $Z \cap D_+(z_i)$ is $S_{(z_i)}$.

*Remark.* Taking $I(Z) \subseteq k[\mathbf{z}]$ instead of $(J : \mathbf{z}^\infty)$ is to coarse and big if one is interested to preserve a possible non-reduced local structure.

16.4. **Special constructions.** The homogeneous coordinate ring is not an invariant of the projective variety, but it depends on its projective embedding, cf. $\nu_{1,2}$:

1) The *Veronese embedding* $\nu_{n,d} : \mathbb{P}^n \hookrightarrow \mathbb{P}(k[\mathbf{z}]_d) = \mathbb{P}^{\binom{d+n}{n}-1}$ is (locally) an isomorphism onto the image. However, $S(\nu_{n,d}(\mathbb{P}^n)) = \oplus_{d|k} k[\mathbf{z}] \subsetneq k[\mathbf{z}] = S(\mathbb{P}^n)$.
*Example:* For $\nu_{1,2} : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2$, $(z_0 : z_1) \mapsto (z_0^2 : z_0z_1 : z_1^2)$ the image is $V(w_0w_2 - w_1^2)$, and the inverse map consists of the two local pieces $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ $(w_0 : w_1 : w_2) \mapsto (w_0 : w_1)$ (not defined in $(0:0:1)$) and $\mapsto (w_1 : w_2)$ (not defined in $(1:0:0)$).

While $\nu_{1,2} : \mathbb{P}^1 \xrightarrow{\sim} V_{\mathbb{P}}(w_0w_2 - w_1^2)$, the map between the homogeneous coordinate rings is $\nu_{1,2}^* : k[w_0, w_1, w_2]/(w_0w_2 - w_1^2) \xrightarrow{\sim} k[z_0^2, z_0z_1, z_1^2] \subset k[z_0, z_1]$, i.e. the quadric yields only the even degrees inside $k[z_0, z_1]$. All non-degenerate quadrics ("conics") in $\mathbb{P}^2_{\mathbb{C}}$ are, via a linear change of coordinates, equal to $V(w_0w_2 - w_1^2)$. In particular, they are isomorphic to $\mathbb{P}^1_{\mathbb{C}}$.

2) The *Segre embedding* $\mathbb{P}^a \times \mathbb{P}^b \hookrightarrow \mathbb{P}^{(a+1)(b+1)-1}$ or, coordinate free, $\mathbb{P}(V)\times\mathbb{P}(W) \hookrightarrow \mathbb{P}(V\otimes W)$ gives $\mathbb{P}^a\times\mathbb{P}^b$ the structure of a projective variety. *Example:* $\mathbb{P}^1\times\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, $(y_0 : y_1), (z_0 : z_1) \mapsto (y_0z_0 : y_0z_1 : y_1z_0 : y_1z_1)$ has the image $V(w_{00}w_{11} - w_{10}w_{01})$. In particular, non-degenerate quadrics in $\mathbb{P}^3$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1 \not\cong \mathbb{P}^2$). Hence, they contain always two infinite families of lines.

On the contrary, a general cubic surface $S \subseteq \mathbb{P}^3$ contains exactly 27 lines, cf. (17.6).

3) *Projective toric varieties:* $M := \mathbb{Z}^n$, $\Delta \subseteq M_{\mathbb{Q}}$ lattice polytope (convex hull of a week 5 (39)
finite subset of $M$) $\rightsquigarrow$ $\boxed{\mathbb{P}(\Delta) \subseteq \mathbb{P}_k^{\#(\Delta\cap M)-1}}$ with equations $\prod_v z_v^{\lambda_v} = \prod_v z_v^{\mu_v}$ resulting from the affine dependencies $\sum_v \lambda_v(v,1) = \sum_v \mu_v(v,1)$ where $v \in \Delta\cap M$, $\lambda_v, \mu_v \in \mathbb{N}$. The $(M\oplus\mathbb{Z})$-graded kernel of $k[z_v \mid v \in \Delta\cap M] \to k[M\oplus\mathbb{Z}]$, $z_v \mapsto x^{(v,1)} = x^v t$ is generated from the above equations, hence $S(\mathbb{P}(\Delta)) = k[\mathbb{N}\cdot(\Delta\cap M, 1)] =: k[\Delta]$.

*Examples:* 3.0) $\Delta^n := \{\mathbf{x} \in \mathbb{Q}^n_{\geq 0} \mid x_1+\ldots+x_n \leq 1\} = \{\mathbf{x} \in \mathbb{Q}^{n+1}_{\geq 0} \mid x_0+\ldots+x_n = 1\}$ $\Rightarrow \mathbb{P}(\Delta^n) = \mathbb{P}^n_k$ (there are no affine dependencies at all, hence no equations).

3.1) Veronese$'$: $d \in \mathbb{Z}_{\geq 1} \rightsquigarrow \mathbb{P}^n_k \cong \mathbb{P}(d\Delta^n) \subseteq \mathbb{P}_k^{\binom{n+d}{d}-1}$, $\underline{z} \mapsto (\underline{z}^r \mid |r| = d)$, but $S(\mathbb{P}(d\Delta^n)) = \oplus_{v\geq 0} S(\mathbb{P}^n_k)_{dv} \subsetneq S(\mathbb{P}^n_k)$. Or, $\mathbb{P}(\Delta) \to \mathbb{P}(d\Delta)$ for normal polytopes.

3.2) Segre$'$: $\mathbb{P}(\Delta_1)\times\mathbb{P}(\Delta_2) = \mathbb{P}(\Delta_1\times\Delta_2)$; the relations $(e_i,e_j)+(e_k,e_l) = (e_i,e_l)+(e_k,e_j)$ yield the equations $\operatorname{rank}(z_{ij})_{0\leq i,j\leq m,n} \leq 1$. There is a natural map $\mathbb{P}(\Delta_1+\Delta_2) \to \mathbb{P}(\Delta_1\times\Delta_2)$.

$\boxed{\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3}$

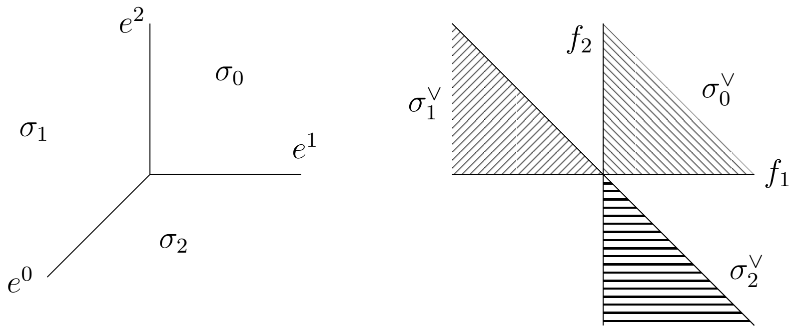
16.5. **Toric varieties.** We recall that affine toric varieties are associated to polyhedral cones and glue this construction afterwards.

16.5.1. *Affine toric varieties.* $N := \mathbb{Z}^n$, $M := \operatorname{Hom}(N,\mathbb{Z})$, $N_{\mathbb{Q}} := N\otimes_{\mathbb{Z}}\mathbb{Q}$, $M_{\mathbb{Q}} := M\otimes_{\mathbb{Z}}\mathbb{Q} \rightsquigarrow$ perfect pairing $\langle\bullet,\bullet\rangle : N_{\mathbb{Q}}\otimes M_{\mathbb{Q}} \to \mathbb{Q}$. Polyhedral cones $\sigma \subseteq N_{\mathbb{Q}}$ with apex $\rightsquigarrow \sigma^\vee := \{r \in M_{\mathbb{Q}} \mid \langle\sigma, r\rangle \geq 0\}$; *polyhedrial duality* $\sigma^{\vee\vee} = \sigma$ and $(\sigma_1\cap\sigma_2)^\vee = \sigma_1^\vee + \sigma_2^\vee$.

$\boxed{\text{Functor } \sigma \mapsto \mathbb{TV}(\sigma)} := \mathbb{TV}(\sigma, N) := \operatorname{Spec} k[\sigma^\vee\cap M] \subseteq \mathbb{A}^H_k$ as in (17.5); if $\tau \leq \sigma$ is a face, then every $r \in \operatorname{int}(\sigma^\vee\cap\tau^\perp)\cap M$ yields $\tau = \sigma\cap r^\perp \Rightarrow \tau^\vee = \sigma^\vee - \mathbb{Q}_{\geq 0}\cdot r$, hence $\mathbb{TV}(\tau) = D(\mathbf{x}^r) \subseteq \mathbb{TV}(\sigma)$, cf. (16.5). *Examples:* $\mathbb{TV}(\mathbb{Q}^n_{\geq 0}) = \mathbb{A}^n_k$; $\mathbb{TV}(\sigma_1)\times\mathbb{TV}(\sigma_2) = \mathbb{TV}(\sigma_1\times\sigma_2)$; $\mathbb{TV}(\mathbb{Q}_{\geq 0}(1,0) + \mathbb{Q}_{\geq 0}(1,2)) = V(z^2 - xy) \subseteq \mathbb{A}^3_k$.

16.5.2. *General toric varieties.* With the notation of (16.5.1): If $\Sigma$ is a $\boxed{\text{fan}}$ of cones in $N_{\mathbb{Q}}$, then we glue $\boxed{\mathbb{TV}(\Sigma, N) := \varinjlim_{\sigma\in\Sigma} \mathbb{TV}(\sigma)}$; this construction is functorial with respect to $f : (N,\Sigma) \to (N',\Sigma')$ meaning a $\mathbb{Z}$-linear map $f : N \to N'$ such that $\forall\sigma\in\Sigma\ \exists\sigma'\in\Sigma'$: $f(\sigma)\subseteq\sigma'$.

*The toric description of* $\mathbb{P}^n$*:* $N := \mathbb{Z}^{n+1}/\sum_i e^i$, hence $M := \operatorname{Hom}(N,\mathbb{Z}) = [\sum e^i = 0] \subseteq \mathbb{Z}^{n+1}$ with basis $f_i := e_i - e_0$ $(i = 1,\ldots,n)$. The cones $\sigma_i := \langle e^0,\ldots,\hat{e}^i,\ldots,e^n\rangle$ $\rightsquigarrow \sigma_i^\vee = \langle e_\bullet - e_i\rangle$ provide $\mathbb{TV}(\sigma_i) = U_i$, and $\tau := \sigma_i\cap\sigma_j$ determines open embeddings $U_i \supseteq D(z_j/z_i) = U_{ij} = D(z_i/z_j) \subseteq U_j$. With $\Sigma := \{\sigma_i \text{ and faces}\}$ we obtain $\mathbb{TV}(\Sigma) = \mathbb{P}^n$.




*The toric description of the blow up:* $\pi : \widetilde{\mathbb{A}^n} \to \mathbb{A}^n$ is the gluing of $k[x_1, \ldots, x_n] \to k[x_1/x_i, \ldots, x_n/x_i,\, x_i]$, hence $k[\mathbb{N}^n] \to k[\langle e_\bullet - e_i,\, e_i \rangle \cap \mathbb{Z}^n]$. Thus, the $i$-th chart corresponds to the inclusion $\sigma_i := \langle e^1, \ldots, \hat{e^i}, \ldots, e^n,\, \sum_\nu e^\nu \rangle \subseteq \mathbb{Q}^n_{\geq 0} =: \sigma$, i.e. $\mathbb{Q}^n_{\geq 0}$ will be subdivided by inserting the inner ray $e := \sum_\nu e^\nu \in \mathbb{Z}^n = N$.


polyhedron $\Delta \subseteq M_\mathbb{Q}$ subdivided cone $\mathbb{Q}^2_{\geq 0} \subseteq N_\mathbb{Q}$

The map $h : \widetilde{\mathbb{A}^n} \to \mathbb{P}^{n-1}_k$ can be obtained from the projection $N \twoheadrightarrow N/\mathbb{Z}e$ with $e := e^1 + \ldots e^n$. (17.2) $\rightsquigarrow \mathcal{O}_{\mathbb{P}^{n-1}}(-1) =$ sheaf of sections of $h$; $\Gamma(\mathbb{P}^{n-1}, \mathcal{O}(-1)) = 0$ is illustrated by the non-existence of global *toric* sections of $h$: There are no hyperplanes meeting all cones of the $\widetilde{\mathbb{A}^n}$-fan at once.

If $\Delta \subseteq M_\mathbb{Q}$ is a lattice polyhedron, then we had defined in (16.4)(3) and (17.5) the toric variety $\mathbb{P}(\Delta)$. Let $\Sigma := \mathcal{N}(\Delta) :=$ (inner) *normal fan* of $\Delta \rightsquigarrow n : \mathbb{TV}(\Sigma) \to \mathbb{P}(\Delta)$ is built from gluing the maps $n_w^* : k[x^{v-w} \mid v \in \Delta \cap M] \to k[\mathbb{Q}_{\geq 0} \cdot (\Delta - w) \cap M]$ for (e.g. vertices) $w \in \Delta \cap M$. This becomes an isomorphism ("$\Delta$ is *ample*") for $\Delta := (\gg 0) \cdot \Delta$.

16.6. **The affine cone and the Hilbert polynomial.** The local structure of $\pi : \mathbb{A}^{n+1}_k \backslash 0 \to \mathbb{P}^n_k$, $(z_0, \ldots, z_n) \mapsto (z_0 : \ldots : z_n)$ is $D_+(z_i) \times (\mathbb{A}^1_k \backslash 0) = D(z_i) \to D_+(z_i)$; on the level of $k$-algebras, this corresponds to $k[\mathbf{z}]_{(z_i)} \otimes k[z_i^{\pm 1}] = k[\mathbf{z}]_{z_i} \supseteq k[\mathbf{z}]_{(z_i)}$.

$\emptyset \neq Z \subseteq \mathbb{P}^n \rightsquigarrow C(Z) := \overline{\pi^{-1}(Z)} = \pi^{-1}(Z) \cup \{0\}$ is called the *affine cone* over $Z$; $\dim C(Z) = \dim Z + 1$. In $A(\mathbb{A}^{n+1}) = k[\mathbf{z}] = S(\mathbb{P}^n)$ we have $I_\mathbb{A}(C(Z)) = I_\mathbb{P}(Z)$. Similarily, if $J \subsetneq k[\mathbf{z}]$ is a homogeneous ideal, then $C(V_\mathbb{P}(J)) = V_\mathbb{A}(J : \mathbf{z}^\infty)$, leading to $A(C(Z)) = S(Z)$.

Homogeneous/projective HNS: Let $k = \overline{k}$ and $Z = V_\mathbb{P}(J) \subseteq \mathbb{P}^n_k$ for a given homogeneous ideal $J \subseteq k[\mathbf{z}]$. Then, if $f \in I_\mathbb{P}(Z)$ is homogeneous with $\deg f > 0 \Rightarrow f = 0$ on $\pi^{-1}(Z)$ and $f(0) = 0$, i.e. $f \in I_\mathbb{A}(Z) \Rightarrow \exists N\colon\ f^N \in J$. In particular, $V_\mathbb{P}(J) = \emptyset$ does only imply that $(\mathbf{z})^N \subseteq J$.

Now, we discuss properties of $Z \subseteq \mathbb{P}^n$ via the local properties of $C(Z)$ in $0 \in \mathbb{A}^{n+1}$:

Let $S = \oplus_{d\geq 0} S_d$ be a finitely generated, graded $(S_0 = k)$-algebra with irrelevant ideal $S_+ := \oplus_{d\geq 1} S_d \Rightarrow$ for $S_{\text{loc}} := (S \setminus S_+)^{-1}S$ it holds true that $\text{Gr}_{S_+}(S_{\text{loc}}) = \oplus_{d\geq 0} S_+^d / S_+^{d+1}$. If $S$ is generated in degree 1 (e.g. $S = S(Z)$) $\Rightarrow \text{Gr}_{S_+}(S_{\text{loc}}) = S \Rightarrow H_S(t) = \chi_{S_+}^{S_{\text{loc}}}(t+1) - \chi_{S_+}^{S_{\text{loc}}}(t) \Rightarrow \deg H_S = \dim S_{\text{loc}} - 1$. In particular, $\deg H_{S(Z)} = \dim Z$, and the (normalized with $(\dim Z)!$) leading coefficient of $H_{S(Z)}$ is $\boxed{\deg Z := \text{mult}(C(Z), 0)}$, cf. (12.3) and (13.1).

*Example:* $\deg V_{\mathbb{P}}(F) (\subseteq \mathbb{P}^n) = \deg F$; $\deg \mathbb{P}(\Delta) = \text{vol}(\Delta)$ where vol is normalized to vol(standard simplex) $= 1$ (quadrics $\nu_2(\mathbb{P}^1)$ and $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$; $\deg \nu_2(\mathbb{P}^2) = 4$).

16.7. **Linear projections.** The map $\pi : \mathbb{A}_k^{n+1} \setminus \{0\} \to \mathbb{P}_k^n$ from (16.6) has the following generalization: Let $L, L' \subseteq \mathbb{P}_k^n$ be disjoint linear subspaces with $\dim L + \dim L' = n - 1 \rightsquigarrow \pi_L : \mathbb{P}_k^n \setminus L \to L'$, $p \mapsto \text{span}(p, L) \cap L'$. Using coordinates, $L = (* : \underline{0})$, $L' = (\underline{0} : *) \Rightarrow \pi_L(\underline{x} : \underline{y}) = (\underline{0} : \underline{y})$. This was already used in (16.4)(1).

16.8. **Global regular functions.** $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = \bigcap_i k[\mathbf{z}]_{(z_i)} = \bigcap_i k[z_0/z_i, \ldots, z_n/z_i] = k$ ("factorial" $\Rightarrow$ the intersection of just two rings is already $k$) $\rightsquigarrow \mathbb{P}^{n\geq 1}$ is not affine!

**Proposition 52.** *Let $Z \subseteq \mathbb{P}_k^n$ be a projective variety* (irreducible) $\Rightarrow \Gamma(Z, \mathcal{O}_Z) = k$.

*Proof.* $f \in \Gamma(Z, \mathcal{O}_Z) = \bigcap_i S(Z)_{(z_i)} \subseteq \text{Quot}\, S(Z) \Rightarrow \exists N : (\mathbf{z})^N f \subseteq (\mathbf{z})^N \Rightarrow (\mathbf{z})^N f^{q\in\mathbb{N}} \subseteq (\mathbf{z})^N \Rightarrow S(Z)[f] \subseteq z_0^{-N} S(Z)$, i.e. $f$ is integral over $S(Z)$. The coefficients of the integrality relation are, w.l.o.g., homogeneous of degree 0, hence $\in k$. □

On the other hand, $z_0, \ldots, z_n$ are global on $\mathbb{P}_k^n$, but they are no functions. Instead, they are global sections of the dual "$\mathcal{O}(1)$" of the locally trivial tautological fibration "$\mathcal{O}(-1)$" on $\mathbb{P}^n$. In general, we define for $d \in \mathbb{Z}$, $\mathcal{O}(-d) := \{(\ell, c) \mid \ell \in \mathbb{P}^n,\ c \in \ell^{\otimes d}\}$ where $\ell$ is understood as a line, i.e. as a 1-dimensional subspace $\ell \subseteq k^{n+1}$, and for $d < 0$ we define $\ell^{\otimes d} := \text{Hom}_k(\ell^{\otimes(-d)}, k)$.

16.9. **The definition of** $\text{Proj}\, S$**.** Let $S = \oplus_{d\geq 0} S_d$ be a ($\mathbb{N}$-)graded ring (e.g. $S = S(Z)$ for $Z \subseteq \mathbb{P}_k^n$, i.e. $S_1 =$ finitely generated $(A := S_0)$-module, the $A$-algebra $S$ is generated from $S_1$) $\rightsquigarrow$ the topological space $\boxed{\text{Proj}\, S} := \{P \in \text{Spec}\, S \mid S_{\geq 1} \not\subseteq P = \text{homogeneous}\} \to \text{Spec}\, A$ (recovering $Z$). Zariski-closed: $V_{\mathbb{P}}(J) \subseteq \text{Proj}\, S$ for homogeneous ideals $J \subseteq S$; open basis $D_+(f) := \text{Proj}\, S \setminus V(f) = \text{Spec}\, S_{(f)}$ for homogeneous $f \in S_{\geq 1}$. The "(affine) *cone*" is $\text{Spec}\, S \setminus V_{\mathbb{A}}(S_{\geq 1}) \to \text{Proj}\, S$, $P \mapsto (P \cap \bigcup_d S_d)$ [Example $x_0(x_1 - c_1) - x_1(x_0 - c_0)$]; locally $D(f) \to D_+(f)$.

*Remark.* While this construction is similar to Spec$(A)$ – what is the analogue to the affine scheme $(\text{Spec}\, A, A)$? The problems are: (i) $S = S(Z)$ depends on the embedding, i.e. different rings $S$ and $T$ might encode the same variety; (ii) $S$ does not provide functions on $\text{Proj}\, S$ – what kind of objects are elements of $S$ at all? (iii) global functions on $\text{Proj}\, S$ are constants.

## 17. Blowing up

**17.1. Blowing up $0 \in \mathbb{A}^n_k$.** (cf. picture [Hart, S.29])

$$\begin{array}{ccc} \widetilde{\mathbb{A}^n_k} := V(x_iy_j - x_jy_i) \subseteq \mathbb{A}^n_k \times \mathbb{P}^{n-1}_k \xrightarrow{\ \pi\ } \mathbb{A}^n_k, & & [(x_1,..,x_n),(y_1:...:y_n)] \overset{\pi}{\mapsto} (x_1,..,x_n) \\ \downarrow h & & \downarrow h \\ \mathbb{P}^{n-1}_k & & (y_1:...:y_n) \end{array}$$

Outside of 0, the map $\pi : \pi^{-1}(\mathbb{A}^n \setminus 0) \xrightarrow{\sim} \mathbb{A}^n \setminus 0$ is an isomorphism; "*exceptional divisor*" $\boxed{E := \pi^{-1}(0) = \mathbb{P}^{n-1}}$; if $\ell$ is a line through $0 \in \mathbb{A}^n \Rightarrow \pi^{\#}(\ell) := \overline{\pi^{-1}(\ell \setminus \{0\})} = \ell \times \{\ell\}$, i.e. $\pi^{\#}(\ell) \cap E = \{\ell\} \subseteq \mathbb{P}^{n-1}$. We consider $\widetilde{\mathbb{A}^n} = \{(c,\ell) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid c \in \ell\}$ with $h(c,\ell) = \ell$ the "*universal line*" over $\mathbb{P}^{n-1}$ (generalizes to the tautological bundle = universal subspace over $\mathrm{Grass}(k,V)$).

**17.2. Local description of the blowing up.** On the $i$-th chart $\mathbb{A}^n_k \times D_+(y_i)$, the space $\widetilde{\mathbb{A}^n_k}$ is given by the equations $\mathbf{x} = x_i \frac{\mathbf{y}}{y_i}$; for the affine coordinate rings this means

$$\begin{array}{c} k[x_i, \mathbf{x}/x_i] = k[x_i, \mathbf{y}/y_i] = k[\mathbf{x}, \mathbf{y}/y_i]/(\mathbf{x} - x_i \frac{\mathbf{y}}{y_i}) \xleftarrow{\quad \pi^* \quad} k[\mathbf{x}] \\ \uparrow h^* \\ k[\mathbf{x}/x_i] = k[\mathbf{y}/y_i] \end{array}$$

and $k[x_i^{\pm 1}, \mathbf{x}/x_i] \xleftarrow{\sim} k[\mathbf{x}]_{x_i}$ for the restriction to $D(x_i) \times D_+(y_i) \to D(x_i)$. While the charts of the blowing up $\widetilde{\mathbb{A}^n_k}$ are obtained from $\mathbb{A}^n_k$ by allowing certain denominators, i.e. while this might remind of a localization procedure, $\pi$ is $\boxed{\text{not flat}}$.

**17.3. Strict transforms.** $X \subseteq \mathbb{A}^n_k \rightsquigarrow \pi^{\#}(X) := \overline{\pi^{-1}(X \setminus 0)} \subseteq \widetilde{\mathbb{A}^n_k}$; the "total transform" splits into $\boxed{\pi^{-1}(X) = \pi^{\#}(X) \cup E}$. The ideal $I_E$ of the exceptional divisor $E = \pi^{-1}(0)$ is locally principal, namely $I_E = (x_i)$ on $h^{-1}(D_+(y_i))$; if $X = V_{\mathbb{A}}(J)$, then the ideal of both the total and strict transform $\pi^{-1}(X)$ and $\pi^{\#}(X)$ in $h^{-1}(D_+(y_i))$ is $J := J\, k[x_i, \mathbf{x}/x_i]$ and $(J : x_i^\infty)$, respectively.
*Example:* $X = V(y^2 - x^3) \rightsquigarrow \pi^{-1}(X) \cap h^{-1}(D_+(x)) = V(t^2x^2 - x^3)$ with $t = y/x$, but $\pi^{\#}(X) = V(t^2 - x)$ is even contained in the $[y \neq 0]$ chart. The morphism $\pi^{\#}(X) \to X$ becomes $\mathrm{Spec}\, k[t] \to \mathrm{Spec}\, k[x,y]/(y^2 - x^3)$ with $x \mapsto t^2$ and $y \mapsto t^3$.

**17.4. Blowing up via** Proj**.** With $I := (\mathbf{x}) \subseteq k[x_1, \ldots, x_n]$, one obtains $\Rightarrow \widetilde{\mathbb{A}^n} = \mathrm{Proj} \oplus_{d \geq 0} I^d \xrightarrow{\pi} \mathrm{Spec}\, k[\mathbf{x}] = \mathbb{A}^n$, namely $\boxed{S := \oplus_{d \geq 0} I^d t^d}$ is a finitely generated, graded $(S_0 = k[\mathbf{x}])$-algebra with $D_+(x_i t) \mathrel{\widehat{=}} S_{(x_i t)} = k[\mathbf{x}][\mathbf{x}/x_i] = k[x_i, \mathbf{x}/x_i]$. Moreover, the closed embedding $\widetilde{\mathbb{A}^n} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$ is realized via the surjection $k[\mathbf{x}][\mathbf{y}] \twoheadrightarrow S$, $y_i \mapsto x_i t$. The exceptional divisor $E$ is recovered via $\pi^{-1}(0) = \mathrm{Proj}\, S/IS = \mathrm{Proj} \oplus_{d \geq 0} I^d/I^{d+1}$. See also (22.5).

17.5. **Toric description of the blowing up.** $\Delta \subseteq M_{\mathbb{Q}}$ lattice polytope $\rightsquigarrow$ the affine charts $D_+(z_v) = \operatorname{Spec} k[\Delta]_{(z_v)}$ are numerated by the $v \in \Delta \cap M$ or just the vertices $v$ of $\Delta$. The affine coordinate rings are the semigroup rings $k[\Delta]_{(z_v)} = k[\mathbb{N} \cdot ((\Delta - v) \cap M)] \subseteq k[\mathbb{Q}_{\geq 0} \cdot (\Delta - v) \cap M]$.

Similarly, $k[x_i, \mathbf{x}/x_i] = k[\mathbb{Q}_{\geq 0} \cdot (\nabla - e^i) \cap \mathbb{Z}^n]$ where $\nabla = \operatorname{conv}\{e^1, \ldots, e^n\} + \mathbb{Q}^n_{\geq 0}$. For those non-compact polyhedra $\Delta = \Delta^c + \operatorname{tail}(\Delta)$, we have a similar construction as in (16.4)(3): $v \in \Delta^c \cap M$ gives rise to a homogeneous coordinate $z_v$; $w \in H \subseteq \operatorname{tail}(\Delta) \cap M$ ($H$ generates $\operatorname{tail}(\Delta) \cap M$) enumerates ordinary coordinates $x_w$. Now, $\boxed{\mathbb{P}(\Delta) \subseteq \mathbb{P}_k^{(\Delta^c \cap M)-1} \times \mathbb{A}_k^H}$ is defined by the binomial equations corresponding to the linear dependencies among $(v, 1)$ and $(w, 0)$ inside $M \oplus \mathbb{Z}$.

*Example:* 0) $\Delta = C = \operatorname{tail}(\Delta)$, i.e. $\Delta^c = \{0\} \Rightarrow \mathbb{P}(C) \subseteq \mathbb{A}^H$, and this equals $\mathbb{TV}(C^\vee) := \operatorname{Spec} k[C \cap M]$. The embedding is induced by $H : C^\vee \to \mathbb{Q}^H_{\geq 0}$.

1) $\nabla$ has the vertices $v^i = e^i$ and the generators of the tail cone $w^i = e^i$. The basic dependencies are $(v^i, 1) + (w^j, 0) = (v^j, 1) + (w^i, 0)$; they lead to the equations of (17.1). Thus, $\boxed{\text{blowing up means cutting off (elementary) corners}}$ of polyhedra.

2) Blowing up $\mathbb{P}^2$ in two points equals blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ once, say $\mathbb{P}^2_{(2)} = (\mathbb{P}^1 \times \mathbb{P}^1)_{(1)}$.

17.6. **Cubic surfaces.** Non-degenerate quadrics in $\mathbb{P}^2$, $\mathbb{P}^3$, and $\mathbb{P}^5$ are isomorphic to $\mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\operatorname{Grass}(2, 4)$, respectively.

The cubic surface $S := V(x^3 + y^3 + z^3 + w^3) \subseteq \mathbb{P}^3_k$ contains exactly 27 lines: Gauß elimination transforms their equations into $x_0 - (a_2x_2 + a_3x_3) = x_1 - (b_2x_2 + b_3x_3) = 0$; substituting $x_0, x_1$ in the original cubic, the vanishing of the coefficients leads to

$$a_2^3 + b_2^3 + 1 = a_3^3 + b_3^3 + 1 = 0 \quad \text{and} \quad a_2^2 a_3 + b_2^2 b_3 = a_3^2 a_2 + b_3^2 b_2 = 0.$$

Considering $c_i := a_i/b_i$ shows that (w.l.o.g.) $b_2 = a_3 = 0$, hence the equations for lines inside $S$ turn into $x_0 + \omega^i x_2 = x_1 + \omega^j x_3 = 0$ with $\omega = \sqrt[3]{1}$ (plus permutations).

Let $L_1, L_2 \subseteq \mathbb{P}^3$ be disjoint lines on a *general* smooth cubic $S = V(g) \subseteq \mathbb{P}^3 \rightsquigarrow f : \mathbb{P}^3 \setminus (L_1 \cup L_2) \to L_1 \times L_2$ such that $p$, $f_1(p) \in L_1$, $f_2(p) \in L_2$ are collinear, i.e. $f_2 = \pi_{L_1} : \mathbb{P}^3 \setminus L_1 \to L_2$. This gives a morphism $f_2 : S \to L_2$ via $f_2(p) := T_pS \cap L_2$ for $p \in L_1$; using coordinates: $L_1 = (* * 0\, 0)$, $L_2 = (0\, 0 * *) \Rightarrow f_2 : (x_0 : x_1 : x_2 : x_3) \mapsto (x_2 : x_3)$ and $T_pS \cap L_2 = (-\frac{\partial g}{\partial x_3} : \frac{\partial g}{\partial x_2})$, see Problem **??**.

The map $f : S \to L_1 \times L_2$ is invertible except in the points $(p, q) \in L_1 \times L_2$ with $\overline{p, q} \subseteq S$ – here, the entire line $\overline{p, q}$ forms the preimage. There are exactly five those points (at least in the above example), hence $f : S \to L_1 \times L_2 = \mathbb{P}^1 \times \mathbb{P}^1$ is the blowing up of five points or, alternatively, the blowing up of six points in $\mathbb{P}^2$. In particular, $\boxed{S = \mathbb{P}^2_{(6)}}$ is rational.

Recovering of the 27 lines in the blowing up $\mathbb{P}^2_{(6)} \to \mathbb{P}^2$: six exceptional divisors, 15 strict transforms of connecting lines, six strict transforms of quadrics through five points. A toric analogon is $\mathbb{P}^2_{(3)}$ – after starting with $\nu_3(\mathbb{P}^2)$ one sees the six toric lines as the six edges of length one of the polytope.

## 18. Sheaves

week 6 (41)

18.1. **Presheaves.** $X$ = topological space $\rightsquigarrow$ "*Presheaf* on $X$":= contravariant functor $\mathcal{F} : \mathcal{O}pen(X)^{\mathrm{opp}} \to \mathcal{A}b/\mathcal{R}ings$; they form a category via $\mathrm{Hom}_{\mathcal{P}reSh}(\mathcal{F}, \mathcal{G}) :=$ {natural transformations $\mathcal{F} \to \mathcal{G}$}.

*Examples:* function sheaves, constant (pre-)sheaf, sections in bundles, restriction $\mathcal{F}|_U$ of presheaves, $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ with $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$.

For an open $U \subseteq X$ and a point $P \in X$ we obtain functors $\mathcal{P}reSh(X, \mathcal{A}b) \to \mathcal{A}b$

$$\mathcal{F} \mapsto \Gamma(U, \mathcal{F}) := \mathcal{F}(U) \text{ ("\textit{sections}") and } \mathcal{F}_P := \varinjlim_{U \ni P} \mathcal{F}(U) \text{ ("\textit{stalk}" in } P)$$

*Example*: $\mathcal{O}^{\mathrm{an}}_{\mathbb{R},0} = \mathbb{R}[|x|]$, but $\mathcal{C}^\infty_{\mathbb{R},0}$ is much bigger.

For sections $s \in \mathcal{F}(U)$ we call $\boxed{s_P \in \mathcal{F}_P \text{ the } \textit{germ}}$ of $s$ in $P \in U$; der *support* $\mathrm{supp}\, s := \{P \in U \mid s_P \neq 0\}$ is automatically closed in $U$. Further operations among presheaves are, e.g., $\ker(\mathcal{F} \to \mathcal{G})$, im, coker, $\mathcal{F}/\mathcal{G}$, $\mathcal{F} \oplus \mathcal{G}$, $\mathcal{F} \otimes \mathcal{G}$; the obvious definitions of injectivity and surjectivity work; $\mathcal{P}reSh(X, \mathcal{A}b)$ becomes an abelian category making the two above functors $\mathcal{P}reSh \to \mathcal{A}b$ exact.

18.2. **Sheaves.** $\mathcal{F}|X$ is called a *sheaf* $:\Leftrightarrow$ $\mathcal{F}(\emptyset) = 0$ and for open $U_i \subseteq X$ the sequence $0 \to \mathcal{F}(\bigcup_i U_i) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j)$ is exact; $\mathcal{S}h(X, \mathcal{A}b) \overset{\iota}{\hookrightarrow} \mathcal{P}reSh(X, \mathcal{A}b)$ is defined as a full subcategory. If $\mathcal{F}, \mathcal{G} \in \mathcal{S}h$, then $\ker(\mathcal{F} \to \mathcal{G}) \in \mathcal{S}h$; similarily $\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{H}om(\mathcal{F}, \mathcal{G}) : U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ stay sheaves.

The essence of $\mathcal{S}h(X)$: [$s \in \mathcal{F}(U)$ vanishes $\Leftrightarrow \forall P \in U$: $s_P = 0$] and [$f : \mathcal{F} \to \mathcal{G}$ is zero/injective/isom $\Leftrightarrow$ $f_P$ is zero/injective/isom $\forall P \in X$].

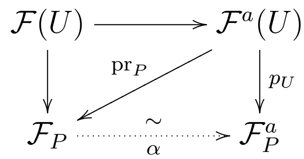
The major problem of $\mathcal{S}h(X)$: The $\mathcal{P}reSh$ notions $\mathrm{im}(\mathcal{F} \to \mathcal{G})$ (and coker and $\otimes$) drop out of $\mathcal{S}h$. *Solution:* Keep ker, but redefine im and coker in (18.6) such that $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ becomes exact $\Leftrightarrow$ $\mathcal{F}_P \to \mathcal{G}_P \to \mathcal{H}_P$ is exact for all $P$. Now, the original problem manifests as the only left-exactness of $\iota$ or the section functors $\Gamma(U, \bullet)$.

week 7 (43)

18.3. **Sheafification.** Let $\mathcal{U} \subseteq \mathcal{O}pen(X)$ be a basis of the topology, i.e. every open subset $U \subseteq X$ is a union of some $U_i \in \mathcal{U}$. The notions of (18.1) make also sense for a functor $\mathcal{F} : \mathcal{U}^{\mathrm{opp}} \to \mathcal{A}b$. To any such $\mathcal{F}$ we associate the $\boxed{\text{sheaf } \mathcal{F}^a}$ defined as

$$\mathcal{F}^a(U) := \{s \in \prod_{P \in U} \mathcal{F}_P \mid \text{locally } s \text{ comes from } s_i \in \mathcal{F}(U_i) \text{ for } U_i \in \mathcal{U}\}$$

and coming with natural isomorphisms $\alpha : \mathcal{F}_P \overset{\sim}{\to} \mathcal{F}^a_P$. (If $P \in U \in \mathcal{U}$, then in the diagram




the $\alpha$ from the universal property of $\mathcal{F}_P$ makes the quadrangle commute; by the local surjectivity of $\mathcal{F}(U) \to \mathcal{F}^a(U)$, everything commutes; hence $\alpha$ is an isomorphism.)

There are two special cases: (1) If $\mathcal{F}|\mathcal{U}$ has the sheaf property of (18.2), but limited to $\mathcal{U}$, then $\mathcal{F}^a$ becomes the unique sheaf with $\mathcal{F}^a|_{\mathcal{U}} = \mathcal{F}$ (the $\mathcal{U}$-sheaf homomorphism $\mathcal{F}^a|_{\mathcal{U}} \leftarrow \mathcal{F}$ is an isomorphism on the stalks).
(2) If $\mathcal{U} = \mathcal{O}pen(X)$, then $\mathcal{F}^a = a(\mathcal{F})$ is called the sheafification of $\mathcal{F}$; it does not change sheaves ($a \circ \iota = \mathrm{id}_{\mathcal{S}h}$), and it comes with natural maps $\mathcal{F} \to \mathcal{F}^a$ making $\boxed{a \dashv \iota}$ into adjoint functors, i.e. $\mathrm{Hom}_{\mathcal{S}h}(\mathcal{F}^a, \mathcal{G}) = \mathrm{Hom}_{\mathcal{P}re\mathcal{S}h}(\mathcal{F}, \iota\mathcal{G})$.
*Example:* The constant sheaf $\underline{A} = (\underline{A}^{\mathrm{pre}})^a$ assigns $U \mapsto A^{\pi_0(U)}$.

18.4. **Famous sheaves.** Famous ring sheaves in the classical topology are $\underline{\mathbb{C}} \subseteq \mathcal{O}^{\mathrm{an}} \subseteq \mathcal{C}^\infty$ or the sheaf of meromorphic funtions $\mathcal{M}^{\mathrm{an}}$ on $\mathbb{C}^n$ (total quotient sheaf of $\mathcal{O}^{\mathrm{an}}$). "(Locally) ringed spaces" $(X, \mathcal{O}_X)$, cf. (19.1). Then, $\mathcal{O}^* \subseteq \mathcal{O}$ (units in $\mathcal{O}$) is a sheaf abelian groups.

On $\mathbb{C}$ there are famous sequences: $0 \to \underline{\mathbb{C}} \to \mathcal{O} \stackrel{d}{\to} \mathcal{O} \to 0$ with $d : f(z) \mapsto f'(z)$ being locally (on the stalks) surjective: $\int_{0\rightsquigarrow z} f(z)dz$ is a preimage of $f$; but there is no global preimage "$\log z$" of $1/z \in \Gamma(\mathbb{C}^*, \bullet)$.
The "exponential sequence" $\boxed{0 \to \underline{\mathbb{Z}} \to \mathcal{O} \stackrel{\exp}{\to} \mathcal{O}^* \to 0}$ with $\exp : f(z) \mapsto e^{f(z)}$; here $\log(g(z))$ yields the local preimage of $g \in \mathcal{O}^*$, but $g(z) = z$ has no global one on $\mathbb{C}^*$.
Examples of *invertible sheaves* from (17.2): $\mathcal{O}(-1)$ := sheaf of regular sections of $\widetilde{\mathbb{A}^n_k} \to \mathbb{P}^{n-1}$; locally $\mathcal{O}(-1)|_{D_+(z_i)} \cong \mathcal{O}|_{D_+(z_i)}$, but $\Gamma(\mathbb{P}^{n-1}, \bullet)$ yields 0 and $\mathbb{C}$, respectively.

18.5. **Sheaves on** $\mathrm{Spec}\, A$. The *structure sheaf* $\mathcal{O}_{\mathrm{Spec}\, A} := \widetilde{A}$ is a special case of the sheaf $\widetilde{M}$ for $A$-modules $M$ given by $\boxed{\Gamma(D(f), \widetilde{M}) := M_f}$ with the natural restriction maps; $\widetilde{M}_P = M_P$. According to (18.3)(1) we check the restricted sheaf properties:

**Proposition 53.** *$\widetilde{M}$ is a sheaf on* $\mathrm{Spec}\, A$.

*Proof. "Injectivity" sheaf property:* If $m \in M$ vanishes in $M_{f_i}$ for a covering of $D(f_i)$, then $m/1 = 0$ in all $M_P$, hence $m = 0$ (consider $\mathrm{Ann}(m)$).
*"Surjectivity" sheaf property:* $m_i/f_i \in M_{f_i}$ (note that $M_{f_i} = M_{f_i^n}$) with $m_i/f_i = m_j/f_j$ in $M_{f_if_j} \Rightarrow m_i f_i^N f_j^{N+1} - m_j f_i^{N+1} f_j^N = (f_if_j)^N(m_if_j - m_jf_i) = 0$ in $M$ for all $i, j$. The $D(f_i^{N+1})$ cover $\mathrm{Spec}\, A \Rightarrow 1 = \sum_j \ell_j f_j^{N+1}$ for some $\ell_j \in A \rightsquigarrow m := \sum_j \ell_j m_j f_j^N$ yields $m/1 = (mf_i^{N+1})/f_i^{N+1} = (m_if_i^N)/f_i^{N+1} = m_i/f_i$. □

$\widetilde{M} \oplus \widetilde{N} = \widetilde{M \oplus N}$ and, if $M$ is finitely presented, $\mathcal{H}om(\widetilde{M}, \widetilde{N}) = \widetilde{\mathrm{Hom}(M, N)}$ (compare both sides on the open subsets $D(f)$).

Analogously: $\boxed{\widetilde{M} \text{ on } \operatorname{Proj} S}$ for graded $S$-modules $M$. If $f \in S$ is homogeneous of positive degree, then, via $D_+(f) = \operatorname{Spec} S_{(f)}$ from (16.9) and Problem **??**, $\widetilde{M}|_{D_+(f)} = \widetilde{M_{(f)}}$. Special cases are $\mathcal{O}_{\operatorname{Proj} S}(k) := \widetilde{S(k)}$. If $\deg f = 1$, then $M_{(f)} \xrightarrow{\cdot f^k} M(k)_{(f)}$ is an isomorphism.

week 8 (45)

18.6. **The abelian category of sheaves.** Operations with sheaves are the usual ones among presheaves with $\boxed{\text{subsequent sheafification}}$, e.g. $\mathcal{F} \otimes_{\mathcal{O}}^{\mathcal{S}h} \mathcal{G} := (\mathcal{F} \otimes_{\mathcal{O}}^{\mathcal{P}re\mathcal{S}h} \mathcal{G})^a$ leads to a canonical $\mathcal{F} \otimes_{\mathcal{O}}^{\mathcal{P}re\mathcal{S}h} \mathcal{G} \to \mathcal{F} \otimes_{\mathcal{O}}^{\mathcal{S}h} \mathcal{G}$ inducing isomorphisms on the stalks. Further examples are $\operatorname{im}(\mathcal{F} \to \mathcal{G})$, $\operatorname{coker}(\mathcal{F} \to \mathcal{G})$, $\mathcal{F}/\mathcal{G}$. Composing several operations gets along with a single sheafification at the end.

*Example:* For $X = \operatorname{Spec} A$, the presheaves $\widetilde{M} \otimes_{\mathcal{O}}^{\mathrm{pre}} \widetilde{N}$ and $\widetilde{M \otimes_A N}$ coincide on the sets $D(f)$, hence $M \mapsto \widetilde{M}$ commutes with $\otimes$. The same holds true for graded $S$-modules and the associated sheaves on $\operatorname{Proj} S$; in particular, $\mathcal{O}_{\operatorname{Proj} S}(a) \otimes \mathcal{O}_{\operatorname{Proj} S}(b) = \mathcal{O}_{\operatorname{Proj} S}(a+b)$.

**Lemma 54.** *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a homomorphism of sheaves of abelian groups. Then*

- (k) *$\mathcal{K} \to \mathcal{F}$ is isomorphic to* $\ker \varphi \Leftrightarrow \forall P \in X$: $0 \to \mathcal{K}_P \to \mathcal{F}_P \to \mathcal{G}_P$ *is exact;*
- (c) *$\mathcal{G} \to \mathcal{C}$ is isomorphic to* $\operatorname{coker} \varphi \Leftrightarrow \forall P \in X$: $\mathcal{F}_P \to \mathcal{G}_P \to \mathcal{C}_P \to 0$ *is exact;*
- (i) $\operatorname{coker}(\ker \varphi) = \ker(\operatorname{coker} \varphi)$ *has* $\operatorname{im} \varphi_P$ *as its stalks.*
- (e) *$\mathcal{S}h(X)$ is an abelian category, and $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ is exact $\Leftrightarrow \forall P \in X$: $\mathcal{F}_P \to \mathcal{G}_P \to \mathcal{H}_P$ is exact.*
- (s) *On $X = \operatorname{Spec} A$, the functor $M \mapsto \widetilde{M}$ is exact. Moreover, $\Gamma(\operatorname{Spec} A, \bullet)$ is exact on "quasi coherent" sheaves, i.e. those of type $\widetilde{M}$.*

*Proof.* (c, $\Rightarrow$) $\mathcal{F} \mapsto \mathcal{F}_P$ is exact on $\mathcal{P}re\mathcal{S}h$; sheafification does not change the stalks. (c, $\Leftarrow$) $\mathcal{F} \to \mathcal{G} \to \mathcal{C}$ is zero $\rightsquigarrow$ there is a map $\operatorname{coker}^{\mathrm{pre}} \varphi \to \mathcal{C}$ inducing isomorphisms on the stalks. □

In general, both the section functors and $\iota$ are left exact functors on $\mathcal{S}h(X)$. If $\mathcal{R}$ is a ring sheaf, then tensorizing with locally free sheaves is exact; isomorphism classes of invertible sheaves (with respect to $\mathcal{R}$) form a group under $\otimes_{\mathcal{R}} \rightsquigarrow \operatorname{Pic}(X, \mathcal{O}_X)$.

18.7. **Changing the topological space.** $f : X \to Y$ continous $\rightsquigarrow f_* : \mathcal{P}re\mathcal{S}h(X) \to \mathcal{P}re\mathcal{S}h(Y)$ and $f_* : \mathcal{S}h(X) \to \mathcal{S}h(Y)$ via $(f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$. This functor is left exact, but is has no good description on the level of stalks.

On the other hand, $f^{-1} : \mathcal{P}re\mathcal{S}h(Y) \to \mathcal{P}re\mathcal{S}h(X)$, $(f^{-1}\mathcal{G})(U) := \varinjlim_{V \supseteq f(U)} \mathcal{G}(V)$ is exact; it requires sheafifying to $f^{-1} : \mathcal{S}h(Y) \to \mathcal{S}h(X)$, but since $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$ it stays exact at the sheaf level.

$\mathrm{Hom}_X(f^{-1}\mathcal{G},\mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$, since both mean a system of compatible maps $\mathcal{G}(V) \to \mathcal{F}(U)$ for $f(U) \subseteq V$, i.e. $U \subseteq f^{-1}(V)$. Hence, $f^{-1} \dashv f_*$.

## 19. Schemes

19.1. **Locally ringed spaces.** $f = (f, f^*) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called a *morphism of locally ringed spaces* $:\Leftrightarrow$ $f : X \to Y$ is continuous and $f^* : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is local, i.e. $f^*_P : \mathcal{O}_{Y,f(P)} \to (f_*\mathcal{O}_X)_{f(P)} \to \mathcal{O}_{X,P}$ satisfies $f^*_P(\mathfrak{m}_{f(P)}) \subseteq \mathfrak{m}_P$. (The latter means $f^*(\varphi) = \varphi \circ f$ if the ring sheaves consist of true functions into the base field; counter example: $\mathrm{Spec}\, k(x) \stackrel{\eta\mapsto 0}{\longrightarrow} \mathrm{Spec}\, k[x]_{(x)}$).

**Proposition 55.** *The full subcategory affSch* $= \{(\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A}) = (\mathrm{Spec}\, A, A)\}$ *coincides with this from* (1.7)*, i.e. with* $\mathcal{R}ings^{\mathrm{opp}}$*; similarily* $\boxed{\mathit{affSch}_k^{\mathrm{opp}} \stackrel{\sim}{\to} \mathcal{A}lg_k}$.

*Proof.* $f : (\mathrm{Spec}\, B,\, \mathcal{O}_B) \to (\mathrm{Spec}\, A,\, \mathcal{O}_A) \rightsquigarrow \varphi := \Gamma(\mathrm{Spec}\, A, f^*) : A \to B \rightsquigarrow g := (\mathrm{Spec}\, \varphi) : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ with $g^{-1}(D_A(a)) = D_B(\varphi(a))$ and $g^* : \mathcal{O}_A \to g_*\mathcal{O}_B$ via $\varphi : A_a \to B_{\varphi(a)}$. Since, for $Q \in \mathrm{Spec}\, B$, the homomorphism $\varphi : A_{\varphi^{-1}(Q)} \to B_Q$ is clearly local, it remains to check that $(f, f^*) = (g, g^*)$:
The original $f$ gives rise to local $A_{f(Q)} \to B_Q$ compatible with $\varphi : A \to B$. Hence $\varphi(A \setminus f(Q)) \subseteq B \setminus Q$ and $\varphi(f(Q)) \subseteq Q$, i.e. $f(Q) = \varphi^{-1}(Q)$. Moreover, since $f^* : A_a \to B_{\varphi(a)}$ is compatible with $\varphi = \Gamma(\mathrm{Spec}\, A, f^*)$, it equals $\varphi = g^*$. ☐

Using $\mathcal{O}_Y \to f_*\mathcal{O}_X$, the push forward functor $f_*$ becomes $f_* : \mathcal{S}h_{\mathcal{O}_X} \to \mathcal{S}h_{\mathcal{O}_Y}$. On the other hand, if $\mathcal{G}$ is a $\mathcal{O}_Y$-module, then $f^{-1}\mathcal{G}$ is just a $f^{-1}\mathcal{O}_Y$-module, and we use $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ to define $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ (including sheafifying again). It remains just right exact, but we still have $f^* \dashv f_*$. week 9 (47)

19.2. **Definition of schemes.** A locally ringed space $(X, \mathcal{O}_X)$ is called *scheme* $:\Leftrightarrow$ $X = \bigcup_i U_i$ with affine schemes $\boxed{(U_i, \mathcal{O}_X|_{U_i}) \cong (\mathrm{Spec}\, A_i, \mathcal{O}_{\mathrm{Spec}\, A_i})}$; gluing maps $\rightsquigarrow$ $\mathrm{Hom}_{\mathcal{S}ch}((X, \mathcal{O}_X), \mathrm{Spec}\, A) = \mathrm{Hom}_{\mathcal{R}ings}(A, \Gamma(X, \mathcal{O}_X))$.
*Example:* $\mathrm{Proj}\, S = \bigcup_{f \in S_{d\geq 1}} \mathrm{Spec}\, S_{(f)}$ with $\mathcal{O}_{\mathrm{Proj}\, S} = \widetilde{S}$.

**Lemma 56.** $\mathrm{Spec}\, A, \mathrm{Spec}\, B \subseteq X$ *open* $\Rightarrow \exists$ *covering* $\{U_\nu\}$ *of* $(\mathrm{Spec}\, A) \cap (\mathrm{Spec}\, B)$ *such that* $U_\nu$ *equals both* $\mathrm{Spec}\, A_{f_\nu}$ *and* $\mathrm{Spec}\, B_{g_\nu}$ *(for some* $f_\nu \in A$, $g_\nu \in B$*).*

*Proof.* w.l.o.g. $\mathrm{Spec}\, A \subseteq \mathrm{Spec}\, B$ (consider an affine covering $\{\mathrm{Spec}\, C_\nu\}$ of the intersection and intersect $(\forall \nu)$ both coverings of $\mathrm{Spec}\, C_\nu$); then, if $\mathrm{Spec}\, B_g \subseteq \mathrm{Spec}\, A$, we have that $\mathrm{Spec}\, B_g = (\mathrm{Spec}\, A) \times_{\mathrm{Spec}\, B} (\mathrm{Spec}\, B_g) = \mathrm{Spec}\, A_g$. ☐

19.3. **Constructions with schemes.** We recall a couple of basic properties mostly being treated in the previous sections for the affine case or in the exercises:

19.3.1. *Morphisms and regular functions.* $A$ is considered the ring of regular functions on $\operatorname{Spec} A$ via $(a \in A)(P \in \operatorname{Spec} A) := \overline{a} \in A/P \subseteq \operatorname{Quot}(A/P) =: K(P)$. If $\varphi : A \to B$ gives rise to $(f = \varphi^{\#}) : \operatorname{Spec} B \to \operatorname{Spec} A$, then for a $Q \in \operatorname{Spec} B$ and $P := f(Q) = \varphi^{-1}(Q) \subseteq A$ we obtain the commutative diagram

$$\begin{array}{ccc} A/P & \hookrightarrow & B/Q \\ \downarrow & & \downarrow \\ K(P) & \hookrightarrow & K(Q), \end{array}$$

i.e. for $a \in A$ we have $a(f(Q)) = a(P) = \varphi(a)(Q) \in K(Q)$ implying that $\varphi(a) = a \circ f$ with both sides understood as maps on the spectra. However, an element $b \in B$ is determined by its values on $\operatorname{Spec} B$ only up to the nilradical $\sqrt{0}$.

19.3.2. *Closed embeddings.* $\varphi^{\#} : \operatorname{Spec} B \to \operatorname{Spec} A$ is a closed embedding $:\Leftrightarrow \varphi : A \twoheadrightarrow B$ is surjective; the special case $A_{\mathrm{red}} := A/\sqrt{0}$ yields a homeomorphism $(\operatorname{Spec} A)_{\mathrm{red}} := \operatorname{Spec} A_{\mathrm{red}} \hookrightarrow \operatorname{Spec} A$ (the "*reduced structure*" on $\operatorname{Spec} A$). (Counterexample: $k \subset K$ fields, but $\operatorname{Spec} K \to \operatorname{Spec} k$ is not a closed embedding.)
Non affine closed embeddings $\iota : Y \hookrightarrow X$ are defined locally on the target; the kernel of $A \twoheadrightarrow B$ is replaced by ideal sheaf $\mathcal{J} = \ker(\mathcal{O}_X \twoheadrightarrow \iota_* \mathcal{O}_Y)$.

19.3.3. *Open embeddings.* $\varphi^{\#}$ is dominant $\Leftarrow \varphi : A \hookrightarrow B$ is injective. The standard open embeddings are $\operatorname{Spec} A_f = D(f) \subseteq \operatorname{Spec} A$. For an open embedding $j : U \hookrightarrow X$ we have $\mathcal{O}_U = j^* \mathcal{O}_X = \mathcal{O}_X|_U$.

19.3.4. *Fiber product.* In the category of affine schemes $\operatorname{Spec} A \times_{\operatorname{Spec} S} \operatorname{Spec} B = \operatorname{Spec}(A \otimes_S B)$ is the fiber product. $\mathbb{A}^m \times_{\mathbb{Z}} \mathbb{A}^n = \mathbb{A}^{m+n}$ has *not* the product topology. $\mathbb{A}^n_A = \mathbb{A}^n \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} A = \mathbb{A}^n_k \times_{\operatorname{Spec} k} \operatorname{Spec} A$ (the latter for $k$-algebras only).
Beyond the affine case, in $\mathcal{S}ch$, fiber products $\boxed{X \times_S Y}$ do also exist – they arise from glueing the affine construction, c.f. Problem **??**.

week 10 (49)

19.3.5. *Preimages.* $f : \operatorname{Spec} A \to \operatorname{Spec} B \Rightarrow f^{-1}(\operatorname{Spec} B/J) = \operatorname{Spec}(A \otimes_B B/J) = \operatorname{Spec} A/JA$ and $f^{-1}(\operatorname{Spec} B_g) = \operatorname{Spec}(A \otimes_B B_g) = \operatorname{Spec} A_{\varphi(g)}$.

19.3.6. *Scheme theoretic image.* $f : \operatorname{Spec} A \to \operatorname{Spec} B$ with $\varphi : B \to A$ induces $B \twoheadrightarrow B/\ker\varphi \hookrightarrow A$, hence $\operatorname{Spec} A \stackrel{\text{domin}}{\longrightarrow} V(\ker\varphi) \subseteq \operatorname{Spec} B$. Thus, $V(\ker\varphi) = \overline{f(\operatorname{Spec} A)}$, and $\operatorname{Spec}(B/\ker\varphi)$ is the "smallest" scheme structure on $V(\ker\varphi)$ such that $f$ factors through.

19.3.7. *Closure.* $\operatorname{Spec}(A/(0 : f^\infty)) = \overline{D(f)} \subseteq \operatorname{Spec} A$ is the scheme theoretic image of $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$. Generalization (for noetherian $A$) to $\overline{\operatorname{Spec} A \setminus V(J)} = \bigcup_{f \in J} \overline{D(f)} = \bigcup_{f \in J} V(0 : f^\infty) = V(\bigcap_{f \in J}(0 : f^\infty)) = \operatorname{Spec}(A/(0 : J^\infty))$.

19.3.8. *Elimination.* $p : V(I) \subseteq \mathbb{A}^{m+n} \twoheadrightarrow \mathbb{A}^n$ corresponds to $p^* : k[\mathbf{y}] \hookrightarrow k[\mathbf{x}, \mathbf{y}] \twoheadrightarrow k[\mathbf{x}, \mathbf{y}]/I \Rightarrow \overline{p(V(I))} = \operatorname{Spec} k[\mathbf{y}]/\ker p^* = \operatorname{Spec} k[\mathbf{y}]/I \cap k[\mathbf{y}]$.

19.3.9. *K-rational points.* $X = \operatorname{Spec} A$; $K = $ field $\Rightarrow X(K) \overset{\text{Yoneda}}{:=} \operatorname{Hom}(\operatorname{Spec} K, X) = \operatorname{Hom}(A, K) = \{(P, i) \mid P \in \operatorname{Spec} A,\ i : K(P) \hookrightarrow K\}$. If $A = k$-algebra and $K \supseteq k$ is an extension field, then $X_k(K) = \operatorname{Hom}_k(A, K) = \{(P, i) \mid k \subseteq K(P) \hookrightarrow K\}$. If $[K : k] < \infty \overset{\text{Prop 24(2)}}{\Longrightarrow} P \in \operatorname{MaxSpec} A$. In particular, $X_k(k) = \{\mathfrak{m} \in \operatorname{MaxSpec} A \mid A/\mathfrak{m} = k\}$, e.g. $\mathbb{A}^n_k(k) = k^n$.

19.3.10. *Tangent directions.* $A = k$-algebra, $X = \operatorname{Spec} A \Rightarrow \operatorname{Hom}(\operatorname{Spec} k[\varepsilon]/\varepsilon^2, X) = \operatorname{Hom}_k(A, k[\varepsilon]/\varepsilon^2) = \{P \in X(k)$ with tangent directions, i.e. derivation $d : A \to k\}$ $(d(fg) = f(P)d(g) + d(f)g(P)$ by the multiplicativity of $f \mapsto f(P) + \varepsilon\, d(f))$.

If $k = \overline{k}$ and $(A, \mathfrak{m}) = $ local with $k \xrightarrow{\sim} A/\mathfrak{m}$, then $T_\mathfrak{m} := \operatorname{Der}_k(A, k) = \operatorname{Hom}_A(\mathfrak{m}, k) = \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$, in particular, $\mathfrak{m}/\mathfrak{m}^2 = T^*_\mathfrak{m}$ is the cotangent space. Thus, $(A, \mathfrak{m})$ is regular $\Leftrightarrow \dim_k T_\mathfrak{m} \geq \dim A$ becomes an equality.

19.4. **Finiteness assumptions.** Special properties of schemes and morphisms are:

(i) (Locally) *noetherian schemes* $X$, i.e. there is a [finite] open covering $X = \bigcup_i \operatorname{Spec} A_i$ with noetherian $A_i \Rightarrow$ every affine open $\operatorname{Spec} A \subseteq X$ has $A = $ noetherian [and $X$ is quasi compact].

This property is bequeathed to open and closed subschemes, and noetherian schemes imply that the underlying topological space is noetherian, i.e. that increasing chains of open subsets terminate.

(ii) $f : X \to Y$ is (locally) of *finite type* $:\Leftrightarrow$ $f$ locally (on $X$ as well as on $Y$) equals $f : \operatorname{Spec} A \to \operatorname{Spec} B$ with $B \to A$ being finitely generated algebras [and $f$ is quasi compact]. (For those $f$, "(locally) noetherian" is beqeathed from $Y$ to $X$.) week 11 (51)

(iii) $f : X \to Y$ is *affine* $:\Leftrightarrow$ the preimages of (a covering of) open, affine $\operatorname{Spec} B \subseteq Y$ are affine open subschemes $\operatorname{Spec} A \subseteq X$.

(iv) $f : X \to Y$ is *finite* $:\Leftrightarrow$ $f$ is affine with $B \to A$ being finite homomorphisms, i.e. $A$ becomes a finitely generated $B$-module.

19.5. **Integral schemes and varieties.** $X$ is *reduced* $:\Leftrightarrow$ all (or a cover of) open $\operatorname{Spec} A \subseteq X$ satisfy $\sqrt{0} = 0$; $X$ is *integral* if it is, additionally, *irreducible*, i.e. if all (or a cover of) open $\operatorname{Spec} A \subseteq X$ are dense with $A$ being integral domains.

Integral schemes $X$ have a unique *generic point* $\eta_X$ (sitting in every non-empty open subset) and give rise to a function field $K(X) := \mathcal{O}_{X,\eta} = \varinjlim_{U \subseteq X} \mathcal{O}_X(U) = \operatorname{Quot} A$ for every such $\operatorname{Spec} A \subseteq X$. If $X = \operatorname{Proj} S$ (with an integral, graded ring $S$), then $K(X) = S_{((0))}$.

A scheme $X = (X, \mathcal{O}_X)$ is called a *variety* over $k$ $:\Leftrightarrow$ $X$ is integral, of finite type over $\operatorname{Spec} k$, and separated (the intersection of affine $U, V \subseteq X$ is again affine, and $\Gamma(U, \mathcal{O}), \Gamma(V, \mathcal{O})$ generate $\Gamma(U \cap V, \mathcal{O})$ as rings). Separation of a morphism $X \to S$ means that the diagonal $\Delta : X \to X \times_S X$ is a closed embedding.

## 20. Separated morphisms

20.1. **Simulating Hausdorff.** $f : X \to Y$ is called "*separated*" $:\Leftrightarrow \Delta : X \hookrightarrow X \times_Y X$ is a closed embedding $\Leftrightarrow \Delta(X) \subseteq X \times_Y X$ is a closed subset (everything is local on $Y$, for affine $X, Y$ the first (and stronger) fact is always true, and for non-affine $X$, we can cover $X \times_Y X$ by $U_i \times_Y U_i$ and $(X \times_Y X) \setminus \Delta(X)$). *Counter example:* [$\mathbb{A}^1_k$ with double origin] $= \mathbb{TV}\big([0,\infty) \cup_{\{0\}} [0,\infty)\big)$, instead of $\mathbb{P}^1 = \mathbb{TV}\big((-\infty, 0] \cup_{\{0\}} [0,\infty)\big)$.

*Properties:* Closed and open *embeddings* are separated ($f : Z \hookrightarrow Y$ is affine; $U \stackrel{\Delta}{\to} U \times_Y U$ is an isomorphism); invariance under *base change*; the *composition* $X \stackrel{f}{\to} Y \stackrel{g}{\to} Z$ of separated $f, g$ is separated ($X \times_Y [Y \stackrel{\Delta}{\to} Y \times_Z Y] \times_Y X = [X \times_Y X \to X \times_Z X]$).

$$X \xrightarrow[\text{separated}]{f} Y \xrightarrow{g} Z, \qquad gf: X \to Z \text{ separated} \;\Rightarrow\; f \text{ separated}$$

[$gf$ separated $\Rightarrow f$ separated]: $X \hookrightarrow X \times_Y X \hookrightarrow X \times_Z X$ is altogether a closed embedding, and the latter map is injective (test $\operatorname{Spec} K \to \ldots$).

"Varieties over $k$" $:\Leftrightarrow$ separated schemes $X \to \operatorname{Spec} k$ of finite type.

20.2. **Intersection of affine sets.** For the absolute separateness (over $\operatorname{Spec} \mathbb{Z}$ or, for $k$-schemes, over $\operatorname{Spec} k$), there is the following criterion:

**Proposition 57.** *$X \to \operatorname{Spec} B$ is separated $\Leftrightarrow$ for open, affine $U, V \subseteq X$ the set $\boxed{U \cap V \text{ is again affine}}$, and $\Gamma(U, \mathcal{O}_X) \otimes_B \Gamma(V, \mathcal{O}_X) \twoheadrightarrow \Gamma(U \cap V, \mathcal{O}_X)$ is surjective.*

*Proof.*

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \times_B V \\ \downarrow{\scriptstyle \text{open}} & & \downarrow{\scriptstyle \text{open}} \\ X & \xrightarrow[\text{closed?}]{\Delta} & X \times_B X \end{array}$$

$U \times_B V = \mathrm{pr}_1^{-1}(U) \cap \mathrm{pr}_2^{-1}(V)$ as open subschemes of $X \times_B X$, hence $(U \cap V) = \Delta^{-1}(U \times_B V)$. On the other hand, $U \times_B V$ is affine, and closedness is a local property. □

*Consequence:* $\mathbb{TV}(\Sigma, N)$, thus in particular $\mathbb{P}^n$, is separated.

20.3. **Maximal domains of definition.** Let $f, g : [X = \text{reduced}] \to [Y = \text{separated}]$ over $S$ with $f = g$ on a dense, open $U \subseteq X \Rightarrow f = g$ on $X$. In particular, rational maps have always a maximal domains of definition:

week 12 (53)

$F|_U$ factorizes over $\Delta(Y) \Rightarrow X' \subseteq X$ is a closed subscheme containing $U$.

$$\begin{array}{ccc} X & \xrightarrow[(f,g)]{F} & Y \times_S Y \\ \uparrow & & \uparrow \\ X' := F^{-1}(\Delta(Y)) & \longrightarrow & \Delta(Y) \end{array}$$

$X, Y = k$-varieties $\rightsquigarrow$ {dominant rational maps $f : X \dashrightarrow Y$} $=$ {$k$-embeddings $K(Y) \hookrightarrow K(X)$}: If $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, then $\operatorname{Quot}(B) \to \operatorname{Quot}(A)$ lifts to $B \hookrightarrow A_f$. Birational $\Leftrightarrow K(Y) = K(X)$.

$k$ = perfect $\Rightarrow$ for each field extension $K = k(\alpha_1, \ldots, \alpha_m) \supseteq k$ there is an $e \subseteq \{\alpha_1, \ldots, \alpha_m\}$ with $K \supseteq k(e) \supseteq k$ (separable|transzendent), cf. [ZS, ch. II, Th 30+31, S.104]. "Satz vom primitiven Element" $\Rightarrow d$-dimensional $k$-varieties are birational equivalent to hypersurfaces in $\mathbb{P}^{d+1}$.

↗ §12

## 21. Quotient singularities and resolutions

week 14 (57)

21.1. **Simplicial cones.** $G$ = [finite abelian group] acts via $\deg : \mathbb{Z}^n \to B := \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^*)$ (characters of $G$) linearly on $\mathbb{A}^n_{\mathbb{C}}$, i.e. $b_i := \deg(e_i) \rightsquigarrow g(x_i) = b_i(g) \cdot x_i$.

$\mathbf{x}^r \in \mathbb{C}[\mathbb{Z}^n]$ is $G$-invariant $\Leftrightarrow \forall g \in G:\ g(\mathbf{x}^r) = \mathbf{x}^r \Leftrightarrow \forall g \in G:\ (\deg r)(g) = 1 \Leftrightarrow \deg r = 1$; i.e. $M := \ker(\deg : \mathbb{Z}^n \to B)$ yields $\mathbb{C}[M] = \mathbb{C}[\mathbb{Z}^n]^G \subseteq \mathbb{C}[\mathbb{Z}^n]$. In particular, $\mathbb{A}^n_{\mathbb{C}}/G = \mathrm{Spec}[\mathbb{Z}^n_{\geq 0}]^G = \mathrm{Spec}\,\mathbb{C}[\mathbb{Q}^n_{\geq 0} \cap M]$.

Let $0 \to M \to \mathbb{Z}^n \to B \to 0$ be exact; dualizing $\rightsquigarrow 0 \to \mathbb{Z}^n \to N \to \mathrm{Ext}^1_{\mathbb{Z}}(B, \mathbb{Z}) \to 0$; the injective resolution $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ shows that $\mathrm{Ext}^1_{\mathbb{Z}}(B, \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{C}^*) = G$, hence $0 \to \mathbb{Z}^n \xrightarrow{p} N \to G \to 0$. (If deg is not surjective, the we replace $B$ by the image and change $G$ accordingly.)

$M_{\mathbb{Q}} \xrightarrow{\sim} \mathbb{Q}^n$ and $p : \mathbb{Q}^n \xrightarrow{\sim} N_{\mathbb{Q}}$ are isomorphisms; $(\mathbb{Q}^n_{\geq 0})^\vee = \mathbb{Q}^n_{\geq 0} \rightsquigarrow \sigma := p(\mathbb{Q}^n_{\geq 0}) \subseteq N_{\mathbb{Q}}$ is simplicial (spanned by the $p(e^i)$) and $\boxed{\mathbb{A}^n_{\mathbb{C}}/G = \mathbb{TV}(\sigma, N)}$; on the other hand, all simplicial cones lead to abelian quotient singularities.

**Example 58.** $\mu_r \subseteq \mathbb{C}^*$ acts on $\mathbb{C}^n$ via $\xi \mapsto \mathrm{diag}(\xi^{a_1}, \ldots, \xi^{a_n})$ with $\mathbf{a} \in \mathbb{Z}^n$ such that $\gcd(\mathbf{a}, r) = 1$. With $\mathrm{Hom}_{\mathbb{Z}}(\mu_r, \mathbb{C}^*) = \mathbb{Z}/r\mathbb{Z}$ this yields $0 \to M \to \mathbb{Z}^n \xrightarrow{\mathbf{a}} \mathbb{Z}/r\mathbb{Z} \to 0$, hence $N = \langle \mathbb{Z}^n, \frac{1}{r}\mathbf{a}\rangle_{\mathbb{Z}} \subseteq \mathbb{Q}^n$ with $\frac{1}{r}\mathbf{a} \mapsto 1 \in \mathbb{Z}/r\mathbb{Z}$. Denote this particular $\mathbb{A}^n_{\mathbb{C}}/\mu_r =: \frac{1}{r}\mathbf{a}$.

*Using coordinates in dimension two:* $\mu_n \subseteq \mathbb{C}^*$ acts on $\mathbb{C}^2$ via $\xi \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix}$; this yields $0 \to \left(M = \mathbb{Z}\begin{bmatrix} -q \\ 1 \end{bmatrix} \oplus \mathbb{Z}\begin{bmatrix} n \\ 0 \end{bmatrix}\right) \to \mathbb{Z}^2 \xrightarrow{(1,q)} \mathbb{Z}/n\mathbb{Z} \to 0$, hence the map $\mathbb{Z}^2 \to N = \mathbb{Z}^2$ is given by the matrix $\begin{pmatrix} -q & 1 \\ n & 0 \end{pmatrix}$, i.e. $X_{n,q} := \frac{1}{n}(1, q) = \mathbb{C}^2/\mu_n = \mathbb{TV}(\sigma, \mathbb{Z}^2)$ with $\boxed{\sigma := \langle (1,0),\, (-q, n)\rangle} \subseteq \mathbb{Q}^2$.

21.2. **CQS in dimension two.** Let $q \in (\mathbb{Z}/n\mathbb{Z})^*$ with $0 \leq q < n$; cone $\sigma := \langle (1,0),\, (-q,n)\rangle \subseteq \mathbb{Q}^2 = N_{\mathbb{Q}}$; let $(1,0) = s^0, \ldots, s^{m+1} = (-q, n)$ be the lattice points on the compact edges of $\nabla := \mathrm{conv}\left((\sigma \cap N) \setminus 0\right) \rightsquigarrow \boxed{s^{i-1} + s^{i+1} = b_i\, s^i}$ with $b_i \in \mathbb{Z}_{\geq 2},\ i = 1, \ldots, m$ ($0, s^i, s^{i+1}$ are vertices of an elementary triangle $\Rightarrow \{s^i, s^{i+1}\}$ are $\mathbb{Z}$-bases of $N$).

**Definition 59.** $c_i \in \mathbb{Z}_{\geq 2} \rightsquigarrow$ continued fraction $[c_1, \ldots, c_\ell] := c_1 - 1/[c_2, \ldots, c_\ell]$.

**Proposition 60.** $n > 1 \Rightarrow n/q = [b_1, \ldots, b_m]$.

*Proof.* Since $(1,0) + s^2 = b_1\,(0,1)$, one obtains $s^2 = (-1, b_1) \Rightarrow s^2$ is the lowest lattice point on $(-1, *)$ above $\mathbb{Q}_{\geq 0}(-1, n/q)$, i.e. $b_1 = \lceil n/q \rceil$ $(= \lfloor n/q \rfloor + 1$ if $q \neq 1)$. Induction: The cone $\sigma' := \langle (0,1), (-q,n)\rangle$ cut off from $\sigma$ along $s^1$ becomes $\sigma' \cong \langle (1,0), (n,q)\rangle$ after the coordinate change $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$; afterwards, the first entry of $(n,q) \Rightarrow (-q', n')$ will be normalized within $(\mathbb{Z}/q\mathbb{Z})^*$ toward $-q' = n - \lceil n/q \rceil q =$

$n - b_1 q \Rightarrow q/(b_1 q - n) = [b_2, \ldots, b_m] \Rightarrow 1/[b_2, \ldots, b_m] = q'/n' = (b_1 q - n)/q = b_1 - n/q$. □

21.3. **Duality.** $\{s^0, \ldots, s^{m+1}\}$ is the Hilbert basis of $\sigma$ (since $\{s^i, s^{i+1}\}$ are $\mathbb{Z}$-bases of $N$ and $\nabla$ is convex); denote by $\{t^0, \ldots, t^{k+1}\}$ the Hilbert basis of $\sigma^\vee = \langle [0,1], [n,q] \rangle \cong \langle [0,1], [n, q-n] \rangle \cong \langle [1,0], [q-n, n] \rangle \rightsquigarrow \boxed{t^{j-1} + t^{j+1} = a_j t^j}$ with $n/(n-q) = [a_1, \ldots, a_k]$. $\rightsquigarrow$ equations $z_{j-1} z_{j+1} = z_j^{a_j}$ of $X_{n,q} \subseteq \mathbb{A}^{k+2}$.

$\ddot{\partial}\nabla := \partial\nabla \setminus \partial\sigma =$ union of the compact edges of $\nabla$ *without* the two extremal vertices, i.e. $\ddot{\partial}\nabla \cap N = \{s^1, \ldots, s^m\}$; analogously $\{t^1, \ldots, t^k\} \subset \ddot{\partial}\Delta \subset \Delta \subset \sigma^\vee$.

**Proposition 61.** 1) $\mathcal{P} := \{(i,j) \in [1,m] \times [1,k] \mid \langle s^i, t^j \rangle = 1\} \subset \big(\mathbb{Z}^2, (\leq, \leq)\big)$ *is totally ordered; it forms a path leading from* $(1,1)$ *to* $(m,k)$ *along horizontal or vertical edges only.*

2) *Length of the horizontal edge* $(\bullet, j)$ *in* $\mathcal{P} = (a_j - 2) =$ *length of* $\nabla \cap [t^j = 1]$.

3) *Length of the vertical edge* $(i, \bullet)$ *in* $\mathcal{P} = (b_i - 2) =$ *length of* $\Delta \cap [s^i = 1]$.

$\rightsquigarrow$ Riemenschneider's point diagram; $\ddot{\partial}\Delta / \ddot{\partial}\nabla$-duality (vertices $\widehat{=}\ a_j / b_i \geq 3$).

*Proof.* (i) $\overline{s^i s^{i+1}} \subseteq$ edge of $\nabla \Rightarrow \overline{s^i s^{i+1}} \subset [t = 1]$ with $t \in \{t^1, \ldots, t^k\} = \ddot{\partial}\Delta \cap M$: $\{s^i, s^{i+1}\}$ is basis $\Rightarrow t \in M$; $[t = 1]$ meets both $\sigma$-edges $\Rightarrow t \in \operatorname{int} \sigma^\vee$; all splittings $t = t' + t''$ in $\sigma^\vee \cap M$ contradict $s^i \in \operatorname{int} \sigma$ (oder $s^{i+1} \in \operatorname{int} \sigma$).

(ii) Every $[t^j = 1]$ cuts off a $\ddot{\partial}\nabla$-face:
Edge $\overline{t^j t^{j+1}} \overset{(i)}{\hookrightarrow} [s^i = 1] \Rightarrow s^i \in [t^j = 1]$; moreover $[t^j \leq 0] \cap \sigma = \{0\}$.

(i)+(ii) $\Rightarrow$ (1) and [length of the horizontal $\mathcal{P}$-edges] = [length of the $\nabla$-edges].

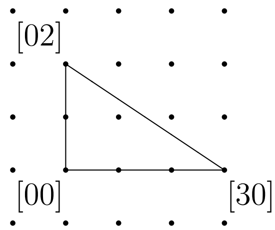
(iii) $\{s^i, \ldots, s^{i+\ell}\} = \nabla \cap [t^j = 1]$-edge with $\ell \geq 1$ in the direction $v := s^{i+1} - s^i \Rightarrow \langle v, t^j \rangle = 0 \Rightarrow \langle v, t^{j-1} \rangle = 1$ ($\{t^{j-1}, t^j\} =$ basis) $\Rightarrow \langle s^{i+\ell}, t^{j-1} \rangle = \langle s^i, t^{j-1} \rangle + \ell$, hence $0 = \langle s^{i+\ell},\ t^{j-1} + t^{j+1} - a_j t^j \rangle = (1 + \ell) + 1 - a_j$. □

Ende Alg II

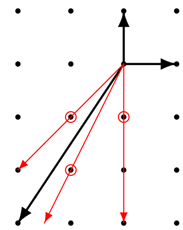
21.4. **Weighted projective spaces.** Let $\mathbf{w} \in \mathbb{Z}^{n+1}$ be primitive $\rightsquigarrow \mathbb{P}(\mathbf{w}) := \mathbb{A}^{n+1} \setminus \{0\} / \mathbb{C}^*$ with $t\,(z_0, \ldots, z_n) := (t^{w_0} z_0, \ldots, t^{w_n} z_n)$, i.e. in the language of (21.1), $\deg : \mathbb{Z}^{n+1} \overset{\mathbf{w}}{\to} \mathbb{Z} = \operatorname{Hom}_{\mathrm{alGr}}(\mathbb{C}^*, \mathbb{C}^*)$, i.e. $\mathbb{P}(\mathbf{w}) = \operatorname{Proj} \mathbb{C}[\mathbf{z}]$ with this grading. The charts are $D_+(z_i) = \operatorname{Spec} k[\ker \mathbf{w} \cap C_i^\vee]$ where $C_i := \partial_i \mathbb{Q}^{n+1}_{\geq 0}$. Thus $\mathbb{P}(\mathbf{w}) = \boxed{\mathbb{TV}(\pi(\partial \mathbb{Q}^{n+1}_{\geq 0}),\ \mathbb{Z}^{n+1}/\mathbf{w}\mathbb{Z})}$. If $\{\mathbf{w}, a^1, \ldots, a^n\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^{n+1}$, then the chart $D_+(z_i)$ has a cyclic quotient singularity of type $\frac{1}{w_i}(a_i^1, \ldots, a_i^n)$.

General procedure: If $\Delta \subseteq M_{\mathbb{Q}}$ is a polyhedron with $\operatorname{cone} \Delta := \mathbb{Q}_{\geq 0}(\Delta, 1) \subseteq M_{\mathbb{Q}} \oplus \mathbb{Q}$, then $(\operatorname{cone} \Delta)^\vee \subseteq N_{\mathbb{Q}} \oplus \mathbb{Q}$ projects to the inner normal fan $\mathcal{N}(\Delta)$. In particular, the fan of $\mathbb{P}(\mathbf{w})$ equals the normal fan of $\Delta_{\mathbf{w}} := [\mathbf{w} = 1] \cap \mathbb{Q}^{n+1}_{\geq 0}$ (or integral multiples).

**Example 62.** The singular charts of $\mathbb{P}(1,2,3)$ are $\operatorname{Spec} \mathbb{C}[z_0^2/z_1,\ z_0 z_2/z_1^2,\ z_2^2/z_1^3]$ and $\operatorname{Spec} \mathbb{C}[z_0^3/z_2,\ z_0 z_1/z_2,\ z_1^3/z_2^2]$ with an $A_1 = \frac{1}{2}(1,-1)$ and an $A_2 = \frac{1}{3}(1,-1)$-singularity, respectively. Projecting $6\Delta_{\mathbf{w}} = \operatorname{conv}\{[600], [030], [002]\} \subseteq \mathbb{Q}^3 \overset{\mathrm{pr}_{23}}{\twoheadrightarrow} \mathbb{Q}^2$ yields




$6\Delta_{(1,2,3)} \subseteq M_{\mathbb{Q}}$



(subdivided) Fan of $\mathbb{P}(1,2,3)$ in $N_{\mathbb{Q}}$.

21.5. **Toric Resolutions.** Let $\sigma \subseteq N_{\mathbb{Q}}$ be a full-dimensional polyhedral cone: Hilbert basis $E \subseteq \sigma^\vee \cap M \rightsquigarrow 0 \in \mathbb{TV}(\sigma) \subseteq \mathbb{A}^E$ corresponds to the ideal $\mathfrak{m}_0 = (z_e \,|\, e \in E) \subseteq k[\mathbf{z}] \twoheadrightarrow k[\sigma^\vee \cap M] \Rightarrow \mathfrak{m}_0/\mathfrak{m}_0^2 = k^E$, but $\dim \mathbb{TV}(\sigma) = \operatorname{rank} N =: n$. In particular, $\mathbb{TV}(\sigma)$ is smooth in $0 \Leftrightarrow (\sigma, N) \cong (\mathbb{Q}^n_{\geq 0}, \mathbb{Z}^n) \Leftrightarrow \mathbb{TV}(\sigma) \cong \mathbb{A}^n$.

(i) If $\sigma$ is as in (21.2), then the subdivision into the fan $\Sigma$ with $\Sigma(1) = \{s^0, \ldots, s^{m+1}\}$ yields a resolution $\pi : \mathbb{TV}(\Sigma) \to \mathbb{TV}(\sigma)$ of the isolated singularity $0 \in \mathbb{TV}(\sigma)$, e.g. the red rays in the right figure in Example 62 (with self intersection numbers $(\overline{\operatorname{orb}}(s^i)^2) = -b_i$ similarily to $(E^2) = -1$ in the blow up of $\mathbb{A}^2$).

(ii) Every $\mathbb{TV}(\sigma)$ allows such a resolution: First, subdivide $\sigma$ into a simplicial fan; afterwards, if $\sigma = \langle a^1, \ldots, a^n \rangle \subseteq \mathbb{Q}^n$ is still not smooth, then there is an $a^* \in \mathbb{Z}^n \cap \sum_{i=1}^n [0,1) a^i$, hence the cones $\tau_i := \langle a^*, a^1, \ldots, \hat{a^i}, \ldots, a^n \rangle \subseteq \sigma$ improve the situation: With $a^* = \sum_{i=1}^n \lambda_i a^i$ we have that $\operatorname{vol}(\tau_i) = \lambda_i \operatorname{vol}(\sigma)$. Eventually, we obtain a "smooth" subdivision $\Sigma \leq \sigma$.

21.6. **Resolutions via Newton polytopes.** Let $f \in \mathbb{C}[\mathbf{x}]$ with $f(0) = 0 \rightsquigarrow$ the hypersurface $V(f) = \operatorname{Spec} \mathbb{C}[\mathbf{x}]/(f)$ is regular (smooth) in $0 \Leftrightarrow x_1, \ldots, x_n$ are linearly dependent in $(\mathbf{x})/(\mathbf{x}^2, f) \Leftrightarrow f'(0) = (\partial_1 f(0), \ldots, \partial_n f(0)) \neq 0$.

Let $g \in \mathbb{C}[\mathbf{x}]$ and $I \subseteq [n] := \{1, \ldots, n\}$ with $J := [n] \setminus I$. Then $V(g)$ is called *transversal* to the coordinate hyperplane $\mathbb{C}^J = V(x_I)$ in $c = (c_I = 0, c_J) \in V(g) \cap \mathbb{C}^J$ $:\Leftrightarrow V(g|_{\mathbb{Z}^J})$ is smooth in $c_J$ (or in any $(*, c_J)$) $\Leftrightarrow \exists\, \partial_{j(\in J)}(g|_{\mathbb{Z}^J})(c) \neq 0$. Since for $I' \subseteq I$ (hence $J' \supseteq J$) the $\mathbb{C}^J$-transversality implies that with $\mathbb{C}^{J'}$ (all monomials of $g|_{\mathbb{Z}^{J'}}$ not in $g|_{\mathbb{Z}^J}$ yield 0 whenever applied to $c$), we obtain:

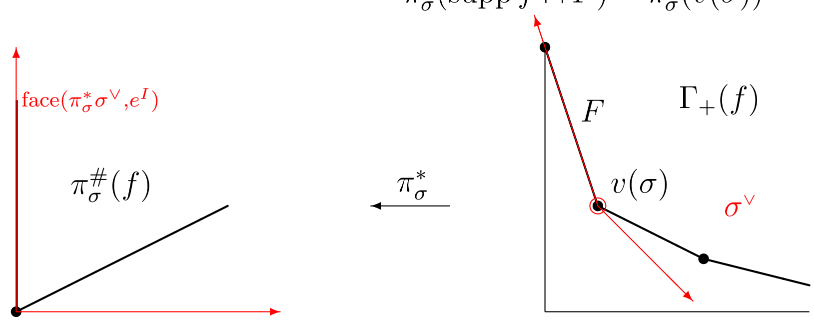
True transversality to the origin ($I = [n]$) is not possible – it can only be obtained via $0 \notin V(g)$, i.e. $g(0) \neq 0$. $V(g)$ is transversal to all coordinate planes in $\mathbb{C}^n$ (hence smooth in $\mathbb{C}^n \setminus (\mathbb{C}^*)^n$) $\Leftrightarrow$ there is no $J \subsetneq [n]$ such that the system $g|_{\mathbb{Z}^J} = \partial_\bullet(g|_{\mathbb{Z}^J}) = 0$ has a solution inside the torus $(\mathbb{C}^*)^n$.

*Varchenko's resolution of hypersurfaces*: $f \in \mathbb{C}[x_1, \ldots, x_n]$ with $f(0) = 0 \rightsquigarrow$ "Newton polyhedra" $\Gamma(f) := \operatorname{conv}(\operatorname{supp} f) \subseteq \mathbb{Q}^n_{\geq 0}$ and $\Gamma_+(f) := \Gamma(f) + \mathbb{Q}^n_{\geq 0}$; let $\Sigma \leq \mathcal{N}(\Gamma_+(f)) \leq \mathbb{Q}^n_{\geq 0}$ be a smooth subdivision $\rightsquigarrow X := \pi^{\#}(V(f))$ = strict transform via $\pi : \mathbb{TV}(\Sigma) \to \mathbb{C}^n$.

**Proposition 63.** *Assume that $0 \in V(f)$ is an isolated singularity and let $f$ be* *non-degenerate* *on the Newton boundary, i.e. for no compact face $F \leq \Gamma_+(f)$, the* *equations $\partial_\bullet(f|_F) = 0$ have a common solution inside $(\mathbb{C}^*)^n$. Then $X$ is smooth in a neighborhood of $E := \pi^{-1}(0) \subsetneq \mathbb{TV}(\Sigma)$, and $X$ is transversal to $E$.*

*Proof.* Every $\sigma = \langle a^1, \dots, a^n\rangle \in \Sigma$ has an associated vertex $v(\sigma) \in \Gamma_+(f)$. The map $\pi_\sigma : \mathbb{C}^n \cong \mathbb{TV}(\sigma) \to \mathbb{TV}(\mathbb{Q}^n_{\ge 0}) = \mathbb{C}^n$ is given on the $N$-level by $A : (\mathbb{Q}^n_{\ge 0}, \mathbb{Z}^n) \xrightarrow{\sim} (\sigma, \mathbb{Z}^n) \stackrel{\text{id}}{\hookrightarrow} (\mathbb{Q}^n_{\ge 0}, \mathbb{Z}^n)$, i.e. $A : \mathbb{Z}^n \to \mathbb{Z}^n$ sends $e^i \mapsto a^i$. Pulling back functions means $\pi^*_\sigma(x^r) = x^s$ with $s = A^T r$, i.e. $\langle e^i, s\rangle = \langle a^i, r\rangle$. In particular, $\pi^*_\sigma(\Gamma_+(f)) \subseteq \pi^*_\sigma(v(\sigma)) + \mathbb{N}^n$, i.e. $\pi^*_\sigma(f) = x^{\pi^*_\sigma(v(\sigma))}\,\pi^\#_\sigma(f)$ with $\pi^\#_\sigma(f)(0) \neq 0$. Moreover, $\pi^{-1}_\sigma(0) \subseteq \mathbb{C}^n \setminus (\mathbb{C}^*)^n$ and

$$\begin{aligned}\operatorname{face}(\pi^*_\sigma\sigma^\vee, e^I) \cap \operatorname{supp}\pi^\#_\sigma(f) &= \pi^*_\sigma\big(\operatorname{face}(\sigma^\vee, A(e^I)) \cap \operatorname{supp} f/x^{v(\sigma)}\big)\\ &= \pi^*_\sigma(\operatorname{supp} f \cap F) - \pi^*_\sigma(v(\sigma))\end{aligned}$$




for some (compact) face $F \le \Gamma_+(f)$. Finally, since we just care about solutions in $(\mathbb{C}^*)^n$, we may use that (i) $\pi_\sigma$ becomes an automorphism, (ii) the monomial $x^{v(\sigma)}$ does not matter, and (iii) we may replace $\partial_i x^r$ by $x_i\,\partial_i x^r = \langle e^i, r\rangle x^r$. □

*Remark:* Logarithmic differentials $df/f = d\log(f)$ perform an altogether linear assignment $(r \in M) \mapsto \mathbf{x}^r \mapsto d\mathbf{x}^r/\mathbf{x}^r$, hence involve the same constant matrix describing their coordinate change. Dually, each $a \in N$ provides in a coordinate free way a derivation $\partial_a : \mathbb{C}[M] \to \mathbb{C}[M]$, $\mathbf{x}^r \mapsto \langle a, r\rangle \mathbf{x}^r$.

## 22. Closed subschemes and quasi coherent sheaves

18.11.23 (59,1)

22.1. **Pull back of sheaves.** Let $f : X \to Y$ continuous $\rightsquigarrow f_* : \mathcal{S}h(X) \to \mathcal{S}h(Y)$; for ringed spaces this even yields $\mathcal{M}od_{\mathcal{O}_X} \to \mathcal{M}od_{\mathcal{O}_Y}$.

*Example:* If $f = \operatorname{Spec}\varphi : \operatorname{Spec} A \to \operatorname{Spec} B$, then $(f_*\widetilde{M})(D(b)) = \widetilde{M}(f^{-1}D(b)) = \widetilde{M}(D(\varphi b)) = M_{\varphi(b)} = M_b$ shows that $f_*\widetilde{M} = \widetilde{M}^{(B\to A)}$ where the latter means $M$ understood as a $B$-module. In general, $f_*$ behaves badly with stalks.

Let $\mathcal{G} \in \mathcal{S}h(Y) \rightsquigarrow \boxed{f^{-1}\mathcal{G} := [U \mapsto \varinjlim_{V\supseteq f(U)} \mathcal{G}(V)]^a}$; since $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$, this functor is exact. Moreover, $f^{-1} \dashv f_*$ on $\mathcal{S}h(X)$ and $\mathcal{S}h(Y)$: If $\mathcal{F} \in \mathcal{S}h(X)$, then elements of $\operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F})$ and $\operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$ both correspond to systems of compatible homomorphisms $\mathcal{G}(V) \to \mathcal{F}(U)$ for $U \subseteq f^{-1}(V)$.

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces $\Rightarrow \mathcal{O}_Y \to f_*\mathcal{O}_X$ provides $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X \rightsquigarrow$ two further variants of $f^{-1}$:

a) $\mathcal{G} = \mathcal{O}_Y$-module $\rightsquigarrow$ $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ becomes an $\mathcal{O}_X$-module with $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$; this functor $f^*$ remains right exact; $\boxed{f^* \dashv f_*}$; there is a canonical $\Gamma(Y,\mathcal{G}) \to \Gamma(X, f^*\mathcal{G})$ (corresponds to $\mathrm{id}_{\mathcal{S}h(Y)} \to f_*f^*$); $f^*(\mathrm{vb}) = \mathrm{vb}$, and $f^*$(globally generated) = [globally generated]. If $f = \operatorname{Spec}\varphi$, then $f^*\widetilde{N} = \widetilde{A \otimes_B N}$. (*Proof:* If $\mathcal{F}|\operatorname{Spec}(A)$, then every $A$-linear $M \to \Gamma(\mathcal{F})$ provides an $\mathcal{O}_A$-linear map $\widetilde{M} \to \mathcal{F}$, e.g. $\widetilde{\Gamma(\mathcal{F})} \to \mathcal{F}$. This can be used for $A \otimes_B N \to \Gamma(\operatorname{Spec} A, f^*(\widetilde{N}))$; on the stalks in $P \in \operatorname{Spec} A$ this becomes an isomorphism.)

b) $\mathcal{J} \subseteq \mathcal{O}_Y$ *ideal sheaf* $\rightsquigarrow$ $\boxed{f^{-1}\mathcal{J} \cdot \mathcal{O}_X}$ $:= \operatorname{im}(f^*\mathcal{J} \to f^*\mathcal{O}_Y = \mathcal{O}_X)$ is an ideal in $\mathcal{O}_X$. If $f = \operatorname{Spec}\varphi$, then $f^{-1}\widetilde{J} \cdot \mathcal{O}_A = \widetilde{JA}$.

*Example:* In (21.6), let $I := (\operatorname{supp} f) \subseteq k[\mathbf{x}]$ be the smallest monomial ideal with $f \in I$. Using $\pi : \mathbb{TV}(\Sigma) \to \mathbb{C}^n$ we get $\pi_\sigma^{-1}\widetilde{I} \cdot \mathcal{O}_{\mathbb{TV}(\sigma)} = (\pi^{-1}\widetilde{I} \cdot \mathcal{O}_{\mathbb{TV}(\Sigma)})|_{\mathbb{TV}(\sigma)} = \widetilde{(x^{v(\sigma)})}$, i.e. the pull back becomes principal on all charts.

22.2. **Quasi coherent sheaves.** $X$ = scheme $\rightsquigarrow$ $\mathcal{O}_X$-module $\mathcal{F}$ is called *quasi coherent* :$\Leftrightarrow$ $\exists$ open, affine covering by some $U_i = \operatorname{Spec} A_i$ with $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for suitable $A_i$-modules $M_i$.

**Proposition 64.** a) *$\mathcal{O}_{\operatorname{Spec} A}$-modules $\mathcal{F}$ equal some $\widetilde{M}$ $\Leftrightarrow$ for all $f \in A$ the maps $\varphi_f : \Gamma(X,\mathcal{F}) \otimes_A A_f \to \Gamma(D(f), \mathcal{F})$ are isomorphisms.* (Consider $\varphi : \Gamma(X,\mathcal{F})^\sim \to \mathcal{F}$.)

b) *Kernel, image, and cokernel of quasi coherent $\mathcal{O}_X$-modules are quasi coherent.*

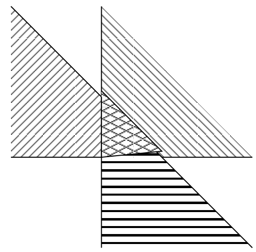
c) *$f : X \to Y \Rightarrow f^*$ and $f_*$ preserve "quasi coherent".*

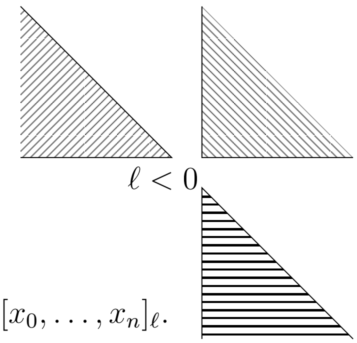
d) *$\mathcal{F}$ = quasi coherent on* $\operatorname{Spec} A \Rightarrow$ *$\mathcal{F}$ equals some $\widetilde{M}$.*

*Proof.* (c) $Y$ = affine; $\mathcal{F}|_{U_i} = \widetilde{M_i}$ for some covering $\psi_i : U_i = \operatorname{Spec} A_i \hookrightarrow X$; let $\phi_{ij\nu} : V_{ij\nu} = \operatorname{Spec} B_{ij\nu} \hookrightarrow (U_i \cap U_j) \hookrightarrow X$ be an affine covering of the intersections. Then $0 \to \mathcal{F} \to \oplus_i (\psi_i)_*\mathcal{F}|_{U_i} \to \oplus_{i,j,\nu}(\phi_{ij\nu})_*\mathcal{F}|_{\operatorname{Spec} C_{ij\nu}}$ is exact, and one applies $f_*$. (d) follows with the same argument for $f = \mathrm{id}$. $\square$

*Example:* $M$ = graded $S$-module $\rightsquigarrow$ $\widetilde{M}$ of (18.5), e.g. $\mathcal{O}_{\operatorname{Proj} S}(\ell) = \widetilde{S(\ell)}$ are quasi coherent on $\operatorname{Proj} S$. With the notation of (16.5), we can look at the example $X = \mathbb{P}^n_k$: Locally, $\mathcal{F} := z_0^{-\ell} \cdot \mathcal{O}_{\mathbb{P}^n}(\ell) \subseteq j_*\mathcal{O}_{(k^*)^n} \subseteq K(\mathbb{P}^n)$ is $(z_i/z_0)^\ell \mathcal{O}_{U_i} = k[\ell \cdot f_i + (\sigma_i^\vee \cap M)]$ (with $f_0 = 0$), hence

$$\Gamma(\mathbb{P}^n_k, \mathcal{F}) = k[\cap_i(\ell f_i + \sigma_i^\vee) \cap M] = \begin{cases} k[x_1/x_0, \ldots, x_n/x_0]_{\le \ell} & \text{if } \ell \ge 0 \\ 0 & \text{otherwise.} \end{cases}$$




$\Rightarrow \Gamma(\mathbb{P}^n_k, \mathcal{O}(\ell)) = x_0^\ell \cdot k[x_1/x_0, \ldots, x_n/x_0]_{\leq \ell} = k[x_0, \ldots, x_n]_\ell$.

22.3. **Closed embeddings.** A morphism $i : Z \to X$ between noetherian schemes is called a *closed embedding* ("$Z$ is a *closed subscheme* of $X$") $:\Leftrightarrow$ the following, mutually equivalent conditions are satisfied:

**Proposition 65.** *$i : Z \hookrightarrow X$ is locally* (with respect to $X$) *isomorphic to* $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$ $\Leftrightarrow$ *topologically, $i$ is a closed embedding plus $\mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_Z$ is surjective.*

*Proof.* First, since this is a special case of finite maps, both local versions agree. Then, for ($\Leftarrow$), the ideal sheaf $\mathcal{I} := \ker(\mathcal{O}_X \twoheadrightarrow i_*\mathcal{O}_Z)$ is quasi coherent, and $\mathcal{I}$ or $i_*\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}$ encode $Z$ completely: $Z' := \operatorname{Spec} \mathcal{O}_X/\mathcal{I}$ and $Z$ equal $\{P \in X \mid (\mathcal{O}_X/\mathcal{I})_P \neq 0\}$ and one applies $i^{-1}$ to $i_*\mathcal{O}_{Z'} = i_*\mathcal{O}_Z$. □

*Examples:* 1) $\mathbb{P}^{n-1} = V(z_0) \hookrightarrow \mathbb{P}^n$ has $\mathcal{I} \cong \mathcal{O}_{\mathbb{P}^n}(-1)$ as its ideal sheaf. More general, $0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{V(F_d)} \to 0$ for a homogeneous $F_d \in \mathbb{C}[\mathbf{z}]_d$.

2) $f : X \to Y$ morphism of schemes, $W \subseteq Y$ closed subscheme with ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_Y \Rightarrow f^{-1}(W) := W \times_Y X \subseteq X$ has the ideal sheaf $f^{-1}\mathcal{J} \cdot \mathcal{O}_X$.

3) $(X, \mathcal{O}_X)$ is called (quasi) affine/*projective* $:\Leftrightarrow$ $X$ is (open in a) closed subscheme of $\mathbb{A}^n_k/\mathbb{P}^n_k$.

4) $\boxed{X_{\text{red}} \subseteq X}$ is the smallest closed subscheme on the topological space $X$.

1.11.23 (61,3) 5) The closed orbits in toric varieties $\mathbb{TV}(\Sigma) = \mathbb{P}(\Delta)$ are $\overline{\operatorname{orb}}(\tau) = \mathbb{TV}(\overline{\Sigma}, N_\tau) = \mathbb{P}(F_\tau)$ with $N_\tau := N/\operatorname{span}(\tau)$, $F_\tau := \operatorname{face}(\Delta, \tau)$, and $\overline{\Sigma} := \{\overline{\sigma} \subseteq (N_\tau)_\mathbb{Q} \mid \Sigma \ni \sigma \supseteq \tau\}$:

$$\begin{array}{ccccccc}
\mathbb{TV}(0, N_\tau) & = & \operatorname{Spec} \mathcal{C}[\tau^\perp \cap M] & = & \operatorname{orb}(\tau) & \xhookrightarrow{\text{closed}} & \mathbb{TV}(\tau) \\
\downarrow{\scriptstyle\text{open}} & & \downarrow{\scriptstyle\text{open}} & & \downarrow{\scriptstyle\text{open}} & & \downarrow{\scriptstyle\text{open}} \\
\mathbb{TV}(\overline{\sigma}, N_\tau) & = & \operatorname{Spec} \mathcal{C}[\sigma^\vee \cap \tau^\perp \cap M] & = & \overline{\operatorname{orb}}(\tau) & \xhookrightarrow{\text{closed}} & \mathbb{TV}(\sigma).
\end{array}$$

Moreover, $\boxed{\mathbb{TV}(\Sigma) = \sqcup_{\sigma \in \Sigma} \operatorname{orb}(\sigma)}$ is a stratification with $\operatorname{orb}(\sigma) \subseteq \overline{\operatorname{orb}}(\tau) \Leftrightarrow \tau \leq \sigma$ and $\dim \operatorname{orb}(\sigma) + \dim \sigma = \operatorname{rank} N$. (For the underlying topological spaces we know that $\mathbb{TV}(\sigma) = \operatorname{Hom}_{\text{sGrp}}(\sigma^\vee \cap M, \mathbb{C})$, and each such map $\varphi : \sigma^\vee \cap M \to \mathbb{C}$ gives rise to a face $\tau \leq \sigma$ with $\sigma^\vee \cap \tau^\perp \cap M = \varphi^{-1}(\mathbb{C}^*)$, i.e. $\varphi : \tau^\perp \cap M \to \mathbb{C}^*$ and $\varphi(\sigma^\vee \setminus \tau^\perp) = 0$.)

22.4. **The scheme theoretic image.** This is the globalization of (19.3.6). Let $Z \subseteq X$ be a closed subscheme with $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I} \Rightarrow$ the "*scheme theoretic image* $\overline{f(Z)}$" of $Z$ along $f: X \to Y$ is given by the ideal sheaf $\mathcal{J} := (f^*)^{-1}(f_*\mathcal{I} \subseteq f_*\mathcal{O}_X) \subseteq \mathcal{O}_Y$; it provides the $\boxed{\text{“smallest” scheme}}$ structure $P$ on $\overline{f(Z)} \subseteq Y$ such that $f|_Z$ factors through it. If $Z$ is reduced, then so is $\overline{f(Z)}$.

*Proof:*
$$\begin{array}{ccccc} \mathcal{O}_Y & \xrightarrow{f^*} & f_*\mathcal{O}_X & & \\ \downarrow & & \downarrow & & \\ \mathcal{O}_{\overline{f(Z)}} := \mathcal{O}_Y/\mathcal{J} & \hookrightarrow & f_*\mathcal{O}_X/f_*\mathcal{I} & \hookrightarrow & f_*(\mathcal{O}_X/\mathcal{I}) \end{array}$$
$\Rightarrow \mathcal{J} := (f^*)^{-1}(f_*\mathcal{I})$ is the maximal ideal sheaf possible for such a $P$

Locally on $Y = \operatorname{Spec} B$: $X = \bigcup_i \operatorname{Spec} A_i \rightsquigarrow A := \prod_i A_i$ and $\operatorname{Spec} A = \coprod_i \operatorname{Spec} A_i \to X \to \operatorname{Spec} B$ via $\varphi: B \to A$; thus $P = \operatorname{Spec} B/\varphi^{-1}(\prod_i I_i)$. Since $\overline{B} \hookrightarrow \overline{A}$ is injective, we see once more that $\operatorname{Spec} \overline{A} \to \operatorname{Spec} \overline{B}$ is dominant. $\square$

*Examples:* $f: X \hookrightarrow Y$ open embedding $\rightsquigarrow X \subseteq \overline{X} \subseteq Y$; the closure of graphs $\Gamma_f$ of rational maps $f: X \dashrightarrow Y$ in $X \times Y$ ($\Gamma_{(\mathbb{A}^n\setminus 0)\to\mathbb{P}^{n-1}} \hookrightarrow \mathbb{A}^n \times \mathbb{P}^{n-1}$ redefines the blowing up); $\overline{V(I) \setminus V(J)} = V(I : J^\infty)$ (this is used for the strict transforms).

25.10.23 (60,

22.5. **The universal property of the blowing up.** Generalizing (17.4) to arbitrary ideals $I \subseteq A \rightsquigarrow$ "Rees-ring" $\oplus_{d\geq 0} I^d$ from (11.5) $\rightsquigarrow \boxed{\operatorname{Bl}_A(I) := \operatorname{Proj} \oplus_{d\geq 0} I^d}$ $= \operatorname{Proj} \oplus_{d\geq 0} I^d t^d \xrightarrow{\pi} \operatorname{Spec} A$. With $I = (g_1, \ldots, g_n)$ we have

$$\operatorname{Bl}_A(I) \supseteq D_+(g_i) = \operatorname{Spec}(\oplus_{d\geq 0} I^d t^d)_{(g_i \in It)} = \operatorname{Spec} A[g_1/g_i, \ldots, g_n/g_i] \xrightarrow{\pi_i} \operatorname{Spec} A$$

where the previous rings are understood as $A[\mathbf{g}/g_i] \subseteq A_{g_i}$ i.e. $f(\mathbf{g}/g_i) = 0$ in $A[\mathbf{g}/g_i]$ $\Leftrightarrow g_i^{\gg 0} f(\mathbf{g}/g_i) = 0$ in $A$. In particular, the ideal sheaf $\pi^{-1}\widetilde{I} \cdot \mathcal{O}_{\operatorname{Bl}_A(I)}$ is locally $I \cdot A[g_1/g_i, \ldots, g_n/g_i] = (g_i)$, i.e. $\boxed{\text{principal; } g_i \text{ is a non-zero divisor}}$ since it is a unit in $A_{g_i}$. Globally, $\mathcal{O}_{\operatorname{Bl}_A(I)}(-E) = \pi^{-1}\widetilde{I} \cdot \mathcal{O}_{\operatorname{Bl}_A(I)} = \mathcal{O}_{\operatorname{Bl}_A(I)}(1)$.

**Proposition 66.** *Let* $J \subseteq I \subseteq A$. *Then* $\operatorname{Bl}_{A/J}(I/J) \hookrightarrow \operatorname{Bl}_A(I) \to \operatorname{Spec} A$ *equals the strict transform of* $\operatorname{Spec} A/J \hookrightarrow \operatorname{Spec} A$.

*Proof.* Locally, $E$ corresponds to the ideal $(g_i) \subseteq A[\mathbf{g}/g_i]$, the full preimage $\pi^{-1}V(J)$ is defined by the ideal $\mathfrak{a} := J \cdot A[\mathbf{g}/g_i]$, and $\operatorname{Bl}_{A/J}(I/J)$ is the vanishing locus of $\mathfrak{b} := \ker\big(A[\mathbf{g}/g_i] \twoheadrightarrow (A/J)[\mathbf{g}/g_i]\big) = \{f \in A[\mathbf{g}/g_i] \mid g_i^{\gg 0} f \in J\}$. Now, the claim follows from $\mathfrak{b} = (\mathfrak{a} : g_i^\infty)$, cf. (19.3.7) and (22.4). $\square$

**Theorem 67.** *Every* $f: X \to \operatorname{Spec} A$ *with an invertible ideal sheaf* $f^{-1}\widetilde{I}$ *factors uniquely via* $\operatorname{Bl}_A(I)$. In particular, everything glues to blowing ups in ideal sheaves.

*Proof.* If $\varphi: A \to B$ has $\varphi(I)B = (\varphi(g_1), \ldots, \varphi(g_n)) = (b)$ with a non-zero divisor $b \in B$, then $\varphi(g_i) = bs_i$ implies $\bigcup_i D(s_i) = \operatorname{Spec} B$, and on $D(s_i)$ we have $\varphi(\mathbf{g})/\varphi(g_i) \in B_{s_i}$ providing $A \to A[\mathbf{g}/g_i] \to B_{s_i}$. $\square$

*Alternativer Beweis mit den Methoden von §25:* $I = (g_1, \ldots, g_n) \Rightarrow \operatorname{Bl}_A(I) \subseteq \mathbb{P}_A^{n-1}$ ist gegeben durch die homogenen Gleichungen $G(\mathbf{x}) \in A[\mathbf{x}]$ mit $G(\mathbf{g}) = G(g_1, \ldots, g_n) = 0 \in A$, z.B. $G_{ij}(\mathbf{x}) := g_i x_j - g_j x_i$.

$$\begin{array}{ccccc} X & \overset{F}{\dashrightarrow} & \mathrm{Bl}_A(I) & \hookrightarrow & \mathbb{P}^{n-1}_A \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi} & \swarrow & \\ & & \operatorname{Spec} A & & \end{array}$$

Falls $F$ vorhanden, so ist $F^{-1}\mathcal{O}_{\mathrm{Bl}_A(I)}(1)\cdot\mathcal{O}_X = F^{-1}(\pi^{-1}I\cdot\mathcal{O}_{\mathrm{Bl}_A(I)})\cdot\mathcal{O}_X = \mathcal{L}$, also existiert eine Surjektion $\rightsquigarrow$ Isomorphismus $F^*\mathcal{O}(1) \overset{\sim}{\to} \mathcal{L}$; dabei entsprechen sich die globalen Schnitte $F^*(x_i) = f^*(g_i)$.

Umgekehrt definieren $\mathcal{L}$ mit $f^*(g_i)\in\Gamma(X,\mathcal{L})$ genau einen Morphismus $F: X\to\mathbb{P}^{n-1}_A$, und dieser geht über $\mathrm{Bl}_A(I)$: Falls $G$ Gleichung, wie oben mit $\deg G = d$, so folgt $G(f^*(g_1),\dots,f^*(g_n)) = f^*G(g_1,\dots,g_n) = 0$ in $\Gamma(X,\mathcal{L}^{\otimes d}) = \Gamma(X,\mathcal{L}^d)$.

## 23. Weil and Cartier divisors

8.11.23 (62,4)

23.1. **Normal rings.** $A =$ noetherian domain; $I\subseteq \mathrm{Quot}(A)$ fractional ideal (finitely generated $A$-submodule) $\rightsquigarrow I^\vee := \mathrm{Hom}_A(I,A)\subseteq\mathrm{Quot}(A)$ with $I\cdot I^\vee\subseteq A$.

**Lemma 68.** 1) *$a\in A\setminus 0$, $P\in\mathrm{Ass}(A/a)\Rightarrow P^\vee\supsetneq A$.*
2) *$A =$ normal, $P^\vee\supsetneq A\Rightarrow P\cdot P^\vee\neq P$.*
3) *$(A,P)$ local, $P\cdot P^\vee = A\Rightarrow P$ is principal ($\rightsquigarrow A$ is regular, 1-dimensional).*

*Proof.* (1) $A/P\hookrightarrow A/a$, $1\mapsto b$ means $P = ((a):(b))$; hence $b/a\in P^\vee\setminus A$.
(2) If $PP^\vee = P\Rightarrow P(P^\vee)^n = P\subseteq A$, every $x\in P^\vee$ implies $A[x]\subseteq\frac{1}{p}A$ for some $p\in P\Rightarrow x$ is integral over $A\Rightarrow x\in A$.
(3) Nakayama $\Rightarrow\exists a\in P\setminus P^2\rightsquigarrow$ ideal $aP^\vee\subseteq A$; from $aP^\vee\not\subseteq P$ (otherwise $a\in a(P^\vee P)\subseteq P^2$) we derive $aP^\vee = A$, hence $1/a\in P^\vee$, i.e., $P = (a)$. □

15.11.23 (63,5) **Proposition 69.** *1) $(A,P)$ local, normal, 1-dimensional $\Rightarrow$ regular* ($\Rightarrow$ normal).
*2) $A$ normal, $a\in A\setminus 0\Rightarrow$ all $P\in\mathrm{Ass}(A/a)$ are minimal, i.e.* $\mathrm{ht}(P) = 1$.

*Proof.* (0) Lemma 68(1,2) $\rightsquigarrow PP^\vee\supsetneq P$ whenever $P\in\mathrm{Ass}(A/a)$ for some $a\in A\setminus 0$.
(1) $\dim A = 1\Rightarrow\forall a\in P$: $P\in\mathrm{Ass}(A/a)\overset{(0)}{\Rightarrow} PP^\vee = A\rightsquigarrow$ Lemma 68(3) applies.
(2) $P\in\mathrm{Ass}(A/a)\overset{(0)}{\Rightarrow} PP^\vee = A$. Lemma 68(3) on $A_P$ yields $\dim(A_P) = 1$. □

**Corollary 70.** *For normal rings $A$ we obtain $A = \bigcap_{\mathrm{ht}(P)=1} A_P$. In particular, $A = \{f\in\mathrm{Quot}(A)\mid \mathrm{div}(f)\geq 0\}$.*

*Proof.* $A = \bigcap_{a\in A\setminus 0}\bigcap_{P\in\mathrm{Ass}(A/a)} A_P$: Let $b/a\in\mathrm{Quot}(A)\rightsquigarrow I := \{x\in A\mid x\cdot b/a\in A\} = ((a):(b)) = \mathrm{Ann}(b\in A/a)$. If $I\subsetneq A$ is not prime $\rightsquigarrow\exists x,y\notin I$, $xy\in I\Rightarrow \mathrm{Ann}(b\in A/a)\subsetneq\mathrm{Ann}(xb\in A/a)\subsetneq A$. This continues until we obtain a prime ideal $P$ of this form, i.e. $I\subseteq P\in\mathrm{Ass}(A/a)$. □

23.2. **The class group.** Let $X$ be an $n$-dimensional variety over $k$. Prime divisors = 1-codimensional, integral subschemes $D\subset X\rightsquigarrow\mathcal{O}_{X,\eta(D)} =$ 1-dimensional, local domains (maybe even normal); Weil divisors $\operatorname{Div} X := Z_{n-1}(X) := \mathbb{Z}^{\oplus\{\text{prime div}\}}\subseteq \operatorname{Div}_{\mathbb{Q}} X$; effective Weil divisors $D\geq 0$ on $X$.

$D$ = prime divisor, $f \in K(X) = \operatorname{Quot}\mathcal{O}_{X,\eta(D)} \rightsquigarrow \boxed{\operatorname{ord}_D(f) \in \mathbb{Z}}$ via extension of $\operatorname{ord}_D(f \in \mathcal{O}_{X,\eta(D)}) := \ell(\mathcal{O}_{X,\eta(D)}/f)$; additivity: $0 \to \mathcal{O}/f \stackrel{g}{\to} \mathcal{O}/fg \to \mathcal{O}/g \to 0$; for *regular* $\mathcal{O}_{X,\eta(D)}$ we have $f = t^{\operatorname{ord}_D(f)} \cdot [\text{unit}]$ with $(t) = \mathfrak{m}_{X,\eta(D)} \subseteq \mathcal{O}_{X,\eta(D)}$.

$f \in K(X)^* \rightsquigarrow \boxed{\text{principal divisors}}$ $\operatorname{PDiv}(X) := \{\operatorname{div}(f) := \sum_D \operatorname{ord}_D(f) \cdot D\} \subseteq \operatorname{Div}(X)$; $\operatorname{Cl}(X) := \operatorname{Div}(X)/\operatorname{PDiv}(X)$, i.e. $K(X)^* \stackrel{\operatorname{div}}{\to} \operatorname{Div}(X) \to \operatorname{Cl}(X) \to 0$. *Example:* $\operatorname{Cl}(\mathbb{P}^1) = \mathbb{Z}$. 22.11.23 (64,

**Proposition 71.** $X = \operatorname{Spec} A \rightsquigarrow$ [*$A$ is factorial* $\Leftrightarrow \operatorname{Cl}(X) = 0$ *and $A$ is normal*].

*Proof.* ($\Rightarrow$) "factorial" $\Rightarrow$ "normal"; $D = V(f \in A)$ prime divisor $\Rightarrow D = \operatorname{div}(f)$. ($\Leftarrow$) $P \subseteq A$ of height 1 $\rightsquigarrow$ prime divisor $D = \operatorname{div}(f \in \operatorname{Quot} A)$. Apply Corollary 70 twice: $\operatorname{ord}_\bullet(f) \geq 0 \Rightarrow f \in A$, and $g \in P \subseteq A \Rightarrow \operatorname{ord}_\bullet(g/f) \geq 0 \Rightarrow P = (f)$. □

23.3. **Cartier divisors.** Let $X$ be a variety. It gives rise to the exact sequence $1 \to \mathcal{O}_X^* \to K(X)^* \to K(X)^*/\mathcal{O}^* \to 1$. Global sections $D \in \Gamma(X, K(X)^*/\mathcal{O}^*)$ are called Cartier divisors $D \in \operatorname{CaDiv}(X)$; they are represented by pairs $(U_i, f_i)$ for some open covering $\{U_i\}$ of $X$ and $f_i \in K(X)^*$ with $f_i/f_j \in \mathcal{O}^*(U_{ij})$. The associated invertible sheaf

$$\mathcal{O}_X(D) := \bigcup_i 1/f_i \cdot \mathcal{O}_{U_i} \subseteq K(X)$$

corresponds to the 1-cocycle $\delta(D) \in \check{Z}^1(\{U_i\}, \mathcal{O}_X^*)$, and all invertible subsheaves of $K(X)$ arise in this way. Principal divisors are Cartier via $\operatorname{PDiv}(X) \hookrightarrow \operatorname{CaDiv}(X)$, $\operatorname{div}(f) \mapsto (X, f)$. Since $D \sim D' \Leftrightarrow \mathcal{O}_X(D) \cong \mathcal{O}_X(D')$, this leads to the identification $\operatorname{CaDiv}(X)/\operatorname{PDiv}(X) = \operatorname{Pic}(X)$.

On the other hand, the maps $\operatorname{div} : K(U_i)^* \to \operatorname{Div} U_i$ glue into $\operatorname{div} : \operatorname{CaDiv}(X) \to \operatorname{Div}(X)$. For normal $X$, this is injective, leading to $\operatorname{Pic}(X) \subseteq \operatorname{Cl}(X)$. For factorial (e.g. regular) $X$, it is surjective, too.

*Example:* (25.7) $\rightsquigarrow$ Weil divisors being not Cartier.

23.4. **Divisors in toric geometry.** If $X = \mathbb{TV}(\Sigma) \supseteq T$, then $\operatorname{Div}_T(X) := \mathbb{Z}^{\Sigma(1)}$ is generated by the $T$-invariant prime divisors $\overline{\operatorname{orb}}(a)$ with $a \in \Sigma(1)$.

**Proposition 72.** $\boxed{\operatorname{div}(\mathbf{x}^r) = \sum_{a\in\Sigma(1)} \langle a, r\rangle \cdot \overline{\operatorname{orb}(a)}}$. *Moreover, if $\Sigma \leq \mathcal{N}(\Delta)$, then $\Delta$ represents a Cartier divisor with associated sheaf $\mathcal{O}_X(\Delta)$ and associated Weil divisor* $\operatorname{div}(\Delta) = -\sum_{a\in\Sigma(1)} \min\langle a, \Delta\rangle \cdot \overline{\operatorname{orb}(a)}$.

*Proof.* Since $\mathbf{x}^r \in k[M]^*$ it remains to check that $\operatorname{ord}_{\overline{\operatorname{orb}(a)}}(\mathbf{x}^r) = \langle a, r\rangle$. Do this in the generic point $\eta_{\overline{\operatorname{orb}(a)}} \in \mathbb{TV}(\mathbb{Q}_{\geq 0} \cdot a) \subseteq \mathbb{TV}(\Sigma)$; one may assume $a = (1, \underline{0})$. □

In particular (if $\Sigma$ is full-dimensional), we obtain the commutative diagram

$$\begin{array}{ccccccc}
M & \xrightarrow{\Sigma(1)^*} & \mathbb{Z}^{\Sigma(1)} & \xrightarrow{\pi} & \operatorname{Cl}(X) & \longrightarrow & 0 \\
\downarrow{\scriptstyle r\mapsto \mathbf{x}^r} & & \downarrow{\scriptstyle a\mapsto\overline{\operatorname{orb}}(a)} & & \| & & \\
K(X)^* & \xrightarrow{\operatorname{div}} & \operatorname{Div}(X) & \xrightarrow{D\mapsto[D]} & \operatorname{Cl}(X) & \longrightarrow & 0
\end{array}$$

One uses that $k[M]$ is factorial and, moreover, that $k[M]^* = k^* \cdot \{\mathbf{x}^r \mid r \in M\}$. It says that $\mathrm{Cl}(\mathbb{TV}(\Sigma))$ is GALE-dual to $\mathbb{Z}^{\Sigma(1)} \to N$. Define the following cones in
29.11.23 (65,7) $\mathrm{Cl}(X)_{\mathbb{Q}} := \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$\mathrm{Eff}(X) := \overline{\mathbb{Q}_{\geq 0} \cdot \{[D] \in \mathrm{Cl}(X) \mid D \geq 0\}} \quad (\text{“pseudo effective cone”})$$

and

$$\mathrm{Amp}(X) := \mathbb{Q}_{\geq 0} \cdot \{[D] \in \mathrm{Pic}(X) \mid D \text{ is ample}\} \quad (\text{“ample cone”}).$$

**Proposition 73.** *Let $\pi : \mathbb{Z}^{\Sigma(1)} \twoheadrightarrow \mathrm{Cl}(X)$ as in the above diagram. Then $\mathrm{Eff}(X) = \pi(\mathbb{Q}_{\geq 0}^{\Sigma(1)})$ and, if additionally $\Sigma = \mathcal{N}(\Delta)$ ($\Rightarrow \mathrm{Pic}(X)_{\mathbb{Q}} = C(\Delta) - C(\Delta)$ by (25.6)), then $\mathrm{Amp}(X) = \mathrm{int}\, C(\Delta) = \mathrm{Pic}(X)_{\mathbb{Q}} \cap \bigcap_{\sigma \in \Sigma} \mathrm{int}\, \pi(\mathbb{Q}_{\geq 0}^{\Sigma(1) \setminus \sigma(1)})$.*

*Proof.* (Eff) $D \geq 0$ on $X \Rightarrow$ choose $f \in K(X)^*$ with $D = \mathrm{div}(f)$ on $T$ and $1 \in$
6.12.23 (66,8) $\mathrm{supp}\, f$. Since $\mathrm{ord}_a(f) \leq 0$ for all $a \in \Sigma(1)$, we have $D - \mathrm{div}(f) \in \pi(\mathbb{Z}_{\geq 0}^{\Sigma(1)})$.
(Amp, 1) $D = \mathrm{div}(\Delta')$ with $\Delta' \in \mathrm{int}\, C(\Delta)$ is ample, and for every $\sigma \in \Sigma$ it can be shifted such that $\Delta'(\sigma) = 0$, i.e, $\mathrm{div}(\Delta') \in \pi(\mathbb{Z}_{>0}^{\Sigma(1) \setminus \sigma(1)})$.
(Amp, 2) Let $h(\sigma, \sigma') = r(\sigma) - r(\sigma') = t(\sigma, \sigma') \cdot (\Delta(\sigma) - \Delta(\sigma'))$ be the 1-cocycle of a Cartier divisor $D$ ($\sigma, \sigma'$ adjacent, top-dimensional). If $[D] \in \mathrm{int}\, \pi(\mathbb{Q}_{\geq 0}^{\Sigma(1) \setminus \sigma(1)}) = \pi(\mathbb{Q}_{>0}^{\Sigma(1) \setminus \sigma(1)})$, then $r(\sigma) = 0$ and $\langle a, r(\sigma') \rangle < 0$ for $a \in \sigma'(1) \setminus \sigma(1)$. The same becomes true for $\Delta(\sigma)$ and $\Delta(\sigma')$ by shifting $\Delta$. Hence, $t(\sigma, \sigma') > 0$.
(Amp, 3) Finally, if $D$ is ample, then some multiple $\mathcal{O}_X(kD)$ is globally generated, and the corresponding $\Delta'$ yields $\Sigma = \mathcal{N}(\Delta')$. □

## 24. Smooth and regular schemes

17.1.24 (70,12) Section §14 characterizes regular local rings – (14.1) via the tangent cone, and (14.2) via the existence of finite free resolutions. Corollary 49 shows that localizations of regular local rings (in prime ideals) stay regular. In (24.1) we have introduced the cotangent sheaves; now we combine both approaches. Let $k$ be a perfect field, cf. (20.3); we will use local $k$-algebras $(A, \mathfrak{m})$ with $k \xrightarrow{\sim} A/\mathfrak{m}$, i.e. we deal with local rings of $k$-rational points.

24.1. **(Co) Tangent sheaves.** Let $A \to B$ be an algebra and $M$ a $B$-module $\rightsquigarrow$ $\mathrm{Der}_A(B, M) := \{A\text{-linear derivations } d : B \to M\}$, i.e. $d(bb') = b\, d(b') + b' d(b) \rightsquigarrow$ universal $A$-derivation $\boxed{B \to \Omega_{B|A}}$ characterized by $\mathrm{Hom}_B(\Omega_{B|A}, M) \xrightarrow{\sim} \mathrm{Der}_A(B, M)$.
*Example:* $\Omega_{k[\mathbf{x}]|k} = \oplus_i\, k[\mathbf{x}]\, dx_i$.

This construction is compatible with localizations ($\Omega_{B_b|A} = \Omega_{B|A} \otimes_B B_b$ and $\Omega_{B|A_a} = \Omega_{B|A}$ if $a \in B^*$) hence glue to a quasi coherent $\mathcal{O}_X$-module $\Omega_{X|Y}$ for a given morphism $X \to Y$ of schemes, e.g. $\Omega_X := \Omega_{X|\mathrm{Spec}\, k}$ for every $k$-scheme $X$.
*Example:* $\Omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$, but $\Omega_E = \mathcal{O}_E$ for smooth $E = \overline{V(y^2 - f_3(x))} \subseteq \mathbb{P}^2$.

The fundamental exact sequences for the universal differentials $\Omega_\bullet$ become $f^*\Omega_{Y|S} \to \Omega_{X|S} \to \Omega_{X|Y} \to 0$ for $X \xrightarrow{f} Y \to S$ and, for closed subschemes $Z \overset{\iota}{\hookrightarrow} Y$,

$$\big(\mathcal{I}/\mathcal{I}^2 = \iota^*\mathcal{I}_{Z\subseteq Y}\big) \to \big(\Omega_{Y|S} \otimes \mathcal{O}_Z = \iota^*\Omega_{Y|S}\big) \to \Omega_{Z|S} \to 0, \quad \text{cf. (22.3).}$$

Dually, with $\mathcal{T} := \mathcal{H}om_{\mathcal{O}}(\Omega, \mathcal{O})$ denoting the "*tangent sheaves*", one obtains the exact sequences $0 \to \mathcal{T}_{X|Y} \to \mathcal{T}_{X|S} \to f^*\mathcal{T}_{Y|S}$ and $0 \to \mathcal{T}_{Z|S} \to \mathcal{T}_{Y|S} \otimes \mathcal{O}_Z \to \mathcal{H}om(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Z)$.

24.2. **The toric** Euler **sequence.** Let $\Sigma$ be a smooth fan; its rays $\Sigma(1)$ gives rise to a surjection $f : \mathbb{Z}^{\Sigma(1)} \to N$. Now, to describe $\boxed{\Omega_X \text{ for } X = \mathbb{TV}(\Sigma)}$, we build the commutative diagram ahead:

(i) $\mathrm{Cl}(\Sigma)^* := \ker f$ ($\cong \mathbb{Z}$ for $\Sigma = \mathbb{P}^n$) followed by $\otimes_{\mathbb{Z}}\mathcal{O}_X$ gives the central row;

(ii) the codimension one orbits $H_a := \overline{\mathrm{orb}}(a)$ provide the central column.

(iii) Locally on $\mathbb{C}^n = \mathbb{TV}(\sigma \in \Sigma)$, the $H_a$ are the coordinate hyperplanes, and we define $\Omega_{\mathbb{C}^n}(\log H) := \oplus_i \mathcal{O}_{\mathbb{C}^n}\, dx_i/x_i \supseteq \Omega_{\mathbb{C}^n}$. The assignment $r \mapsto dx^r/x^r$ shows that $M \otimes_{\mathbb{Z}} \mathcal{O}_X = \Omega_X(\log H)$.

(iv) The cokernel of the left hand column is checked locally via $\oplus_i\, \mathbb{C}[\mathbf{x}]\, dx_i/x_i \twoheadrightarrow \oplus_i\, \mathbb{C}[\mathbf{x}]/(x_i)\, e_i$ sending $dx_i/x_i \mapsto \overline{e_i}$ (residuum map).

(v) The top line follows by diagram chasing and is called the toric EULER sequence. For $X = \mathbb{P}^n$, it turns into the exact sequence $0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_X & \xrightarrow{dx^r \mapsto \sum_a \langle a,r\rangle x^r e_a} & \oplus_{a\in\Sigma(1)}\mathcal{O}_X(-H_a) & \longrightarrow & \mathrm{Cl}(\Sigma)\otimes\mathcal{O}_X \to 0 \\
 & & \downarrow & & \downarrow & & \| \\
0 & \to & \big(M\otimes\mathcal{O}_X = \Omega_X(\log H)\big) & \xrightarrow{r \mapsto \sum_a \langle a,r\rangle e_a} & \mathcal{O}_X^{\Sigma(1)} & \longrightarrow & \mathrm{Cl}(\Sigma)\otimes\mathcal{O}_X \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \oplus_{a\in\Sigma(1)}\mathcal{O}_{H_a} & = & \oplus_{a\in\Sigma(1)}\mathcal{O}_{H_a} & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

In particular, $\omega_X := \det \Omega_X = \otimes_{a\in\Sigma(1)}\mathcal{O}_X(-H_a)$, e.g. $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$. (This again shows that $\Omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ and, using adjunction, $\Omega_E = \mathcal{O}_E$ from (24.1).) 24.1.24 (71,1

24.3. **Regular implies factorial.** While regular rings are automatically integral (Problem 65) and "factorial" means "regular in codimension one", we have

**Proposition 74.** *Regular local rings are factorial.*

*Proof.* Let $P$ be a height one prime in the regular $(A, \mathfrak{m})$. If $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, then $A/(x)$ is regular, hence integral, hence $(x) \subseteq A$ is prime.

If $x \in \mathfrak{m} \setminus (\mathfrak{m}^2 \cup P)$, then $P_x \subseteq A_x$ is locally free of rank 1: Every $Q \in \operatorname{Spec} A_x$ over $P$ leads to a factorial $A_Q$ (induction), hence $P_Q A_Q$ is principal. Now, if $F_\bullet \to P$ is a free $A$-resolution of $P$, then $P_x = \otimes_i (\det F_i)^{\pm 1} = f \cdot A_x$. If $f \in A \setminus (x)$, then this implies $P = (f)$ (if $p \in P$ satisfies that $p\,x^k = fg$, then the primality of $x$ implies $x|g$, and one can lower $k$). $\square$

24.4. **The cotangent space.** Let $(A, \mathfrak{m})$ be a local $k$-algebra with $k \xrightarrow{\sim} A/\mathfrak{m}$. Then $\boxed{\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} \Omega_{A|k} \otimes_A k}$ becomes an isomorphism ("cotangent space"): Injectivity follows from the surjectivity of $\mathrm{Der}_k(A, M) \twoheadrightarrow \mathrm{Hom}_A(\mathfrak{m}/\mathfrak{m}^2, M)$ for $A/\mathfrak{m}$-modules $M$ (extend $\mathfrak{m}/\mathfrak{m}^2 \to M$ by $(c \in k) \mapsto 0$).

**Theorem 75.** 1) *Assume that $k$ is a perfect field. Then, $\Omega_{A|k}$ is a free $A$-module of rank* $\dim A \Leftrightarrow$ *$A$ is a regular ring.*
2) *Let $X$ be a variety over $k = \overline{k}$. Then $\Omega_{X|k}$ is locally free of rank* $\dim(X)$ ("$X$ is non-singular") $\Leftrightarrow$ *all local rings of $X$ are regular* ("$X$ is regular").

*Proof.* First, $\Omega_{A|k}$ = free of rank $\dim A$ implies $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$, hence $A$ is regular, hence a domain.
Now, we may use Problem 28. Since $k$ is perfect, $Q(A)|k$ is separably generated, hence $\dim_{Q(A)} \Omega_{Q(A)|k} = \text{tr-deg}_k\, Q(A) = \dim A$: If $k \hookrightarrow (K = k(x_1, \ldots, x_d)) \hookrightarrow (L = K(s))$ is a tower of fields, then $0 \to \Omega_{K|k} \otimes_K L \to \Omega_{L|k} \to \Omega_{L|K} \to 0$ is exact, and the latter vanishs because of $m_s'(s)\, ds = d\, m_s(s) = 0$. $\square$

31.1.24 (72,14)

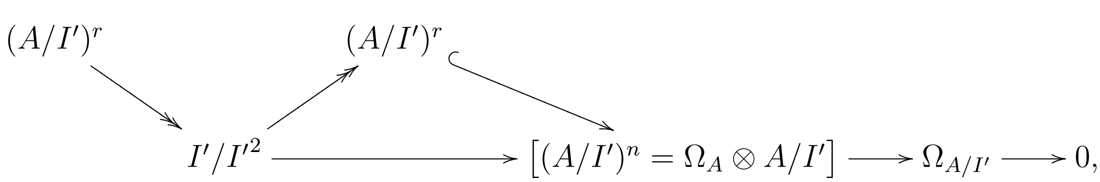
**Corollary 76.** *If $X$ is a $\overline{k}$-variety, then $X_{\text{smooth}} \subseteq X$ is open and dense.*

*Proof.* Consider $\Omega_{K(X)|k} = (\Omega_{X|k})_{\eta(X)}$. $\square$

24.5. **Smooth subvarieties.** Let $X$ be an $n$-dimensional, smooth $(k = \overline{k})$-variety; let $Y \subseteq X$ be an irreducible closed subscheme of codimension $r$ with ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$.

**Proposition 77.** *$Y$ is smooth $\Leftrightarrow$ $\Omega_{Y|k}$ is locally free, and the conormal sequence* $0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{X|k} \otimes \mathcal{O}_Y \xrightarrow{\varphi} \Omega_{Y|k} \to 0$ *of* (24.1) *is left exact, too.*
*In this case, $\mathcal{I}$ is locally generated by $r$ elements, and $\mathcal{I}/\mathcal{I}^2$ is locally free of rank $r$.*

*Proof.* $q := \operatorname{rank} \Omega_{Y|k} \Rightarrow \ker \varphi$ is locally free of rank $n - q$.
($\Leftarrow$) Nakayama $\rightsquigarrow \mathcal{I}$ is locally generated by $n - q$ elements ($I/I^2 \twoheadrightarrow I/\mathfrak{m}I$), hence $\dim Y \geq q$. On the other hand, for $y \in Y$, $q = \dim_k \mathfrak{m}_y/\mathfrak{m}_y^2 \geq \dim Y$.
($\Rightarrow$) Denote $A := \mathcal{O}_{X,y}$ with $y \in Y$ and $I = \mathcal{I}_y$. Since $\Omega_{A/I}$ is free of rank $q = n - r$, we have a split embedding $\ker \varphi \cong (A/I)^r \hookrightarrow (A/I)^n$. Let $\mathcal{I}' := \langle x_1, \ldots, x_r \rangle \subseteq \mathcal{I}$ locally generate the $\ker \varphi$ and $Y' := V(\mathcal{I}') \supseteq Y$, Nakayama implies that we can lift the splitting




hence $I'/I'^2 \xrightarrow{\sim} (A/I')^r$ and $\Omega_{A/I'}$ is free, too. In particular, $Y \subseteq Y'$ are both smooth of the same dimension, hence equal. $\square$

In this situation, as in (24.1), we can dualize the above sequence into the exact $0 \to \mathcal{T}_Y \to \mathcal{T}_X \otimes \mathcal{O}_Y \to \mathcal{N}_{Y|X} \to 0$ as well as take the determinant to obtain the adjunction formula $\omega_Y = \omega_X \otimes \det \mathcal{N}_{Y|X}$.

24.6. **Geometric genus.** We define the geometric genus of a projective, smooth $k$-variety $X$ (with $k = \overline{k}$) as $p_g(X) := \dim_k \Gamma(X, \omega_X)$. 7.2.24 (73,15

**Proposition 78.** *Let $X$, $X'$ be two birationally equivalent, smooth, projective $k$-varieties. Then $p_g(X) = p_g(X')$.*

*Proof.* If $U \subseteq X$, then the restriction map $\Gamma(X, \omega_X) \hookrightarrow \Gamma(U, \omega_X)$ is injective; if $\operatorname{codim}_X(X \setminus U) \geq 2$, then it is even bijective (to be checked locally, since $X$ is normal: $A = \bigcap_{\operatorname{ht} P=1} A_P$, cf. Corollary 70).
On the other hand, the range of definition of $X \dashrightarrow X'$ is of the latter type $X \supseteq U \stackrel{f}{\to} X'$ (see Problem **??**). Since $f = \mathrm{id}$ on a smaller $U \supseteq W \subseteq X'$, the pull back map $\Gamma(X', \omega_{X'}) \to \Gamma(U, \omega_U) = \Gamma(X, \omega_X)$ (induced from $f^*\omega_{X'} \to \omega_U$) takes place inside $\Gamma(W, \omega_W)$, hence is injective. □

## 25. Invertible sheaves

On affine schemes: Invertible sheaves $\leftrightarrow$ projective modules of rank one.

25.1. **Morphisms by sections.** Fix a base ring $A$ and an $A$−scheme $X \rightsquigarrow$ [$A$-morphisms $X \stackrel{\varphi}{\to} \mathbb{A}^n_A$] $\widehat{=}$ [$A[\mathbf{x}] \to \Gamma(X, \mathcal{O}_X)$] $\widehat{=}$ [$\varphi_1, \ldots, \varphi_n \in \Gamma(X, \mathcal{O}_X)$].

**Proposition 79.** [$A$-morphisms $X \stackrel{\varphi}{\to} \mathbb{P}^n_A$] $\widehat{=}$ [Invertible sheaves $\mathcal{L}|X$ with generating sections $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$]/Iso. *The ideal sheaf of the scheme theoretical image of such a $\varphi$ is then induced from* $\ker\big(A[s_0, \ldots, s_n] \to \oplus_{d\geq 0}\Gamma(X, \mathcal{L}^d)\big)$.

*Proof.* $\varphi : X \to \mathbb{P}^n_A \rightsquigarrow \mathcal{L} := \varphi^*\mathcal{O}_{\mathbb{P}^n}(1)$, $s_i := \varphi^*(z_i)$. Conversely, we define $X \to \mathbb{P}^n_A$ by $P \mapsto (s_0(P) : \ldots : s_n(P))$. The local description uses trivializations like $\mathcal{L}|_{X_s} = s \cdot \mathcal{O}_{X_s}$ on $X_s \subseteq X$ for $s \in \Gamma(X, \mathcal{L})$. In particular, this yields $X_{s_\nu} \to D_+(z_\nu) \subseteq \mathbb{P}^n_A$ via $z_i/z_\nu \mapsto s_i/s_\nu \in \Gamma(X_{s_\nu}, \mathcal{O}_X)$. □

*Special situations:* (i) $\mathcal{L}$ is called $\boxed{\text{very ample}}$, if there are $s_i$ such that $X \to \mathbb{P}^n_A$ becomes an immersion as a locally closed subset. An invertible sheaf is called "*ample*" if some power is very ample.
(ii) If $\Gamma(X, \mathcal{L})$ is a finitely generated $A$-module and $\mathcal{L}$ is globally generated $\rightsquigarrow$ 20.12.23
$\Phi_{\mathcal{L}} : X \to \overline{X} \subseteq \mathbb{P}^n_A$, and $\mathcal{L} = \Phi^*(\overline{\mathcal{L}})$ with $\overline{\mathcal{L}}|\overline{X}$ very ample. (68,10)

*Examples:* $(\mathbb{P}^n, \mathcal{O}(d))$, $(\mathbb{P}^m \times \mathbb{P}^n, \mathcal{O}(1,1))$; among them $(\mathbb{P}^1, \mathcal{O}(2))$ and $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1))$ are quadrics in $\mathbb{P}^2$ and $\mathbb{P}^3$, respectively.

**25.2.** *Beispiel für ample Garben:* Kubische Kurve = elliptische Kurve; kubische Fläche – $\mathbb{P}^2$ in 6 Punkten aufblasen: Sieht man wenigstens die Geraden? Ja: Auf den exzeptionellen Divisoren ist die Garbe $3H - E$ genau $\mathcal{O}(1)$ (sieht man z.B. torisch); die strikten Transformierten von Verbindungsgeraden gehen auch so: $\mathcal{O}(3-1-1)$. Siehe [GrHa, S.480 ff.]: $3H - E$ ist sehr ample (mit dem üblichen Verfahren der Trennung von Punkten); die globalen Schnitte haben Dimension 4.

25.3. **Automorphisms of $\mathbb{P}^n$.** $B$ = factorial $\Rightarrow \operatorname{Pic}(\operatorname{Spec} B) = 0$ (via 1-cocycles on the open covering $\{D(g_i)\}$, cf. Problem 104).

$A$ factorial $\Rightarrow A[\mathbf{z}]$ factorial $\Rightarrow \operatorname{Pic}\mathbb{P}^n_A = \mathbb{Z}$ (if $h_{ij} \in A[\mathbf{z}/z_i, \mathbf{z}/z_j]^*$ is a 1-cocycle $\Rightarrow h_{ij} = [u_{ij} \in A^*] \cdot (z_i/z_j)^{k_{ij}}$, and $k_{ij} \in \mathbb{Z}$ cannot depend on $i, j$).

*$A$-automorphisms of $\mathbb{P}^n_A$:* $\varphi \in \operatorname{Aut}_A \mathbb{P}^n_A \Rightarrow \varphi^*(\mathcal{O}(1)) = \mathcal{O}(d)$; since $\Gamma(\mathbb{P}^n_A, \mathcal{O}(-1)) = 0$ we know that $\varphi^*(\mathcal{O}(1)) = \mathcal{O}(1)$. Thus, $\varphi^*(x_i) = \sum_j a_{ij} x_j$, i.e. $\operatorname{Aut}_A \mathbb{P}^n_A = \operatorname{PGL}(n, A)$.

25.4. **Resolving indeterminancies again.** We describe an instance of the general graph method of (22.4). Let $\mathcal{L}$ be invertible on the $A$-scheme $X$; let $\Phi_{\mathcal{L}} : X \dashrightarrow \mathbb{P}^n_A$ be induced from $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$, i.e. $\Phi_{\mathcal{L}}$ is defined on $\bigcup_i X_{s_i} = X \setminus B$ with $B := V_{\mathbb{P}}(s_0, \ldots, s_n)$; its ideal sheaf is $\mathcal{J} := \sum_i s_i \mathcal{L}^{-1} \subseteq \mathcal{O}_X$.

$\mathcal{J} \otimes \mathcal{L} = \sum_i s_i \mathcal{O}_X \subseteq \mathcal{L}$ is, by definition, generated by the global sections $s_0, \ldots, s_n$ – but it is not invertible anymore.

Let $\pi : \widetilde{X} \to X$ be the blowing up in $\mathcal{J}$ with $\widetilde{\mathcal{J}} := \pi^{-1}\mathcal{J} \cdot \mathcal{O}_{\widetilde{X}}$; then $\widetilde{\mathcal{L}} := \widetilde{\mathcal{J}} \otimes \pi^*\mathcal{L}$ is invertible, generated by $\pi^*(s_0), \ldots, \pi^*(s_n)$, and, $\widetilde{\mathcal{L}} = \mathcal{L}$ holds true on $\widetilde{X} \setminus \pi^{-1}(B) = X \setminus B$. Hence, we obtain $\varphi$ with $\varphi^*\mathcal{O}(1) = \widetilde{\mathcal{L}}$.

$$\begin{array}{ccc} X & \dashrightarrow & \mathbb{P}^n_A \\ {\scriptstyle\pi}\uparrow & \nearrow_{\varphi} & \\ \widetilde{X} & & \end{array}$$

*Example:* Projection $\pi : \mathbb{P}^n_k \dashrightarrow \mathbb{P}^{n-1}_k$, $(x_0 : \ldots : x_n) \mapsto (x_1 : \ldots : x_n)$ or, locally in the chart $U_0 \subseteq \mathbb{P}^n_k$, $\pi : \mathbb{A}^n_k \setminus \{0\} \to \mathbb{P}^{n-1}_k$, $(x_1, \ldots, x_n) \mapsto (x_1 : \ldots : x_n)$. Then, $\mathcal{L} = \mathcal{O}_{\mathcal{A}^n}$ and $\mathcal{J} = (x_1, \ldots, x_n) \subseteq k[x_1, \ldots, x_n]$, hence $\pi : \widetilde{\mathbb{A}}^n_k = \operatorname{Bl}_{k[\mathbf{x}]}(\mathbf{x}) \to \mathbb{A}^n_k$ with $\widetilde{\mathcal{J}} = [\mathcal{O}_{\widetilde{\mathbb{A}}^n}(1)$ associated to the closed embedding $\widetilde{\mathbb{A}}^n_k \subseteq \mathbb{P}^{n-1}_{k[\mathbf{x}]} = \mathbb{P}^{n-1}_k \times \mathbb{A}^n_k]$.

25.5. **The Picard group of affine toric varieties.** Generalizing the case of $\mathbb{A}^n_k$,
10.1.24 (69,11) we show that $\boxed{\operatorname{Pic} \mathbb{TV}(\sigma) = 0}$ for *affine* toric varieties (while, e.g., $k[x, y, z]/(xz - y^2)$ is not factorial by $xz = y^2$). Let $S := k[\sigma^\vee \cap M]$.

**Lemma 80.** *Let $L \subseteq k[M]$ be an $S$-submodule with $L \otimes_S k[M] = k[M]$ (i.e. $L$ contains monomials). Then $L^\vee$ is $M$-graded, i.e. it is generated by monomials. Moreover, if $L$ is also invertible $\Rightarrow L = x^r \cdot S$ for a unique $r \in M$.*

*Proof.* $L^\vee = \operatorname{Hom}_S(L, S) = \{f \in \operatorname{Quot}(S) \mid f \cdot L \subseteq S\} \subseteq k[M]$. Let $L = \langle \ell^1, \ldots, \ell^m \rangle$. For $f \in L^\vee$ and $a \in \sigma$ we know that $\deg_a f + \deg_a \ell^i \geq 0$ (with $\deg_a := \min\langle a, \operatorname{supp}\rangle$) $\Rightarrow$ for every $f$-monomial $x^r$ we have $x^r \cdot \ell^i \in [a \geq 0] \Rightarrow x^r \cdot \ell^i \in S$, i.e. $x^r \in L^\vee$.

Now, let $L$ be *invertible*. Corollary 7(1) in (2.7) $\rightsquigarrow$ the Nakayama lemma applies also to the graded case $(S, S_+) = (\oplus_{d\geq 0} S_d, \oplus_{d>0} S_d) \Rightarrow$ as in (8.1) it follows that, if $S_0 = k$, then minimal, homogeneous generating systems of graded, *projective* $S$-modules of finite presentation are automatically free.

*Alternatively:* $L = \langle \mathbf{x}^{r^i} \rangle$, $L^{-1} = \langle \mathbf{x}^{s^j} \rangle \Rightarrow \langle \mathbf{x}^{r^i + s^j} \rangle = S \Rightarrow$ w.l.o.g. $r^1 + s^1 = 0$ and $r^i + s^j \in \sigma^\vee$ otherwise $\Rightarrow r^i - r^1,\ s^j - s^1 \in \sigma^\vee$, i.e. $L = \mathbf{x}^{r^1} \cdot S$. □

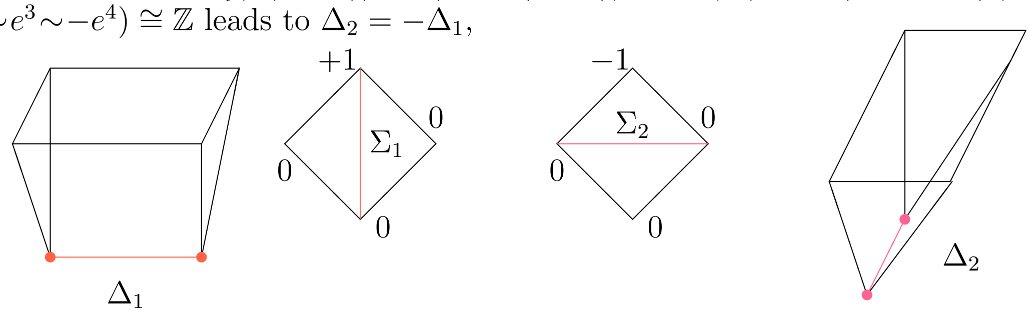
25.6. **The Picard group of general toric varieties.** $\Sigma =$ fan in $N_\mathbb{Q}$; $\Delta \subseteq M_\mathbb{Q}$ lattice polyhedron with $\boxed{\Sigma \leq \mathcal{N}(\Delta)}$. The normal fan consists of the linearity regions of $(a \in \mathrm{tail}(\Delta)^\vee) \mapsto \min\langle a, \Delta\rangle$; in particular $|\Sigma| = |\mathcal{N}(\Delta)| = \mathrm{tail}(\Delta)^\vee$, and we have $[\Sigma^{\mathrm{top}} \to \Delta\text{-vertices}, \sigma \mapsto \Delta(\sigma)]$ with $\mathcal{N}_{\Delta(\sigma)}(\Delta) \supseteq \sigma \in \Sigma$, i.e. $\min\langle a, \Delta\rangle = \langle a, \Delta(\sigma)\rangle$ for $a \in \sigma$. Hence, $\Delta(\sigma) - \Delta(\sigma')$ is orthogonal to $\sigma \cap \sigma'$, and we may also define (non-unique) $\Delta(\sigma)$ for non-maximal cones $\sigma$. This shows that the globally generated sheaf $\boxed{\mathcal{O}_{\mathbb{TV}(\Sigma)}(\Delta)} := \sum_{\sigma\in\Sigma} x^{\Delta(\sigma)} \cdot \mathcal{O}_{\mathbb{TV}(\Sigma)}$ locally equals $\mathcal{O}_{\mathbb{TV}(\sigma)}(\Delta) = x^{\Delta(\sigma)} \cdot \mathcal{O}_{\mathbb{TV}(\sigma)}$. This sheaf induces the map $\Phi_{\mathcal{O}(\Delta)} : \mathbb{TV}(\Sigma) \to \mathbb{P}(\Delta)$ discussed in (16.5).

Conversely, let $\mathcal{L}$ be invertible on $\mathbb{TV}(\Sigma, N) \rightsquigarrow h =$ 1-cocycle of $\mathcal{O}^*$ with respect to the standard covering $\{\mathbb{TV}(\sigma)\}$ as in Problem 95: $\sigma, \tau \in \Sigma^{\mathrm{top}} \rightsquigarrow h_{\sigma\tau} = u_{\sigma\tau} \cdot \mathbf{x}^{h(\sigma,\tau)} \in k[(\sigma\cap\tau)^\vee \cap M]^* \subseteq k[M]^* = k^* \cdot \{\mathbf{x}^M\}$, i.e. $h =$ 1-cocycle $\{h(\sigma,\tau) \in (\sigma\cap\tau)^\perp \cap M\}$. This is equivalent to $h = \{h(\sigma,\tau) \in (\sigma\cap\tau)^\perp \cap M \mid \sigma, \tau \in \Sigma^{\mathrm{top}} \text{ adjacent}\}$ with the additional condition that $\sum_i h(\sigma_i, \sigma_{i+1}) = 0$ along cycles around 2-codimensional cones. Moreover, since $k[\sigma^\vee \cap M]^* = k^*$, there are no 1-coboundaries.

On the other hand, since $k[M] =$ factorial, the embedding $j : T \hookrightarrow \mathbb{TV}(\Sigma)$ yields $\mathcal{L} \subseteq j_*\mathcal{L}|_T \xrightarrow{\sim} j_*\mathcal{O}_T$. Hence, by Lemma 80, $L_\sigma := \mathcal{L}(U_\sigma) \hookrightarrow \mathcal{O}_T(U_\sigma \cap T) = k[M]$ provides elements $r(\sigma) \in M$ which locally trivialize $\mathcal{L} \rightsquigarrow h(\sigma,\tau) = r(\sigma) - r(\tau)$, and $\{r(\sigma) \mid \sigma \in \Sigma^{\mathrm{top}}\}$ is, up to a common shift along $M$, uniquely determined. *Example:* $\mathcal{O}_\Sigma(\Delta)$.

Let $\boxed{\Sigma = \mathcal{N}(\Delta)} \rightsquigarrow$ For every $r = r(\mathcal{L})$ the sum $R := r + (N \gg 0) \cdot \Delta$ is a 1-cocycle with $R(\sigma,\tau) \in \mathbb{Q}_{\geq 0} \cdot (\Delta(\sigma) - \Delta(\tau))$ i.e.. it corresponds to Minkowski summand $\Delta_R$ of $\mathbb{Q}_{\geq 0} \cdot \Delta$ ($\Delta$ is "*ample*") $\rightsquigarrow \mathcal{L} \cong \mathcal{O}(\Delta_R) \otimes \mathcal{O}(\Delta)^{-N}$, and $\mathrm{Pic}\,\mathbb{P}(\Delta) = \{\Delta' - \Delta''\}$ consists of the lattice points in the so-called Grothendieck group of the convex cone $C(\Delta)$ of $(\mathbb{Q}_{\geq 0} \cdot \Delta)$-Minkowski summands.

25.7. **A toric flop.** Let $\mathbb{TV}(\Sigma_i) \to X = V(xy - zw) = \mathbb{TV}(\langle a^1, \ldots, a^4\rangle)$ be the two small resolutions; $\mathcal{O}_{\Sigma_i}(\Delta) \mapsto (\langle a^1, \Delta\rangle, \ldots, \langle a^4, \Delta\rangle) \in A_2(X) = \mathbb{Z}^4/M = \mathbb{Z}^4/(e^1 \sim -e^2 \sim e^3 \sim -e^4) \cong \mathbb{Z}$ leads to $\Delta_2 = -\Delta_1$,



i.e. under the natural identification $\operatorname{Pic}\mathbb{TV}(\Sigma_1) \xrightarrow{\sim} \mathbb{Z} \xleftarrow{\sim} \operatorname{Pic}\mathbb{TV}(\Sigma_2)$ we obtain that $\mathcal{L}$ is globally generated on $\mathbb{TV}(\Sigma_1) \Leftrightarrow \mathcal{L}^{-1}$ ist globally generated on $\mathbb{TV}(\Sigma_2)$.

## 26. Weil divisors and reflexive sheaves on normal schemes

26.1. **Reflexive modules and sheaves.** Let $A$ = normal ring; denote $M^\vee := \operatorname{Hom}_A(M,A)$ for $A$-modules $M$. A finitely generated $A$-module $L$ is reflexive $:\Leftrightarrow$
6.12.23 (66,8) $L = M^\vee$ for some $A$-module $M$. This implies that $L$ is torsion free, i.e. $P \in D(f) \overset{j}{\hookrightarrow} X = \operatorname{Spec} A \Rightarrow L \to L_f \to L_P \to L \otimes \operatorname{Quot}(A)$ is injective. All restriction maps in $\widetilde{L}$ are injective, and torsionfreeness implies that there is an open $U \overset{j}{\hookrightarrow} X$ with $\operatorname{codim}_X(X \setminus U) \geq 2$ such that $\widetilde{L}|_U$ is locally free.

**Proposition 81.** (i) *If $L$ is reflexive, then $L = \bigcap_{\operatorname{ht} P=1} L_P$. In particular, if $U \subseteq X$ is any open subset with $\operatorname{codim}_X(X \setminus U) \geq 2$, then $\widetilde{L} = j_* j^* \widetilde{L}$.*

(ii) *On the other hand, for finitely generated, torsion free $A$-modules $L$ with $L = \bigcap_{\operatorname{ht} P=1} L_P$ the adjunction map $L \to L^{\vee\vee}$ is an isomorphism.*

*Proof.* (i) $L = \operatorname{Hom}(M,A) \Rightarrow L_P = \operatorname{Hom}_{A_P}(M_P, A_P) = \operatorname{Hom}_A(M, A_P)$; then use Corollary 70.
(ii) $L \to L^{\vee\vee}$ induces isomorphisms $L_P \to L_P^{\vee\vee}$ for $\operatorname{ht} P = 1$. Now, consider

$$\begin{array}{ccc} L & \longrightarrow & L^{\vee\vee} \\ \downarrow & & \downarrow\wr \\ \bigcap_{\operatorname{ht} P=1} L_P & \xrightarrow{\sim} & \bigcap_{\operatorname{ht} P=1} L_P^{\vee\vee}. \end{array}$$

□

*Example*: $L = (x,y) \subseteq k[x,y]$ is not reflexive (since $(x,y)^\vee = k[x,y]$).

26.2. **Sheaves for Weil divisors.** Let $X$ = normal; consider affine charts $\operatorname{Spec} A \subseteq X$. Weil divisors $D \in \operatorname{Div} X \rightsquigarrow \mathcal{O}_X(D) := \{f \in K(X) \,|\, \operatorname{div}(f) + D \geq 0\}$ is a (coherent) fractional ideal sheaf, i.e. the global version of a fractional ideal from (23.1): $\operatorname{supp} D^+|_{\operatorname{Spec} A} \subseteq V(g) \Rightarrow g \cdot \mathcal{O}_A(D) = \mathcal{O}_A(D - \operatorname{div}(g)) \subseteq A$ is finitely generated. *Example:* $\mathcal{O}_X(0) = \mathcal{O}_X$ or $\mathcal{O}_X(D)$ from (23.3).

**Lemma 82.** *Let $X = \operatorname{Spec} A$ be affine. If $D = \sum_i \lambda_i D_i$, then $\forall i \; \exists f \in \mathcal{O}_X(D)$:* $\operatorname{ord}_{D_i}(f) = -\lambda_i$ (instead just "$\geq$").

*Proof.* $i = 1 \rightsquigarrow h \in I(D_1) \setminus I(D_1)^2 \subseteq A$ (assume $\operatorname{supp}(\operatorname{div}(h)) \subseteq \bigcup_j D_j$) and $g_j \in I(D_j) \setminus I(D_1)$ yield $f := h^{-\lambda_1} \prod_{j \geq 2} g_j^{\gg 0}$. □

**Proposition 83.** (1) *For $D, D' \in \operatorname{Div} X$ we have $\mathcal{O}_X(D) \subseteq \mathcal{O}_X(D') \Leftrightarrow D \leq D'$.*
(2) $\mathcal{O}_X(-D) = \mathcal{O}_X(D)^\vee := \mathcal{H}om_X\big(\mathcal{O}_X(D), \mathcal{O}_X\big)$*; in particular, $\mathcal{O}_X(D)$ is* $\boxed{\textit{reflexive}}$.
(3) $\mathcal{O}_X(D + D') = \big(\mathcal{O}_X(D) \cdot \mathcal{O}_X(D')\big)^{\vee\vee}$ = *"reflexive hull".*

*Proof.* (1) $\mathcal{O}_X(\sum_i \lambda_i D_i) \subseteq \mathcal{O}_X(\sum_i \lambda'_i D_i)$ with $\lambda_1 > \lambda'_1$ contradicts Lemma 82.
(2) $\mathcal{H}om_X\big(\mathcal{O}_X(D), \mathcal{O}_X\big) \subseteq K(X)$ via $\varphi \mapsto \varphi(f)/f$ (does not depend on $f$), i.e. we obtain $\mathcal{H}om_X\big(\mathcal{O}_X(D), \mathcal{O}_X\big) = \{g \in K(X) \mid g \cdot \mathcal{O}_X(D) \subseteq \mathcal{O}_X\}$. Since $g \cdot \mathcal{O}_X(D) = \mathcal{O}_X(D - \operatorname{div}(g))$, the claim follows from (1).
(3) Let $\varphi : \mathcal{O}_X(D) \cdot \mathcal{O}_X(D') \to \mathcal{O}_X$; we check that $\varphi(\mathcal{O}_X(D + D')) \subseteq \mathcal{O}_X$: For $g \in \mathcal{O}_X(D')$ we obtain $g\varphi : \mathcal{O}_X(D) \to \mathcal{O}_X$, hence $g\varphi \in \mathcal{O}_X(-D)$, i.e. altogether one has $\varphi : \mathcal{O}_X(D') \to \mathcal{O}_X(-D)$, hence $\mathcal{O}_X(D' - \operatorname{div}(\varphi)) \subseteq \mathcal{O}_X(-D)$. □

As for Cartier divisors, we still have $D \sim D' \Leftrightarrow \mathcal{O}_X(D) \cong \mathcal{O}_X(D')$: The isomorphisms are always given by elements of $K(X)^*$. Note that $1 \in \Gamma(X, \mathcal{O}(D)) \Leftrightarrow D \geq 0$.

26.3. **Effective divisors.** Let $X$ be normal. There is a bijection $\{$(effective) divisors $D\} \leftrightarrow \{$reflexive (non-fractional) ideal sheaves $J \subseteq K(X)\}$:

**Proposition 84.** *$D \mapsto \mathcal{O}_X(-D)$ and $J \overset{\operatorname{div}}{\mapsto} \sum_\nu \ell(\mathcal{O}_{X,\eta(D_\nu)}/J)\, D_\nu$ are mutually inverse. In particular, if $D \geq 0$, then $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ is exact.*

*Proof.* First, let (w.l.o.g.) $D \geq 0$ be given. Since $\ell(\mathcal{O}_{X,\eta(D_\nu)}/J) = \min_{f\in J} \operatorname{ord}_{D_\nu}(f)$, we get $\ell(\mathcal{O}_{X,\eta(D_\nu)}/\mathcal{O}_X(-D)_{\eta(D_\nu)}) = \lambda_\nu$ from Lemma 82. Second, beginning with $J$, we obviously have that $J \subseteq \mathcal{O}_X(-\operatorname{div} J)$. Moreover, if $\varphi : J \to \mathcal{O}_X$, i.e. if for $g \in J$ we have $\operatorname{ord}_{D_\nu}(\varphi \cdot g) \geq 0$, then $\varphi \in \mathcal{O}_X(\operatorname{div} J)$. □ 13.12.23 (67,

Another point of view is to fix a reflexive sheaf $\mathcal{F}|X$ of rank 1 and vary the embeddings of $\mathcal{F}$ into $K(X)$; they are parametrized by rational sections of $\mathcal{F}^\vee$:

**Proposition 85.** *There is a bijection $\mathcal{F}_\eta \setminus \{0\} \leftrightarrow \{$embeddings $\mathcal{F}^\vee \hookrightarrow K(X)\}$, and $s, t \in \mathcal{F}_\eta$ induce the same subsheaf $\Leftrightarrow$ they differ by $\Gamma(\mathcal{O}_X^*)$. Hence $(\mathcal{F}_\eta \backslash 0)/\Gamma(\mathcal{O}_X^*) \leftrightarrow \{$divisors with $\mathcal{O}_X(D) \cong \mathcal{F}\}$. If $\mathcal{F} = \mathcal{O}_X(E)$, then $s \mapsto D(s) = \operatorname{div}(s) + E$.*
*Moreover, $\big(\Gamma(X, \mathcal{F}) \setminus 0\big)/\Gamma(\mathcal{O}_X^*) = \{\mathcal{O}_X \subseteq \mathcal{F}\} = \{\mathcal{F}^\vee \subseteq \mathcal{O}_X\} = \{D \geq 0\}$ with $D(s) = \operatorname{supp}(\operatorname{coker} s)$.*

*Proof.* A section $s \in \mathcal{F}(U) \subseteq \mathcal{F}_\eta$ gives $\mathcal{F}^\vee|_U \to \mathcal{O}_U$, hence $\mathcal{F}^\vee \to j_* j^* \mathcal{F}^\vee \to j_* \mathcal{O}_U \subseteq K(X)$. This map is automatical injective, since it is an isomorphism in $\eta$. Or, shorter, $s \mapsto [\varphi \mapsto \varphi(s)]$. On the other hand, every $\mathcal{F}^\vee \overset{\iota}{\hookrightarrow} K(X)$ can be represented as some $\mathcal{F}^\vee|_U \to \frac{1}{g}\mathcal{O}_U$ (with $g \in \Gamma(U, \mathcal{O}_X) \subseteq K(X)$), i.e. $g\iota \in \operatorname{Hom}(\mathcal{F}^\vee|_U, \mathcal{O}_U) = \mathcal{F}(U)$. Finally, we note that $\mathcal{O}(-D) = s \cdot \mathcal{O}(-E) = \mathcal{O}(-E - \operatorname{div}(s))$. □

**Problem 86.** Let $A = \mathbb{C}[x, y, z]/(xz - y^2)$ and recall that it is an affine toric variety where the dual cone $\sigma^\vee \subset M_\mathbb{R} = \mathbb{R}^2$ is generated by the lattice points $A = [1, 0]$, $B = [1, 1]$, and $C = [1, 2]$. Then, $x = \chi^A$, $y = \chi^B$, and $z = \chi^C$.

Denote by $P$ and $Q$ the two toric prime divisors given by the two rays of $\sigma$, say $P = V(x, y)$ and $Q = (y, z)$. In particular, $(x, y) = \mathcal{O}_A(-P)$ and $(y, z) = \mathcal{O}_A(-Q)$. Both ideals are reflexive. Finally, recall that $2P = \operatorname{div}(x)$ is a principal divisor.

Compare the ideal $(x)$ with $(x, y)^2 = (x^2, xy, y^2)$. Draw the lattice points corresponding to the monomials of $A$ and both ideals as figures in $\mathbb{Z}^2$. Which of the ideals $(x)$ or $(x^2, xy, y^2)$ is reflexive? What is their reflexive hulls (meaning the double duals) – can one observe this in the picture?

26.4. **Linear systems.** Let $X$ be a complete $k$-variety; $D=$ Cartier divisor, $\mathcal{L}:=\mathcal{O}_X(D)\rightsquigarrow$ the coordinate free version of Proposition 79 in (25.1) is $\Phi_{\mathcal{L}}: X \dashrightarrow \mathbb{P}(V):=(V^*\setminus 0)/k^*$, $P\mapsto [s\overset{\Phi(P)}{\mapsto} s(P)]$ with $V\subseteq \Gamma(X,\mathcal{O}_X(D))$. We may identify $\Phi(P)$ with $\Phi(P)^\perp=\{s\in V\mid s(P)=0\}\subseteq V$.

$|D|:=\{0\le D'\sim D\}=\big(\Gamma(X,\mathcal{O}_X(D))\setminus\{0\}\big)/k^*$ "general linear system" $\rightsquigarrow V$ induces a "special linear system" $|D|_V:=(V\setminus 0)/k^*\subseteq |D|$, and $\Phi(P)^\perp=\{D'\in |D|_V\mid D'\ni P\}$. The base locus of $V$ is $\mathcal{B}(V)=\{x\in X\mid \forall D'\in|D|_V\colon x\in D'\}$, i.e. $\Phi(P)^\perp$ is a hyperplane in $|D|_V\Leftrightarrow P\notin\mathcal{B}(V)$.

## 27. Fractional toric ideals

27.1. **Tailed subsets.** $X=\mathbb{TV}(\sigma,N)\rightsquigarrow$ Lemma 80 shows that $D\in\operatorname{Div}_T(X)$ provide $M$-graded $\mathcal{O}_X(D)\subseteq j_*\mathcal{O}_T$, i.e. $M$-graded pieces $\mathcal{O}_{\mathbb{TV}(\sigma)}(D)\subseteq k[M]$.

$J=\sum_i \mathbf{x}^{r^i}k[\sigma^\vee\cap M]\subseteq k[M]$ monomial, fractional ideal $\leftrightarrow$ finitely generated $(\sigma^\vee\cap M)$-"module" $\Delta(J)\subseteq M$; denote $\boxed{J=\mathbf{x}^\Delta}$.

**Definition 87.** $\Delta\subseteq M$ is called *polyhedral* $:\Leftrightarrow \Delta=P\cap M$ for some lattice polyhedron $P\subseteq M_{\mathbb{Q}}$ (implying that $P=\operatorname{conv}(\Delta)$, i.e. discrete polyhedral sets correspond 1-to-1 to lattice polyhedra).

Even rational polyhedra $P$ suffice to make $\Delta=P\cap M$ polyhedral. Moreover, $\Delta$ becomes then a $\boxed{\text{finitely generated}}$ ($\operatorname{tail}(\Delta)$-module where $\operatorname{tail}(\Delta):=\{r\in M\mid r+\Delta\subseteq\Delta\}$ is the lattice points of the tail cone: $\Delta=M\cap(P^c+\operatorname{tail}_{\le 1}(P))+\operatorname{tail}(\Delta)$.

*Example:* If $D=\sum_a\lambda_a\overline{\operatorname{orb}(a)}\in\operatorname{Div}_T X\rightsquigarrow \Delta_\sigma^{\mathbb{Q}}(D):=\{r\in M_{\mathbb{Q}}\mid \langle a,r\rangle\ge -\lambda_a \text{ for } a\in\sigma^{(1)}\}$ does not need to be a lattice polyhedron. Nevertheless, $\Delta_\sigma(D):=\Delta_\sigma^{\mathbb{Q}}(D)\cap M$ is polyhedral, and it provides $\mathcal{O}_X(D)=\mathbf{x}^{\Delta_\sigma(D)}$.

{Polyhedra $\Delta\subseteq M_{\mathbb{Q}}$ with tail $\sigma^\vee$} $\leftrightarrow$ $\{j:\sigma\to\mathbb{Q}\mid$ fanwise linear, concave$\}$ via $\Delta\mapsto j(\Delta):=\min\langle\bullet,\Delta\rangle$ and $j\mapsto\Delta_{\ge j}:=\{r\in M_{\mathbb{Q}}\mid\langle\bullet,r\rangle\ge j \text{ on } \sigma\}$. Taking into account the lattice structure, one obtains $\{J=\mathbf{x}^\Delta\subseteq k[M] \text{ with tail } =\sigma^\vee\}\leftrightarrow\{j:\sigma\cap N\to\mathbb{Z}\mid$ fanwise linear, concave$\}$ via $\boxed{\operatorname{ord}_J(a):=\min_{\mathbf{x}^r\in J}\langle a,r\rangle}$ (similarly to $\operatorname{ord}_{D_\nu}(J)$ from (26.3)).

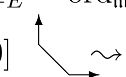
*Weil divisors:* $D=\sum_a\lambda_a\overline{\operatorname{orb}(a)}\Rightarrow\operatorname{ord}_{\mathcal{O}_X(D)}(a\in\sigma^{(1)})=-\lambda_a$, and $\operatorname{ord}_D$ is the smallest concave fanwise linear function with these boundary values ("concave interpolation") – this characterizes "reflexive".
*Example:* The non-reflexive $(x,y)\subseteq k[x,y]$ yields $\operatorname{ord}(1,0)=\operatorname{ord}(0,1)=0$ and $\operatorname{ord}(1,1)=1$, but the concave interpolation equals 0.

*Cartier divisors* correspond to $\Delta_\sigma(D)=R_\sigma+(\sigma^\vee\cap M)$; for simplicial cones we have $\operatorname{Div}_T X\subseteq\operatorname{CaDiv}_{\mathbb{Q}}X$. (Examples $k[x,y,z]/(xz-y^2)$ and $k[x,y,z]/(xz-yw)$.)

27.2. **Pulling back fractional ideals.** Let $\mathcal{J} \subseteq j_*\mathcal{O}_T = k[M]$ be a monomial sheaf of fractional ideals ($j^*\mathcal{J} = \mathcal{O}_T$); locally it corresponds to fractional $(\sigma^\vee \cap M)$ ideals $J_\sigma = \mathbf{x}^{\Delta_\sigma(J)} \subseteq k[M]$; the global sections are $\Gamma(\mathbb{TV}(\Sigma), \mathcal{J}) \widehat{=} \Delta(J) := \bigcap_{\sigma\in\Sigma} \Delta_\sigma(J)$. Note that $0 \in \Delta(D) := \Delta(\mathcal{O}_X(D)) \Leftrightarrow D \geq 0$.

Let $f : (\Sigma', N') \to (\Sigma, N)$; via $f : T' \to T$ we obtain $j'^*(f^*\mathcal{J}) = f^*(j^*\mathcal{J}) = \mathcal{O}_{T'}$. Define $\mathcal{J}' = f^{-1}\mathcal{J} \cdot j'_*\mathcal{O}_{T'} := \operatorname{im}(f^*\mathcal{J} \to j'_*\mathcal{O}_{T'})$ ($= f^*\mathcal{J}$ for invertible $\mathcal{J}$: $A$-linear surjections $A \twoheadrightarrow A$ are isomorphisms).

Locally, with $f(\sigma') \subseteq \sigma$, this means $J'_{\sigma'} = (J_\sigma \subseteq k[M]) \odot_{k[\sigma^\vee\cap M]} k[\sigma'^\vee \cap M'] \subseteq k[M']$ is generated by $\{\mathbf{x}^{f^*r}\}$ with $\{\mathbf{x}^r\}$ generating $J_\sigma$, i.e. $\Delta_{\sigma'}(J') = f^*\Delta_\sigma(J) + (\sigma'^\vee \cap M')$. Alteratively, the order functions glue to maps $|\Sigma| \cap N \to \mathbb{Z}$. Then, $\boxed{\operatorname{ord}_{\mathcal{J}'} = \operatorname{ord}_{\mathcal{J}} \circ f}$.

*Example:* Blowing up $\mathcal{J}$ via subdividing $\sigma$ into the linearity regions of $\operatorname{ord}_{\mathcal{J}}$. For instance, $J = (x, y)$ leads to the invertible $\mathcal{J} := \pi^{-1}\mathfrak{m} \cdot \mathcal{O}_{\widetilde{\mathbb{A}^2}}$, and $\operatorname{ord}_{\mathcal{J}} : (1,0), (0,1) \mapsto 0,\ (1,1) \mapsto 1$ shows that $\mathcal{J} = \mathcal{O}_{\widetilde{\mathbb{A}^2}}(-E)$. Concavity of $\operatorname{ord}_{-E} = \operatorname{ord}_{\mathfrak{m}} \Rightarrow \mathcal{O}_{\widetilde{\mathbb{A}^2}}(-E)$ is globally generated; $\Delta(-E)$ has the vertices $[0,1]$ and $[1,0]$  $\rightsquigarrow \Phi_{\mathcal{O}(-E)} : \widetilde{\mathbb{A}^2} \to \mathbb{P}^1 \times \mathbb{A}^2$ yields the embedding; on the other hand, $\Phi_{\mathcal{O}}$ provides the contraction $\widetilde{\mathbb{A}^2} \to \mathbb{A}^2$.

31.1.24 (72,1

27.3. **The canonical sheaf.** From (24.2) we know that $\omega_X = \mathcal{O}_X(-\sum_{a\in\Sigma(1)} H_a)$ for smooth toric varieties $X = \mathbb{TV}(\Sigma)$ (with $H_a = \overline{\operatorname{orb}}(a)$). In general, we define $\boxed{\omega_X := (\det \Omega_X)^{\vee\vee}} \subseteq j_*j^*\omega_X = j_*\omega_T$. On $T$ we know that $\Omega_T = k[M] \otimes_{\mathbb{Z}} M$ with $d\mathbf{x}^r = \mathbf{x}^r \otimes r$, hence $\omega_T = k[M] \otimes_{\mathbb{Z}} \det M$ with $\frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n} = 1 \otimes (e_1 \wedge \ldots \wedge e_n)$. This is coordinate independent, and $\omega_X$ ist $T$-invariant.

**Proposition 88.** $\boxed{\operatorname{ord}_{\omega_X} = 1 \text{ on } \Sigma^{(1)}}$, *i.e.* $\boxed{K_X = -\sum_{a\in\Sigma^{(1)}} \overline{\operatorname{orb}}(a)}$ *is a* (the nicest) *so-called canonical divisor on $X$.*

*Proof.* The fact that $\mathbf{x}^r \cdot (d\mathbf{x}/\mathbf{x})^{\wedge n} \in \bigcap_{a\in\Sigma^{(1)}} j_{a*}j_a^*\omega_X \Leftrightarrow \operatorname{ord}_a \mathbf{x}^r \geq 1$ for all $a \in \Sigma^{(1)}$ follows as in Proposition 72: $\omega_{\mathbb{TV}((1,0)\cdot\mathbb{Q}_{\geq 0})}$ is generated by $dx_1 \wedge \frac{dx_2}{x_2} \wedge \ldots \wedge \frac{dx_n}{x_n} = x_1\,(d\mathbf{x}/\mathbf{x})^{\wedge n}$. □

In particular, $\boxed{\Delta_\sigma(K_X) = (\operatorname{int}\sigma^\vee) \cap M}$. $X$ is "*Gorenstein*" $:\Leftrightarrow \omega_X$ is invertible ($K_X$ is Cartier) $\Leftrightarrow \forall\sigma \in \Sigma\ \exists m_\sigma \in M$ with $(\operatorname{int}\sigma^\vee) \cap M = m_\sigma + (\sigma^\vee \cap M) \Leftrightarrow \ldots \langle\sigma(1), m_\sigma\rangle = 1 \Leftrightarrow$ all $\sigma$ are cones over lattice polytopes (in height 1, namely in $[m_\sigma = 1]$). For the condition "$\mathbb{Q}$-*Gorenstein*" one relaxes the previous condition by asking just for $m_\sigma \in M_{\mathbb{Q}}$.

$X$ is (weakly) $\boxed{\text{CY}}$ $:\Leftrightarrow \omega_X \cong \mathcal{O}_X \Leftrightarrow \exists m \in M \colon \forall\sigma \in \Sigma \ldots \Leftrightarrow$ the above $m_\sigma = m$ do not depend on $\sigma \Leftrightarrow \Sigma$ is the cone over a complex of lattice polyhedra (in height 1).

*Example:* There are no complete toric CYs; the easiest non-affine (and smooth) one is the small resolution of $\operatorname{cone}(\mathbb{P}^1 \times \mathbb{P}^1)$, cf. (25.7).

**Theorem 89** (Serre duality)**.** *If $X$ is a $d$-dimensional, smooth, projective variety and $\mathcal{F}$ is a locally free $\mathcal{O}_X$-module, then* $\mathrm{H}^i(X, \mathcal{F}) = \mathrm{H}^{d-i}(X,\, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X)^\vee$.

27.4. **Toric Fano varieties.** While $\Delta^{\mathbb{Q}}(K_X) := \bigcap_{\sigma\in\Sigma} \Delta^{\mathbb{Q}}_{\sigma}(K) = \emptyset$ for complete toric varieties $X = \mathbb{TV}(\Sigma)$, we will investigate $-K_X$ instead. Assume that $|\Sigma| =$ convex; then $\Delta^{\mathbb{Q}}(-K_X) = \{r \in M_{\mathbb{Q}} \mid \langle \Sigma(1), r\rangle \geq -1\}$ contains 0 as an interior lattice point.

**Definition 90.** $X$ is Fano $:\Leftrightarrow -K_X$ is ample (i.e. in particular $\mathbb{Q}$-Cartier). In the toric situation, this means that $\mathcal{N}(\Delta_{\mathbb{Q}}(-K_X)) = \Sigma$.

For rational polyhedra there is a duality theory extending this of polyhedral cones. Defining $P^{\vee} := \{a \mid \langle a, P\rangle \geq -1\}$, this construction interchanges tail $P$ and $\mathbb{Q}_{\geq 0}\cdot P$. The main tool for investigating this duality is the equality of the polyhedral cones $\overline{\mathbb{Q}_{\geq 0}\cdot(P^{\vee},1)} = \overline{\mathbb{Q}_{\geq 0}\cdot(P,1)}^{\vee}$. Note that taking the closure of $\mathbb{Q}_{\geq 0}\cdot(P,1)$ means to add the cone $(\operatorname{tail} P, 0)$.

In particular, if $\Sigma$ is complete, then $\Delta_{\mathbb{Q}}(-K_X)^{\vee} = \operatorname{conv}(\Sigma(1)) \ni 0$.

**Lemma 91.** *For a polytope $P$ mit $0 \in \operatorname{int} P$, the normal fan $\mathcal{N}(P)$ equals the face fan of $P^{\vee}$.*

*Proof.* $\operatorname{facefan}(P^{\vee}) = \operatorname{pr}\left[\partial\,\mathbb{Q}_{\geq 0}(P^{\vee},1)\right] = \operatorname{pr}\left[\partial\,\mathbb{Q}_{\geq 0}(P,1)^{\vee}\right] = \mathcal{N}(P)$. □

Hence, a complete toric $\boxed{\mathbb{TV}(\Sigma) \text{ is Fano} \Leftrightarrow \operatorname{facefan}(\operatorname{conv}(\Sigma(1))) = \Sigma}$. In particular, we may construct all toric Fano varieties as follows: Let $P :=$ lattice polytope with $0 \in \operatorname{int} P$ and primitive vertices $\rightsquigarrow \Sigma := \operatorname{facefan}(P)$.

27.5. **Reflexive polytopes.** A polytope $P$ is called *reflexive* $:\Leftrightarrow P$ and $P^{\vee}$ are lattice polytopes. If so, then 0 is the only interior lattice point of both.

*Classification:* In every dimension there are only finitely many; there are exactly 16 two-dimensional ones.

A toric Fano $\mathbb{TV}(\Sigma)$ is Gorenstein $\Leftrightarrow \Delta^{\mathbb{Q}}(-K_X) = \operatorname{conv}(\Sigma(1))^{\vee}$ is a lattice polytope $\Leftrightarrow \Delta := \Delta^{\mathbb{Q}}(-K_X)$ (or $\operatorname{conv}(\Sigma(1))$) is reflexive. Recall that then $\Sigma = \mathcal{N}(\Delta)$.

If $f \in k[M]$ is a Laurent polynomial with $\Delta := \operatorname{conv}(\operatorname{supp} f))$ and $\Sigma \leq \mathcal{N}(\Delta)$, then $Y := \overline{V(f) \subseteq T} \subseteq \mathbb{TV}(\Sigma) =: X$ is locally a hypersurface. If $\mathcal{J} \subseteq \mathcal{O}_X$ denotes its ideal sheaf, then $\mathcal{J} = f\cdot\mathcal{O}_X(-\Delta)$ (locally on $\mathbb{TV}(\sigma)$, we have $\mathcal{J} = (f/\mathbf{x}^{\Delta(\sigma)}) \subseteq \mathcal{O}_{\mathbb{TV}(\sigma)}$). If the adjunction formula applies (e.g. if $Y$ and $X$ are smooth), then $\omega_Y = \omega_X \otimes \mathcal{O}_Y(\Delta)$. In particular, if $\Delta$ is reflexive, then $\omega_Y \cong \mathcal{O}_Y$.

27.6. **Discrepancies.** To make pull backs of canonical divisors possible, we always assume the $\mathbb{Q}$-Gorenstein property in this section.

**Definition 92.** Let $\pi : \widetilde{X} \to X$ be a resolution of singularities; we denote $K_{\widetilde{X}} = \pi^* K_X + \sum_i \lambda_i E_i$ with $\{E_i\}$ being the exceptional divisors. Then, the $\{\lambda_i\}$ are called the *discrepancies* of $\pi$, and the singularities of $X$ are called *canonical/terminal* $:\Leftrightarrow$ $\lambda_i \geq 0/\ \lambda_i > 0$ for all $i$.

**Proposition 93.** *Let $\sigma$ be a $\mathbb{Q}$-Gorenstein cone; let $m_\sigma \in M_\mathbb{Q}$ with $\langle \sigma(1), m_\sigma \rangle = 1$.*
*1) For a subdivision $\Sigma \le \sigma$ we have $K_\Sigma = \pi^* K_\sigma + \sum_{a \in \Sigma(1) \setminus \sigma(1)} \big( \langle a, m_\sigma \rangle - 1 \big) \overline{\operatorname{orb}(a)}$.*
*2) $\mathbb{TV}(\sigma)$ has canonical singularites $\Leftrightarrow \sigma \cap N \subseteq [m_\sigma = 1] \cup \{0\}$; $\mathbb{TV}(\sigma)$ has terminal singularities $\Leftrightarrow \sigma \cap N \subseteq \sigma(1) \cup \{0\}$.*

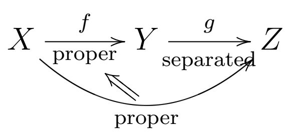
*Proof.* (1) For $a \in \Sigma(1)$ we have $\operatorname{ord}_{K_\Sigma}(a) = 1$, but $\operatorname{ord}^{\mathbb{Q}}_{\pi^* K_\sigma}(a) = \langle a, m_\sigma \rangle$.
(2) If $a$ is (properly) below $[m_\sigma = 1]$, then one may consider a subdivision involving $a$. Alternatively, every smooth subdivision contains vertices below $[m_\sigma = 1]$. □

Let $X = \mathbb{TV}(\operatorname{facefan}(P))$ be Fano as in (27.4). Then $X$ has at most *canonical* singularities $\Leftrightarrow$ 0 is the only interior lattice point of $P$ (being equivalent to "reflexive" in dim $= 2$, but strictly weaker than it in dim $\ge 3$). $X$ has at most *terminal* singularities $\Leftrightarrow$ the vertices and $0 \in P$ are the only lattice points of $P$.

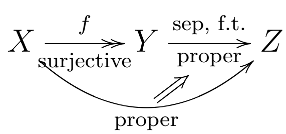
## 28. Proper morphisms

28.1. **Simulating compactness.** $f : X \to Y$ is called "*proper*" :⇔ separated, finite type. and universally closed; this notion is local on $Y$. *Example:* closed embeddings, finite morphisms; *counter example*: open embeddings. Absolute version of "proper": "*complete*". 7.2.24 (73,15

*Properties:* Invariance under *base change*; the *composition* $X \xrightarrow{f} Y \xrightarrow{g} Z$ of proper $f, g$ is proper ($S \to Z \Rightarrow X \times_Z S \to Y \times_Z S \to S$ are closed);




[$gf$ proper, $g$ separated $\Rightarrow f$ proper]: $S \to Y$ new basis $\Rightarrow X \times_Y S \hookrightarrow X \times_Z S$ is a closed embedding, and $X \times_Z S \to S$ is closed;




[$gf$ proper, $f$ surjective, $g$ separated, finite type $\Rightarrow g$ proper]: $f$ remains surjective under $S \to Y$, since $\operatorname{Spec} K \to S \to Y$ induces points $(P \in X) \mapsto (Q \in Y)$ with $K \supseteq K(Q) \subseteq K(P)$, i.e. $\operatorname{Spec} K(P) \otimes_{K(Q)} K \to \operatorname{Spec} K$ provides pre-images.

28.2. **The projective space.** The standard example for complete varieties is

**Proposition 94.** $\mathbb{P}^n_\mathbb{Z} \to \operatorname{Spec} \mathbb{Z}$ *is proper.*

*Proof.* It remains to show that $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$ is a closed map; let $Z = (f_1, \ldots, f_m)$ with homogeneous $f_i \in A[\mathbf{x}]$ of degree $d_i$. For $d \gg 0$ the map $\beta_d : \oplus_i f_i \cdot A[\mathbf{x}]_{d-d_i} \to A[\mathbf{x}]_d$ is given by a matrix with entries in $A$; denote by $m_\nu(d) \in A$ the $\binom{d+n}{n}$-minors and by $m(d) \subseteq A$ the ideal generated by them. For $P \in \operatorname{Spec} A$ we obtain:
$P \notin \pi(Z) \Leftrightarrow \emptyset = Z \cap \pi^{-1}(P) = V(f_1, \ldots, f_m)$ in $\mathbb{P}^n_{K(P)} \Leftrightarrow \exists d\colon (f_1, \ldots, f_m) \supseteq (\mathbf{x})^d$ in $K(P)[\mathbf{x}] \Leftrightarrow \exists d\colon \beta_d \otimes_A K(P)$ is surjective $\Leftrightarrow \exists d, \nu\colon m_\nu(d) \neq 0$ in $K(P)$, i.e. $m_\nu(d) \in A_P^* \Leftrightarrow \exists d\colon m(d) = (1)$ in $A_P \Leftrightarrow \exists d\colon P \notin V(m(d)) \subseteq \operatorname{Spec} A$. □

In particular, "projective morphisms" $Z \stackrel{\text{abg}}{\hookrightarrow} \mathbb{P}^n_Y := \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} Y \stackrel{\text{pr}}{\to} Y$ are proper.
*Example:* Blowing up $\operatorname{Proj} \oplus_{d\geq 0} I^d \to \operatorname{Spec} A$.

28.3. **Toric situation.** In toric geometry we can exactly describe the difference between projective and complete $k$-varieties:

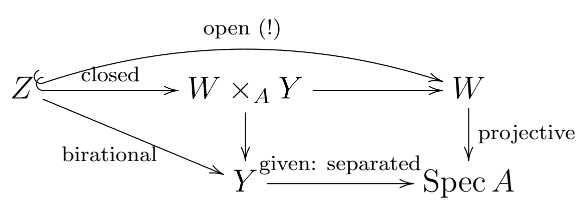
**Proposition 95.** 1) *$\mathbb{TV}(\Sigma, N)$ is projective over $k$ $\Leftrightarrow$ $\exists$ polytope $\Delta \subseteq M_{\mathbb{R}}$ with $\Sigma = \mathcal{N}(\Delta)$; $\mathbb{TV}(\Sigma, N)$ is proper over $k$ $\Leftrightarrow$ $|\Sigma| = N_{\mathbb{Q}}$* (whirlpool example)*.*
14.2.24 (74,16) 2) *An equivariant morphism $\mathbb{TV}(\Sigma, N) \to \mathbb{TV}(\Sigma', N')$ is proper $\Leftrightarrow$ $\varphi : N \to N'$ satisfies* $\boxed{\varphi^{-1}|\Sigma'| = |\Sigma|}$.

*Proof.* (1) $\Sigma = \mathcal{N}(\Delta) \Rightarrow \mathbb{TV}(\Sigma) = \mathbb{P}((\gg 0) \cdot \Delta) \subseteq \mathbb{P}^N$ is projective; the reverse implication follows from knowing the ample sheaves on $\mathbb{TV}(\Sigma) \rightsquigarrow$ *coming up soon!*
$|\Sigma| \subsetneq N_{\mathbb{Q}} \Rightarrow \exists\, \Sigma \hookrightarrow \overline{\Sigma} \rightsquigarrow$ open embedding $\mathbb{TV}(\Sigma) \hookrightarrow \mathbb{TV}(\overline{\Sigma})$, and this is non-proper.
However, if $|\Sigma| = N_{\mathbb{Q}}$, then one extends all codimension one walls to hyperplanes $\rightsquigarrow \Sigma \geq \Sigma' = \text{HypPlArrangement} = \mathcal{N}(\text{Zonotop}) \Rightarrow$ projective $= \mathbb{TV}(\Sigma') \to \mathbb{TV}(\Sigma)$ birational, proper (by 28.1; "Chow-Lemma") $\Rightarrow$ surjektive $\Rightarrow \mathbb{TV}(\Sigma)$ is complete (again by 28.1).
(2) The latter arguments do also work in the relative case, i.e. for non-compact polyhedra. □

*Example:* (Toric) resolutions of singularities in (21.5) were proper and birational.

28.4. **Non-toric Chow Lemma.** The "Chow Lemma" argument of the prevoius proof does also have a non-toric, i.e. a general version:

**Lemma 96.** *$Y \,|\operatorname{Spec} A$ irreducible, separated $\Rightarrow \exists\, W \,|\operatorname{Spec} A$ projective, $Z \subseteq W \times_A Y$ closed: $Z \stackrel{\pi_W}{\hookrightarrow} W$ is an open embedding, and $\pi : Z \stackrel{\pi_Y}{\to} Y$ is birational* (*and proper*)*.*




*Proof.* $Y = \bigcup_i Y_i$ open, affine covering, $Y_i \hookrightarrow \overline{Y}_i$ projective; $U := \bigcap_i Y_i$ affine $\Rightarrow U \stackrel{\text{cl}}{\hookrightarrow} U \times_A \ldots \times_A U \stackrel{\text{op}}{\hookrightarrow} \prod_i Y_i \stackrel{\text{op}}{\hookrightarrow} \prod_i \overline{Y}_i \Rightarrow U \stackrel{\text{op}}{\hookrightarrow} \overline{U} =: W$; $Z := \overline{\Gamma_{U \hookrightarrow Y}} \subseteq W \times_A Y$.
$Z \supseteq U \subseteq Y \Rightarrow \text{pr}_Y$ is birational. Moreover, let $Z_i := Z \cap (W \times_A Y_i) \stackrel{\text{op}}{\hookrightarrow} Z$; since $Z_i = \overline{\Gamma_{U \hookrightarrow Y_i}} \subseteq \Delta_i \subseteq (\overline{Y}_1 \times \ldots \times Y_i \times \ldots \times \overline{Y}_m) \times Y_i$ ($i$-th diagonal), we have $Z_i \stackrel{\sim}{\to} W_i := W \cap (\overline{Y}_1 \times \ldots \times Y_i \times \ldots \times \overline{Y}_m) \stackrel{\text{op}}{\hookrightarrow} W$. □

**Corollary 97** (Chow-Lemma)**.** *$Y$ as above + proper $\Rightarrow \text{pr}_W$ proper $\Rightarrow Z = W$ is projective over $A$.*

28.5. **Valuative criteria.** Let $R = \nu^{-1}(A_{\geq 0}) \subseteq K$ be a valuation ring, i.e. $\nu : K^* \to$ [ordered, abelian group $A$] with $\nu(xy) = \nu(x)+\nu(y)$ and $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$; denote $\operatorname{Spec} K = \{\eta\}$, $\operatorname{Spec} R = \{\eta, \xi\}$, i.e. $\operatorname{Spec} R$ is an abstract curve, but $\xi$ should rather be seen as a codimension one subvariety in variety with function field $K$.

**Proposition 98.** $$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{g} & X \\ \downarrow & \overset{s}{\nearrow} & \downarrow f \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$ *Let $f$ be proper $\Rightarrow$ there exists a unique $s : \operatorname{Spec} R \to X$ making the diagram commutative.*

*Proof.* The uniquenes follows from (20.3); for proving the existence, we may suppose that $Y = \operatorname{Spec} R$ and $g =$ dominant (base change/schemetheoretic image). Let $f(x) = \xi$; the local rings form a chain

$$R = \mathcal{O}_{\operatorname{Spec} R,\xi} \overset{f^*}{\hookrightarrow} \mathcal{O}_{X,x} \subseteq \operatorname{Quot} \mathcal{O}_{X,x} = K(X) \subseteq K.$$

Since $K = \operatorname{Quot} R$ we have $K(X) = K$; the valuation $\nu$ ensures $R = \mathcal{O}_{X,x}$ ($q \in \mathcal{O}_{X,x} \setminus R \Rightarrow \nu(q) < 0$, i.e. $\forall x \in K\ \exists N \gg 0 : \nu(x/q^N) \geq 0$, i.e. $x/q^N \in R \rightsquigarrow x \in \mathcal{O}_{X,x}$). Hence, we obtain $s$ out of $\operatorname{Spec} R = \operatorname{Spec} \mathcal{O}_{X,x} \to X$. □

*Remark.* By [Hart, Theorem II.4.7], the opposite direction is true, too. I.e. a map $f$ is proper if all (!) codimension one gaps in lifting of maps $Z \to Y$ toward $Z \to X$ can be filled.

28.6. **Global functions on proper schemes.** Let $X$ be scheme over $k = \overline{k}$. We are going to generalize Proposition 52.

**Proposition 99.** *$X =$ reduced, connected, complete $\Rightarrow \Gamma(X, \mathcal{O}_X) = k$.*

*Proof.* $g \in \Gamma(X, \mathcal{O}_X) \Rightarrow (\mathrm{id}, g^{-1}) : X_g \xrightarrow{\sim} V(g\,t-1) \subseteq X \times \mathbb{A}^1_k$ is a closed embedding ($\mathcal{O}_X[t] \twoheadrightarrow \mathcal{O}_X[t]/(gt-1) \xrightarrow{\sim} \mathcal{O}_{X_g}$), but does also factorize over $X \times (\mathbb{A}^1_k \setminus 0)$ via $t^{-1} \mapsto g$. $X =$ complete $\Rightarrow X \times \mathbb{A}^1_k \to \mathbb{A}^1_k$ is closed $\Rightarrow$ the image of $X_g$ is a point in $\mathbb{A}^1_k \setminus 0$, i.e. understood as a function, $g$ is constant. Since closed points of $\mathbb{A}^1_k$ are $k$-rational, we are done. □

28.7. **Applications of properness.** Rather shortly, we mention the following important applications:

28.7.1. *Rigidity.* (Georg Heins Gast-VL 5.7.2011): Rigidity theorem; commutativity of complete group schemes, all morphisms among them with $1_G \to 1_H$ are group homomorphisms. Example: Group structure on elliptic curves.

28.7.2. *Direct images stay coherent.* In the spirit of Proposition 99, one has the very general theorem: $\boxed{f : X \to Y \text{ proper} \Rightarrow f_*(\text{coherent}) \text{ is coherent}}$.

28.7.3. *Stein factorization.* a) $f : X \to Y$ birational, proper; everything noetherian and integral; $Y$ normal $\Rightarrow f_*\mathcal{O}_X = \mathcal{O}_Y$ (follows from (28.7.2).

b) $f : X \to Y$ proper, $f_*\mathcal{O}_X = \mathcal{O}_Y \Rightarrow$ all $\boxed{\text{fibers are connected}}$. ("ZMT": [Hart, Th III.11.1/3] for projective morphisms).

c) $f : X \to Y$ proper $\Rightarrow$ it factorizes into $X \xrightarrow{g} \operatorname{Spec} f_*\mathcal{O}_X \xrightarrow{h} Y$, where the proper $g$ has connected fibers, and $h$ is finite.

28.7.4. *Quasifiniteness.* Quasifinite, proper morphisms $f : X \to Y$ are finite: Stein factorization $\rightsquigarrow f_*\mathcal{O}_X = \mathcal{O}_Y$, and $f$ is bijective.

## 29. The dimension of fibers and intersections

29.1. **Geometric notions.** $X = k$-variety $\rightsquigarrow \overline{X} := X \times_k \overline{k} \rightsquigarrow X =$ "*geometrically* irreducible/reduced", if $\overline{X}$ inherits this property (Ex: $\mathbb{R}[x]/(x^2+1)$, $k[x]/(x^p - \alpha^p)$ with $\operatorname{char} k = p$ and $\alpha^p \in k$, $\alpha \notin k$).

**Proposition 100.** *$X = k$-variety, $Y :=$ irreducible $\overline{X}$-component (with reduced structure) $\Rightarrow \dim X = \dim Y$.*

*Proof.* $k[y_1, \ldots, y_d] \subseteq A$ finite $\Rightarrow \overline{k}[y_1, \ldots, y_d] \subseteq A \otimes_k \overline{k}$ finite, and $Y$ corresponds to a minimal prime ideal of $A \otimes_k \overline{k}$. □

29.2. **The image of morphisms.** Let $X, Y$ be integral, affine schemes; let $f : X \to Y$ of finite type and dominant. Then there is an open $V \subseteq Y$ wher $f$ factorizes into $f^{-1}V \overset{\text{finite}}{\twoheadrightarrow} V \times_{\mathbb{Z}} \mathbb{A}^e \overset{\text{pr}}{\twoheadrightarrow} V$.

($f^* : B \hookrightarrow A \Rightarrow$ $\operatorname{Quot}(B) \subseteq \operatorname{Quot}(B)[y_1, \ldots, y_e] \overset{\text{finite}}{\hookrightarrow} A \otimes_B \operatorname{Quot}(B) \subseteq \operatorname{Quot}(A)$ Noether normalization; $s :=$ common denominator of the integrality relations of the $A|B$ generators $\Rightarrow$ $B_s \subseteq B_s[y_1, \ldots, y_e] \overset{\text{finite}}{\hookrightarrow} A \otimes_B B_s = A_{f^*(s)}$ .)

**Proposition 101** (Chevalley). *Let $f : X \to Y$ of finite type between noetherian schemes $\Rightarrow f(X) \subseteq Y$ is $\boxed{\text{constructible}}$*, i.e. a (disjoint) finite union of [open $\cap$ closed].

*Proof.* W.l.o.g. everything is affine and irreducible, and $f$ is dominant $\Rightarrow$ with the above notation we have $f(X) = V \cup f(X \setminus f^{-1}V)$. □

29.3. **Almost complete intersections.** (Reduced) schemes are called "*pure dimensional*" :$\Leftrightarrow$ all irreducible components have the same dimension.

**Lemma 102.** *Let $Z \subseteq X$ be a closed embedding of affine, pure dimensional $k$-schemes $\Rightarrow \exists f_1, \ldots, f_r$: all components of $V(f_\bullet)$ have dimension $\dim X - r$, and $Z \subseteq V(f_\bullet)$ is a union of some of these components.*

*Proof.* Induction by $\dim X$: Let $P_i \subseteq \Gamma(X, \mathcal{O}_X)$ the minimal prime ideals $\Rightarrow I(Z) \not\subseteq P_i \Rightarrow$ choose $f \in I(Z) \setminus \bigcup_i P_i$. □

29.4. **Fiber dimensions.** $X \supseteq W$ closed, irreducible $\Rightarrow \operatorname{codim}_X W := \dim \mathcal{O}_{X,W} = \operatorname{ht} I(W)$; for $k$-varieties we have $\operatorname{codim}_X W = \dim X - \dim W$ (Proposition 51(2)).

**Proposition 103.** *Let $X, Y = k$-varieties, $f : X \to Y$ (dominant) morphism.*
*1) $y \in Y \Rightarrow \overline{f^{-1}(y)} \subseteq f^{-1}(\overline{y})$ consists of exactly those components $Z$ being mapped dominantly to $\overline{y}$, i.e. with $f(\eta_Z) = y$.*
*2) $W \subseteq f^{-1}(y)$ component $\Rightarrow \operatorname{codim}_X \overline{W} \le \operatorname{codim}_Y(\overline{y})$ and $\boxed{\dim W \ge e} := \dim X - \dim \overline{f(X)}$ (with "=" for flat $f$ – see (31.2)).*
*3) $x \mapsto \dim_x f^{-1}f(x)$ is upper semicontinous, i.e. $F_d := [\dim \ge d] \subseteq X$ is closed.*
*4) $F_e = X$ and $F_{e+1} \subsetneq X$.*

(The upper semicontinouity does not ask for irreducible varieties – just take the maximum of the single functions.)

*Proof.* (1) $Z \subseteq \overline{f^{-1}(y)}$ component $\Rightarrow Z \cap f^{-1}(y) \ne \emptyset \Rightarrow \exists z \in Z : f(z) = y \Rightarrow Z \to \overline{y}$ is dominant. On the other hand, if $Z \subseteq f^{-1}(\overline{y})$ is irreducible and $Z \to \overline{y}$ is dominant, then $f(\eta_Z) = y$, hence $Z = \overline{Z \cap f^{-1}(y)} \subseteq \overline{f^{-1}(y)}$.

(2) W.l.o.g., everything is affine; $y \in Y$, $b := \operatorname{codim}_Y(\overline{y}) \overset{\text{Lemma 102}}{\Longrightarrow} \exists g_1, \ldots, g_b \in K[Y]$: $\overline{y}$ is component of $V(g_1, \ldots, g_b) \Rightarrow \overline{W} \subseteq \overline{f^{-1}(y)} \subseteq f^{-1}(\overline{y}) \subseteq V(f^*g_1, \ldots, f^*g_b)$ is a component (Problem **??**) $\Rightarrow \operatorname{codim}_X \overline{W} \le b$.
It remains to check: $\dim \overline{W} = \dim W + \dim \overline{y}$ ($W \subseteq \overline{W}$ is not open!): With $\overline{W} = \operatorname{Spec} A$ and $\overline{y} = \operatorname{Spec} B$ the map $f|_{\overline{W}}$ corresponds to an inclusion $B \hookrightarrow A$, and $y = \operatorname{Spec}(\operatorname{Quot} B)$ and $W = \operatorname{Spec}(A \otimes_B \operatorname{Quot} B) \rightsquigarrow$ transzendental degrees.

(3+4) First, $F_e = X$ follows from (2). If everything is affine, then (29.2) provides an open $V \subseteq Y$ with $f^{-1}(V) \subseteq F_e \setminus F_{e+1} \Rightarrow F_{d>e} \subseteq f^{-1}(Y \setminus V) \rightsquigarrow$ induction. □

29.5. **Intersections in $\mathbb{A}^n_k$ and $\mathbb{P}^n_k$.** We show that in $\mathbb{P}^n$ varieties of the right dimension do always intersect.

**Proposition 104.** 1) *$Y, Z \subseteq \mathbb{A}^n_k$ irreducible $\Rightarrow$ for all irreducible components $W_i$ of $Y \cap Z$ we have $\operatorname{codim} W_i \le \operatorname{codim} Y + \operatorname{codim} Z$.*
2) *The same claim applies for irreducible, closed $Y, Z \subseteq \mathbb{P}^n_k$. Additionally, one knows that $Y \cap Z \ne \emptyset$, provided $\operatorname{codim} Y + \operatorname{codim} Z \le n$.*

*Proof.* $Z = V(f_1, \ldots, f_k) \Rightarrow \operatorname{codim} W_i \le \operatorname{codim} Y + k$ by (15.1), and this is exactly the claim if $\operatorname{codim} Z = k$, i.e. if $Z$ is a c.i.. Hence, let $\Delta := V(x_i - y_i \mid i = 1, \ldots, n) \subseteq \mathbb{A}^n_k \times_k \mathbb{A}^n_k = \mathbb{A}^{2n}_k$ be the diagonal: It provides $Y \cap Z = (Y \times Z) \cap \Delta$ (even scheme theoretical), and $\Delta \subseteq \mathbb{A}^{2n}_k$ is a linear subspace ($\rightsquigarrow$ c.i.).
Projective case: $0 \in \operatorname{Cone}(Y) \cap \operatorname{Cone}(Z) = \operatorname{Cone}(Y \cap Z) \subseteq \mathbb{A}^{n+1}_k$. □

## 30. (proper)$_*$ of cycles

30.1. **Generalized orders.** We are going to generalize $\operatorname{ord}_D(f) = \ell(\mathcal{O}_{X,\eta(D)}/f)$ of (23.2). If $A$ is a noetherian domain, essentially of finite type over a field $k$, then we define $\boxed{e_A(\varphi, M) := \ell(\operatorname{coker}\varphi) - \ell(\ker\varphi)}$ for finitely generated $A$-module $M$ with some given $\varphi \in \operatorname{End}_A(M)$.

*Properties:* Additivity in short exact sequences; $e_A(\varphi, \text{finite length}) = 0$; $e_A(\varphi\psi) = e_A(\varphi) + e_A(\psi)$ $(0 \to \ker\psi \to \ker(\varphi\psi) \xrightarrow{\psi} \ker\varphi \to \operatorname{coker}\psi \to \ldots)$; $e_A$ is also additive w.r.t. all localizations of $M$ (we apply for $\operatorname{coker}\varphi$ and $\ker\varphi$ that the length of these length modules has this additivity – just use the $A/P$-filtration from (3.3)), and $e_A$ for a one-dimensional domain $A$ is defined $\Leftrightarrow \varphi \otimes \mathrm{id}_{\mathrm{Quot}(A)} = \text{iso}$.

**Lemma 105.** *Let $A$ be a one-dimensional domain. Then $e_A(\varphi, M) = \operatorname{ord}\left[\det(\varphi \otimes \mathrm{id}_{\mathrm{Quot}\,A})\right]$, and $\det(\varphi \otimes \mathrm{id}_{\mathrm{Quot}\,A})) \in A$ whenever $A = $ normal.*

*Proof.* We may assume that $A = (A, P)$ is local. $A = $ normal $\Rightarrow$ principal ideal domain $\Rightarrow \ell_A(T(M)) < \infty$, and $M/T(M)$ is free $\rightsquigarrow$ Elementarteilersatz.

In general if $A \hookrightarrow \bar{A}$ is the normalization $\Rightarrow$ ker and coker of $M \to M \otimes_A \overline{A}$ have finite length (normalization is birational and finite). Moreover, if $Q_i \in \operatorname{MaxSpec}\overline{A}$, then $A/P \hookrightarrow \overline{A}/Q_i$ is a finite field extension of degree $d_i$, and, for finite length $\overline{A}$-modules $N$, we obtain $\ell_A(N) = \sum_i d_i\, \ell_{\overline{A}_{Q_i}}(N_{Q_i})$. □

30.2. **Rational equivalence of cycles.** $Z_k(X) :=$ group of $k$-dimensional cycles in an $n$-dimensional $k$-scheme $X$, i.e. the free abelian group generated by the $k$-dimensional subvarieties; in particular $Z_{n-1}(X) = \operatorname{Div}(X)$.

"Rational equivalence" of $k$-cycles is defined via $\operatorname{PDiv}(W^{k+1}) \subseteq \operatorname{Div}(W^{k+1}) = Z_k(W^{k+1}) \hookrightarrow Z_k(X)$ for $(k+1)$-dimensional subvarieties $W^{k+1} \subseteq X \rightsquigarrow Z_k(X) \twoheadrightarrow A_k(X)$ ("Chow groups"; note that $A_{n-1}(X) = \operatorname{Cl}(X)$).

*Examples:* $A_k(\mathbb{P}^n) = \mathbb{Z} \cdot [L^k]$: $T$-action $\rightsquigarrow$ linear subspaces, or rather see (31.6); $A_{\mathrm{top}}(X) = \mathbb{Z}^{\oplus \mathrm{topComp}}$.

30.3. **Proper push forward of cycles.** Let $p : X \to Y$ be proper between schemes $\rightsquigarrow p_* : Z_k(X) \to Z_k(Y)$ via

$$D \subseteq X \Rightarrow p_*(D) := \begin{cases} 0 & \text{if } \dim p(D) < \dim D, \\ [K(D) : K(p(D))] \cdot p(D) & \text{if } \dim p(D) = \dim D. \end{cases}$$

**Theorem 106.** *$p_*$ factorizes via $A_k(X) \to A_k(Y)$.* More detailed: *If $p : X \to Y$ is a surjective map between $k$-varieties and $f \in K(X) \Rightarrow \boxed{p_* \operatorname{div}(f) = \operatorname{div}(N(f))}$ if $\dim X = \dim Y$, and $p_* \operatorname{div}(f) = 0$ otherwise.*

(Recall that $N(f) := \det f$ w.r.t. the $K(Y)$-linear $K(X) \xrightarrow{\cdot f} K(X)$.)

*Proof.* (i) If $p : \mathbb{P}^1_K \to \operatorname{Spec} K$ (arbitrary fields are necessary – even in the case that $k = \overline{k}$) $\rightsquigarrow p_* \operatorname{div}(f) = 0$.

(ii) Let $p$ be a *finite* morphism: Let $(A, P) := \mathcal{O}_{Y,\eta(E)}$ for a prime divisor $E \subseteq Y$ $\rightsquigarrow$ finite $A$-algebra $B :=$ "$A \otimes_Y X$" with maximal ideals $Q_i \subseteq B$ over $P \subseteq A \rightsquigarrow$ for $f \in B$ we have to show that $\sum_i \ell_B(B_{Q_i}/f) \cdot [B/Q_i : A/P] = \ell_A(A/N(f))$.
As already used in the proof of Lemma 105, the left hand side equals $\ell_A(B/f)$, and now the equality follows from this very lemma.

(iii) $\dim X = \dim Y$: Stein factorization (28.7.3)(c) $\rightsquigarrow$ we may assume $p_*\mathcal{O}_X = \mathcal{O}_Y$, and this implies "birational": The $\mathbb{A}^{e=0}$ separation done in (29.2) yields open $U \subseteq X$, $V \subseteq Y$ and a finite $p|_U : U \to V$; by (28.1) the properness of $p|_U$ implies that $U = p^{-1}(V)$. Thus $p_*\mathcal{O}_U = \mathcal{O}_V$; since $p$ is finite, this means that $p$ is an isomorphism.
Now replace $X$, $Y$ by their normalizations (finite!) $\Rightarrow p^{-1}$ is defined in all codimension one loci $\Rightarrow p_* \operatorname{div}(f) = \operatorname{div}(f)$; the contracted divisors do not hurt.

(iv) $\dim X > \dim Y$: W.l.o.g. $Y = \operatorname{Spec} K$ (with $K := K(Y)$) and $X$ = curve over $K$. Normalization $\rightsquigarrow \exists$ finite $q : X \to \mathbb{P}^1_K$ (locally by Noether Normalization, Proposition 26, then globalize by **??**). ☐

30.4. **The degree of curves.** $C|k$ = curve, $p \in C$ closed point, i.e. prime divisor $\rightsquigarrow \deg p := [K(p) : k]$ (always $= 1$, if $k = \overline{k}$) $\rightsquigarrow \deg : \operatorname{Div}(C) \to \mathbb{Z}$.

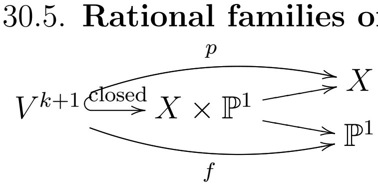
**Corollary 107.** *$C|k$ = complete curve* $\Rightarrow \deg(\operatorname{PDiv} C) = 0 \rightsquigarrow \deg : \operatorname{Cl}(C) \to \mathbb{Z}$.

*Bézout in* $\mathbb{P}^2_k$: Intersection multiplicity $i(P; V(F), V(G)) := \ell(\mathcal{O}_{\mathbb{P}^2,P}/(f,g))$ for properly intersecting, plane curves $\Rightarrow \boxed{\sum_{P\in\mathbb{P}^2} i(P) \deg P = \deg F \cdot \deg G}$.

*Proof.* Proper intersection $\Rightarrow f \in \mathcal{O}_{\mathbb{P}^2,P}/g$ is a non-zerodivisor $\rightsquigarrow$ w.l.o.g. $F$ = irreducible $\Rightarrow$ for $\deg G = \deg G'$ we may consider $G/G' \in K(V(F))$ with $\operatorname{ord}_P(G/G') = i(P; F, G) - i(P; F, G')$. By Corollary 107, this means that $\sum_{P\in\mathbb{P}^2} i(P) \deg P$ depends only on $\deg G$, $\deg F \rightsquigarrow$ w.l.o.g. $\deg = 1$. ☐

**Example 108.** Let $X := V(z_1, z_2) \cup V(z_3, z_4)$ and $Y := V(z_1 - z_3, z_2 - z_4)$ in $\mathbb{A}^4_k$; then $Z = \{0\}$ is the only intersection. $J(X) = (z_1, z_2) \cap (z_3, z_4) = (z_1z_3, z_1z_4, z_2z_3, z_2z_4) \subseteq k[(z_0), z_1, \ldots, z_4]$ has the resolution $0 \to S(-4) \to S^4(-3) \xrightarrow{f} S^4(-2) \to S \to 0$ with $f_1 = z_4e_{13} - z_3e_{14}$, $f_2 = z_4e_{23} - z_3e_{24}$, $f_3 = z_2e_{13} - z_1e_{23}$, $f_4 = z_2e_{14} - z_1e_{24}$ and $z_2f_1 - z_1f_2 - z_4f_3 + z_3f_4 = 0 \rightsquigarrow \deg \overline{X} = 2$, $\deg \overline{Y} = 1$ in $\mathbb{P}^4_k$.
$X \cap Y$ corresponds to the ideal $(z_1^2, z_1z_2, z_2^2) \subseteq k[z_1, z_2]$, hence $\dim \operatorname{Tor}_0 = 3$. The monomials $z_1z_4, z_2z_3 \in J(X)$ become $z_1z_2 = z_2z_1$ in $\overline{J(X)} \subseteq K[Y] = k[z_1, z_2]$ – the missing relation "$e_{14} = e_{23}$" in [resolution] $\otimes\, k[z_1, z_2]$ yields $\dim \operatorname{Tor}_1 = 1$, hence altogether $i(X, Y; Z) = 2$ when defined as $\sum_{i\geq 0}(-1)^i \ell(\operatorname{Tor}_i)$ instead of just $\ell(\operatorname{Tor}_0)$.

30.5. **Rational families of cycles.** They correspond to linear equivalence:




A dominant $f \rightsquigarrow p_* \operatorname{div}(f) = V(0) - V(\infty)$ with $V(P) := V \cap (X \times P) := p_* \operatorname{cyc}[f^{-1}(P)]$ (cf. (31.3) and Problem **??**) $\Rightarrow$ all $V(P) \in Z_k(X)$ are mutually equivalent.

Conversely: $W^{k+1} \subseteq X$, $f \in K(W) \Rightarrow \operatorname{div}(f) \in Z_k(W) \subseteq Z_k(X)$ can be pressed in the above pattern via $f : W - \to \mathbb{P}^1$. Indeed, $\overline{W} := \overline{\Gamma(f)} \subseteq W \times \mathbb{P}^1 \subseteq X \times \mathbb{P}^1$ provides a dominant $\overline{f} : \overline{W} \to \mathbb{P}^1$ and a proper, birational $\pi : \overline{W} \to W$ with $\pi_* \operatorname{div}(\overline{f}) = \operatorname{div}(f)$.

## 31. (FLAT)$^*$ OF CYCLES

Examples of flat morphisms: Open embeddings, locally trivial bundles, dominant maps from varieties onto smooth curves.

31.1. **Flat implies open.** Let $\varphi : A \to B$ be a local, flat ring homomorphism. Then $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective: Since $\varphi$(non-zero divisors) $\subseteq$ {non-zero divisors}, we obtain a pre-image of w.l.o.g. $(0) \in \operatorname{Spec} A$ from every prime ideal $Q$ in the localization $\{\text{non-zero divisors}\}^{-1} \cdot B$.

**Proposition 109.** *Flat morphisms between $k$-schemes of finite type are open.* (Counter example: $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$.)

*Proof.* If $Z$ is the complement of the image of an open subset, then we have just seen that it is closed under specialization ($y \in \overline{z}$ with $z \in Z$ implies $y \in Z$), and it is constructible by Chevalley's Proposition 101. The latter means that it contains a $U \subseteq Z$ which is open and dense in $\overline{Z}$. Thus, this $U$ contains the generic points of all irreducible components of $\overline{Z}$. □

31.2. **Flat fiber dimensions.** Scheme theoretical specification of (the proof of) Proposition 102 of (29.3): Let $Z^{n-r} \subseteq X^n$ be a closed embedding of affine, equidimensional $k$-schemes $\Rightarrow \exists f_1, \ldots, f_r \in A(X)$ (with $V(f_\bullet)$ being $(n-r)$-dimensional and $Z$ being the union of some components) and ideals $I_1, \ldots, I_r \subseteq A(X)$:

(i) $I_v$ is the nilradical modulo $(f_1, \ldots, f_{v-1}) + I_1 + \ldots + I_{v-1}$,
(ii) $f_v$ is a non-zero divisor modulo $(f_1, \ldots, f_{v-1}) + I_1 + \ldots + I_v$.

In particular, we obtain that $\sqrt{(f_1, \ldots, f_r)} = \sqrt{(f_1, \ldots, f_r) + I_1 + \ldots + I_r}$. The property "nilpotent" (for elements and ideals) is preserved under base change; similarily "non-zero-divisor" is preserved und *flat* base change. In particular, Proposition 103(2) from (29.4) can be extended to

If $f : X \to Y$ is a *flat* ($\Rightarrow$ dominant) morphism between $k$-varieties and $y \in Y$, then for all components $W \subseteq f^{-1}(y)$ we have $\boxed{\dim W = e} := \dim X - \dim Y$.

31.3. **Cycles of subschemes.** $Z \subseteq Y$ purely $k$-dimensional, closed subscheme $\rightsquigarrow$ with $V \in$ {$k$-dim subvarieties of $Y$} we define $\operatorname{cyc}_k(Z) := \sum_V \ell_{\mathcal{O}_{Y,\eta(V)}}(\mathcal{O}_{Z,\eta(V)}) \cdot [V] \in Z_k(Y)$.

$f : X \to Y$ *flat* between $k$-varieties with relative dimension $e \rightsquigarrow f^*\colon Z_\bullet(Y) \to Z_{\bullet+e}(X)$ (in particular $\operatorname{Div} Y \to \operatorname{Div} X$) via $f^*\colon [V] \mapsto \operatorname{cyc}_{k+e}(f^{-1}V)$; for subschemes $Z \subseteq Y$ one obtains then that $\boxed{f^* \operatorname{cyc}(Z) = \operatorname{cyc}(f^{-1}Z)}$:

Local rings $B \leftarrow A$ in prime cycles $W \subseteq f^{-1}V \subseteq X$ and $V \subseteq Y \rightsquigarrow Z$ corresponds to an $A$-module $M := A/I(Z)$, and the claim follows from $\ell_B(M \otimes_A B) = \ell_A(M) \cdot \ell_B(B/\mathfrak{m}_A B)$ (by tensorizing the composition series of $M$ with $B$).

**Theorem 110.** *$f^*$ is compatible with rational equivalence* $\rightsquigarrow$ $f^*\colon A_\bullet(Y) \to A_{\bullet+e}(X)$.

*Proof.* $\begin{array}{ccccc} W & \xrightarrow{f} & V & \xrightarrow{h} & \mathbb{P}^1 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \\ X & \xrightarrow{f} & Y & & \end{array}$ $q_* \operatorname{div}(h) = q_* h^*(0) - q_* h^*(\infty) \in Z_k(Y)$; $t \in \mathbb{P}^1 \Rightarrow f^* q_* h^*(t) = p_*(hf)^*(t)$; with $\operatorname{cyc}(W) = \sum_i m_i [W_i]$, we have $\operatorname{cyc}(hf)^{-1}(t) = \sum_i m_i \operatorname{cyc}(hf)_i^{-1}(t)$:

$A := \mathcal{O}_{W,\eta}$ = local ring in a prime divisor $\rightsquigarrow P_i \subseteq A$ minimal prime ideals associated to $W_i$; $A \ni s$ = local equation of $(hf)^{-1}(t) \in A \Rightarrow \ell_A(A/s) = \sum_i m_i \, \ell_A(A/(s + P_i))$: $m_i = \ell_{A_{P_i}}(A_{P_i}) = \#\{A/P_i \text{ in } A\text{-filtration of } A\} = \#\{A/(s+P_i) \text{ in } A/s\}$ because of $\operatorname{Tor}_1^A(A/P_i, A/s) = 0$, and since $A/\mathfrak{m}_A$-factors do not contribute: $0 \to M \to N \to A/\mathfrak{m}_A \to 0 \Rightarrow \ell_A(N/sN) - \ell_A(M/sM) = \sum_{v \geq 0} (-1)^v \ell(\operatorname{Tor}_v^A(A/\mathfrak{m}_A, A/s)) = 0$ since $s \in A$ is a non-zero divisor, $M, N \subseteq A$ are ideals, and $A/\mathfrak{m}_A$ is of finite length. □

31.4. **Flat and finite.** $\pi : X' \to X$ flat and finite $\Rightarrow$ $\boxed{\pi_* \pi^* = \deg \pi}$ $\in \operatorname{End} A_\bullet(X)$: $A :=$ local ring in a prime cycle of $X \Rightarrow B := A \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$ is a free $A$-algebra of rank $\deg \pi$; let $P_i \subseteq B$ be the maximal ideals (over $\mathfrak{m}_A$) $\rightsquigarrow$ for an $A$-module $M$ $(= A/\mathfrak{m}_A)$ we have $\sum_i \ell_{B_{P_i}}(M \otimes_A B_{P_i}) \cdot [B/P_i : A/\mathfrak{m}_A] = \ell_A(M \otimes_A B) = \deg \pi \cdot \ell_A(M)$.

This implies Corollary 107 again: $f : C \to \mathbb{P}^1 \Rightarrow f_* \operatorname{div}(f) = f_* f^*(0) - f_* f^*(\infty) = (\deg f) \cdot ([0] - [\infty])$ has degree 0.

**Corollary 111.** *Let $C$ be a smooth, complete curve.*

1) *If there are $k$-rational points $P, Q \in C$ with $[P] - [Q] = \operatorname{div}(f)$, then $f : C \to \mathbb{P}^1$ is an isomorphism.* ($f$ exists as a map; $f^*(0) = P \Rightarrow \deg f = 1$.)

2) *Every non-constant $\mathbb{P}^1 \xrightarrow{g} C$ is an isomorphism.* (Consider $g_*([P] - [Q])$.)

*Example:* Elliptic curves – see [Hart, S.139/140].

31.5. **Stratification.** $Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$ complementary (closed/open) $\rightsquigarrow$ exact sequence $A_\bullet Z \xrightarrow{i_*} A_\bullet X \xrightarrow{j^*} A_\bullet U \to 0$. Example: $\operatorname{Cl}(\mathbb{P}^n \setminus V(F_d)) = \mathbb{Z}/d\mathbb{Z}$.

*Mayer-Vietoris:* $\begin{array}{ccc} Z' & \overset{i'}{\hookrightarrow} & X' \\ \downarrow & & \downarrow{\scriptstyle p} \\ Z & \overset{i}{\hookrightarrow} & X \end{array}$ cartesian with closed embedding $i$, proper $p$ and an isomorphism $p : (X' \setminus Z') \xrightarrow{\sim} (X \setminus Z)$ (e.g. blowing up or $X = Z \cup X'$, $Z' = Z \cap X'$)

$\Rightarrow$ $A_\bullet Z' \to A_\bullet Z \oplus A_\bullet X' \to A_\bullet X \to 0$ is exact:

Vertical arrows with $A_\bullet(X' \setminus Z') \xrightarrow{\sim} A_\bullet(X \setminus Z) \Rightarrow$ diagram chasing reduces everything onto the surjectivity of $\ker i'_* \twoheadrightarrow \ker i_*$. Let $\sum_i \operatorname{div} f_i$ be then a $Z$-cycle with $f_i \in K(W_i)$ for some $W_i \subseteq X$ – w.l.o.g. with $W_i \not\subseteq Z \Rightarrow W'_i := \overline{W_i \setminus Z} \subseteq \overline{X' \setminus Z'} = X'$ is birational to $W_i \Rightarrow f_i \in K(W'_i)$, and $\sum_i \operatorname{div} f_i \in \ker i'_*$.

31.6. **Affine bundles.** Locally trivial $\mathbb{A}^e$-bundles $p : \widetilde{X} \to X$ are flat, and $p^*$ provides an isomorphism $p^* : A_\bullet(X) \xrightarrow{\sim} A_{\bullet+e}(\widetilde{X})$ between the Chow groups ($\rightsquigarrow A_\bullet(\mathbb{A}^e) = \mathbb{Z}[-e]$). Via stratification of the base, this follows from

**Lemma 112.** *Denote by $p : X \times \mathbb{A}^1 \to X$ the projection. Then, $p^* : A_\bullet(X) \xrightarrow{\sim} A_{\bullet+1}(X \times \mathbb{A}^1)$, $\alpha \mapsto \alpha \times \mathbb{A}^1$ is an isomorphism.*

*Proof.* Stratification $\rightsquigarrow$ w.l.o.g. $X = \operatorname{Spec} A$ is affine and irreducible.

*Surjectivity:* $Z \subseteq X \times \mathbb{A}^1 = \operatorname{Spec} A[t]$ prime cycle $\Rightarrow$ w.l.o.g. $X = \overline{p(Z)}$. If $Z = X \times \mathbb{A}^1$, then $Z = p^*(X)$ – we are left with the case [$\dim Z = \dim X$]: Within $\operatorname{Spec} Q(A)[t]$, the subvariety $Z$ is given by an equation $h(t) \in A[t] \rightsquigarrow Z := Z - \operatorname{div}(h)$ shrinks the support of $p(Z)$.

*Injectivity for divisors:* From $D \times \mathbb{A}^1 = \operatorname{div}(f(t))$ with $f \in Q(A)(t)$ we conclude that $f \in Q(A)$, and $D = \operatorname{div}(f)$ on $X$. □

31.7. **Toric cycles.** Torus $T = \mathbb{G}_m^n$ is open in $\mathbb{A}_k^n \Rightarrow A_\bullet T = \mathbb{Z}[-n]$. If $\Sigma$ is a fan in $N = \mathbb{Z}^n$, then we obtain $\mathbb{Z}^{\Sigma(1)} = \oplus_{a \in \Sigma(1)} A_{n-1}(\overline{\operatorname{orb}}(a)) \twoheadrightarrow A_{n-1}\mathbb{TV}(\Sigma)$ in codimension one; its kernel is $M$ by (23.4).

If $X = \bigcup_i X_i$ is the decomposition into irreducible components, then there is a surjection $\oplus_i A_\bullet(X_i) \twoheadrightarrow A_\bullet(X)$, and the kernel is covered by $\oplus_{i \neq j} A_\bullet(X_i \cap X_j)$. Hence, via $\sigma \in \Sigma \rightsquigarrow [\overline{\operatorname{orb}}(\sigma)] \in A_{n-\dim\sigma}(\overline{\operatorname{orb}}(\sigma)) \to A_{n-\dim\sigma}(\mathbb{TV}(\Sigma))$, we obtain inductivly a surjection $\oplus_{\sigma\in\Sigma}\mathbb{Z}[\dim\sigma - n] \twoheadrightarrow A_\bullet(\mathbb{TV}(\Sigma))$. Moreover, using the sequence from (23.4), every cone $\tau$ does also provide a contribution to the kernel via $(\tau^\perp \cap M) \to \oplus_{\sigma\supseteq\tau}\mathbb{Z}[\dim\sigma - n]$ where the sum is taken over all $\sigma \supseteq \tau$ with $\dim\sigma = 1 + \dim\tau$.

Intersection theory on toric varieties: Either an $n$-dimensional $X = \mathbb{TV}(\Sigma)$ is $\mathbb{Q}$-factorial ($\Sigma$ is simplicial), then one can consider products like $\overline{\operatorname{orb}(a^1)} \cdot \ldots \cdot \overline{\operatorname{orb}(a^n)}$, or, in general, one can intersect $k$-cycles $\overline{\operatorname{orb}(\tau)}$ with Cartier divisors, e.g. with $\mathcal{O}(\Delta)$.

## 32. Satz von Bézout und (proj)$_*$(coh)=coh

**32.1.** Sei $X =$ noethersches Schema, $\mathcal{L}|X$ invertierbar, $\mathcal{F}|X$ quasikohärent.

**Lemma 113.** *$s \in \Gamma(X, \mathcal{F})$, $f \in \Gamma(X, \mathcal{L})$ mit $s|_{X_f} = 0 \Rightarrow \exists n\colon 0 = f^n s \in \Gamma(\mathcal{F} \otimes \mathcal{L}^n)$. Umgekehrt gilt: $t \in \Gamma(X_f, \mathcal{F}) \Rightarrow \exists n$: $f^n t$ kommt von $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^n)$.*

*Proof.* Endliche affine Überdeckung $\{U_i\} \rightsquigarrow \exists n: 0 = f^n s \in \Gamma(U_i, \mathcal{F} \otimes \mathcal{L}^n)$. Für $t$ gibt es $s_i \in \Gamma(U_i, \mathcal{F} \otimes \mathcal{L}^m)$ mit $s_i = f^m t = s_j$ auf $(U_i \cap U_j) \cap X_f \Rightarrow \exists n$: $f^n s_i = f^n s_j$ auf $U_i \cap U_j \rightsquigarrow$ Verklebung. □

**32.2.** Sei $A = S_0$ noethersch und $S = \oplus_{d\geq 0} S_d$ von $S_1$ (= endlicher $A$-Modul) als $A$-Algebra erzeugt. Seien $X := \operatorname{Proj} S \to \operatorname{Spec} A$ und $\mathcal{F}|X$ quasikohärent $\rightsquigarrow$ graduierter $S$-Modul $\Gamma_*(\mathcal{F}) := \oplus_{d\in\mathbb{Z}}\Gamma(X, \mathcal{F}(d)) \rightsquigarrow$ Adjunktionsabb. $\widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$ und $M \to \Gamma_*(\widetilde{M})$.

**Satz 114.** *1) $\widetilde{\Gamma_*(\mathcal{F})} = \mathcal{F}$, und für kohärente $\mathcal{F}$ gibt es einen endlich erzeugten, graduierten $S$-Modul $M \subseteq \Gamma_*(\mathcal{F})$ mit $\widetilde{M} = \mathcal{F}$. Insbesondere sind alle $\mathcal{F}(\gg 0)$ durch endlich viele globale Schnitte erzeugt.*

*2) $M =$ endlicher, graduierter $S$-Modul $\Rightarrow M_d \hookrightarrow \Gamma(X, \widetilde{M}(d))$ ist injektiv für $d \gg 0$.*

*Proof.* 1) $f \in S_1 \to \Gamma(X, \mathcal{O}_X(1)) \Rightarrow \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$ gibt wegen Lemma 113 Isomorphismen auf $\Gamma(D_+(f), \bullet)$. Für kohärente $\mathcal{F}$ ist $\Gamma_*(\mathcal{F})_{(f)}$ ein endlicher $S_{(f)}$-Modul – und die (für alle $S_1$-Erzeuger $f$) benötigten Erzeuger geben dann $M$.

2) $K := \ker(M \to \Gamma_*(\widetilde{M}))$ ist endlich erzeugt (mit $\deg \leq e$) $\Rightarrow \exists n\colon S_+^n K = 0$ und $K_{\geq(n+e)} \subseteq S_+^n K_{\geq e}$. □

**32.3.** Seien $A$ = endlich erzeugte $k$-Algebra, $X = \operatorname{Proj} S$ wie oben und $M$ = endlicher, graduierter $S$-Modul. Insbesondere ist wieder alles noethersch.

**Satz 115.** *1)* $M_d \xrightarrow{\sim} \Gamma(X, \widetilde{M}(d))$ *ist ein* $A$*-Modul-Isomorphismus für* $d \gg 0$.
*2)* $\Gamma(X, \widetilde{M}(d))$ *ist ein endlich erzeugter* $A$*-Modul für alle(!)* $d \in \mathbb{Z}$.

*Proof.* Filtration von $M \rightsquigarrow M = S/P \rightsquigarrow$ o.B.d.A. $M = S$ = Integritätsbereich; $S_1 = \langle s_0, \ldots, s_n \rangle \Rightarrow \Gamma(X, \mathcal{O}(d)) \overset{\cdot s_0}{\hookrightarrow} \Gamma(X, \mathcal{O}(d+1))$ injektiv $\Rightarrow$ (1) impliziert (2).
$S \subseteq T := \oplus_{d\geq 0} \Gamma(X, \mathcal{O}(d)) \subseteq \Gamma_*(\mathcal{O}_X) \subseteq \bigcap S_{s_i} \subseteq \operatorname{Quot} S$; für homogene $t \in T$ gilt dann: $\exists n\colon S_+^n\, t \subseteq S \rightsquigarrow S_+^n\, t \subseteq S_+^n \Rightarrow S_+^n\, t^k \subseteq S_+^n \subseteq S$ für alle $k \in \mathbb{N}$. Wegen $s_0^n \in S_+^n$ impliziert das $S[t] \subseteq s_0^{-n}\, S$, also die Ganzheit von $t$ über $S$.
$k \subseteq A \subseteq S \Longrightarrow$ Normalisierung $\overline{S}$ ist endlicher $S$-Modul $\Rightarrow T$ ist endlicher $S$-Modul; seien die Erzeuger vom Grad $\leq e$. Wie oben folgt $S_+^n \cdot [\text{Erzeuger}] \subseteq S$, also $T_{\geq n+e} \subseteq S_+^n \cdot [\text{Erzeuger}] \subseteq S$. □

**Folgerung 116.** *Seien* $f : X \to Y$ *ein projektiver Morphismus von* $k$*-Schemata und* $\mathcal{F}|X$ *ein kohärenter* $\mathcal{O}_X$*-Modul* $\Rightarrow f_*\mathcal{F}$ *ist ein kohärenter* $\mathcal{O}_Y$*-Modul.*

**32.4.** $S = \oplus_{d\geq 0} S_d$ graduierte, durch $S_1$ endlich erzeugte $(S_0 = k)$-Algebra $\Rightarrow X := \operatorname{Proj} S \subseteq \mathbb{P}_k^N$ hat Hilbertpolynom $\boxed{H_X(t) = \dim_k \Gamma(X, \mathcal{O}(t)) \text{ für } t \gg 0}$; $\deg H_X = n := \dim X$ (wegen **??**) $\rightsquigarrow H_X(t) = c_n\binom{t+n}{n} + c_{n-1}\binom{t+n-1}{n-1} + \ldots$ mit $c_k \in \mathbb{Z}$ nach **??**; der höchste Koeffizient heißt "*Grad* von $X$" $\boxed{\deg X := c_n \in \mathbb{Z}_{\geq 1}}$; $\deg(X \times_k \overline{k}) = \deg X$ wegen $\dim_{\overline{k}}(S_d \otimes_k \overline{k}) = \dim_k S_d$.
Beispiele: $\deg \mathbb{P}_k^N = 1$; *Hyperfläche* $V(F) \subseteq \mathbb{P}_k^N$ mit $F \in k[\mathbf{x}]$ homogen vom Grad $d \Rightarrow 0 \to k[\mathbf{x}](-d) \to k[\mathbf{x}] \to k[\mathbf{x}]/(F) \to 0$ gibt $H_{V(F)}(t) = \binom{t+N}{N} - \binom{t+N-d}{N} = \frac{d\,N}{N!} t^{N-1} + \ldots$, also $\deg(V(F)) = d$; *Linearer Unterraum* $L = \operatorname{Proj} k[\mathbf{z}]/(\ell_1, \ldots, \ell_k) \subseteq \mathbb{P}_k^N$ mit linearen $\ell_i \in \operatorname{span}_k(\mathbf{z}) \Rightarrow \deg L = 1$; *abgeschlossener Punkt* $P = \operatorname{Spec} K(P) \subseteq \mathbb{P}_k^N$ (ist linearer Unterraum nur für $K(P) = k$) $\Rightarrow \mathcal{O}(1)$ ist trivial auf $P \Rightarrow H_P(t)$ ist konstant und $\deg P = \dim_k \Gamma(P, \mathcal{O}_P) = [K(P) : k]$.

$\boxed{p_a(X) := (-1)^{\dim X}(H_X(0) - 1)}$ "arithmetisches Geschlecht"; für $d$-dimensionale Hyperflächen ist $H_{V(F)}(t) = \binom{t+N}{N} - \binom{t+N-d}{N} \Rightarrow p_a(V(F)) = \binom{d-1}{N}$.

**32.5.** Lokale Situation: $P$ sei glatte $k$-Varietät (siehe **??**, z.B. $P = \mathbb{A}_k^N$); $X, Y \subseteq P$ abgeschlossene Untervarietäten mit irreduzibler Komponente $Z \subseteq X \cap Y$ der "richtigen Dimension": $\operatorname{codim}_P Z = \operatorname{codim}_P X + \operatorname{codim}_P Y \rightsquigarrow$ "*Schnittmultiplizität*" $\boxed{i(X, Y; Z) := \sum_{v\geq 0} (-1)^v \ell\big(\operatorname{Tor}_v^{\mathcal{O}_{P,Z}}(\mathcal{O}_{P,Z}/J(X), \mathcal{O}_{P,Z}/J(Y))\big)}$ ist wohl-definiert: Tor ist endlich erzeugter $\mathcal{O}_{P,Z}/(J(X) + J(Y))$-Modul, hat also endliche Länge; Hilberts

Syzygiensatz **??** $\Rightarrow$ die Summe bricht ab. (Der "Hauptterm" $i_0 := \ell(\mathrm{Tor}_0 = \otimes)$ entspricht genau dem schematheoretischen Durchschnitt $X \cap Y$.)

Beispiele: 1) $Y = V(f) \Rightarrow 0 \to \mathcal{O}_{P,Z} \xrightarrow{\cdot f} \mathcal{O}_{P,Z} \twoheadrightarrow \mathcal{O}_{P,Z}/J(Y)$ ist Auflösung $\Rightarrow$ $\mathrm{Tor}_1 = \ker\big(\mathcal{O}_{P,Z}/J(X) \xrightarrow{\cdot f} \mathcal{O}_{P,Z}/J(X)\big)$, also $i(X, V(f); Z) = \ell(\mathrm{Tor}_0) = \ell\big(\mathcal{O}_{P,Z}/(J(X)+(f))\big) = \ell\big(\mathcal{O}_{X,Z}/f \cdot \mathcal{O}_{X,Z}\big)$, falls $f$ kein Nullteiler in $\mathcal{O}_{X,Z}$ ist.

2) Siehe Beispiel 108 in (30.4).

**32.6.** Sei $X \subseteq \mathbb{P}^N_k$ projektiv und rein-dimensional. Falls $Y := V(F)$ keine $X$-Komponente enthält $\Rightarrow$ alle Komponenten $Z_j \subseteq X \cap Y$ erfüllen $\dim Z_j = \dim X - 1$, bzw. $\mathrm{codim}(Z_j) = \mathrm{codim}(X) + 1$.

**Satz 117** (Bézout für Hyperflächen)**.** *Seien $X$ und $Y = V(F)$ wie oben. Dann gilt* $(\deg X)\,(\deg Y) = \sum_j i(X, Y; Z_j) \cdot \deg Z_j$. *Falls $X$ keine eingebetteten Komponenten besitzt (z.B. reduziert oder Hyperfläche), so ist außerdem $i(X, Y; Z_j) = i_0(X, Y; Z_j)$.*

*Proof.* $S := k[z_0, \ldots, z_N]$, $d := \deg F$, $I := J(X) \Rightarrow$ tensorieren der $S/F$-Auflösung liefert die exakte Sequenz $0 \to \mathrm{Tor}_1^S(S/I, S/F) \to S/I\,(-d) \xrightarrow{\cdot F} S/I \to S/(I+F) \to 0$; mit $T_v := \mathrm{Tor}_v^S(S/I, S/F)$ erhalten wir analog zu 32.4: $H_{T_0}(t) - H_{T_1}(t) = H_X(t) - H_X(t-d) = d\,(\deg X) \cdot \frac{t^{\dim X - 1}}{(\dim X - 1)!} + \ldots$.
Andererseits haben die $T := T_v$ Filtrationen mit Faktoren $\cong S/P$, und nur die minimalen Primideale $P_i = I(Z_i)$ tragen zum $t^{\dim X - 1}$-Koeffizienten (mit $\frac{\deg Z_i}{(\dim X - 1)!}$) bei. Die Anzahl der Faktoren $\cong S/P_i$ ist $\ell(T_{(P_i)}) = \ell(T_{P_i})$, denn $(S/P_i)_{(P_i)} \subseteq (S/P_i)_{P_i}$ sind Körper, haben also Länge $= 1$.
$p \in D_+(x_0)$ (affine Karte) $\Rightarrow F \in \Gamma(X, \mathcal{O}_X(d))$ gibt die Funktion $f := F/x_0^d \in \mathcal{O}_{X,p}$; Schnittbedingung $\rightsquigarrow f$ liegt in keinem minimalen, also assoziierten Primideal $\Rightarrow$ $\mathcal{O}_{X,p} \xrightarrow{\cdot f} \mathcal{O}_{X,p}$ ist injektiv, und wir sind im Beispiel 32.5(1).
*Alternativ:* $F : \mathcal{O}_X \to \mathcal{O}_X(d)$ ist injektiv $\Rightarrow F : S/I(-d) \to S/I$ (mit $S/I = S_X$) ist injektiv für hohe Grade. $\square$

Beispiel: $X = V(x^2, xy)$, $Y = V(y - z)$, $Y' = V(y) \subseteq \mathbb{P}^2_k \Rightarrow H_X = t + 2 = H_{\mathbb{P}^1} + 1$, $i(X, Y; 0) = 1$, $i(X, Y'; 0) = 2 - 1 = 1$.

**32.7.** $X, Y \subseteq \mathbb{P}^N_k$ (projektiv, und rein-dimensional) liegen in "*eigentlicher Schnittposition*" $:\Leftrightarrow$ alle Komponenten $Z_j \subseteq X \cap Y$ erfüllen $\mathrm{codim}(Z_j) = \mathrm{codim}(X) + \mathrm{codim}(Y)$.

**Satz 118** (Bézout)**.** *Seien $X, Y \subseteq \mathbb{P}^N_k$ in "eigentlicher Schnittposition" und $Z_j \subseteq X \cap Y$ die Komponenten. Dann gilt* $(\deg X)\,(\deg Y) = \sum_j i(X, Y; Z_j) \cdot \deg Z_j$.

*Proof.* $S = k[z_0, \ldots, z_N] \twoheadrightarrow S_X, S_Y$; **??** $\rightsquigarrow$ endliche, $S$-freie, graduierte Auflösungen $M_\bullet \to S_X$, $N_\bullet \to S_Y \Rightarrow \mathrm{tot}(M_\bullet \otimes_S N_\bullet)$ hat Homologie $\mathrm{Tor}_v^S(S_X, S_Y)$ (Spektralsequenz: $\mathrm{H}^I_p \mathrm{H}^{II}_q = 0$ für $q \neq 0$, $\mathrm{H}^I_p \mathrm{H}^{II}_0 = \mathrm{Tor}_p$) $\Rightarrow$ analog zum Beweis von Satz 117 erhalten wir die rechte Seite der Behauptung als die alternierende Summe der (binomial normierten) höchsten, also $t^{\dim(X \cap Y)}$-Koeffizienten der Hilbertpolynome der

Homologie von $\mathrm{tot}(M_\bullet \otimes_S N_\bullet)$.

**??** $\leadsto$ für (endlich erzeugte, graduierte) $S$-Moduln $L$ mit $\operatorname{supp} L \subseteq Z$ ist $(\dim Z)! \cdot [t^{\dim Z}$-Koeffizient von $H_L(t)] = (1-t)^{1+\dim Z} P(L,t)|_{t=1}$. Speziell, wegen der Additivität von $P(\bullet,t)$: $\deg X = (1-t)^{1+\dim X} P(S_X,t)|_{t=1} = (1-t)^{1+\dim X} P(M_\bullet,t)|_{t=1}$, analog für $Y$, und [rechte Seite] $= (1-t)^{1+\dim(X\cap Y)} P(M_\bullet \otimes_S N_\bullet, t)|_{t=1}$.

Schließlich hat für freie, graduierte $S$-Moduln $M, N$ die Differenz $P(M\otimes N,t)\cdot P(S,t) - P(M,t)\cdot P(N,t)$ eine $(t=1)$-Polordnung $\leq (N+1)$: Das folgt aus $P(S(d),t) = t^{-d}P(S,t) +$ [endlich viele Terme] und o.B.d.A. $M = S(a)$, $N = S(b)$. Wegen $1+\dim(X\cap Y) > 0$, also $(1+\dim X)+(1+\dim Y) = (1+\dim(X\cap Y)) + (N+1) > (N+1)$ verschwinden die Störterme kleiner Polordnung in $t=1$. $\square$

**32.8.** $X =$ rein-$d$-dimensionales, noethersches Schema mit irreduziblen Komponenten $X_i := (X_i)_{\mathrm{red}} \leadsto$ "$d$-Zykel" $\operatorname{cyc}(X) := \sum_j \ell(\mathcal{O}_{X,X_i})\cdot[X_i]$. Eingebettete Zyklen $Z = \sum_i c_i[Z_i]$ in $\mathbb{P}^N_k \leadsto$ "*Grad*" $\deg Z := \sum c_i \deg Z_i$. Bézout für $Y := \mathbb{P}^N_k$ gibt $\deg(\operatorname{cyc} X) = \deg X$, d.h. $\{d\text{-dim Unterschemata}\} \xrightarrow{\mathrm{cyc}} A_d(\mathbb{P}^N_k) \xrightarrow{\deg} \mathbb{Z}$.

*Produkt*: $[X]\cdot[Y] := \sum_j i(X,Y;Z_j)\cdot[Z_j]$ für sich (in den $Z_j$) eigentlich schneidende Zyklen $\Rightarrow \deg\big(\operatorname{cyc}(X)\cdot\operatorname{cyc}(Y)\big) = \deg(X)\cdot\deg(Y)$. Es gilt aber nicht $\operatorname{cyc}(X\cap Y) = \operatorname{cyc}(X)\cdot\operatorname{cyc}(Y)$; dafür muß $X\cap Y$ durch $\mathcal{O}_X \otimes^{\mathbb{L}} \mathcal{O}_Y \in \mathcal{D}^b(\mathbb{P}^N_k)$ ersetzt werden.

**32.9.** $X \subseteq \mathbb{P}^N_k$ rein-dimensional $\Rightarrow$ *allgemeine* lineare $L$ mit $\dim L = c := \operatorname{codim} X$ schneiden $X$ eigentlich in abgeschlossenen Punkten $P_j$

$\leadsto$ $\deg X = \sum_j i(X,L;P_j)\deg P_j = \sum_j i(X,L;P_j)\,[K(P_j):k]$. Siehe Aufgabe **??**.

**32.10.** $(A,\mathfrak{m})$ lokal, $d$-dimensional $\Rightarrow$ $\operatorname{mult}(A) := d!\cdot[t^d -$ Koeffizient von $\chi^A_{\mathfrak{m}}]$; speziell: $X\subseteq \mathbb{P}^n_k$ projektiv $\Rightarrow \deg X = \operatorname{mult}(\operatorname{Cone} X, 0)$.

Hyperfläche $A = k[\mathbf{x}]/f$ mit $f\in\mathfrak{m}^d\setminus\mathfrak{m}^{d+1} \Rightarrow \chi^A_{\mathfrak{m}}(t) = \dim_k k[\mathbf{x}]/(\mathfrak{m}^t + (f)) = \dim_k k[\mathbf{x}]/\mathfrak{m}^t - \dim_k k[\mathbf{x}]/\mathfrak{m}^{t-d}$ für $t \gg 0$ (Multiplikation mit $f$ ist vom Grad $d$ und injektiv auf $\mathrm{Gr}_{\mathfrak{m}}\, k[\mathbf{x}]$) $\Rightarrow \operatorname{mult} A = d$.

$(A,\mathfrak{m}) =$ 0-dimensionale $k$-Algebra (d.h. $(0)$ ist $\mathfrak{m}$-primär) $\Rightarrow \operatorname{mult}(A) = \dim_k A$; Beispiele: Ideale $(x,y), (x^2,y), (x^2,xy,y^2) \subseteq k[x,y]$ liefern $\operatorname{mult} = 1,2,3$.

**32.11.** 1) $X\subseteq\mathbb{P}^N$ projektive Varietät, $\deg X \leq \operatorname{codim} X \Rightarrow X$ liegt in einer Hyperebene.

Problem: Schneide $L$ (mit $\dim L = \operatorname{codim} X$) das $X$ eigentlich und transversal $\Rightarrow X\cap L = \{P_1,\ldots,P_d\}$; sei $P\in X\setminus L$ beliebig $\Rightarrow \exists L' \supseteq \{P_1,\ldots,P_d,P\}$, und, falls auch $L'$ eigentlichen Schnitt hat (?), so sind das zu viele Punkte.

Lösung: Über Sekanten durch glatte Punkte, siehe Aufgabe [Hart, (7.7), S.55].

Für reduzible $X$ ist die Behauptung falsch: Die Vereinigung zweier disjunkter Geraden in $\mathbb{P}^3$ ist vom Grad 2, aber nicht eben.

2) Grad, Mult der Sings, SchnittMult – gemischt: z.B. Aufgabe [Hart, I.7.5, S.55].

3) [Gath, S.97 ff] gibt Anwendungen: Pascals Theorem, Abschätzung der Anzahl singulärer Punkte auf Kurven.

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VL "(Comm) Algebra I"
FU Berlin, Winter 2022/23

## 1. Aufgabenblatt zum 24.10.2022

**Problem 1.** Show that $\mathbb{R}[x]/(x^2+1)$ is isomorphic to $\mathbb{C}$ as an $\mathbb{R}$-algebra.

*Solution:* Assigning $x \mapsto i$ yields a surjective $\mathbb{R}$-algebra homomorphism. Its kernel equals the ideal $(x^2+1)$: The inclusion $\supseteq$ is clear, and the other direction follows similarly to the example $\mathbb{R}[x,y]/(y^2-x^3)$ in class.

**Problem 2.** a) An ideal $P$ in a ring $R$ is called prime (ideal) if and only if the set $R \setminus P$ is closed under multiplication. Show directly that $(0)$ and $(3)$ are prime ideals in $R = \mathbb{Z}$ and that $(10)$ is not.

b) Show that an ideal $P \subseteq R$ is prime if and only if $R/P$ is a domain, i.e. lacks zero-divisors. Revisit the three examples of (a) under this aspect.

c) Let $I, J \subseteq R$ be ideals and let $P$ be a prime ideal in $R$. Show that $[P \supseteq I$ or $P \supseteq J]$ if and only if $P \supseteq I \cap J$ if and only if $P \supseteq IJ$.

*Solution:* (a) The product of $x, y \in \mathbb{Z} \setminus \{0\}$ is non-zero. The product of two integers not divisible by 3 is not divisible by 3. $2, 5 \in \mathbb{Z} \setminus (10)$, but $2 \cdot 5 \in (10)$.

(b) $P \subseteq R$ is prime $\Leftrightarrow$ $(0) \subseteq R/P$ is prime $\Leftrightarrow$ $(R/P \setminus \{0\})$ is multiplicatively closed. $\mathbb{Z}$ is a domain; $\mathbb{Z}/(3) = \mathbb{F}_3$ is a field, hence a domain, and $\mathbb{Z}/(10)$ has $2 \cdot 5 = 0$ as zero-divisors.

(c) The implications $[P \supseteq I \text{ or } P \supseteq J] \Rightarrow [P \supseteq I \cap J] \Rightarrow [P \supseteq IJ]$ are clear. On the other hand, if, $[P \not\supseteq I \text{ and } P \not\supseteq J]$, then there are $a \in I \setminus P$ and $b \in J \setminus P$, hence $ab \in IJ \setminus P$.

**Problem 3.** Show that (a) the sum of two nilpotent elements is again nilpotent and (b) that the sum of a nilpotent element and a unit is a always unit.

*Solution:* (a) $\sqrt{0}$ is an ideal. (b) Let $e \in R^*$ and $a^n = 0$. Then, $(e-a)(e^{n-1} + e^{n-2}a + \ldots + a^{n-1}) = e^n - a^n = e^n \in R^*$.

**Problem 4.** a) Recall (or consult a textbook or wikipedia) the notion of a category $\mathcal{C}$. Roughly speaking, it is a collection of objects $\mathrm{Ob}(\mathcal{C})$ (e.g. sets or groups or rings), and for every $A, B \in \mathrm{Ob}(\mathcal{C})$ there is a set $\mathrm{Mor}(A,B)$ of so-called morphisms with a couple of axioms. In particular, there is always provided a distinguished element $\mathrm{id}_A \in \mathrm{Mor}(A,A)$ and a so-called composition map $\mathrm{Mor}(A,B) \times \mathrm{Mor}(B,C) \to \mathrm{Mor}(A,C)$, $f, g \mapsto g \circ f$.
In any category there is a well defined notion of isomorphisms. Moreover, $f \in \mathrm{Mor}(A,B)$ are often written as $f : A \to B$.

b) Call an $A \in \mathrm{Ob}(\mathcal{C})$ to be an *initial object*, if for any $B \in \mathrm{Ob}(\mathcal{C})$ the set $\mathrm{Mor}(A,B)$ consists of exactly one element. Check if the category of sets, the category of abelian groups, or the category of commutative rings with 1 have initial objects.

c) While initial objects might not exist at all (example?), show that whenever they exist they are uniquely determined. I.e. show that if $A, B \in \mathcal{C}$ are two initial objects, then there exists a unique isomorphism $f \in \mathrm{Mor}(A, B)$.

d) Let $\mathcal{C}$ be the category with

$$\mathrm{Ob}(\mathcal{C}) := \{(R, r) \mid R = \text{commutative ring with 1, and } r \in R\}.$$

A morphism $f \in \mathrm{Mor}\big((R, r), (S, s)\big)$ is defined to be a ring homomorphism $f : R \to S$ with $f(r) = s$. Determine the initial object in $\mathcal{C}$ (if it exists at all).

*Solution:* (b) $\emptyset$ is an initial set, $\{0\}$ is an initial abelian group, and $\mathbb{Z}$ is the initial commutative ring with 1.

(c) The category of non-empty sets does not have an initial object. If $A, B \in \mathcal{C}$, then there are $f : A \to B$ and $g : B \to A$ ($\#\,\mathrm{Mor} = 1$), but then $g \circ f$ and $\mathrm{id}_A$ are both contained in $\mathrm{Mor}(A, A)$. Since $\#\,\mathrm{Mor}(A, A) = 1$, we obtain $g \circ f = \mathrm{id}_A$, and similarily with $f \circ g$.

(d) It is $(\mathbb{Z}[x], x)$. That is, for every ring $R$ with a given element $r \in R$ there is exactly one ring homomorphism $\mathbb{Z}[x] \to R$ with $x \mapsto r$.

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# 2. Aufgabenblatt zum 31.10.2022

**Problem 5.** Let $Z \subseteq \mathbb{A}^n_k$ be a closed algebraic subset. Give a clean proof for the following claim discussed in class: $Z$ is a point if and only if $I(Z) \subseteq k[x_1, \ldots, x_n]$ is a maximal ideal.

*Solution:* If $Z = \{c\}$ with $c = (c_1, \ldots, c_n)$, then $I(Z) = \mathfrak{m}_c := (x_1 - c_1, \ldots, x_n - c_n)$: Obviously, we have $I(Z) \supseteq \mathfrak{m}_c$; on the other hand, $\mathfrak{m}_c$ is a maximal ideal, which follows from looking at $\varphi_c : k[x_1, \ldots, x_n] \twoheadrightarrow k$ sending $x_i \mapsto c_i$ and having $\mathfrak{m}_c$ as its kernel.

If we know that $I(Z)$ is a maximal ideal, then $Z \neq \emptyset$, and we may choose some $z \in Z$. Then $I(Z) \subseteq I(z) \subsetneq k[\underline{x}]$, hence $I(Z) = I(z)$. Applying the operator $V(\bullet)$ yields $Z = V(I(Z)) = V(I(z)) = \{z\}$.

**Problem 6.** a) Show that the Zariski topology on $\mathbb{A}^2_k = k^2$ is not equal to the *product topology* (consult a textbook or Wikipedia if necessary) of the Zariski topologies on both factors $k^1$.

b) Let $Z \subseteq k^n$ be a Zariski closed subset; let $f \in A(Z) := k[x_1, \ldots, x_n]/I(Z)$. Show that $f : Z \to \mathbb{A}^1_k$ is a continous function with respect to the Zariski topology on both sides. (Note that the Zariski topology on $Z \subseteq k^n$ is defined as the topology being induced from the Zariski topology on $k^n$ – consult a textbook or Wikipedia to see what this means).

c) Prove or disprove (by giving a counter example): Every bijective map $\varphi : k^1 \to k^1$ is continous with respect to the Zariski topology on both sides.

*Solution:* (a) The open (or closed) subsets of the product topology of $X \times Y$ are generated (via the usual operations) by the products $U \times V$ for open $U \subseteq X$, $V \subseteq Y$ (or $F \times G$ for closed $F \subseteq X$, $G \subseteq Y$, respectively). Note that $(X \setminus U) \times (Y \setminus V) = (X \times Y) \setminus \big((X \times V) \cup (U \times Y)\big)$.

Hence, the product topology on $k^2$ contains only the following non-trivial closed subsets: Finite unions of points $(x, y)$ or vertical or horizontal lines, i.e. $c \times k^1$ or $k^1 \times d$ for $c, d \in k$. In contrast, the Zariski topology of $k^2$ contains subsets like $V(y^2 - x^3)$ or the "diagonal" $V(y - x) = \{(c, c) \mid c \in k\}$.

(b) Let $f \in k[x_1, \ldots, x_n]$ be a polynomial lifting $f \in A(Z)$. Then it is enough to check that $f : k^n \to k$ is continous with respect to the Zariski topology on both sides. However, since (up to $k^1$ itself) it is only the finite subsets of $k^1$ being Zariski closed, we have just to consider sets like $f^{-1}(c)$ for $c \in k^1$. And since $f^{-1}(c) = V(f - c)$, we are done.

(c) This is true: As in (b), we just have to check the sets $\varphi^{-1}(c)$ – but these are single points.

**Problem 7.** A topological space $X$ is called irreducible if it cannot be written as $X = X_1 \cup X_2$ with some proper closed subsets $X_i \subsetneq X$ $(i = 1, 2)$. Show that this is equivalent to the fact that all non-empty open subsets $U \subseteq X$ are dense in $X$, i.e. fulfill $\overline{U} = X$.

*Solution:* $X = X_1 \cup X_2 \Rightarrow X \setminus X_1$ is open but not dense (since $\subseteq X_2$). For the reverse implication assume that $\emptyset \neq U \subset X$ is not dense. Then $X = \overline{U} \cup (X \setminus U)$ provides a decomposition.

**Problem 8.** Recall (or consult a textbook or wikipedia) the notion of covariant and contravariant functors between categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a (covariant) functor between two categories; let $A, A_i \in \mathrm{Ob}(\mathcal{A})$ $(i = 1, 2)$.

a) Show that if $f : A_1 \to A_2$ is an iomorphism, then $F(f) : F(A_1) \to F(A_2)$ is an iomorphism, too.

b) Show that $\mathrm{Aut}(A) \to \mathrm{Aut}(F(A))$ is a group homomorphism (where $\mathrm{Aut}(A) := \{\varphi \in \mathrm{Hom}_{\mathcal{A}}(A, A) \,|\, \varphi \text{ is an isomorphism}\}$).

c) Assume that $F$ is *fully faithful*, i.e. $\mathrm{Hom}(A_1, A_2) \to \mathrm{Hom}(FA_1, FA_2)$ is bijective for all $A_1, A_2 \in \mathrm{Ob}\mathcal{A}$. Show that then the reverse implication of (a) is true, too. That is, if $F(f)$ is an isomorphism, then so is $f$.

d) Provide an example showing that in (c) the injectivity of

$$\mathrm{Hom}(A_1, A_2) \to \mathrm{Hom}(FA_1, FA_2)$$

does not suffice.

e) ("Yoneda-Lemma") Let $\mathcal{C}$ be a category. Show that the functor

$$\Phi : \quad \mathcal{C} \longrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{opp}}, \mathcal{S}et)$$
$$Y \longmapsto \mathrm{Hom}_{\mathcal{C}}(\bullet, Y)$$

is fully faithful.

(The latter contains the covariant functors $\mathcal{C}^{\mathrm{opp}} \to \mathcal{S}et$, i.e. the contravariant functors $\mathcal{C} \to \mathcal{S}et$ as objects and the natural transformations between them as morphisms. The functors $F = \Phi(Y)$ are called "represented by $Y$". They come with a distinguished element $\xi \in F(Y)$.)

*Hint:* Show $\mathrm{Hom}_{\mathrm{Fun}}(\Phi Y, F) = F(Y)$ for any contravariant functor $F : \mathcal{C}^{\mathrm{opp}} \to \mathcal{S}et$.

*Solution:* (a) If $g = f^{-1}$, then $F(g) = F(f)^{-1}$.

(b) This follows from $F(f \circ g) = F(f) \circ F(g)$.

(c) Let $f : A_1 \to A_2$ be such that $F(f)$ is an isomorphism. Denote by $g \in \mathrm{Hom}(A_1, A_2)$ the (unique) pre-image of $F(f)^{-1}$. Then both $f \circ g$ and $g \circ f$ map to id under $F$, i.e. they are already equal to id.

(d) Let $\mathcal{A}$ the category of free abelian groups of finite rank, i.e. of those being isomorphic to some $\mathbb{Z}^n$, and $\mathcal{B} := \mathcal{V}ect_{\mathbb{Q}}$ = the category of $\mathbb{Q}$-vector spaces with $F$ being the functor $A \mapsto A \otimes_{\mathbb{Z}} \mathbb{Q}$ (i.e. $\mathbb{Z}^n \mapsto \mathbb{Q}^n$). It is injective on the Hom-groups, but, in general, not surjective. Choosing $f := (\cdot 2) : \mathbb{Z} \to \mathbb{Z}$ as the multiplication with 2, then this is not an isomorphism, but $F(f)$ is.

Another example is the embedding of metric or topological spaces (with continous maps) into the category of sets.

(e) We are supposed to show that $\mathrm{Hom}_{\mathbb{C}}(Y,Z) \stackrel{\sim}{\to} \mathrm{Hom}_{\mathrm{Fun}}(\Phi Y, \Phi Z)$ is bijective. If $F := \mathrm{Hom}_{\mathcal{C}}(\bullet, Z)$, then this is a special case of the bijectivity of

$$F(Y) \to \mathrm{Hom}_{\mathrm{Fun}}(\Phi Y, F), \quad \xi \mapsto \left[ \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(\bullet, Y) & \to & F(\bullet) \\ \varphi & \mapsto & F(\varphi)(\xi) \end{array} \right]$$

for any contravariant functor $F : \mathcal{C}^{\mathrm{opp}} \to \mathcal{S}et$. The latter, however, follows directly from presenting its inverse:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Fun}}(\Phi Y, F) & \longrightarrow & F(Y) \\ \Psi \longmapsto \left[\Psi(Y) : \quad \underset{\mathrm{Hom}_{\mathcal{C}}(Y,Y)}{\overset{\Phi Y(Y)}{\parallel}} \longrightarrow F(Y)\right] & \mapsto & \Psi(Y)(\mathrm{id}_Y). \end{array}$$

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## 3. Aufgabenblatt zum 7.11.2022

**Problem 9.** a) Construct two non-trivial, open subsets $D(f), D(g) \subseteq \mathbb{A}^2_{\mathbb{C}}$, such that $D(f) \cup D(g) = \mathbb{A}^2_{\mathbb{C}}$.
b) Construct an open covering of the $\mathbb{A}^2_{\mathbb{C}}$ by three subsets $D(f), D(g), D(h)$ such that any choice of only two of them does not cover the whole plane.

*Solution:* (a) The only condition to $f, g \in \mathbb{C}[x, y]$ is $(f, g) = (1)$. Thus, e.g., $f = x$ and $g = x - 1$ do the job.

(b) $f = x,\ g = y,\ h = x + y - 1$.

**Problem 10.** Let $f \in k[x_1, \ldots, x_n] =: k[\mathbf{x}]$. Then, we have obtained in Subsection (1.3) the bijective map $p : Z_f \to D(f)$. We are going to show that it is a homeomorphism, i.e. that both $p$ and $p^{-1}$ are continous (with respect to the Zariski topologies on both sides):

a) Denote by $\iota_Z$ and $\iota_D$ the embeddings $Z \hookrightarrow k^{m+1}$ and $D \hookrightarrow k^m$, respectively. Then $\iota_D \circ p = \mathrm{pr} \circ \iota_Z$ is a continous map $Z_f \to k^m$. Conclude that then $p$ has to be continous, too.
(*Reminder:* A map between topological spaces is continous if the preimages of closed subsets are closed.)

b) It remains to show that the map $\phi : D(f) \to k^{m+1}$, $\mathbf{x} \mapsto (\mathbf{x}, t := 1/f(\mathbf{x}))$ is continous, too. Let $J \subseteq k[\mathbf{x}, t]$ be an ideal. For each $g \in k[\mathbf{x}, t]$ we define $\widetilde{g} \in k[\mathbf{x}]$ to be

$$\widetilde{g}(\mathbf{x}) := f(\mathbf{x})^N \cdot g(\mathbf{x}, \tfrac{1}{f(\mathbf{x})})$$

where $N \gg 0$ is sufficiently large such that $\widetilde{g}$ becomes a polynomial. Note that $N$ depends on $g$ and that it is not uniquely determined at all – just choose and fix one for each $g$.

Finally, we define $\widetilde{J} := \{\widetilde{g} \mid g \in J\}$ – or likewise the ideal generated from this set. Then show that $\phi^{-1}(V(J)) = V(\widetilde{J}) \cap D(f)$.

*Solution:* (a) If $S \subseteq D(f)$ is closed, then it looks like $S = \overline{S} \cap D(f)$ for a closed subset $\overline{S} \subseteq k^m$. Hence, $p^{-1}(S) = (\iota_D \circ p)^{-1}(\overline{S})$ has to be closed, too.

(b) Let $p \in D(f) \subseteq k^m$. Then, $p \in \phi^{-1}(V(J))$ iff $\phi(p) \in V(J)$ iff

$$g(p, \frac{1}{f(p)}) = g(\phi(p)) = 0$$

for all $g \in J$. However, since $f(p) \neq 0$, this vanishing is equivalent to $\widetilde{g}(p) = 0$.

**Problem 11.** Let $k$ be an algebraically closed field, i.e. you may use the HNS saying that $I(V(J)) = \sqrt{J}$ for ideals $J \subseteq k[\mathbf{x}] := k[x_1, \ldots, x_n]$. Show that for Zariski closed subsets $Z_i \subseteq k^n$ one has then $I(\bigcap_i Z_i) = \sqrt{\sum_i I(Z_i)}$.

*Solution:* With $J_i := I(Z_i)$ we have $Z_i = V(J_i)$. Hence $\bigcap_i Z_i = \bigcap_i V(J_i) = V(\sum_i J_i)$. Denoting $J := \sum_i J_i$ this means that $\bigcap_i Z_i = V(J)$, i.e. $I(\bigcap_i Z_i) = I(V(J)) = \sqrt{J} = \sqrt{\sum_i J_i} = \sqrt{\sum_i I(Z_i)}$.

**Problem 12.** A $k$-algebra $k \to R$ is called finitely generated if there are finitely many elements $r_1, \ldots, r_n \in R$ such that there is no proper subalgebra $k \to S \subsetneq R$ containing $r_1, \ldots, r_n$, i.e. $r_1, \ldots, r_n \in S$.

a) Show that $k \to R$ is a f.g. $k$-algebra if and only if it is of the form, i.e. isomorphic to $k[x_1, \ldots, x_n]/J$ for some ideal $J \subseteq k[\mathbf{x}]$. In particular, there is then a surjection $k[\mathbf{x}] \twoheadrightarrow R$ of $k$-algebras.

b) Find such a representation for $R = k[t^2, t^3] = k \oplus t^2 \cdot k[t]$.

c) If $f : R \to S$ is a $k$-algebra-homomorphism between f.g. $k$-algebras, i.e. $f$ is compatible with the "structure homomorphisms $k \to R$ and $k \to S$, then we know from (a) that there are $k$-algebra surjections $k[\mathbf{x}] \twoheadrightarrow R$ and $k[\mathbf{y}] \twoheadrightarrow S$. Show that there is a $k$-algebra homomorphism $F : k[\mathbf{x}] \to k[\mathbf{y}]$ such that

$$\begin{array}{ccc} k[\mathbf{x}] & \xrightarrow{F} & k[\mathbf{y}] \\ \downarrow & & \downarrow \\ R & \xrightarrow{f} & S \end{array}$$

commutes. Is $F$ uniquely determined?

d) Do (c) explicitely for $R = k[t^2, t^3] \hookrightarrow k[t] = S$.

e) What is the geometric counterpart of (c) and (d)?

*Solution:* (a) $R = k[x_1, \ldots, x_n]/J$ is generated by the images of the $x_i$ in $R$, i.e. by their classes. On the other hand, if some $R$ is generated by $r_1, \ldots, r_n$, then we obtain a surjection $\varphi : k[x_1, \ldots, x_n] \twoheadrightarrow R$ via $x_i \mapsto r_i$. Then, take $J := \ker \varphi$.

(b) $k[x, y] \to k[t]$ with $x \mapsto t^2$ and $y \mapsto t^3$ has $R$ as its image, and the kernel is $(x^3 - y^2)$.

(c) Let $\varphi : k[\mathbf{x}] \twoheadrightarrow R$ and $\psi : k[\mathbf{y}] \twoheadrightarrow S$. We define $F : k[\mathbf{x}] \to k[\mathbf{y}]$ by mapping $x_i$ to *some* lift of $f(\varphi(x_i)) \in S$ into $k[\mathbf{y}]$ via $\psi$. That is, $\psi(F(x_i)) = f(\varphi(x_i))$. On the other hand, this means that $\psi \circ F = f \circ \varphi$ – corresponding to the commutativity of the diagram in question.
Since the lifts along $\psi$ are not unique, neither is $F$.

(d) $\varphi : k[x, y] \to k[t^2, t^3]$ with $\varphi(x) = t^2$ and $\varphi(y) = t^3$ has to be combined with $(\psi = \mathrm{id}) : k[t] \to k[t]$. Here, the lifts are even unique, and we are forced to define $F : k[x, y] \to k[t]$ via $F(x) := t^2$ and $F(y) := t^3$. That is, in this example, $F$ does, more or less, coincide with $\varphi$.

(e) The map $F : k[x_1, \ldots, x_n] \to k[y_1, \ldots, y_m]$ of (c) corresponds to an regular map $\Phi : k^m \to k^n$. If $I \subseteq k[x_1, \ldots, x_n]$ and $J \subseteq k[y_1, \ldots, y_m]$ are the kernels of the surjections, then $\Phi$ factors via $V(J) \to V(I)$. In the case of (d), $\Phi$ maps $k^1 \to k^2$ $(t \mapsto (t^2, t^3)$, but this is not surjective – the image is $V(x^3 - y^2)$.

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## 4. Aufgabenblatt zum 14.11.2022

**Problem 13.** Let $\varphi : A \to B$ be a ring homomorphism. Show that

a) the associated $(f = \operatorname{Spec}\varphi) : \operatorname{Spec} B \to \operatorname{Spec} A$ (defined via $f : Q \mapsto \varphi^{-1}Q$) is continous. That is, with respect to the Zariski topology on both sides, show that the pre-images of closed subsets are closed.

b) Give an alternative proof of (a) by showing that the pre-image of the so-called elementary open subsets $D(f) \subseteq \operatorname{Spec} A$ (for $f \in A$) are open in $\operatorname{Spec} B$. Why does it suffice to consider these special open subsets instead of all ones?

c) Recall that, for every $P \in \operatorname{Spec} A$, we denote by $K(P) := \operatorname{Quot}(A/P)$ the associated residue field of $P$. Show that $\varphi$ and $f$ from (a) provide a natural embedding $\overline{\varphi} : K\big(f(Q)\big) \hookrightarrow K(Q)$ for each $Q \in \operatorname{Spec} B$.

d) Recall that elements $a \in A$ can be understood as functions on $\operatorname{Spec} A$ via assigning each $P$ its residue class $\overline{a} \in K(P)$. Show that, in this context, the map $\varphi : A \to B$ can be understood as the pull back map (along $f$) for functions, i.e. that, under use of (c), $\varphi(a) \mathrel{\widehat{=}} a \circ f$.
(A maybe confusing remark: Making the last correspondence more explicit – but maybe less user friendly – one is tempted to write $\varphi(a) = \overline{\varphi} \circ a \circ f$. However, this is even less correct, since there is no "general map" $\overline{\varphi}$; even the domain and the target of $\overline{\varphi}$ depend on $Q$.)

*Solution:* (a) If $J \subseteq A$, then $Q \in f^{-1}(V(J)) \Leftrightarrow f(Q) \in V(J) \Leftrightarrow \varphi^{-1}(Q) \supseteq J \Leftrightarrow Q \supseteq \varphi(J) \Leftrightarrow Q \supseteq \varphi(J) \cdot B$. Thus, $f^{-1}(V(J)) = V(\varphi(J) \cdot B)$.

(b) If $a \in A$, then $Q \in f^{-1}(D(a)) \Leftrightarrow f(Q) \in D(a) \Leftrightarrow a \notin \varphi^{-1}(Q) \Leftrightarrow \varphi(a) \notin Q$. Thus, $f^{-1}(D(a)) = D(\varphi(a))$. Checking these special "elementary" open subsets suffices since every open subset is a union of those. Moreover, the operator "$\cup$" is compatible with $f^{-1}$.

(c) $K(Q) = \operatorname{Quot} B/Q$ and $K(f(Q)) = \operatorname{Quot} A/\varphi^{-1}(Q)$. Hence, the inclusion $\overline{\varphi} : A/\varphi^{-1}(Q) \hookrightarrow B/Q$ induces an inclusion $\overline{\varphi} : K(f(Q)) \hookrightarrow K(Q)$ among their respective quotient fields.

(d) We have to compare two functions on $\operatorname{Spec} B$. Accordingly, we take an element $Q \in \operatorname{Spec} B$, i.e. a prime ideal $Q \subseteq B$.
Now, $\big(\varphi(a)\big)(Q)$ was defined as the residue class $\overline{\varphi(a)}$ of $\varphi(a) \in B$ in $B/Q \subseteq \operatorname{Quot}(B/Q) = K(Q)$.
On the other hand, $(a \circ f)(Q) = a(f(Q)) = a(\varphi^{-1}(Q)$. And this equals the residue class $\overline{a}$ of $a \in A$ in $A/\varphi^{-1}(Q) = K(f(Q))$.

**Problem 14.** a) Let $A$ be a ring. Describe the set of elements $a \in A$ with $D(a) = \emptyset$.

b) Let $\varphi : A \to B$ be a surjective ring homomorphism. Show that $\operatorname{Spec}\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is injective.

c) Let $\varphi : A \to B$ be an injective ring homomorphism. Show that $\operatorname{Spec}\varphi : \operatorname{Spec} B \to \operatorname{Spec} A$ is dominant, i.e. that the image is dense.
(*Hint:* You might use that a subset $S \subseteq X$ of a topological space $X$ is *not* dense iff there exists a non-empty open $U \subseteq X$ being disjoint to $S$.)

d) Give an example for the situation of (c) where $\operatorname{Spec}\varphi$ is *not* surjective.

*Solution:* (a) $D(a) = \emptyset \Leftrightarrow a \in P$ for all prime ideals $P \subseteq A$, i.e. $a \in \bigcap_P P = \sqrt{0}$. That is, $D(a) = \emptyset \Leftrightarrow a$ is nilpotent.

(b) Surjective ring homomorphisms are always of the form $A \twoheadrightarrow A/J = B$. And we know that $\operatorname{Spec} A/J = V(J) \subseteq \operatorname{Spec} A$.

(c) If there were a nonempty $U \subseteq \operatorname{Spec} A$ being disjoint to $f(\operatorname{Spec} B)$ with $f = \operatorname{Spec}\varphi$, then we can assume that $U$ is of the form $U = D(a)$ (because those gadgets form a basis of the topology). Now, the non-emptyness means that $a \in A$ is not nilpotent – but this means that $\varphi(a) \in B$ is not nilpotent either. Thus, $\emptyset \neq D(\varphi(a)) = f^{-1}D(a)$, i.e. $D(a)$ couldn't be disjoint from the image.

(d) $\mathbb{Z} \hookrightarrow \mathbb{Q}$ gives the embedding $\operatorname{Spec}\mathbb{Q} \hookrightarrow \operatorname{Spec}\mathbb{Z}$ which is not surjective (the image consists of the single point $(0)$), but the image, i.e. this single point, is dense.

**Problem 15.** Show that $\operatorname{Spec} A$ is quasicompact, i.e. that every open covering admits a finite subcovering. (Note that we avoid the name "compact" for this property because $\operatorname{Spec} A$ is not HAUSDORFF.)
(*Hint:* Try to use the "elementary" open subsets $D(a)$ whenever you can.)

*Solution:* Given a covering $\{U_\nu\}_{\nu\in\Lambda}$, we may assume w.l.o.g. that $U_\nu$ is of the form $U_\lambda = D(f_\nu)$ for some elements $f_\nu \in A$. (This is possible because the open subsets $D(f)$ form a basis of the topology.) But this means that the ideal $(f_\nu \mid \nu \in \Lambda)$ equals $(1)$, i.e. that 1 is a $A$-linear combination of finitely many $f_\nu$ $(\nu \in \Lambda_0 \subseteq \Lambda)$. But then $(f_n u \mid \nu \in \Lambda_0) = (1)$ shows that the finite subfamily $\{U_\nu\}_{\nu\in\Lambda_0}$ is still a covering.

**Problem 16.** Let $R_1, \ldots, R_m$ be (commutative) rings (with 1) and denote by $R := \prod_i R_i$ their product.
a) Show that the units $1_i \in R_i$ induce so-called "orthogonal idempotents" $e_i \in R$, i.e. elements having the property $e_i e_j = \delta_{i,j} e_i$. Moreover, show that each choice of orthogonal idempotents $\{e_1, \ldots, e_m\}$ in a ring $R$ gives rise of a decomposition $R = \prod_i R_i$ of $R$ into a product of rings.
b) Do we have natural ring homomorphisms $\varphi_i : R_i \to R$ or $\psi_i : R \to R_i$? Show that the right choice induces a homeomorphism between the topological spaces $\coprod_i \operatorname{Spec} R_i$ and $\operatorname{Spec} R$. What is the geometric interpretation of $\varphi_i/\psi_i$ when $\operatorname{Spec} R$ is identified with $\coprod_i \operatorname{Spec} R_i$?

*Solution:* (a) $e_i := (0. \ldots, 1_i, \ldots, 0) \in R$ and, for the opposite direction, $R_i := e_i R$.
(b) There is no good embedding $\varphi_i : R_i \hookrightarrow R$, since the only natural choice would be $\varphi_i(1_i) = e_i$, but this violates $1_i \mapsto 1$ which should be always satisfied for rings with a unit. However, the projection $\psi_i : R \to R_i,\ r \mapsto re_i$ works well.
If $P \subseteq R$ is a prime ideal, then for $i \neq j$ we have $e_i e_j \in P$, thus $e_i \in P$ or $e_j \in P$. Hence, $P$ contains one of the ideals $Q_i := \prod_{j\neq i} R_j \subseteq R$. On the other hand, we

know that $R/Q_i = R_i$. This does also show that $\psi_i : R \twoheadrightarrow R_i$ induces the closed embedding $\operatorname{Spec} R_i \hookrightarrow \coprod_i \operatorname{Spec} R_i$.

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## 5. Aufgabenblatt zum 21.11.2022

**Problem 17.** a) Let $\varphi : A \to B$ be a ring homomorphism where $A$ and $B$ are even fields. Show that $\varphi$ is then automatically injective.

b) Give counter examples for the cases that either $A$ or $B$ is not a field.

*Solution:* (a) If $\varphi$ was not injective, then $\ker(\varphi) \neq (0)$. Hence, if $A$ is a field, this implies that $\ker(\varphi) \neq (1)$ – just because there is no other ideals at all. Thus, $\varphi = 0$, implying that $1_B = \varphi(1_A) = 0_B$ which is not allowed when $B$ is a field.

(b) Counter examples: $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \to 0$.

**Problem 18.** In Problem 14(c) it had to be exploited that injective ring homomorphisms $\varphi : A \to B$ send non-nilpotent elements to non-nilpotent elements. Do those $\varphi$ also send non-zero divisors to non-zero divisors? (Proof/counter example)

*Solution:* Consider $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]/(2x)$. Then $\mathbb{Z}$ is a domain, but 2 becomes a zero divisor in $\mathbb{Z}[x]/(2x)$.

**Problem 19.** a) Let $0 \to K \stackrel{f}{\to} L \stackrel{g}{\to} M \to 0$ be an exact sequence of $R$-modules. Show this sequence is a split exact sequence (i.e. it is isomorphic to the sequence $0 \to K \to K \oplus M \to M \to 0$) $\Leftrightarrow$ the map $g$ has a section, i.e. if there is an ($R$-linear) map $s : M \to L$ such that $gs = \mathrm{id}_M$.

b) In class we have shown that the short exact sequence $0 \to \mathbb{Z} \stackrel{\cdot 2}{\to} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ does not split. Give an alternative proof of this via using (a).

c) Show that short exact sequences of vector spaces (i.e. $R$ is a field) do always split.

*Solution:* (a) If the sequence splits, then $M \hookrightarrow K \oplus M \stackrel{\sim}{\to} L$ gives the section. On the other hand, if $s : M \to L$ exists, then $(f + s) : K \oplus M \to L$ establishes an isomorphism compatible with the embedding of $K$ and the projection onto $M$.

(b) Any candidate for a section of $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ has to be a $\mathbb{Z}$-linear map $s : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$. But this leads immediately to $s = 0$.

(c) Every surjection $\pi : V \twoheadrightarrow W$ of $k$-vector spaces admits a section – just choose a basis $B$ of $W$ and assign to every $w \in B$ an arbitrary element of $\pi^{-1}(w)$.

**Problem 20.** Give an example of an injection $M \hookrightarrow M'$ of abelian groups, i.e. $\mathbb{Z}$-modules, and an abelian group $N$ such that

$$M \otimes_{\mathbb{Z}} N \neq 0 \qquad \text{but} \qquad M' \otimes_{\mathbb{Z}} N = 0.$$

*Hint:* Do your search among the usual suspects $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/2\mathbb{Z}$...

*Solution:* $M = \mathbb{Z}$, $M' = \mathbb{Q}$, and $N = \mathbb{Z}/2\mathbb{Z}$.

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## 6. Aufgabenblatt zum 28.11.2022

**Problem 21.** Let $F : R\text{-mod} \to S\text{-mod}$ be a covariant, additive functor from the category of $R$-modules into the category of $S$-modules. (Additivity here just means that for $R$-modules $M, N$ the map $F : \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_S(FM, FN)$ is additive, i.e. $\mathbb{Z}$-linear.)

a) Check the additivity of the functors $F =$ tensor $\otimes_R N$, $\operatorname{Hom}(M, \bullet)$, and localization $M \mapsto S^{-1}M$.

b) Show that $F$ preserves the exactness of arbitrary sequences ("*exact functors*") $\Leftrightarrow$ of sequences of the form $M' \to M \to M''$ $\Leftrightarrow$ of "short" exact sequences $0 \to M' \to M \to M'' \to 0$.

(*Hint:* Decompose $M' \to M \to M''$ into two "short" exact sequences.)

c) $F$ preserves the exactness of sequences of the form $0 \to M' \to M \to M''$ ("*left exact functors*") $\Leftrightarrow$ short exact sequences $0 \to M' \to M \to M'' \to 0$ yield exact sequences $0 \to FM' \to FM \to FM''$.

*Solution:* (b) An exact sequence $M' \xrightarrow{f} M \xrightarrow{g} M''$ is decomposable into $0 \to \ker f \to M' \to N \to 0$ and $0 \to N \to M \to \operatorname{im} g \to 0$ with $\operatorname{im} f = N = \ker g$.

(c) For an exact $0 \to M' \to M \to M''$ we define $N := \operatorname{im}(M \to M'') \subseteq M''$. That is, we obtain two exact sequences $0 \to M' \to M \to N \to 0$ and $0 \to N \to M' \to C \to 0$ for some $R$-module $C$ (namely, $C := M''/N$). Thus, $0 \to FM' \to FM \to FN$ and $0 \to FN \to FM' \to FC$ are exact. The latter implies that $FN \to FM'$ is injective, hence we may replace $FN$ in the former sequence by $FM'$.

**Problem 22.** Calculate the Dehn invariant $D(S) = \sum_{e \in S_1} \ell(e) \otimes a(e) \in \mathbb{R} \otimes_{\mathbb{Q}} (\mathbb{R}/\pi\mathbb{Q})$ for the following three solids (all of volume 1):

(i) $S_1 =$ unit cube,

(ii) $S_2 =$ prism $[0, 1] \times A$ where $A$ is a triangle with angles $\alpha$, $\beta$, $\gamma$ and area 1, and

(iii) $S_3 =$ is a regular tetrahedron with edge length $s$ that $\operatorname{vol}(S_3) = 1$ (what is $s$?).

Finally, check which of the results in (i), (ii), (iii) are equal, and which differ from each other. (*Hint:* Use that for $k$-vector spaces $V, W$ with bases $B \subset V$ and $C \subset W$, the set $B \otimes C := \{b \otimes c \mid b \in B,\ c \in C\}$ forms a basis of $V \otimes_k W$.)

*Solution:* (i) All inner angles are $\pi/2$, hence $D(S_1) = 0$.

(ii) Forgetting the angles $\pi/2$, it remains

$$D(S_2) = 1 \otimes \alpha + 1 \otimes \beta + 1 \otimes \gamma = 1 \otimes (\alpha + \beta + \gamma) = 1 \otimes \pi/2 = 0.$$

(iii) Denote by $h$ the height of a triangle of the boundary of $S_3$ and by $H$ its total height. Then: $h^2 + (\frac{s}{2})^2 = s^2$ and $H^2 + (\frac{h}{3})^2 = h^2$ and $H^2 + (\frac{2h}{3})^2 = s^2$. Thus, $h = \frac{\sqrt{3}}{2}s$ and $H = \frac{\sqrt{8}}{3}h = \frac{\sqrt{2}}{\sqrt{3}}s$. In particular, $\operatorname{vol}(S_3) = \frac{1}{3} \cdot \frac{s}{2} \cdot h \cdot H = \frac{\sqrt{2}}{12}s^3$. Since this volume was supposed to be 1, we obtain $s = \sqrt[3]{2} \cdot \sqrt[2]{3}$.

For the Dehn invariant, we obtain $D(S_3) = 6s \otimes \theta$ with $\cos\theta = \frac{1}{3}$. This tensor vanishes if and only if $\theta \in \pi\mathbb{Q}$, which is not the case (but requires a proof which is, by the way, not so easy).

**Problem 23.** a) Recall that $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$. What about $\mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/8\mathbb{Z}$? Can you generalize this into a description of $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z}$?

b) Detemine a basis of the $\mathbb{Q}$-vector space $V = \mathbb{Q}^2 \otimes_{\mathbb{Q}} \mathbb{Q}^3$. What is the difference of this space to the abelian group $\mathbb{Q}^2 \otimes_{\mathbb{Z}} \mathbb{Q}^3$?

c) Detemine a basis of the $\mathbb{C}$-vector space $V = \mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^3$. What is it dimension as an $\mathbb{R}$-vector space? What is its difference to the $\mathbb{R}$-vector space $\mathbb{C}^2 \otimes_{\mathbb{R}} \mathbb{C}^3$?

d) What is $R/I \otimes_R R/J$?

e) Determine $\mathbb{R}[x,y] \otimes_{\mathbb{R}} \mathbb{C},\ \mathbb{R}[x,y]/(y^2 - x^3) \otimes_{\mathbb{R}} \mathbb{C},\ \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[y],\ \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[x]$.

*Solution:* (a) $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z} = \mathbb{Z}/(a,b) = \mathbb{Z}/\gcd(a,b)\mathbb{Z}$.

(b) $\mathbb{Q}^2 \otimes_{\mathbb{Q}} \mathbb{Q}^3 = \mathbb{Q}^2 \otimes_{\mathbb{Z}} \mathbb{Q}^3 = \mathbb{Q}^6$

(c) $\mathbb{C}^2 \otimes_{\mathbb{C}} \mathbb{C}^3 = \mathbb{C}^6 = \mathbb{R}^{12}$, but $\mathbb{C}^2 \otimes_{\mathbb{R}} \mathbb{C}^3 = \mathbb{R}^4 \otimes_{\mathbb{R}} \mathbb{R}^6 = \mathbb{R}^{24}$.

(d) $R/I \otimes_R R/J = R/(I+J)$.

(e) $\mathbb{C}[x,y],\ \mathbb{C}[x,y]/(y^2 - x^3),\ \mathbb{C}[x,y],\ \mathbb{C}[x,y]$.

**Problem 24.** a) Let $F : \mathcal{C} \to \mathcal{D}$ und $G : \mathcal{D} \to \mathcal{C}$ be functors. We call $F$ to be "*left adjoint*" to $G$ (or $G$ to be "*right adjoint*" to $F$; written as $F \dashv G$) if $\mathrm{Hom}_{\mathcal{D}}(FA, B) = \mathrm{Hom}_{\mathcal{C}}(A, GB)$ for $A, B$ being objects of $\mathcal{C}$ and $\mathcal{D}$, respectively. Here the equality sign means a bijection that is functorial in both arguments. Explain what is meant by the last sentence.

b) In the situation of (a) show that $F \dashv G$ is equivalent to the existence of the so-called adjunction maps, i.e. of natural transformations $FG \to \mathrm{id}_{\mathcal{D}}$ and $\mathrm{id}_{\mathcal{C}} \to GF$ with certain compatibility properties (describe them). How do these maps look like for all examples of mutually adjoint functors you have heard of in the past?

c) Let $\varphi : R \to T$ be a ring homomorphism, i.e. let $T$ be an $R$-algebra. Then there are the following functors between the module categories $F : \mathcal{M}od_R \to \mathcal{M}od_T$, $M \mapsto M \otimes_R T$ and $G : \mathcal{M}od_T \to \mathcal{M}od_R$, $N \mapsto N$, where $N$ becomes an $R$-module via $rn := \varphi(r)n$. Show that $F \dashv G$.

d) Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor between two categories of modules over some rings. Show that the existence of a right adjoint for $F$ implies right exactness of $F$. Does the existence of a left adjoint have a comparable impact?

*Solution:* (b) $F \to F(GF) = (FG)F \to F$ and $G \to (GF)G = G(FG) \to G$ have to be the identity maps.

Our only (or at least most prominent) example of adjoint functors was $(\otimes_R N) \dashv \mathrm{Hom}_R(N, \bullet)$. That is, we are looking for natural transformations $\mathrm{Hom}_R(N, \bullet) \otimes_R N \to \mathrm{id}_{\mathcal{M}od_R}$ and $\mathrm{id}_{\mathcal{M}od_R} \to \mathrm{Hom}_R(N, \bullet \otimes_R N)$. Indeed, if $M$ is an $R$-module, then we have natural $R$-linear maps $\mathrm{Hom}_R(N, M) \otimes_R N \to M$ and $M \to \mathrm{Hom}_R(N, M \otimes_R N)$.

(c) $\mathrm{Hom}_T(M \otimes_R T, N) = \mathrm{Hom}_R(M, N)$ for $R$-modules $M$ and $T$-modules $N$.

(d) is similar to (2.2). The existence of a left adjoint implies left exactness.

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# 7. Aufgabenblatt zum 5.12.2022

**Problem 25.** a) Determine the localizations $(\mathbb{Z}/6\mathbb{Z})_2$, $(\mathbb{Z}/6\mathbb{Z})_3$, $(\mathbb{Z}/6\mathbb{Z})_{(2)}$, $(\mathbb{Z}/6\mathbb{Z})_{(3)}$. Is there respective localization maps $\mathbb{Z}/6\mathbb{Z} \to \ldots$ injective or surjective?

b) Let $M$, $N$ be two $R$-modules. Show that $M \oplus N$ is flat over $R \Leftrightarrow$ both $M$ and $N$ are flat $R$-modules.

c) Give two different proofs for the flatness of $\mathbb{Z}/6\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$. (A ring homomorphism $f : R \to S$ is called flat if $S$ becomes, via $f$, a flat $R$-module.)

*Solution:* (a) $(\mathbb{Z}/6\mathbb{Z})_2 = (\mathbb{Z}/6\mathbb{Z})_{(3)} = \mathbb{Z}/3\mathbb{Z}$ and $(\mathbb{Z}/6\mathbb{Z})_3 = (\mathbb{Z}/6\mathbb{Z})_{(2)} = \mathbb{Z}/2\mathbb{Z}$. The localization maps coincide with the surjective quotient maps. They are defintely not injective.
(b) $P_\bullet \otimes (M \oplus N) = (P_\bullet \otimes M) \oplus (P_\bullet \otimes N)$ and $P_\bullet \oplus Q_\bullet$ is exact $\Leftrightarrow$ both $P_\bullet$ and $Q_\bullet$ are exact (ker and im commute with direct sums).
(c) $\mathbb{Z}/2\mathbb{Z}$ is both a localization and a direct summand of $\mathbb{Z}/6\mathbb{Z}$.

**Problem 26.** a) Let $R$ be a (commutative) ring and $f : R^m \to R^n$ an $R$-linear map given by a matrix $A$ with $R$-entries. If $\varphi : R \to S$ is a ring homomorphism, then $R$-modules $M$ turn into $S$-modules $M \otimes_R S$. Since $R^m \otimes_R S = S^m$, the map $f$ turns into $(f \otimes_R \mathrm{id}_S) : S^m \to S^n$. What is the associated matrix over $S$?

b) Let $R$ be a (commutative) ring and $f : R^m \twoheadrightarrow R^n$ a surjective, $R$-linear map. Show that $m \geq n$.

c) Let $g : R^m \hookrightarrow R^n$ be injective. Under the assumption that $R$ is an integral domain, show that $m \leq n$. Does this claim still hold true if $R$ has zero divisors?

*Solution:* (a) The new matrix is $\varphi(A)$. Moreover, tensor with $R/\mathfrak{m}$ in (b), and with $\operatorname{Quot} R$ in (c).
(c) [Nikola Sadovek] If $\alpha : R^m \hookrightarrow R^{m-1}$ was injective, then we can compose it with the standard embedding $\beta : R^{m-1} \hookrightarrow R^m$ to obtain an injective $\varphi := \beta \circ \alpha : R^m \to R^m$ yielding always zero at, e.g., the last coordinate.
Cayley-Hamilton yields a polynomial $f = \sum_{i=1}^n \lambda_i t^i \in R[t]$ killing $\varphi$ with $\lambda_n = 1$. Now, $f$ is representable as $f(t) = t^k \cdot g(t)$ with $k \geq 0$ and $g(0) \neq 0$. Hence, $\varphi^k \circ g(\varphi) = 0$ with $\varphi^k$ being injective. This implies $g(\varphi) = 0$ on $R^m$. Thus, for every $v \in R^m$, $g(0) \cdot v = \big[ -g(\varphi) + g(0) \big](v)$ has always 0 as its last entry – yielding a contradiction.

**Problem 27.** a) Let $R = (R, \mathfrak{m})$ be a local ring; let $f : M \to N$ be $R$-linear. Decide which of the possible four implications ($\Rightarrow$/$\Leftarrow$) holds true: $f : M \to N$ is injective/surjective $\Leftrightarrow$ $\overline{f} : M/\mathfrak{m}M \to N/\mathfrak{m}N$ is injective/surjective? Is it important whether $M$, $N$ are finitely generated?

b) If $I \subseteq J \subseteq R$ are ideals, then show that the ideal $(J/I)^2 \subseteq R/I$ equals $(J^2 + I)/I$.

c) Let $\mathfrak{m} := (x, y, z) \subseteq R$ with

$$R := \{f/g \mid f, g \in \mathbb{C}[x, y, z]/(xyz + x + y + z) \text{ mit } g(0, 0, 0) \neq 0\}.$$

Determine a basis of the $R/\mathfrak{m}$-vector space $\mathfrak{m}/\mathfrak{m}^2$ and a minimal generating system of the ideal $\mathfrak{m}$. Express $x, y, z \in \mathfrak{m}$ as $R$-linear combinations of this system.
(*Hint:* Use that the space $\mathfrak{m}/\mathfrak{m}^2$ equals $(x, y, z)/(x, y, z)^2$ where $(x, y, z)$ is understood as an ideal in the ring $\mathbb{C}[x, y, z]/(xyz + x + y + z)$, i.e. one can use (b) now.)

*Solution:* (a) Surjectivity "$\Rightarrow$", e.g. by right-exactness of tensor product; surjectivity "$\Leftarrow$" if $N$ is finitely generated (Nakayama). Injectivity neither: $f$ multiplication by non-zero-divisor $x \in \mathfrak{m}$; $f\colon R \to R/\mathfrak{m}$.
(c) $x, y, z$ generate $\mathfrak{m}$, hence $\mathfrak{m}/\mathfrak{m}^2$. However,

$$\mathfrak{m}/\mathfrak{m}^2 = (x, y, z)/\big((x, y, z)^2 + (xyz + x + y + z)\big) = (x, y, z)/\big((x, y, z)^2 + (x + y + z)\big)$$

by (b). That is, in $\mathfrak{m}/\mathfrak{m}^2$ we have the equation $x+y+z = 0$. Thus, $x, y$ are sufficient to generate $\mathfrak{m}/\mathfrak{m}^2$ (in fact they form a basis), hence $x, y$ generate the ideal $\mathfrak{m}$ within $R$ by the Lemma of Nakayama.
In $R$ we have $z(xy + 1) = -(x + y)$, hence $z = -\frac{x+y}{xy+1}$.

**Problem 28.** Let $(R, \mathfrak{m})$ be a local integral domain; denote by $k := R/\mathfrak{m}$ and $K := \operatorname{Quot} R$ its residue and quotient field, respectively. If $M$ is a finitely generated $R$-module, then show that $M$ is free $\Leftrightarrow \dim_k(M \otimes_R k) = \dim_K(M \otimes_R K)$. (*Hint:* Choose a surjection $R^n \twoheadrightarrow M$ with minimal $n$ and tensorize.)

*Solution:* $R^n \twoheadrightarrow M$ minimal $\Rightarrow \dim_K(M \otimes_R K) = \dim_k(M \otimes_R k) = n$ (Nakayama); $[0 \to N \to R^n \to M \to 0] \Rightarrow [0 \to N \otimes K \to K^n \to M \otimes K \to 0]$ is exact, hence $N \subseteq N \otimes K = 0$.

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## 8. Aufgabenblatt zum 12.12.2022

**Problem 29.** For $R := k \oplus x^2\, k[x] \subseteq k[x]$ and $S := k \oplus xy\, k[x,y] \subseteq k[x,y]$ check if they are finitely generated $k$-algebras, and check if they are noetherian.

*Solution:* $R = k[x^2, x^3] = k[y,z]/(y^3 - z^2)$ is a finitely generated $k$-algebra, and hence it is noetherian, too.
$S$ is not noetherian (thus not a finitely generated $k$-algebra): The ideal $S_+ := xy\, k[x,y]$ is not finitely generated: If it was so, then there would be finitely many monomial generators – but this does not work for combinatorial reasons.

**Problem 30.** Construct a filtration of $R := k[x,y]/(x^2y, x^3)$ where all factors are isomorphic to $R/P_i$ for some $P_i \in \operatorname{Spec} R$. In particular, identify the $P_i$ for all factors.

*Solution:* One possibility is to choose

$$0 \subset x^2 \cdot k[x,y]/(x^2y,x^3) \subset x \cdot k[x,y]/(x^2y,x^3) \subset k[x,y]/(x^2y,x^3)$$

with factors (i) $x^2 \cdot k[x,y]/(x^2y,x^3) \cong k[x,y]/(x,y)$ (via $x^2 \widehat{=} 1$),
(ii) $\big(x \cdot k[x,y]/(x^2y,x^3)\big)/\big(x^2 \cdot k[x,y]/(x^2y,x^3)\big) = x \cdot k[x,y]/x^2 \cdot k[x,y] \cong k[x,y]/(x)$,
and (iii) $\big(k[x,y]/(x^2y,x^3)\big)/\big(x \cdot k[x,y]/(x^2y,x^3)\big) = k[x,y]/(x)$.

**Problem 31.** a) Let $I := (I, \leq)$ be a poset. It turns into a category via objects $:= I$ and $\operatorname{Hom}_I(a,b) := \begin{cases} \{(a,b)\} & \text{if } a \leq b, \\ \emptyset & \text{otherwise}. \end{cases}$ A "*directed system* on $I$ with values in a category $\mathcal{C}$" is a (covariant) functor $I \to \mathcal{C}$; the "direct limit" $\varinjlim X_i$ of such a system $X = (X_i \mid i \in I)$ is defined via the following universal property: $\operatorname{Hom}_{\mathcal{C}}(\varinjlim X_i, Z) = \{\varphi \in \prod_i \operatorname{Hom}(X_i, Z) \mid i \leq j \Rightarrow \varphi_i = \varphi_j \circ [X_i \to X_j]\}$. In particular, there are canonical maps $X_j \to \varinjlim X_i$ (as the image of $\mathrm{id} \in \operatorname{Hom}_{\mathcal{C}}(\varinjlim X_i, \varinjlim X_i)$). Translate the notion of the direct limit into that of an initial object in some category.

b) What is $\varinjlim X_i$ if $I$ contains a maximum? What is $\varinjlim X_i$ if all elements of $I$ are mutually non-comparable, i.e. if $i \leq j \Leftrightarrow i = j$?

c) Let $\mathcal{C} = \mathcal{M}od_R$ be the category of modules over some ring $R$. For an element $m_j \in M_j$ we will use the same symbol $m_j$ for its canonical image in $M := \oplus_{i \in I} M_i$, too. Using this notation, show that $\varinjlim M_i = M/N$ where the submodule $N \subseteq M$ is generated by all differences $m_j - \varphi_{jk}(m_j)$ with $m_j \in M_j$, $j \leq k$, and $\varphi_{jk} : M_j \to M_k$ being the associated $R$-linear map.

d) Assume $(I, \leq)$ to be *filtered*, i.e. for $i, j \in I$ there is always a $k = k(i,j) \in I$ with $i, j \leq k$. If $\mathcal{C} = \mathcal{M}od_R$, then $\varinjlim M_i = \coprod_i M_i/\sim$, where $\coprod$ means the disjoint union (as sets) and "$\sim$" is the equivalence relation generated by $[\varphi_{ij}(m_i) \sim m_i$ for $i \leq j]$ (with $\varphi_{ij} : M_i \to M_j$). (*Hint:* First, define an $R$-module structure of the right hand side. Then check that an element $x \in M_i$ turns into $0 \in \varinjlim M_i$ if and only if there is a $j \geq i$ with $\varphi_{ij}(x) = 0 \in M_j$.)

*Solution:* (a) For a fixed directed system $X = (X_i \mid i \in I)$ which includes compatible maps $\psi_{ij} : X_i \to X_j$ for $i \leq j$, we define the category

$$\mathcal{C}^X := \big\{ (Z, \varphi_i \mid i \in I) \,\big|\, Z \in \mathcal{C}, \varphi_i = \varphi_j \circ \psi_{ij} \big\}$$

with the obvious morphisms. Then, $\widetilde{Z} := \varinjlim X_i$ together with the maps $\widetilde{\varphi}_j : X_j \to \varinjlim X_i$ is the initial object of the category $\mathcal{C}^X$.

(b) $\varinjlim X_i = X_{\max I}$ and $\varinjlim X_i =$ coproduct (being the direct sum in $\mathcal{M}od_R$ and the disjoint union in $\mathcal{S}et$).

(c) For a directed system $(M_i \mid i \in I)$ (including compatible maps $\phi_{ij} : X_i \to X_j$ for $i \leq j$), we define $M := \oplus_{i \in I} M_i$ and $N$ as in the problem. Then, we have natural maps $\iota_j : M_j \hookrightarrow M \twoheadrightarrow M/N$. The quotient construction with $N$ ensures $\iota_k = \iota_j \circ \varphi_{jk}$ for $j \leq k$. Finally, the universal property follows directly from this construction: If we have compatible $R$-linear maps $f_j : M_j \to L$, then we obtain, e.g. by the universal property of the direct sum, a map $M \to L$. And the compatibilities among the maps $f_i$ ensures that $N$ is sent to 0 via this map.

(d) Denote $C := \coprod_{i \in I} M_i$. Then, if $m_i \in M_i \subseteq C$ is a representative of $\overline{m_i} \in C/\sim$ and $r \in R$, then it is clear how to obtain $r \cdot \overline{m_i} := \overline{rm_i}$. Moreover, this construction is compatible with $\varphi_{ij}(m_i) \sim m_i$ – just because $varphi_{ij}$ is linear.
More intersting is the addition – this is exactly the part where the filtering becomes essential: If we were supposed to add $\overline{m_i}$ and $\overline{m_j}$, then we may choose a $k = k(i,j)$ with $i, j \leq k$. But then, by the definition of $\sim$, we obtain $\overline{m_i} = \overline{\varphi_{ik}(m_i)}$ and $\overline{m_j} = \overline{\varphi_{jk}(m_j)}$, i.e. both summands are represented by elements in $M_k$. There, we can add them, and this solves the problem.

**Problem 32.** a) Let $P \in \operatorname{Spec} R$ be a prime ideal and $M$ an $R$-module. Show that the localisation $M_P$ is the direct limes of modules $M_f$ with distinguished elements $f \in R$. What is the associated poset $(I, \leq)$? Is it filtered?

b) Show that infinite direct sums are filtered direct limits of finite direct sums. What is the underlying poset?

*Solution:* (a) The underlying poset is $I := R \setminus P$ with $f \leq g$ being defined via the relation $f|g$. If this is the case, we have natural maps $R_f \to R_g$. This poset is filtered because of $k(f,g) := fg$. Now, we can check the universal property for the compatible system of maps $\{M_f \to M_P \mid f \in I\}$.

(b) Let $\Lambda$ be a set, and we consider $R$-modules $M_\lambda$ for $\lambda \in \Lambda$. The basic poset $I$ is defined as

$$I := \{S \subseteq \Lambda \mid \#S < \infty\} \subseteq 2^\Lambda$$

with the inclusion relation. For each $S \in I$ we define $M_S := \oplus_{\lambda \in S} M_\lambda$ which induces natural embeddings $M_S \hookrightarrow M_{S'}$ whenever $S \subseteq S'$. They are compatible with the overall embedding $M_S \hookrightarrow \oplus_{\lambda \in \Lambda} M_\lambda$.

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## 9. Aufgabenblatt zum 2.1.2023

**Problem 33.** Let $M_1, M_2 \subseteq M$ be submodules of a finitely generated module over a noetherian ring $R$. Show that $\mathrm{Ass}\big(M/(M_1 \cap M_2)\big) \subseteq \mathrm{Ass}(M/M_1) \cup \mathrm{Ass}(M/M_2)$.
*Hint:* Try to exploit Propsition 13, i.e. to look for exact sequences relating, e.g., $M/(M_1 \cap M_2)$ and $M/M_1$.

*Solution:* $0 \to M_1/(M_1 \cap M_2) \to M/(M_1 \cap M_2) \to M/M_1 \to 0$ and $M_1/(M_1 \cap M_2) \subseteq M/M_2$.

**Problem 34.** Show that $I := (x, y) \subseteq k[x, y] =: R$ is not "clean", i.e. there is no "nice filtration" (i.e. with factors $\cong R/P_i$) of $I$ (not of $R/I$) with an exclusive use of primes associated to $I = (x, y)$ (really to $I$, not to $R/I$).

*Solution:* (0) is the only associated prime of $(x, y)$. On the other hand, $k[x, y] \hookrightarrow (x, y)$ can never become an isomorphism ($(x, y)$ is not principal) and, moreover, all possible cokernels must be torsion.

**Problem 35.** In the category of directed systems of $R$-modules on a poset $I := (I, \leq)$ (cf. Problem 31) define kernels, images, and cokernels in a natural way, e.g.

$$\ker\big(\varphi : (M_i \mid i \in I) \to (N_i \mid i \in I)\big) := \big(\ker[\varphi_i : M_i \to N_i] \mid i \in I\big).$$

This leads to the notion of exact sequences of directed systems.

a) Show that $\varinjlim$ is right exact (by constructing a right adjoint functor).

b) Show that *filtered* direct limits with values in $\mathcal{M}od_R$ are even exact.

c) Consider the set $I := \{m, a, b\}$ with $m < a$ and $m < b$. Show that the direct limit over this $I$ (even with values in $\mathcal{M}od_R$) is not left exact.

*Solution:* (a) The right adjoint functor is $Z \mapsto$ [constant system $Z$]. Indeed, the universal property of the direct limit says $\mathrm{Hom}(\varinjlim_i M_i, N) = \mathrm{Hom}(\{M_i\}, N) = \mathrm{Hom}(\{M_i\}, \{N_i := N\})$.
(c) Consider $(0, M, M) \hookrightarrow (M, M, M)$. The direct limits of both systems are $M \oplus M$ and $M^3/\sim$ with $(m, 0, 0) \sim (0, m, 0)$ and $(m, 0, 0) \sim (0, 0, m)$, i.e. the latter becomes isomorphic to $M$. The map $\varinjlim(0, M, M) \to \varinjlim(M, M, M)$ becomes the addition $M \oplus M \to M$. It is not injective at all.

**Problem 36.** Analogous to Problem 31, we define the "*inverse*" limit $\varprojlim M_i$ of a directed system of $R$-modules as the *terminal* object of a certain category, namely via $\mathrm{Hom}_R(P, \varprojlim M_i) = \{\varphi \in \prod_i \mathrm{Hom}(P, M_i) \mid i \leq j \Rightarrow \varphi_j = [M_j \leftarrow M_i] \circ \varphi_i\}$. In particular, there are canonical maps $\varprojlim M_i \to M_i$.

a) Realize $\varprojlim_i M_i$ as a submodule of $\prod_i M_i$ and derive from this that the projective limit is left exact.

b) Let $p \in \mathbb{Z}$ be a prime and $I := \mathbb{N}$. Show that $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^i\mathbb{Z}$ ("$p$-adic numbers" – not to be confused with the localization $\mathbb{Z}_p$) is a local ring without zero divisors. Show further that it contains $\mathbb{Z}$ and the localization $\mathbb{Z}_{(p)}$.

*Solution:* (a) $\varprojlim_i M_i = \{\underline{m} \in \prod_i M_i \mid m_i \mapsto m_j \text{ via } M_i \to M_j\}$

(b) For each $a \in \mathbb{Z}/p^i\mathbb{Z}$ not divisible by $p$ there is a *unique* $b \in \mathbb{Z}/p^i\mathbb{Z}$ with $ab = 1$. This shows that $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus (p)$, i.e. $(p)$ is the only maximal ideal.

Merry Christmas and a Happy New Year 2023!

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## 10. Aufgabenblatt zum 9.1.2023

**Problem 37.** Let $R$ be noetherian and $P \subseteq R$ a prime ideal. Let $M$ be a finitely generated $R$-module. Show that that $M_P$ is a free $R_P$-module if and only if there is an element $f \in R \setminus P$ such that $M_f$ is a free $R_f$-module.

*Solution:* ($\Leftarrow$) Just tensor any isomorphism $R_f^n \xrightarrow{\sim} M_f$ with $\otimes_{R_f} R_P$.

($\Rightarrow$) Let $0 \to C \to F \to M \to C' \to 0$ with a free $F = R^n$ such that $F_P \xrightarrow{\sim} M_P$ is an isomorphism. Then, $C$ is finitely generated, $C_P = 0$, hence there is an $f \in R \setminus P$ with $fC = 0$, i.e., $C_f = 0$. Do the same with $C'$.

**Problem 38.** Recall Problem 26(c): For $R$ being a noetherian ring we were given $m, n \in \mathbb{N}$. Then, the existence of an injective $R$-linear map $g : R^m \hookrightarrow R^n$ had implied that $m \leq n$.

a) Give an alternative proof of this fact in the case of $R$ being an Artinian ring, i.e., if $\ell_R(R) < \infty$.

b) Provide a proof of the general claim (without assuming that $R$ is Artinian) under use of Part (a). (*Hint*: Show and use that, for a minimal prime $P$, the localization $R_P$ is artinian.)

*Solution:* (a) Denote by $\ell := \ell_R(R) \geq 1$ the length of the Artinian ring $R$. Then $g$ provides the inequality $m\ell = \ell(R^m) \leq \ell(R^n) = n\ell$.

(b) Choose a minimal prime $P \subseteq R$ and localize, i.e. apply $(\otimes_R R_P)$. Since this functor is exact (localization is flat), the resulting $g \otimes \mathrm{id}_{R_P}$ stays injective, and one can apply (a).

**Problem 39.** a) What is the length of the ring $\mathbb{Z}/30\mathbb{Z}$? Provide a composition series and describe the factors.

b) Write down a composition series of the ring $k[t]/t^3$ and identify its factors as fields.

*Solution:* (a) $0 \subseteq 15\mathbb{Z}/30\mathbb{Z} \subseteq 5\mathbb{Z}/30\mathbb{Z} \subseteq \mathbb{Z}/30\mathbb{Z}$ has the factors $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, and $\mathbb{Z}/5\mathbb{Z}$. Hence, the length is 3.

(b) $0 = t^3k[t]/t^3 \subset t^2k[t]/t^3 \subset tk[t]/t^3 \subset k[t]/t^3$. The $k$-th factor is isomorphic to $k$ via

$$\big(t^k k[t]/t^3\big)/\big(t^{k+1}k[t]/t^3\big) \xleftarrow{\cdot t^k} k.$$

**Problem 40.** What are the minimal and what are the associated primes $P$ of $R = \mathbb{C}[x,y]/(x^2, xy^2)$? For the latter provide always an embedding $R/P \hookrightarrow R$. Which of the localizations $R_P$ have finite length – and what is this length then? Visualize a monomial base of $R$ and all $R/P$ – how does this reflect the previous information about the lenghths?

*Solution:* $P_1 := (x) = \sqrt{(x^2, xy^2)}$ is the only minimal prime. Moreover, $P_2 := (x, y)$ is associated, but not minimal. There is not more associated primes – this can be obtained either by looking for "nice" filtrations, or by the following argument: Localizing in points $(0, c)$, i.e. in maximal ideals $(x, y - c)$ for $c \neq 0$, turns $y^2$ into a unit. Hence, $(x^2, xy^2)$ becomes equal to $(x)$ then.

Examples for embeddings $R/P \hookrightarrow R$ are $1 \mapsto y^2$ for $P_1 = (x)$ and $1 \mapsto xy$ for $P_2 = (x, y)$.

The non-minimal prime $P_2$ leads to an infinite length. This follows because the ring $R_{P_2}$ has infinite $\mathbb{C}$-dimension compared to $\dim_{\mathbb{C}} R/(x, y) = 1$. For instance, the monomials $\{y^k \mid k \in \mathbb{N}\}$ are linearily independent. An alternative argument is that $R_{P_2}$ has $P_1$ (meaning $(P_1)_{P_2} = (R \setminus P_2)^{-1} P_1$) as a non-maximal prime ideal, hence is not artinian.

The length of $R_{P_1}$ is one – this follows from

$$\Big(\mathbb{C}[x, y]/(x^2, xy^2)\Big)_{(x)} = \Big(\mathbb{C}[x, y]/(x)\Big)_{(x)} = \mathbb{C}[x]_{(0)} = \operatorname{Quot} \mathbb{C}[x] = \mathbb{C}(x)$$

using that $y^2$ becomes a unit after localization. Hence, the result is a field.

Alternatively, this length equals the number of occurences of $R/P_1$ in "nice" filtrations: Using the embedding from above, we obtain

$$\operatorname{coker}(R/P_1 \hookrightarrow R) = \mathbb{C}[x, y]\big/(x^2, xy^2, y^2) = \mathbb{C}[x, y]\big/(x^2, y^2).$$

Since this is a finite-dimensional $\mathbb{C}$-vector space, only $R/P_2$ will occur as further factors. Thus, the length equals 1.

$R/(x)$ is a vertical line of lattice points, and $R/(x, y)$ is just the origin. The length $\ell(R_{(x)}) = 1$ corresponds to the *single* unbounded vertical line of lattice points visualizing the basic monomials of $R$; the infinite length of $R_{(x,y)}$ corresponds to the infinite number of lattice points in this picture.

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VL "(Comm) Algebra I"
FU Berlin, Winter 2022/23

## 11. Aufgabenblatt zum 16.1.2023

**Problem 41.** Find reduced, i.e. non-redundant primary decompositions of the ideals

$$I = (xy^5, x^3y^4, x^6y^2) \subset k[x,y] \quad \text{and} \quad J = (x^5, x^3yz, x^4z) \subset k[x,y,z].$$

Download the software Singular or Macaulay2 and check the result by one of these computer algebra systems.

*Solution:* $I = (x) \cap (y^2) \cap (x^6, x^3y^4, y^5)$ and $J = (x^3) \cap (x^4, y) \cap (x^5, z)$.

Singular (for the ideal $J$):

```
ring r=0,(x,y,z),(dp(3));
ideal i = x^3; ideal j = x^5, z; ideal k = x^4, y;
ideal m = intersect(i,j,k);
LIB "primdec.lib";
primdecGTZ(m);
```

> You can learn about the usage of the computeralgebra system Singular by attending the two weeks compact course "Computeralgebra" in early March. It is a BA-course within the so-called ABV part. That is, as master students, you cannot earn any formal credit for this – but, nevertheless, it might be useful. And it is fun, anyway.

**Problem 42.** Show directly that $\alpha := t^2 + 1 \in \mathbb{C}[t]$ is integral over the subring $\mathbb{C}[t^3]$.

*Solution:* $(\alpha - 1) = t^2$, hence $(\alpha - 1)^3 = t^6 = (t^3)^2$. This leads to the polynomial $f(x) := (x-1)^3 - (t^3)^2 \in \mathbb{C}[t^3][x]$. It has 1 as leading coefficient and it vanishes for $x = \alpha$.

**Problem 43.** Let $A \subseteq B$ be two rings and assume that all elements of $B$ are integral over $A$. Show that this implies $B^* \cap A = A^*$. Is the reverse implication true as well?

*Solution:* $a \in B^* \cap A \Rightarrow 1/a \in B$, hence $1/a$ is integral over $A$. Multiplying the polynomial $f(x) = x^n + \sum_{i=0}^{n-1} a_i\, x^i \in A[x]$ indicating the integrality of $1/a$ by $a^n$ one obtains

$$0 = a^n\,(1/a)^n + a^n \sum_{i=0}^{n-1} a_i\,(1/a)^i = 1 + a \cdot (\sum_{i=0}^{n-1} a_i\, a^{n-1-i})$$

showing that $a$ is a unit.

For domains $A$ iwe have that $A[x]^* = A^*$, hence $A[x]^* \cap A = A^*$. But $A[x]$ is not integral over $A$.

**Problem 44.** A ring homomorphism $f : A \to B$ is called *integral* if all elements $b \in B$ are integral over $f(A)$.

a) Show that the integrality of $f$ implies the integrality of $f \otimes \mathrm{id}_C : C \to B \otimes_A C$ for every $A$-algebra $C$. In particular, localizing $f$ via multiplicative subsets $S \subseteq A$

is compatible with integrality.

b) Let $(f_1, \ldots, f_k) = (1)$ in $A$. Thus, the open subsets $D(f_i) = \operatorname{Spec} A_{f_i}$ cover $\operatorname{Spec} A$. Show that an $A$-module $M$ is finitely generated if and only if all $M_{f_i}$ are finitely generated $A_{f_i}$-modules.

c) Assume again that $(f_1, \ldots, f_k) = (1)$ in $A$. Show that an element $b \in B$ is integral over $A \Leftrightarrow b/1 \in B_{f_i}$ is integral over $A_{f_i}$ for all $i$.

d) Let $M$ be an $A$-module such that the localizations $M_P$ are finitely generated over $A_P$ for all $P \in \operatorname{Spec} A$. Show that $M := \oplus_{P \in \operatorname{MaxSpec} A} A_P/PA_P$ is an example demonstrating that the original $M$ does not need to be finitely generated though.

e) Show that an element $b \in B$ is integral over $A \Leftrightarrow b/1 \in B \otimes_A A_P$ is integral over $A_P$ for all $P \in \operatorname{Spec} A$. (*Hint:* For each $P$ construct an element $f \notin P$ such that $b/1 \in B_f$ is integral over $A_f$.)

*Solution:* (a) It is sufficient to check integrality for elements of the form $b \otimes c$.
(b) Let $N$ be generated by all numerators $m_{ij}$ of the generators $m_{ij}/f_i^{e_{ij}}$ of $M_{f_i}$. Then $N \subseteq M$ becomes an equality after localising by $f_i$.
(c) Consider $A[b] \subseteq B$ and its localizations via $f_i$.
(d) $M := \oplus_{P \in \operatorname{Spec} A} A_P/PA_P$. The localizations $M_P$ equal $A_P/PA_P$.
(e) Take $f$ as the lowest common denominator of the $A_P$-coefficients of an integrality relation for $b$.

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## 12. Aufgabenblatt zum 23.1.2023

**Problem 45.** Let $R$ be a domain such that for every $q \in \operatorname{Quot} R$ one has $q \in R$ or $1/q \in R$ ($R$ is called a "valuation ring"). Show that this property implies that $R$ is local and normal, i.e. integrally closed in its quotient field. (*Hint:* Show that $R \setminus R^*$ is an ideal; for the additivity consider $x/y$ for given $x, y \in R \setminus R^*$.)

*Solution:* If $x, y \in R \setminus R^*$, then either $x/y$ or $y/x$ belong to $R$; assume that $x/y = z \in R$. Then $x + y = yz + y = y(1 + z)$. In particular, if $x + y$ was a unit, then $y$ has to be a unit, too.
To show normality, assume that $x \in \operatorname{Quot} R$ is integral over $R$. If $x^n + a_1 x^{n-1} + \ldots + a_n = 0$ is an integrality relation of minimal degree, the $x \notin R$ implies $1/x \in R$, hence $x^{n-1} + a_1 x^{n-2} + \ldots + a_{n-1} + a_n/x = 0$ is an integrality relation of one degree less.

**Problem 46.** For a semigroup $H$ with neutral element $0 \in H$ we define the associated "semigroup algebra" $\mathbb{C}[H] := \oplus_{h \in H} \mathbb{C} \cdot \chi^h$ with multiplication $\chi^h \cdot \chi^{h'} := \chi^{h+h'}$ among the basis vectors.

a) Describe $\mathbb{C}[H]$ explicitely for the examples $H = \mathbb{N}$, $H = \mathbb{Z}$, $H = \mathbb{N}^2$, and $H = \mathbb{N} \times \mathbb{Z}$.

b) Assume that $H \subseteq \mathbb{Z}^n$ is finitely generated with $\mathbb{Z}^n = H - H := \{h - h' \mid h, h' \in H\}$. Show that $\mathbb{C}[H]$ is a normal ring if and only if $H = \mathbb{Z}^n \cap (\mathbb{Q}_{\geq 0} \cdot H)$ inside $\mathbb{Q}^n$ ("$H$ is saturated"). Give an example where this condition does not hold true.

(*Hint:* For the part ($\Leftarrow$) write $H$ as an intersection of half spaces. Hence, the claim can be reduced to the special case of $H = \mathbb{N} \times \mathbb{Z}^{n-1}$.)

*Solution:* (a) One obtains $\mathbb{C}[x]$, $\mathbb{C}[x, x^{-1}] = \mathbb{C}[x]_x$, $\mathbb{C}[x, y]$, and $\mathbb{C}[x, y, y^{-1}] = \mathbb{C}[x, y]_y$, respectively.

(b) ($\Rightarrow$) Let $s \in \mathbb{Z}^n$ with $N \cdot s \in H$ and $s = h - h'$ ($h, h' \in H$). Then, $\chi^s := \chi^h / \chi^{h'} \in \operatorname{Quot} \mathbb{C}[H] \setminus \mathbb{C}[H]$, but $(\chi^s)^N = \chi^{Ns} \in \mathbb{C}[H]$.
($\Leftarrow$) $H$ is a finite intersection of semigroups like $H_a := \{r \in \mathbb{Z}^n \mid \langle a, r \rangle \geq 0\}$ with $a \in \mathbb{Z}^n$. Now, on the one hand, $\mathbb{C}[H] = \mathbb{C}[H_{a_1}] \cap \ldots \cap \mathbb{C}[H_{a_k}]$, and, on the other, $H_a \cong \mathbb{N} \times \mathbb{Z}^{n-1}$, hence $\mathbb{C}[H_a] \cong \mathbb{C}[x_1^{\pm 1}, x_2, \ldots, x_n]$. This ring is normal.

The example for a non-saturated semigroup: $\{0\} \cup \mathbb{Z}_{\geq 2}$ yielding the semigroup algebra $k[t^2, t^3]$.

**Problem 47.** a) Let $A \subseteq B$ be a finite extension of domains, i.e. the $A$-algebra $B$ becomes a finitely generated $A$-*module*. Further denote by $F : \operatorname{Spec} B \to \operatorname{Spec} A$ the associated map on the geometric side. Show that $F$ is quasi-finite, i.e. that $F$ has finite fibers, i.e. that for each prime ideal $P \subset A$ the set $F^{-1}(P)$ is finite.
(*Hint:* Exploit the usual localization/quotient constructions on the $A$-side to improve the situation.)

b) Determine the fibers of $P = (x, y)$ and of $P' = (x-1, y-1)$ with respect to the situation $A = \mathbb{C}[x, y]$ and $B = \mathbb{C}[x, y, z]/(xy - z^2)$.

c) What is the description of $F$ and $P, P', Q_i$ from (b) within the classical geometric language, i.e. understanding $\operatorname{Spec} \mathbb{C}[x, y] = \mathbb{A}^2_{\mathbb{C}}$ as $\mathbb{C}^2$?

*Solution:* (a) Down to earth, we have to show that for each prime ideal $P \subseteq A$ there is only finitely many prime ideals $Q_i \subset B$ such that $P = Q_i \cap A$.

We may, w.o.l.g., assume that $(A, P)$ is a local ring, i.e. that $P$ is a maximal ideal and that it is the only one. Indeed, localizing by $S := (A \setminus P)$, we obtain the still finite, injective map $A_P = S^{-1}A \hookrightarrow S^{-1}B$. Moreover, each $Q_i \subset B$ with $P = Q_i \cap A$ is disjoint to $S$, i.e., corresponds to the prime ideal $S^{-1}Q_i \subset S^{-1}B$ still satisfying $S^{-1}Q_i \cap A_P = P_P = S^{-1}P$.

Now, we tensor with $A/P$. This gives a still finite map $k := A/P \to B \otimes_A A/P = \overline{B} := B/(P \cdot B)$. That is, $\overline{B}$ is a finite $k$-algebra. While we cannot apply Proposition 24(1) because $\overline{B}$ fails to be a domain, we, nevertheless, recognize that $\overline{B}$ is a finite-dimensional $k$-vector space. Thus, $\overline{B}$ is an Artinian ring, having only finitely many prime ideals (being all maximal).

(b) $\mathbb{C}[x, y] \hookrightarrow \mathbb{C}[x, y, z]/(xy - z^2)$ becomes $\mathbb{C} \hookrightarrow \mathbb{C}[z]/(z^2)$ or $\mathbb{C} \hookrightarrow \mathbb{C}[z]/(1 - z^2)$ after tensorizing with $\mathbb{C}[x, y]/P$ or $\mathbb{C}[x, y]/P'$, respectively. Thus, the preimages are
Case $P = (x, y)$: $(z) \subset \mathbb{C}[z]/(z^2)$ leading to $(x, y, z) \subset \mathbb{C}[z]/(xy - z^2)$ and
Case $P' = (x-1, y-1)$: $(1 - z)$ and $(1 + z)$ inside $\mathbb{C}[z]/(1 - z^2)$ leading to $(x - 1, y - 1, z \pm 1) \subset \mathbb{C}[z]/(xy - z^2)$.

(c) $F$ is the map $V(xy - z^2) \hookrightarrow \mathbb{C}^3 \xrightarrow{\text{pr}_3} \mathbb{C}^2$ where the latter sends $(x, y, z) \mapsto (x, y)$. The prime ideals $P$ and $P'$ correspond to the points $(0, 0)$ and $(1, 1) \in \mathbb{C}^2$, respectively. Their pre-images are $(0, 0, 0)$ and $(1, 1, \pm 1) \in V(xy - z^2)$, respectively.

**Problem 48.** a) Let $f : A \to B$ be an integral ring homomorphism, i.e. $B$ is integral over the subring $f(A)$. Show that $\operatorname{Spec}(f) : \operatorname{Spec} B \to \operatorname{Spec} A$ is then a closed map, i.e. the images of closed subsets are always closed.
(*Hint:* Identify first the natural candidate for the closed subset of $\operatorname{Spec} A$ forming the image of some $\operatorname{Spec} B/J = V(J) \subseteq \operatorname{Spec} B$ under $F = \operatorname{Spec}(f)$. Then show that $F$ does indeed map $V(J)$ surjectively onto this candidate.)

b) Show, in the situation of (a), that for $A$-algebras $C$, i.e. for ring homomorphisms $A \to C$, the map $\operatorname{Spec}(f \otimes \mathrm{id}) : \operatorname{Spec}(B \otimes_A C) \to \operatorname{Spec} C$ is closed, too.

c) Give an example for a (non-integral) $f : A \hookrightarrow B$ and some $A$-algebra $C$, such that $\operatorname{Spec}(f) : \operatorname{Spec} B \to \operatorname{Spec} A$ is a closed map, but $\operatorname{Spec}(f \otimes \mathrm{id}) : \operatorname{Spec}(B \otimes_A C) \to \operatorname{Spec} C$ is not.

*Solution:* (a) Let $V(J) \subseteq \operatorname{Spec} B$ be closed. Then, $A/f^{-1}(J) \hookrightarrow B/J$ is both injective and integral. Actually, the injectivity of $A/f^{-1}(J) \hookrightarrow B/J$ was the reason for choosing $f^{-1}(J) \subseteq A$ – just because of Problem 14(c). Now, we have, additionally, the integrality of this ring map. Hence, by the going up theorem, the restriction of $F = \operatorname{Spec}(f)$ to $V(J) = \operatorname{Spec} B/J \to \operatorname{Spec} A/f^{-1}(J) = V(f^{-1}(J))$ is surjective.

(b) If $f : A \hookrightarrow B$ is integral, then $(f \otimes \mathrm{id}) : C = A \otimes_A C \to B \otimes_A C$ is integral, too.

(c) $\iota : \mathbb{C} \hookrightarrow \mathbb{C}[x]$ is not integral, but, since the target is just a point, the map $\operatorname{Spec} \iota : \operatorname{Spec} \mathbb{C}[x] \to \operatorname{Spec} \mathbb{C}$ is nevertheless a closed map. If $C = \mathbb{C}[y]$, then the map $\operatorname{Spec} \mathbb{C}[x, y] \to \operatorname{Spec} \mathbb{C}[y]$ (being the second projection $\mathbb{A}^2_{\mathbb{C}} \twoheadrightarrow \mathbb{A}^1_{\mathbb{C}}$) is no longer closed: Just take the hyperbola $V(xy - 1)$; its image is $D(y) = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$.

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## 13. Aufgabenblatt zum 30.1.2023

**Problem 49.** As introduced in Problem 46, denote by $\mathbb{C}[H]$ the semigroup algebra of an (in our case abelian) semigroup $H$.

a) For $H_1 := \mathbb{N}^2$ and $H_2 := \{(a, b) \in \mathbb{N}^2 \mid b \le 2a\}$ present $\mathbb{C}[H_i]$ as a quotient of polynomial rings by an ideal. Which geometric objects are described by $\operatorname{Spec}\mathbb{C}[H_1]$ and $\operatorname{Spec}\mathbb{C}[H_2]$? Show that both contain $(\mathbb{C}^*)^2 = \operatorname{Spec}\mathbb{C}[\mathbb{Z}^2]$ as an open subset.

b) Verify the Noether normalization lemma explicitely for the example $\mathbb{C}[H_2]$.

c) Do (a) and (b) with the example $H_3 := \{(a, b, c) \in \mathbb{N}^3 \mid a, b \le c\}$, too.
Is it possible to choose the subalgebra $\mathbb{C}[\mathbf{y}] \subseteq \mathbb{C}[H_3]$ (where $\mathbb{C}[H_3]$ is finite over) such that all $y_i$ are monomials in $\mathbb{C}[H_3]$?

*Solution:* (a) $\mathbb{C}[H_1] = \mathbb{C}[\mathbb{N}^2] = \mathbb{C}[x, y]$ via $x := \chi^{(1,0)}$ and $y := \chi^{(0,1)}$. This leads to $\operatorname{Spec}\mathbb{C}[H_1] = \mathbb{A}^2$.
Since the semigroup $H_2 = \{(a, b) \in \mathbb{N}^2 \mid b \le 2a\}$ is generated (as a semigroup) by the elements $(1, 0)$, $(1, 1)$, and $(1, 2)$ (the ray generators $(1, 0)$ and $(1, 2)$ alone do not suffice), we may write $\mathbb{C}[H_2] = \mathbb{C}[x, xy, xy^2] \subset \mathbb{C}[x, y]$. On the other hand, denoting $A := x$, $B := xy$, abd $C := xy^2$, then these new variables satisfy $AC = B^2$, and this is the generating relation among them. Hence, $\mathbb{C}[x, xy, xy^2] = \mathbb{C}[A, B, C]/(AC - B^2)$, i.e. $\operatorname{Spec}\mathbb{C}[H_2]$ becomes $V(AC - B^2) \subset \mathbb{A}^3$.

(b) This is exactly the situation of the concrete example being treated in Problem 47.

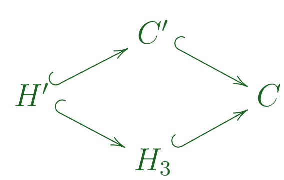
(c) $\mathbb{C}[H_3] = \mathbb{C}[z, xz, yz, xyz]$, $A := \mathbb{C}[z, (x + y)z, xyz] \subseteq \mathbb{C}[H_3] \Rightarrow \mathbb{C}[H_3]$ is finite over $A$: Since $(xz)^2 - ((x + y)z)(xz) + z(xyz) = 0$, the element $xz$ is integral over the ring $A$.
There is no monomial solution: If $B = \mathbb{C}[y_1, y_2, y_3] \subseteq \mathbb{C}[H_3]$ were such an algebra, then we knew that $y_k = \chi^{r_k}$ where $r_k \in H_3$. Denote by $H'$ the semigroup generated by $\{r_1, r_2, r_3\}$. On the other hand, we consider the polyhedral cones $C'$ and $C$ being generated by $H'$ and $H_3$, respectively. That is

$$C' = \mathbb{R}_{\ge 0} r_1 + \ldots + \mathbb{R}_{\ge 0} r_3$$

and

$$C = \{(a, b, c) \in \mathbb{R}^3_{\ge 0} \mid b \le 2a\}.$$

Since the latter has four generating rays, i.e., it is *not* simplicial, we obtain that $C' \subsetneq C$ is a proper subcone. Moreover, note that $H_3 = C \cap \mathbb{Z}^3$. Summarizing the situation, we have that



and

$$B = \mathbb{C}[H'] \subseteq \mathbb{C}[C' \cap \mathbb{Z}^3] \subseteq \mathbb{C}[C \cap \mathbb{Z}^3] = \mathbb{C}[H_3].$$

The overall extension $\mathbb{C}[H'] \hookrightarrow \mathbb{C}[H_3]$ is supposed to be integral. Thus, $\mathbb{C}[C' \cap \mathbb{Z}^3] \subseteq \mathbb{C}[C \cap \mathbb{Z}^3]$ has to be integral, too. On the other hand, both $C' \cap \mathbb{Z}^3$ and $C \cap \mathbb{Z}^3$ are saturated, i.e., the associated semigroup algebras are normal, i.e., integrally closed in their respective quotient fields. However, these fields coincide – they are $\operatorname{Quot} \mathbb{C}[\mathbb{Z}^3]$. This gives the contradiction we were looking for.

**Problem 50.** Let $R := \mathbb{C}[x, y]/(y^2 - x^3)$. For a point $(a, b) \in \mathbb{C}^2$ let $\mathfrak{m}_{(a,b)} := (x - a,\, y - b) \subseteq R$.
a) For which points is $\mathfrak{m}_{(a,b)} = (1)$?
b) For which points is $\mathfrak{m}_{(a,b)}$ a projective $R$-module?
c) Draw the curve $E := \{(a, b) \in \mathbb{R}^2 \mid b^2 = a^3\}$ and mark the points where $\mathfrak{m}_{(a,b)}$ is not projective.

*Solution:* (a) $\mathfrak{m} := \mathfrak{m}_{(a,b)} \neq (1) \Leftrightarrow b^2 = a^3$.
(b) If $b^2 = a^3$, then all localizations are still equal to (1) – with the only exception of $\mathfrak{m}_\mathfrak{m} = \mathfrak{m}R_\mathfrak{m} \subseteq R_\mathfrak{m}$. Here we know that $(y-b)(y+b) = y^2-b^2 = x^3-a^3 = (x-a)f(x)$, i.e. for $b \neq 0$ we have that $\mathfrak{m}_\mathfrak{m} = (y - b)$. Hence all localizations are free, implying that the ideal $\mathfrak{m}_{(a,b)}$ is projective.
(c) On the other hand, let $(a, b) = (0, 0)$. Then $\mathfrak{m} = (x, y)$ is not a principal ideal, i.e. $\mathfrak{m}$ is not free in $R_\mathfrak{m}$: Using Nakayama, it suffices to calculate the $\mathbb{C}$-dimension of $\mathfrak{m} \otimes (R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m}) = \mathfrak{m} \otimes R/\mathfrak{m} = \mathfrak{m}/\mathfrak{m}^2$. We obtain $(x, y)/((x, y)^2 + (y^2 - x^3)) = (x, y)/(x, y)^2$, i.e. the dimension in question is 2.

**Problem 51.** Let $I, J \subseteq A$ be ideals.
a) Determine the kernel of $f$ such that

$$0 \to (???) \to I \oplus J \xrightarrow{f} I + J \to 0$$

becomes an exact sequence of $A$-modules.

b) Assume that $I + J = A$. Show that this implies that $IJ \oplus A \cong I \oplus J$.

c) Present explicitely $(2, 1 + \sqrt{-5})$ as a direct summand of a free $\mathbb{Z}[\sqrt{-5}]$-module.

*Solution:* (a) $0 \to I \cap J \to I \oplus J \to I + J \to 0$

(b) Chinese remainder says that $IJ = I \cap J$. Thus, $0 \to IJ \to I \oplus J \to A \to 0$, and this sequence splits.

(c) Use (b) with $I := (2, 1 + \sqrt{-5})$ and $J := (3, 1 + \sqrt{-5})$. They are obviously coprime, and one has $IJ = (1 + \sqrt{-5}) \cong \mathbb{Z}[\sqrt{-5}]$ (isomorphic as modules).
Let's present an isomorphism $\Phi : \mathbb{Z}[\sqrt{-5}]^2 \xrightarrow{\sim} (2, 1+\sqrt{-5}) \oplus (3, 1+\sqrt{-5}) \subsetneq \mathbb{Z}[\sqrt{-5}]^2$ explicitely: For this we exploit the multiplication by $(1 + \sqrt{-5})$ yielding $\mathbb{Z}[\sqrt{-5}] \xrightarrow{\sim} (1 + \sqrt{-5})$ and the section $\iota : \mathbb{Z}[\sqrt{-5}] \hookrightarrow (2, 1 + \sqrt{-5}) \oplus (3, 1 + \sqrt{-5})$ of $f$ given by $1 \mapsto (-2, 3)$. Altogether this yields $\Phi = \begin{pmatrix} 1 + \sqrt{-5} & -2 \\ -1 - \sqrt{-5} & 3 \end{pmatrix}$.

**Problem 52.** a) Let $0 \to A_\bullet \xrightarrow{f} B_\bullet \xrightarrow{p} C_\bullet \to 0$ be an exact sequence of complexes. Show that the projection $\mathrm{pr}_B : \mathrm{Cone}\,(f) \to B_\bullet$ (despite it is not a map complexes itself) induces a map complexes $\Phi = (p \circ \mathrm{pr}_B) : \mathrm{Cone}\,(f) \to C_\bullet$. Show further that $\Phi$ is a quasiisomorphism. In particular, we almost obtain a map $\mathrm{pr}_A \circ \Phi^{-1} : C_\bullet \to A_\bullet[1]$. What does the word "almost" refer to?

b) If all sequences $0 \to A_i \to B_i \to C_i \to 0$ from (a) split, then $\Phi$ is even a homotopy equivalence. (*Hint:* One constructs the "inverse" $\Psi$ of $\Phi$ as $\Psi_i(c_i) := (s(c_i), \ldots)$ where the second entry is chosen such that $\Psi$ commutes with the differentials.) What does this change about the word "almost" from (a)?

c) A sequence $A_\bullet \to B_\bullet \to C_\bullet$ of complexes is called a *distinguished triangle* if it is isomorphic to the sequence $N_\bullet \to \mathrm{Cone}\,(f)_\bullet \to M_\bullet[1]$ obtained from some map of complexes $f : M_\bullet \to N_\bullet$. Assume now that $A_\bullet \to B_\bullet \to C_\bullet$ is such an object in the homotopy category $K(\mathcal{A})$ (for $\mathcal{A} = \mathcal{M}od_R$ or, more general, $\mathcal{A} =$ abelian category). Show that it gives rise to a new distinguished triangle $B_\bullet \to C_\bullet \to A_\bullet[1]$.

*Solution:* (a) The obstruction for $\mathrm{pr}_B$ becoming a map of complexes is killed when dividing out the image of $f$. Afterwards, use the 5-lemma to compare the long exact sequences associated to $0 \to A_i \to B_i \to C_i \to 0$ and $0 \to B_\bullet \to \mathrm{Cone}\,(f)_\bullet \to A_\bullet[1] \to 0$.
"Almost": Despite the fact that $\Phi$ is a Qis, it is not an isomorphism (in the category $K_{\mathcal{A}}$) – hence $\Phi^{-1}$ does not need to exist.
(b) Denote by $s_i : C_i \hookrightarrow B_i$ the section of $B_i \twoheadrightarrow C_i$. Then, we define $\Psi : C_\bullet \to \mathrm{Cone}\,(f)$ via $\Psi_i(c_i) := (s(c_i), s(dc_i) - ds(c_i))$. First, one checks, that $\Psi$ is compatible with the differentials. Then, it is obvious that $\Phi \circ \Psi = \mathrm{id}_C$, and it remains to show tah $\Psi \circ \Phi \sim \mathrm{id}_{\mathrm{Cone}\,(f)}$. For this, one uses the homotopy $H_i : B_i \oplus A_{i-1} \to B_{i+1} \oplus A_i$ given by $(b, a) \mapsto (0, b - s(\bar{b}))$.
"Almost": Now, $\Phi : \mathrm{Cone}\,(f) \to C_\bullet$ becomes a true isomorphism in the homotopy category $K_{\mathcal{A}}$. And $\Psi$ becomes its inverse.
(c) Everything follows from the following diagram:

$$
\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & B & \longrightarrow & C & \dashrightarrow & A[1] \\
\| \wr & & \| \wr & & \| \wr & & \| \wr \\
N & \xrightarrow{\alpha} & \mathrm{Cone}\,(f) & \longrightarrow & M[1] & \xrightarrow{f[1]} & N[1] \\
 & & \| & & \wr \uparrow \Phi & & \| \\
 & & \mathrm{Cone}\,(f) & \longrightarrow & \mathrm{Cone}\,(\alpha) & \longrightarrow & N[1]
\end{array}
$$

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VL "(Comm) Algebra I"
FU Berlin, Winter 2022/23

## 14. Aufgabenblatt zum 6.2.2023

**Problem 53.** Let $f : M_\bullet \to N_\bullet$ be a complex homomorphism and $A_\bullet$ be a bounded complex. Show that

$$\mathrm{Hom}_\bullet(A_\bullet, \mathrm{Cone}\,(f)_\bullet) = \mathrm{Cone}\,(\mathrm{Hom}(A_\bullet, f)),$$

i.e. the Hom functor commutes with the mapping cone construction. (Note that $\mathrm{Hom}(A_\bullet, f)$ denotes the complex homomorphism $\mathrm{Hom}(A_\bullet, M_\bullet) \to \mathrm{Hom}(A_\bullet, N_\bullet)$ being induced from $f$.)

*Solution:* A rather informal way (i.e. not taking too much care about the signs) of understanding this is that both complexes are the total complex of the following *three-dimensional* complex: It results from the two double complexes $(i,j) \mapsto \mathrm{Hom}(A_{-i}, M_j)$ and $(i,j) \mapsto \mathrm{Hom}(A_{-i}, N_j)$ being written within two parallel planes and being connected by the map $f$ at all spots.

Now, the trick is that taking the total complex of a three-dimensional complex could be decomposed into two steps via reaching a two-dimensional ("double") complex in between. However, this decomposition can be performed in several ways – and the choices correspond to the left and right hand side of the claim, respectively.

Alternatively, the $n$-th module of the left hand complex is

$$X_n := \oplus_{i\in\mathbb{Z}}\, \mathrm{Hom}(A_{i-n},\ M_{i-1} \oplus N_i)$$

with differential $d : X_n \to X_{n-1}$ working as

$$\begin{array}{rcl} \Phi = (\phi^M_{i-1}, \psi^N_i) & \mapsto & d^{\mathrm{Cone}} \circ \Phi - \Phi \circ d^A \\ & = & (-d^M \phi^M_{i-1},\ d^N \psi^N_j + f\phi^M_{i-1}) - (\phi^M_{i-1} d^A, \psi^N_i d^A). \end{array}$$

The right hand complex $Y_\bullet$ is given by

$$Y_n := \oplus_{i\in\mathbb{Z}}\, \mathrm{Hom}(A_{i-(n-1)}, M_i) \oplus\ \oplus_{j\in\mathbb{Z}}\, \mathrm{Hom}(A_{j-n}, N_j)$$

with differential

$$(\phi^M_i, \psi^N_j)) \mapsto \big( - d^M \phi^M_i + \phi_i d^A,\ (d^N \psi^N_j - \psi^N_j d^A) + f\phi^M_i \big)$$

Actually, this does also contain a sign mistake…

**Problem 54.** Consider the free resolution $\mathbb{Z}^2 \xrightarrow{\alpha} \mathbb{Z}^2 \xrightarrow{\beta} \mathbb{Z}/3\mathbb{Z}$ given by $\alpha = \begin{pmatrix} 3\,1 \\ 0\,1 \end{pmatrix}$ and $\beta = (1\,2)$. Construct a homotopy equivalence between this free (hence projective) resolution and the usual one $\mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$. (*Note that the resolved* $\mathbb{Z}$*-module* $\mathbb{Z}/3\mathbb{Z}$ *is* <u>*not*</u> *part of the resolution.*)

*Solution:* First, we have the following maps between both complexes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow{\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}} & \mathbb{Z}^2 & \xrightarrow{(1\,2)} & \mathbb{Z}/3\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (1\,1)} & & \downarrow{\scriptstyle (1\,2)} & & \| & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{3} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z}/3\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \begin{pmatrix} 1 \\ 0 \end{pmatrix}} & & \downarrow{\scriptstyle \begin{pmatrix} 1 \\ 0 \end{pmatrix}} & & \| & & \\
0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow{\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}} & \mathbb{Z}^2 & \xrightarrow{(1\,2)} & \mathbb{Z}/3\mathbb{Z} & \longrightarrow & 0
\end{array}$$

The composition $\mathbb{Z}_\bullet \to \mathbb{Z}^2_\bullet \to \mathbb{Z}_\bullet$ equals id. The other one $\varphi_\bullet : \mathbb{Z}^2_\bullet \to \mathbb{Z}_\bullet \to \mathbb{Z}^2_\bullet$ (the vertical map in the previous diagram) is not, but the vertical map $(\varphi_\bullet - \mathrm{id}_{\mathbb{Z}^2})$ is homotopic to zero via the homotopy $H = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow{\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}} & \mathbb{Z}^2 & \longrightarrow & 0 \\
 & & {\scriptstyle \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}}\downarrow & \overset{H}{\swarrow} & \downarrow{\scriptstyle \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}} & & \\
0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow[\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}]{} & \mathbb{Z}^2 & \longrightarrow & 0.
\end{array}$$

**Problem 55.** a) Let $R$ be a commutative ring and $a \in R$ a non-zerodivisor. Determine all $\mathrm{Tor}_i^R(R/(a), M)$ ($M = R$-module).

b) Find *free* resolutions of $\mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}/4\mathbb{Z}$- and as $\mathbb{Z}/6\mathbb{Z}$-Modul, respectively.

c) Compute $\mathrm{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, $\mathrm{Tor}_i^{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ and $\mathrm{Tor}_i^{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$.

d) Is $\mathbb{Z}/2\mathbb{Z}$ a projective $\mathbb{Z}/4\mathbb{Z}$- or $\mathbb{Z}/6\mathbb{Z}$-module?

*Solution:* (a) Using the resolution $R \xrightarrow{\cdot a} R \to R/(a)$, one obtains $\mathrm{Tor}_0^R(M, R/(a)) = M \otimes_R R/(a) = M/aM$, $\mathrm{Tor}_1^R(M, R/(a)) = \mathrm{Ann}_M(a)$, and $\mathrm{Tor}_{i \geq 2}^R(M, R/(a)) = 0$.

(b) The $R$-module $\mathbb{Z}/2\mathbb{Z}$ (with $R = \mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$) has the periodic resolutions $\ldots \to R \xrightarrow{\cdot 2} R \xrightarrow{\cdot 2} R \to \mathbb{Z}/2\mathbb{Z}$ and $\ldots \to R \xrightarrow{\cdot 2} R \xrightarrow{\cdot 3} R \xrightarrow{\cdot 2} R \to \mathbb{Z}/2\mathbb{Z}$, respectively.

(c) Tensorizing with $\mathbb{Z}/2\mathbb{Z}$ yields zero maps in the first case and an exact sequence of alternating [0/isomorphism] in the second. Thus, $\mathrm{Tor}_i^{\mathbb{Z}/4\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ for all $i \geq 0$ and $\mathrm{Tor}_i^{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0$ for all $i \geq 1$.

(d) Hence, $\mathbb{Z}/2\mathbb{Z}$ cannot be a projective $\mathbb{Z}/4\mathbb{Z}$-module. However, since $\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$, this is the case over the base ring $\mathbb{Z}/6\mathbb{Z}$.

**Problem 56.** Let $I, J \subseteq R$ be ideals. Show then that $\mathrm{Tor}_0^R(R/I, R/J) = R/(I + J)$ and $\mathrm{Tor}_1^R(R/I, R/J) = (I \cap J)/IJ$.

*Solution:* $M \otimes_R R/J = M/JM$, hence $R/I \otimes_R R/J = (R/I)/J(R/I) = R/(I + J)$. For $\mathrm{Tor}_1$ one considers $0 \to I \to R \to R/I \to 0$. This yields the exact sequence $0 \to \operatorname{Tor}_1^R(R/I, R/J) \to I \otimes R/J \to R/J \to R/I \otimes_R R/J \to 0$, hence $\operatorname{Tor}_1^R(R/I, R/J) = \ker(I/IJ \to R/J)$.

This was the last series of problems at the present semester "Algebra I". I hope you had fun. This class continues at the summer semester 2023 in a similar style – and I hope that I will meet many of you there.

See my homepage for details of how we run the written exam. In short: It takes place on Monday, 2/20 12 - 2 at the "Großer Hörsaal" in Takustr. 9. It is *not* allowed to use any of your notes, nor a prepared sheet of paper. (Note that this is in contrast to some exams in earlier classes of mine.)

A week before, on Monday, 2/13, we will have our last Zentralübung dealing with the content of the classes from the 15th week.

# 1st Exam Algebra I, February 20, 2023

**Problem 1.** Consider the ideals $I_5 := (5)$, $I_6 := (6)$, $I_9 := (9)$ in $\mathbb{Z}$ and their images $J_5 := (5)$, $J_6 := (6)$, $J_9 := (9)$ under $\phi : \mathbb{Z} \to \mathbb{Z}/12\,\mathbb{Z}$.

a) Which of the 3 ideals $I_\nu \subseteq \mathbb{Z}$ ($\nu = 5, 6, 9$) are prime ideals, and which are not?

b) What are the ideals $\phi^{-1}(J_\nu) \subseteq \mathbb{Z}$ for $\nu = 5, 6, 9$?

c) Which of the 3 ideals $J_\nu \subseteq \mathbb{Z}/12\,\mathbb{Z}$ ($\nu = 5, 6, 9$) are prime ideals, and which are not?

*Solution:* (a) In $\mathbb{Z}$, an ideal $(k)$ with $k \neq 0$ is prime if and only if $k$ is a prime number. Thus, only $I_5 = (5)$ is a prime ideal.

(b) The ideals $\phi^{-1}(J_\nu)$ equal $(5, 12) = (1)$, $(6, 12) = (6)$, and $(9, 12) = (3)$, respectively. Here, only the latter $\phi^{-1}(J_9)$ is prime (which was, however, not asked for).

(c) An ideal $J \subseteq \mathbb{Z}/12\,\mathbb{Z}$ is prime if and only if $I := \phi^{-1}(J) \subseteq \mathbb{Z}$ is a prime ideal; we have then $J = I/(12)$. Thus, using (b), we obtain that exactly $J_9 = (9)$ is prime.

**Problem 2.** Let $I \subseteq A$ be an ideal in a commutative ring with 1. Then, write the open subset $\operatorname{Spec}(A) \setminus V(I)$ of $\operatorname{Spec}(A)$ as a union of certain distinguished open subsets $D(f)$ with suitable $f \in A$.

*Solution:* $\operatorname{Spec}(A) \setminus V(I) = \bigcup_{f \in I} D(f)$. Either one explains that some $P \in \operatorname{Spec}(A)$ belongs to the LHS iff $P \not\supseteq I$ and to RHS iff there is an $f \in I$ such that $f \notin P$ – or, alternatively, one compares the complements $V(I) = \bigcap_{f \in I} V(f)$.

**Problem 3.** Let $R$ be a finitely generated $\mathbb{Z}$-algebra and $f \in R$.

a) Let $I, J \subseteq R$ be ideals. Recall the definition of the ideal quotient $(I : J)$. What kind of structure is this (set, ring, field, ideal, group, module. . . )?

b) Show that there is a $k \in \mathbb{N}$ such that $(0 : f^k) = (0 : f^{k+1})$. (Note that we write $(a : b) := \big((a) : (b)\big)$, i.e., we omit the parentheses indicating principal ideals.)

c) Show that for any such $k$ from (b) we even get $(0 : f^k) = (0 : f^\ell)$ for all $\ell \geq k$. We call this ideal $(0 : f^\infty)$.

d) Show that $(0 : f^\infty) = \ker\big(R \to R_f\big)$.

*Solution:* (a) $(I : J) = \{r \in R \,|\, r \cdot J \subseteq I\}$ is an ideal in $R$.

(b) We have $(0 : f^k) \subseteq (0 : f^{k+1})$ for all $k$. Since $R$ is a finitely generated $\mathbb{Z}$-algebra, it is noetherian, i.e., ascending sequences of ideals terminate.

(c) We show that $(0 : f^{k+1}) = (0 : f^{k+2})$; then the claim follows by induction. Let $r \in (0 : f^{k+2})$, i.e., $0 = f^{k+2} \cdot r = f^{k+1} \cdot fr$, i.e., $fr \in (0 : f^{k+1}) = (0 : f^k)$. This, however, means that $0 = f^k \cdot fr = f^{k+1} \cdot r$, i.e., $r \in (0 : f^{k+1})$.

(d) Part (c) implies that $(0 : f^\infty) = \{r \in R \,|\, \exists \ell \in \mathbb{N} : f^\ell \cdot r = 0\}$. This however, is exactly the condition that $r/1 = 0/1$ within the localization $R_f$.

**Problem 4.** Let $R := \mathbb{C}[x, y]/(x^3, x^2y, y^3)$.

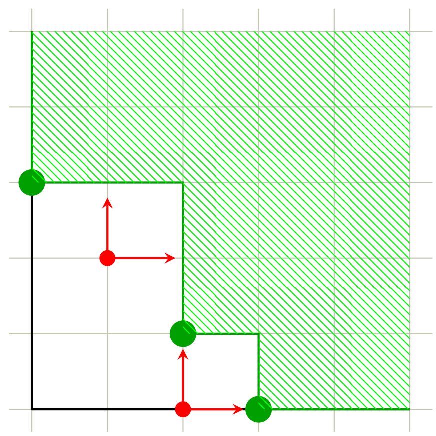
(i) Draw a picture visualizing the monomials of $R$. What is $\dim_{\mathbb{C}} R$?

(ii) Name two examples for anihilators of non-vanishing monomials which are non-prime ideals, and

(iii) name all monomials with anihilators being prime ideals.

(iv) What are the associated primes for $R$? Does $R$ have embedded, i.e., associated, but non-minimal primes?

*Solution:* The green shaded area indicates the monomial ideal $I = (x^3, x^2y, y^3)$.



The lattice points left and below are the "standard monomials" yielding a $\mathbb{C}$-basis of $R = \mathbb{C}[x,y]/(x^3, x^2y, y^3)$. Thus, for our example, the $\mathbb{C}$-dimension is 7. The two red dots $(2,0)$ and $(1,2)$ represent $x^2$ and $xy^2$, respectively. Their anihilators are the maximal ideal $(x, y)$.

The remaining standard monomials have non-prime anihilators, e.g., $\text{Ann}(xy) = (x, y^2)$ or $\text{Ann}(1) = (x^3, x^2y, y^3)$.

Altogether, we see that $(x, y)$ is the only asociated prime of $R$. Thus, it is also a minimal one; there is no embedded primes.

**Problem 5.** Consider $R := \mathbb{C}[x,y]/(x^3 + x^2y + xy^2)$ as a (finitely generated)

(i) $\mathbb{C}[x]$-algebra and

(ii) $\mathbb{C}[y]$-algebra.

That is, consider the ring homomorphisms (i) $\alpha : \mathbb{C}[x] \to R$ and (ii) $\beta : \mathbb{C}[y] \to R$ sending $\alpha : x \mapsto x$ and $\beta : y \mapsto y$, respectively.

a) Which of the algebras (i) and (ii) are finite, which are not?

b) Translate the algebra homomorphisms into geometric maps $a$ and $b$ both running as $V(x^3 + x^2y + xy^2) \hookrightarrow \mathbb{C}^2 \to \mathbb{C}^1$ and ending with projections to the $x$- or $y$-coordinate, respectively. Which of them have only finite fibers, which of them have some infinite fibers?

*Solution:* (a)(ii) The polynomial $f(x,y) = x^3 + x^2y + xy^2$ is an integrality equation for $x$ over $\mathbb{C}[y]$. Thus, $\mathbb{C}[y] \to R$ is finite.

(b)(ii) It follows from part (a) that $b$ is a map with only finite fibers. This, however follows also directly: If some value $y = c \in \mathbb{C}$ is fixed, then all pre-images $(x, c)$ arise from the at most three solutions of $x^3 + cx^2 + c^2x = 0$.

(b)(i) For $x = 0$ we obtain $\{0\} \times \mathbb{C}$ as the associated pre-image. It is infinite.

(a)(i) Considered as an element of $\mathbb{C}[x][y]$, the polynomial $f$ has $x$ instead of 1 as the highest coefficient, i.e., in front of $y^2$. This is an indication that $\alpha : \mathbb{C}[x] \to R$ might be not finite. However, taking it serious, one has still to check that there is no other equation doing the job instead.
Alternatively, the non-integrality follows immediately from (b)(i).

**Problem 6.** Denote by $A_\bullet$ the following (exact) complex of $\mathbb{C}[x]$-modules

$$0 \to \mathbb{C}[x] \xrightarrow{\cdot x} \mathbb{C}[x] \xrightarrow{\mathrm{ev}_0} \mathbb{C} \to 0.$$

where the right most object $\mathbb{C}$ obtains its $\mathbb{C}[x]$-module structure by declaring the multiplication with $x$ as zero, and the right most map is $\mathrm{ev}_0(f) := f(0)$.

a) Give a general argument (without calculations) why there should exist a $\mathbb{C}$-linear (being weaker than $\mathbb{C}[x]$-linear) homotopy between $\mathrm{id} : A_\bullet \to A_\bullet$ and $0 : A_\bullet \to A_\bullet$.

b) Construct such a homotopy $h : A_\bullet \to A_\bullet[1]$ explicitely.

*Solution:* (a) The first complex $A_\bullet$ consists of objects which are projective $\mathbb{C}$-modules (vector spaces); the second complex $A_\bullet$ is exact. Therefore, by Proposition 29, id has to be 0-homotopic.

(b) Denote $A_2 = A_1 = \mathbb{C}[x]$ and $A_0 = \mathbb{C}$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & A_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & \swarrow{\scriptstyle h_1} & \downarrow{\scriptstyle \mathrm{id}} & \swarrow{\scriptstyle h_0} & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & A_0 & \longrightarrow & 0
\end{array}$$

Then, we take $h_0 : A_0 \to A_1$ to be the natural embedding $\mathbb{C} \hookrightarrow \mathbb{C}[x]$ and $h_1 : A_1 = \mathbb{C}[x] \to \mathbb{C}[x] = A_2$ to be $f(x) \mapsto \frac{f(x) - f(0)}{x}$.

# 2nd Exam Algebra I, April 3rd, 2023

**Problem 1.** Let $I := (xyz,\ x^2 + y^2 + z^2) \subseteq \mathbb{C}[x, y, z]$. Give two examples for maximal ideals $\mathfrak{m} \subseteq \mathbb{C}[x, y, z]$ containing $I$ and two examples for maximal ideals $\overline{\mathfrak{m}} \subseteq \mathbb{C}[x, y, z]/I$.

*Solution:* The maximal ideals in $\mathbb{C}[x, y, z]$ are always of the form

$$\mathfrak{m} = \mathfrak{m}_{(a,b,c)} := (x - a, y - b, z - c) \quad \text{for } (a, b, c) \in \mathbb{C}^3.$$

The condition $I \subseteq \mathfrak{m}_{(a,b,c)}$ is equivalent to $f(a, b, c) = 0$ for all $f(x, y, z) \in I$, or just for a generating set. Thus, we may choose $(a, b, c) = (0, 0, 0)$ or $(1, i, 0)$.

The maximal ideals $\overline{\mathfrak{m}}$ correspond to the $\mathfrak{m} \supseteq I$ via $\overline{\mathfrak{m}} = \mathfrak{m}/I$.

**Problem 2.** Let $\varphi : R \to S$ be a homomorphism of commutative rings with 1. Show that the associated map $f : \operatorname{Spec} S \to \operatorname{Spec} R$ is continous (Zariski topology), and give examples showing that it does not need to be open, neither closed.

*Solution:* If $I \subseteq R$ is an ideal, then $f^{-1}(V(I)) = V\big(\varphi(I)S\big)$. The reason for this is that, for a $Q \in \operatorname{Spec} S$,

$$Q \in LHS \Leftrightarrow f(Q) \supseteq I \Leftrightarrow \varphi^{-1}(Q) \supseteq I \Leftrightarrow Q \supseteq \varphi(I) \Leftrightarrow Q \supseteq \varphi(I)S.$$

Here are the examples: $\mathbb{C}[x] \twoheadrightarrow \mathbb{C}$ gives the non-open embedding $\{0\} \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$. On the other hand, $\mathbb{C}[x] \hookrightarrow \mathbb{C}[x]_x$ gives the non-closed embedding $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \hookrightarrow \mathbb{A}^1_{\mathbb{C}}$.

**Problem 3.** Let $R = \mathbb{Z}/12\,\mathbb{Z}$. Write down some "nice" filtration $R = M_0 \supset M_1 \supset \ldots \supset M_k = 0$ with $R$-modules $M_i$ such that each $M_i/M_{i+1}$ is isomorphic to $R/P_i$ for some $P_i \in \operatorname{Spec} R$ $(i = 0, \ldots, k - 1)$. Is your filtration a composition series? Is $R$ an Artinian ring? What is its length?

*Solution:* The anihilator of $6 \in \mathbb{Z}/12\,\mathbb{Z}$ is the prime ideal $(2)$, hence, sending $1 \mapsto 6$, we obtain an injection

$$\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/12\,\mathbb{Z}$$

with cokernel $\mathbb{Z}/6\mathbb{Z}$. In other words, we have a short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 6} \mathbb{Z}/12\,\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/6\mathbb{Z} \to 0$$

which can be understood as a filtration of $M_0 := R$ by $M_1 := 6 \cdot \mathbb{Z}/12\,\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ and $M_2 := 0$. The only interesting factor is $M_0/M_1 = \mathbb{Z}/6\mathbb{Z}$ which has to be processed further. It fits into an exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z}/6\,\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$$

meaning a filtration $\mathbb{Z}/6\,\mathbb{Z} \supset 3 \cdot \mathbb{Z}/6\,\mathbb{Z} \supset 0$. Pulling this back via $\pi$ yields

$$\mathbb{Z}/12\,\mathbb{Z} \supset 3 \cdot \mathbb{Z}/12\,\mathbb{Z} \supset 6 \cdot \mathbb{Z}/12\,\mathbb{Z} \supset 0$$

Since the factors are $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, and $\mathbb{Z}/2\mathbb{Z}$ again, and since all of them are fields, it is a composition series of length 3. Thus, $R$ is Artinian.

**Problem 4.** a) Is $t \in \mathbb{C}[t]$ integral over $\mathbb{C}[t^7]$?

b) Is $2t^3 - 3t^2 + 5 \in \mathbb{C}[t]$ integral over $\mathbb{C}[t^7]$?

*Solution:* (a) Since $t$ is a zero of the polynomial $f(x) = x^7 - t^7 \in \mathbb{C}[t^7][x]$, it is integral over $\mathbb{C}[t^7]$.

(b) By (a), the ring $\mathbb{C}[t] \supset \mathbb{C}[t^7]$ is a finitely generated $\mathbb{C}[t^7]$-module. Actually, it is free with $\{1, t, \ldots, t^6\}$ working as a basis. But this implies that all elements of $\mathbb{C}[t]$ are integral over $\mathbb{C}[t^7]$ – and so does $2t^3 - 3t^2 + 5$.

**Problem 5.** Let $R := \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} \mid a, b \in \mathbb{Z}\}$. Let $P := (2, \sqrt{5}+1)$ and note that $\sqrt{5} - 1 \in P$.

a) Compare the three ideals

$$P^2, \quad (2) \cdot P, \quad \text{and } (\sqrt{5}+1) \cdot P,$$

by presenting generators for each. Is any of these ideals contained in another one?

b) Show that $P$ is a prime ideal. Is it maximal?

c) Are the ideals $(2)$ and $(\sqrt{5}+1)$ primary ideals? What are their respective radicals?

*Solution:* (a) $P^2 = (2) \cdot P = (\sqrt{5}+1) \cdot P = (4,\ 2\sqrt{5} \pm 2)$.

(b) $\mathbb{Z}[\sqrt{5}]/P = \mathbb{Z}[x]/(x^2-5,\ 2,\ x+1) = \mathbb{Z}[x]/(2,\ x+1) = \mathbb{F}_2$, hence $P$ is a maximal ideal.

(c) Since $P$ is a maximal ideal and $P = \sqrt{P^2} = \sqrt{(2) \cdot P} \subseteq \sqrt{(2)}$, it follows that both $P^2$ and $(2)$ are $P$-primary. Alternatively, one can directly see that $(\sqrt{5}+1)^2 \in (2)$, i.e., $\sqrt{5}+1 \in \sqrt{(2)}$. Similarly, we obtain that $\sqrt{(\sqrt{5}+1)} = P$, i.e., $(\sqrt{5}+1)$ is $P$-primary, too.
The property "primary" could also be seen as follows: $\mathbb{Z}[\sqrt{5}]/(2) = \mathbb{Z}[x]/(x^2-5, 2) = \mathbb{F}_2[x]/(x-1)^2 = \mathbb{F}_2[t]/t^2$ (with $t = x - 1$). Thus, all zero divisors are nilpotent. Similarly, we obtain that $\mathbb{Z}[\sqrt{5}]/(\sqrt{5}+1) = \mathbb{Z}[x]/(x^2-5,\ x+1) = \mathbb{Z}/4\mathbb{Z}$.

**Problem 6.** Denote by $A_\bullet$ the following (exact) complex of $\mathbb{C}[x]$-modules

$$0 \to \mathbb{C}[x] \xrightarrow{\cdot x} \mathbb{C}[x] \xrightarrow{\mathrm{ev}_0} \mathbb{C} \to 0.$$

where the right most object $\mathbb{C}$ obtains its $\mathbb{C}[x]$-module structure by declaring the multiplication with $x$ as zero, and the right most map is $\mathrm{ev}_0(f) := f(0)$.

a) Give an example for an additive functor destroying the exactness of the complex $A_\bullet$ and derive from this why there cannot exist a $\mathbb{C}[x]$-linear homotopy between $\mathrm{id} : A_\bullet \to A_\bullet$ and $0 : A_\bullet \to A_\bullet$.

b) Show explicitely at this example that such a $\mathbb{C}[x]$-linear homotopy $h : A_\bullet \to A_\bullet[1]$ does not exist.

*Solution:* (a) We can take $F(M) := M \otimes_{\mathbb{C}[x]} \mathbb{C}$; then $F(A_\bullet)$ becomes $0 \to \mathbb{C} \to \mathbb{C} \to \mathbb{C} \to 0$ which cannot be exact, independently on the shape of the differentials.

(b) Denote $A_2 = A_1 = \mathbb{C}[x]$ and $A_0 = \mathbb{C}$.

$$\begin{tikzcd}
0 \arrow[r] & A_2 \arrow[r] \arrow[d, "\mathrm{id}"'] & A_1 \arrow[r] \arrow[d, "\mathrm{id}"'] & A_0 \arrow[r] \arrow[d, "\mathrm{id}"'] \arrow[dl, "h_0"'] & 0 \arrow[d, "\mathrm{id}"'] \arrow[dl, "h_{-1}"'] \\
0 \arrow[r] & A_2 \arrow[r] & A_1 \arrow[r] & A_0 \arrow[r] & 0
\end{tikzcd}$$

Then, any $\mathbb{C}[x]$-linear map $h_0 : A_0 = \mathbb{C} \to \mathbb{C}[x] = A_1$ has to vanish. In particular, there is already no chance of $\mathrm{id} = dh_0 + h_{-1}d$.

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## 1. Aufgabenblatt zum 24.4.2023

**Problem 57.** In a noetherian ring $R$ we define for ideals $I, J \subseteq R$ the quotient $(I : J) := \{x \in R \,|\, xJ \subseteq I\}$.

a) Show that this yields an increasing (hence terminating) chain of ideals $I = (I : J^0) \subseteq (I : J^1) \subseteq \ldots \subseteq (I : J^k) = (I : J^{k+1}) =: (I : J^\infty)$.

b) Calculate the quotient ideals $(I : J)$, $(J : I)$, $(I : J^\infty)$, and $(J : I^\infty)$ for $I = (x^2-1)$ and $J = (x-1)^2$ in the ring $\mathbb{C}[x]$.

c) Let $J = (f_1, \ldots, f_r)$. Show that $(I : J^\infty) = \{x \in R \,|\, \exists n : xJ^n \subseteq I\} = \{x \in R \,|\, \forall y \in J\; \exists n\colon xy^n \in I\} = \{x \in R \,|\, \exists n \forall \nu\colon x f_\nu^n \in I\} \subseteq (\sqrt{I} : J)$.

d) In $\operatorname{Spec} R$ it holds true that $V(I) \setminus V(J) = V(I : J^\infty) \setminus V(J)$ and $\overline{V(I) \setminus V(J)} = V(I : J^\infty)$. (*Hint:* W.l.o.g. $I = 0$ and $(0 : J) = (0)$.)

*Solution:* (b) $\big((x^2-1) : (x-1)^2\big) = \big((x^2-1) : (x-1)^\infty\big) = (x+1)$ and $\big((x-1)^2 : (x^2-1)\big) = (x-1)$, but $\big((x-1)^2 : (x^2-1)^\infty\big) = (1)$.

(d) It remains to show that $V(I : J^\infty) \setminus V(J)$ is dense in $V(I : J^\infty)$. With $I := I : J^\infty$ and $R := R/I$ it remains to show the following fact: "Let $(0 : J) = (0)$, then $\operatorname{Spec} R \setminus V(J)$ is dense in $\operatorname{Spec} R$". Otherwise, let $\emptyset \neq D(f) \subseteq V(J)$; since $\operatorname{Spec} R \setminus V(J) = \bigcup_\nu D(f_\nu)$ this implies $D(ff_\nu) = \emptyset$, hence $(ff_\nu)^{N \gg 0} = 0$ for all $\nu$. Thus, $f^N \in (0 : J^\infty) = (0)$, i.e. $D(f) = \emptyset$.

**Problem 58.** Let $M = \oplus_{i\in\mathbb{Z}} M_i$ be a graded module over a graded ring $S = \oplus_{i\in\mathbb{N}} S_i$. For an $m = \sum_i m_i$ with $m_i \in M_i$ we call $m_i$ the "homogeneous component" of degree $i$ of $m$. Let $N \subseteq M$ be an $S$-submodule. Show that $N = \oplus_{i\in\mathbb{Z}}(M_i \cap N)$ ("$N$ is a *graded* submodule of $M$") $\Leftrightarrow$ for all $m \in N \subseteq M$ the homogeneous components $m_i \in M$ are contained in $N$, too $\Leftrightarrow$ $N$ is generated by homogeneous elements of $M$, i.e. by certain elements from $\bigcup_i M_i$.

*Solution:* The first equivalence is almost a tautology, and the direction ($\Rightarrow$) of the latter one was shown in class. We turn to ($\Leftarrow$):
Let $N = \langle k^1, \ldots, k^n \rangle$ with homogeneous $k^j \in M_{d(j)}$. If $\sum_i m_i = m = \sum_j c^j k^j \in N$ with $m_i \in M_i$, then $m_i = \sum_j c^j_{i-d(j)}\, k^j \in N$. This holds true for all kinds of gradings.

**Problem 59.** Let $S = \oplus_{d\in\mathbb{N}} S_d$ be an $\mathbb{N}$-graded ring. Note that for homogeneous ideals $I \subseteq S$ (i.e. graded submodules of $S$) the ring $S/I$ becomes graded, too.

a) Show that a homogenous ideal $P \subseteq S$ is prime $\Leftrightarrow$ for all homogenous $a, b \in S$ the membership $ab \in P$ implies that $a \in P$ or $b \in P$.

b) Does Statement (a) remain true for gradings over more general groups like $\mathbb{N}^2$ or $\mathbb{Z}^2$ or $\mathbb{Z}/2\mathbb{Z}$ instead of just $\mathbb{N}$?

*Solution:* (a) If $a = \sum_i a_i \notin P$ and $b = \sum_i b_i \notin P$ (the sums mean the homogeneous decompositions), we are supposed to show that $ab \notin P$ as well. W.l.o.g. we may assume that all $a_i, b_j \notin P$. But then, $a_{\min} \cdot b_{\min} \notin P$, and this is the homogeneous component of $ab$ of smallest degree.

(b) is certainly ok for grading semigroups allowing a total order that is compatible with the semigroup structure, e.g. like a term order in $\mathbb{N}^2$. However, in the $\mathbb{Z}/2\mathbb{Z}$-graded $S := \mathbb{C}[x]/(x^2-1)$ ($\deg x := 1$), Statement (b) is false for $P = (0)$.

**Problem 60.** Let $M = \oplus_{e\in\mathbb{Z}} M_e$ be a $\mathbb{Z}$-graded module over the $\mathbb{N}$-graded ring $S = \oplus_{d\in\mathbb{N}} S_d$ which is supposed to be finitely generated as an algebra over $S_0$. Show that the finite generation of $M$ implies that all $M_e$ are finitely generated $S_0$-modules. Give an example that the opposite implication fails.

*Solution:* First, let $s_1, \ldots, s_k$ be homogeneous generators of $S$ as an $S_0$-algebra with degrees $d_1, \ldots, d_k \in \mathbb{N}$. Second, let $m_1, \ldots, m_N$ be homogeneous generators of $M$ with degrees $e_1, \ldots, e_N \in \mathbb{Z}$. Then, $M_\ell$ is generated, as an $S_0$-module, by all products $s_1^{\nu_1} \cdot \ldots \cdot s_k^{\nu_k} \cdot m_i$ with $\sum_j \nu_j d_j + e_i = \ell$. And there is only finitely many of them.

$S = \mathbb{C}[x]$ and $M = \oplus_{k\geq 0} S(-k) = \oplus_{k\geq 0}(x^k) \cdot t^k$ where all summands are still graded with $\deg x := 1$ and $\deg t = 0$, i.e., the meaning of $t$ is just to mark the summands. Then, the part of degree $d$ equals $M_d = \oplus_{k=0}^d x^{d-k} \cdot x^k \cdot t^k\mathbb{C} = \oplus_{k=0}^d x^d \cdot t^k\mathbb{C}$.

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## 2. Aufgabenblatt zum 1.5.2023

**Problem 61.** In class, see (11.3), we have claimed that $\widetilde{R} := k[t,\mathbf{x}]/I^h$ is flat over $k[t]$. For this, we have used that $k[t]$-modules $M$ are flat if and only they are torsion free. On the other hand, we had just checked that $(\cdot t) : \widetilde{R} \to \widetilde{R}$ was injective (which was equivalent to the $t$-saturation of the ideal $I^h$). Conclude the proof.

(Hint: Exploit the knowledge $p_X^{-1}(\mathbb{A}^1\backslash 0) \cong X \times (\mathbb{A}^1\backslash 0)$ over $\mathbb{A}^1\backslash 0$ where $X = \operatorname{Spec} R$ with $R = k[\mathbf{x}]/I$. While the LHS corresponds to $\widetilde{R}_t = \widetilde{R} \otimes_{k[t]} k[t,t^{-1}]$, try to write the RHS as a tensor product, too.)

*Solution:* The map $p_X$ corresponds to a ring homomorphism $\varphi : k[t] \to \widetilde{R}$. We know that $(\cdot t) : \widetilde{R} \to \widetilde{R}$ is injective and that $\widetilde{R}_t = \widetilde{R} \otimes_{k[t]} k[t,t^{-1}]$ is isomorphic to $k[t,\mathbf{x}]/I$, i.e., to $R \otimes_k k[t,t^{-1}]$, as a $k[t,t^{-1}]$-algebra.

The latter is flat over $k[t,t^{-1}]$ – just because for any $k$-algebra $A$ the tensor product $R \otimes_k A$ is a flat $A$-algebra. To check this, one takes an arbitrary $A$-module $M$ and uses $(R \otimes_k A) \otimes_A M = R \otimes_k M$. Over $k$, everything is flat.

Now, we could retranslate the flatness of $\widetilde{R}_t$ over $k[t,t^{-1}]$ as the injectivity of the multiplications with $(t-c)$ for all $c \in k \setminus \{0\}$. And here, it is not important if we understand this as maps $\widetilde{R} \to \widetilde{R}$ or $\widetilde{R}_t \to \widetilde{R}_t$.

Alternatively, we could recall that flatness (of $\widetilde{R}$ as an $k[t]$-module) is local. First, we look at the localization $k[t]_{(t)}$ in the ideal $(t)$. Here, flatness means indeed just the injectivity of $(\cdot t)$. And all the other localizations are taken care of via considering $k[t]_t = k[t,t^{-1}]$. Over this, however, we have seen that the algebra is free.

**Problem 62.** For a fixed ideal $\mathfrak{m} \subseteq A$ in a ring, e.g. if $(A,\mathfrak{m})$ is a local ring, we define $\operatorname{Gr}_{\mathfrak{m}}(A) := \oplus_{\nu\ge 0}\, \mathfrak{m}^\nu/\mathfrak{m}^{\nu+1} = \oplus_{\nu\ge 0}\, \mathfrak{m}^\nu/\mathfrak{m}^{\nu+1} \cdot t^\nu$. Check that this is a graded $A/\mathfrak{m}$-algebra.

For an element $f \in \mathfrak{m}^\nu \setminus \mathfrak{m}^{\nu+1}$, we set $\operatorname{in}(f) := \overline{f} \in \mathfrak{m}^\nu/\mathfrak{m}^{\nu+1} = \operatorname{Gr}_{\mathfrak{m}}(A)_\nu$. And, since $\bigcap_{\nu\ge 0} \mathfrak{m}^\nu = 0$, there is a (unique) $\nu = \nu(f)$ for every $f \in A \setminus 0$. For an ideal $I \subseteq A$ we define $\operatorname{in}(I) := (\operatorname{in}(f) \mid f \in I \setminus 0) \subseteq \operatorname{Gr}_{\mathfrak{m}}(A)$.

If $I = (f_1,\ldots,f_k)$, then compare $\operatorname{in}(I)$ with $(\operatorname{in}(f_1,\ldots,\operatorname{in}(f_k))$ and give an example where they do not coincide.

*Solution:* $\operatorname{Gr}_{\mathfrak{m}}(A)$ becomes a graded ring because $\mathfrak{m}^\nu\mathfrak{m}^\mu \subseteq \mathfrak{m}^{\nu+\mu}$ induces a bilinear map

$$\mathfrak{m}^\nu/\mathfrak{m}^{\nu+1} \times \mathfrak{m}^\mu/\mathfrak{m}^{\mu+1} \to \mathfrak{m}^{\nu+\mu}/\mathfrak{m}^{\nu+\mu+1}.$$

The algebra structure results from the fact that $A/\mathfrak{m}$ appears as the component of degree 0.

Moreover, we clearly have $(\operatorname{in}(f_1,\ldots,\operatorname{in}(f_k)) \subseteq \operatorname{in}(I)$, but they might be different: Take $A = k[x,y,z]_{(x,y,z)}$ with $\mathfrak{m} = (x,y,z)$. Then, the ideal $I = (x-y^2,\, x-z^2)$ leads to $\operatorname{in}(x-y^2) = x$ and $\operatorname{in}(x-y^2) = x$, but in $I$ does also contain $y^2 - z^2$.

**Problem 63.** Show that the family $V(f_1, f_2, f_3) \xrightarrow{t} \mathbb{A}^1$ with $f_i = x_i x_{i+1} - t$ ($i \in \mathbb{Z}/3\mathbb{Z}$) is not flat in a neighborhood of $t = 0$, i.e. check that, with $R := \mathbb{C}[t]_{(t)}$, the $R$-algebra $A := R[x_1, x_2, x_3]/(f_1, f_2, f_3)$ is not a flat one. Can you visualize what is going on when $t \to 0$?

*Solution:* We have to show that the multiplication with $t = x_i x_{i+1}$ is not injective within $B := \mathbb{C}[x_1, x_2, x_3]/(x_1x_2 = x_2x_3 = x_3x_1)$. Since $x_1x_2 - x_2x_3 = x_2(x_1 - x_3)$, we see that $t$ kills $(x_1 - x_3)$.

For $c \neq 0$, the fibers $t^{-1}(c)$ consist of two points each, namely $\pm(\sqrt{c}, \sqrt{c}, \sqrt{c})$. For $t = 0$, however, we obtain the three coordinate lines in $\mathbb{C}^3$.

**Problem 64.** Give an example for an ideal $I \subseteq R$ and a pair of $R$-modules $M' \subseteq M$ such that $I(I^k M \cap M') \subsetneq I^{k+1} M \cap M'$ for some $k$.

*Solution:* $I \subseteq R$ ideal and $M := R$, $M' := I^2$, and $k = 1 \rightsquigarrow$ the inclusion $I^3 = I(I^1 R \cap I^2) \subsetneq I^2 R \cap I^2 = I^2$ is usually strict.

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# 3. Aufgabenblatt zum 8.5.2023

**Problem 65.** a) Let $I \subseteq A$ be an ideal with $\bigcap_\nu I^\nu = 0$, e.g. $I \neq (1)$ in a noetherian local ring. Show that the lack of zero divisors in the graded ring $\mathrm{Gr}_I(A) := \oplus_{d\geq 0} I^d/I^{d+1}$ implies that the original ring $A$ was an integral domain, too.

b) Present an example indicating the necessity of the assumption $\bigcap_\nu I^\nu = 0$.

c) Give an example of an integral domain $A$ and an ideal $I \subseteq A$ with $\bigcap_\nu I^\nu = 0$ such that $\mathrm{Gr}_I(A)$ has zero divisores.

*Solution:* (a) Let $a, b \in A$ – then there are $k, l \in \mathbb{N}$ such that $a \in I^k \setminus I^{k+1}$ and $b \in I^l \setminus I^{l+1}$. We define $0 \neq \overline{a} \in I^k/I^{k+1}$ and $0 \neq \overline{b} \in I^l/I^{l+1}$ to be the images of $a, b$, respectively. Then their product $\overline{a} \cdot \overline{b} \in I^{k+l}/I^{k+l+1}$ within $\mathrm{Gr}_I(A)$ equals the image of $ab \in I^{k+l}$. And, since $\overline{a} \cdot \overline{b} \neq 0$, this means $ab \notin I^{k+l+1}$, hence $ab \neq 0$.

(b) If $I = (2)$ in $A = \mathbb{Z}/6\mathbb{Z}$, then $I = I^k$ for $k \geq 1$. In particular, $\mathrm{Gr}_I(A) = \mathbb{Z}/2\mathbb{Z}$ is even a field. It is concentrated in degree 0.

(c) Take $A := \mathbb{C}[x, y]/(y^2 - x^3)$. Since $y^2 - x^3$ is irreducible in $\mathbb{C}[x, y]$ and the latter is factorial, $(y^2 - x^3)$ is a prime ideal, i.e., $A$ is a domain. Since it is contained in the localization $A_{(x,y)}$ and $\bigcap_\nu (x, y)^\nu = 0$ in this local ring, we have the same equation in $A$. On the other hand, in $A$ we have the following interesting feature: $y \in (x, y) \setminus (x, y)^2$, but $y^2 = x^3 \in (x, y)^3$. Thus, the square of $y \in (x, y)/(x, y)^2 = \mathrm{Gr}^1_{(x,y)}(A)$ equals $y^2 \in (x, y)^2/(x, y)^3 = \mathrm{Gr}^2_{(x,y)}(A)$, i.e., zero.

**Problem 66.** Let $A = \mathbb{C}[x, y]$ and consider the ideal $I := (x, y)$. Write the ring $\widetilde{A} := \oplus_{\nu\geq 0} I^\nu$ as a polynomial ring over $\mathbb{C}$ modulo some ideal. Moreover, express the embedding $A \hookrightarrow \widetilde{A}$ within this language.

*Solution:* $\widetilde{A} = \oplus_{\nu\geq 0}(x, y)^\nu \cdot \xi^\nu$ is generated, as an $A$-algebra from $\widetilde{A}_1 = \mathrm{span}_\mathbb{C}\{x\xi, y\xi\}$, i.e. from the elements $X := x\xi$ and $Y := y\xi$. The only relation is

$$y\,X = y \cdot x\xi = x \cdot y\xi = x\,Y,$$

hence $\widetilde{A} = \mathbb{C}[x, y, X, Y]/(yX - xY)$ with $\deg(x) = \deg(y) = 0$ and $\deg(X) = \deg(Y) = 1$. In particular, the generator $yX - xY$ is homogeneous of degree 1. The subring $\mathbb{C}[x, y]$ of degree 0 equals $A$ inside $\widetilde{A}$.

**Problem 67.** In Subsection 16.1.1 we took a fan $\Sigma$ and have interpreted the affine toric varieties corresponding to the cones $\sigma \in \Sigma$ and their mutual open embeddings coming from the face relation among the cones of $\Sigma$.

Now, play the same game with $\Sigma := \{\sigma_1, \sigma_2, \tau\}$ where the $\sigma_i$ are the 2-dimensional cones

$$\sigma_1 := \langle (1, 0),\ (1, 1)\rangle \qquad \sigma_2 := \langle (1, 1),\ (0, 1)\rangle \qquad \tau := \mathbb{R}_{\geq 0} \cdot (1, 1).$$

(Strictly speaking, this is not a fan, since $\tau = \sigma_1 \cap \sigma_2$ is only one face of the cones $\sigma_i$. The other 1-dimensional faces and the 0-cone is missing – but they are not important, hence we simplify everything by just forgetting about them.)

a) Draw these three cones.

b) Determine and draw the dual cones $\sigma_1^\vee$, $\sigma_2^\vee$, and $\tau^\vee$. Determine the semigroups obtained by intersecting with $\mathbb{Z}^2$. What is their isomorphy type understood as abstract semigroups?

c) Describe the associated semigroup rings by generators and relations.

d) Describe the ring homomorphisms among these rings and express what this means for the associated affine varieties $X_i = \mathbb{TV}(\sigma_i)$ and $U = \mathbb{TV}(\tau)$.

e) Show that there exist (natural) morphisms of schemes $X_i \to \mathbb{A}^2 = \mathbb{C}^2$ which coincide on $U$.

*Solution:* (b) The dualization yields

$$\sigma_1^\vee = \langle [1,-1],\ [0,1] \rangle, \quad \sigma_2^\vee = \langle [-1,1],\ [1,0] \rangle, \quad \tau^\vee = \{[a,b] \in \mathbb{R}^2 \mid a+b \geq 0\}.$$

The semigroup $\sigma_1^\vee \cap \mathbb{Z}^2$, for instance, is isomorphic to $\mathbb{N}^2$: It is freely generated by two elements. Namely, by $[1,-1]$ and $[0,1]$. The semigroup $\tau^\vee \cap \mathbb{Z}^2$ is isomorphic to $\mathbb{N} \times \mathbb{Z}$.

(c) We denote $x := \chi^{[1,0]}$, $y := \chi^{[0,1]}$, $s := \chi^{[-1,1]}$, $t := \chi^{[1,-1]}$. Then,

$$\mathbb{C}[\sigma_1^\vee \cap \mathbb{Z}^2] = \mathbb{C}[t,y], \quad \mathbb{C}[\sigma_2^\vee \cap \mathbb{Z}^2] = \mathbb{C}[s,x], \quad \mathbb{C}[\tau^\vee \cap \mathbb{Z}^2] = \mathbb{C}[s,t,x]/(st-1),$$

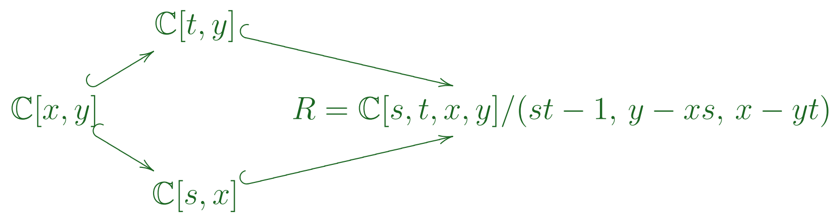
For $\mathbb{C}[\tau^\vee \cap \mathbb{Z}^2]$ we could have taken $\mathbb{C}[s,t,y]/(st-1)$ as well – or, alternatively and more symmetric, $R := \mathbb{C}[s,t,x,y]/(st-1,\ y-xs,\ x-yt)$. Since $s,t$ are mutually inverse units, the equations $y = xs$ and $x = yt$ are equivalent to each other. One can see that the embedding $\mathbb{C}[t,y] \hookrightarrow R$ coincides (is isomorphic to) the embedding $\mathbb{C}[t,y] \hookrightarrow \mathbb{C}[t,y]_t$. And similarily with $\mathbb{C}[s,x]_s = R$.

(d) We obtain the following maps which are naturally given by the names of the variables:

$$\begin{array}{ccc} \sigma_1^\vee \cap \mathbb{Z}^2 & & \\ & \searrow & \\ & & \tau^\vee \cap \mathbb{Z}^2 \\ & \nearrow & \\ \sigma_2^\vee \cap \mathbb{Z}^2 & & \end{array} \qquad \begin{array}{ccc} \mathbb{C}[t,y] & & \\ & \searrow & \\ & & R = \mathbb{C}[s,t,x,y]/(st-1,\ y-xs,\ x-yt) \\ & \nearrow & \\ \mathbb{C}[s,x] & & \end{array}$$

That is, $X_1$ and $X_2$ are isomorphic to $\mathbb{C}^2$ with coordinates $(t,y)$ and $(s,x)$, respectively. Their open subsets $D(s)$ and $D(t)$ are identified with each other via $t = 1/s$ and $y = xs$.

(e) We know that $x = yt \in \mathbb{C}[t,y]$ and $y = xs \in \mathbb{C}[s,x]$. Thus, both rings contain the polynomial ring $\mathbb{C}[x, y]$ yielding a commutative diagram of $\mathbb{C}$-algebras




To become familiar with the Spec notation, let us repeat Problem 13 from Algebra I. I recommend *not* using the published solution if you still have them somewhere in your notes.

**Problem 68.** Let $\varphi : A \to B$ be a ring homomorphism. Show that

a) the associated $(f = \operatorname{Spec}\varphi) : \operatorname{Spec} B \to \operatorname{Spec} A$ (defined via $f : Q \mapsto \varphi^{-1}Q$) is continous. That is, with respect to the Zariski topology on both sides, show that the pre-images of closed subsets are closed.

b) Give an alternative proof of (a) by showing that the pre-image of the so-called elementary open subsets $D(f) \subseteq \operatorname{Spec} A$ (for $f \in A$) are open in $\operatorname{Spec} B$. Why does it suffice to consider these special open subsets instead of all ones?

c) Recall that, for every $P \in \operatorname{Spec} A$, we denote by $K(P) := \operatorname{Quot}(A/P)$ the associated residue field of $P$. Show that $\varphi$ and $f$ from (a) provide a natural embedding $\overline{\varphi} : K\big(f(Q)\big) \hookrightarrow K(Q)$ for each $Q \in \operatorname{Spec} B$.

d) Recall that elements $a \in A$ can be understood as functions on $\operatorname{Spec} A$ via assigning each $P$ its residue class $\overline{a} \in K(P)$. Show that, in this context, the map $\varphi : A \to B$ can be understood as the pull back map (along $f$) for functions, i.e. that, under use of (c), $\varphi(a) \mathrel{\widehat{=}} a \circ f$.
(A maybe confusing remark: Making the last correspondence more explicit – but maybe less user friendly – one is tempted to write $\varphi(a) = \overline{\varphi} \circ a \circ f$. However, this is even less correct, since there is no "general map" $\overline{\varphi}$; even the domain and the target of $\overline{\varphi}$ depend on $Q$.)

*Solution:* (a) If $J \subseteq A$, then $Q \in f^{-1}(V(J)) \Leftrightarrow f(Q) \in V(J) \Leftrightarrow \varphi^{-1}(Q) \supseteq J \Leftrightarrow Q \supseteq \varphi(J) \Leftrightarrow Q \supseteq \varphi(J) \cdot B$. Thus, $f^{-1}(V(J)) = V(\varphi(J) \cdot B)$.

(b) If $a \in A$, then $Q \in f^{-1}(D(a)) \Leftrightarrow f(Q) \in D(a) \Leftrightarrow a \notin \varphi^{-1}(Q) \Leftrightarrow \varphi(a) \notin Q$. Thus, $f^{-1}(D(a)) = D(\varphi(a))$. Checking these special "elementary" open subsets suffices since every open subset is a union of those. Moreover, the operator "$\cup$" is compatible with $f^{-1}$.

(c) $K(Q) = \operatorname{Quot} B/Q$ and $K(f(Q)) = \operatorname{Quot} A/\varphi^{-1}(Q)$. Hence, the inclusion $\overline{\varphi} : A/\varphi^{-1}(Q) \hookrightarrow B/Q$ induces an inclusion $\overline{\varphi} : K(f(Q)) \hookrightarrow K(Q)$ among their respective quotient fields.

(d) We have to compare two functions on $\operatorname{Spec} B$. Accordingly, we take an element $Q \in \operatorname{Spec} B$, i.e. a prime ideal $Q \subseteq B$.

Now, $\big(\varphi(a)\big)(Q)$ was defined as the residue class $\overline{\varphi(a)}$ of $\varphi(a) \in B$ in $B/Q \subseteq \operatorname{Quot}(B/Q) = K(Q)$.
On the other hand, $(a \circ f)(Q) = a(f(Q)) = a(\varphi^{-1}(Q)$. And this equals the residue class $\overline{a}$ of $a \in A$ in $A/\varphi^{-1}(Q) = K(f(Q))$.

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# 4. Aufgabenblatt zum 15.5.2023

**Problem 69.** Let $k$ be a field, and let $P_1, \ldots, P_5 \in \mathbb{P}^2_k$ be five points with no three of them being on a common line. Show that there is then exactly one conic in $\mathbb{P}^2$ containing these points, i.e. there is (up to a constant factor) exactly one homogenous polynomial $Q(z_0, z_1, z_2)$ of degree 2 such that $P_1, \ldots, P_5 \in V(Q)$.

(*Hint:* First, use linear coordinate changes to assume that all points are in the affine chart $\mathbb{A}^2_k \subseteq \mathbb{P}^2_k$ and, moreover, that $P_3 = (0,0)$, $P_4 = (1,0)$, $P_5 = (0,1)$ in affine coordinates. Then, we may deal with $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$.)

*Solution:* We have to show that

$$\operatorname{rank}\begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & y_1^2 & x_1y_1 \\ 1 & x_2 & y_2 & x_2^2 & y_2^2 & x_2y_2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} = 5, \quad \text{i.e., via Gauß elimination,} \quad \operatorname{rank}\begin{pmatrix} x_1(x_1-1) & y_1(y_1-1) & x_1y_1 \\ x_2(x_2-1) & y_2(y_2-1) & x_2y_2 \end{pmatrix} = 2.$$

However, with $x_i, y_i \neq 0$, the condition "rank $\leq 1$" implies $[x_1 = x_2,\ y_1 = y_2]$ or $[x_1 + y_1 = x_2 + y_2 = 1]$, i.e. $P_1 = P_2$ or $P_1, P_2 \in$ line $V(x + y - 1) \subseteq \mathbb{A}^2$.

**Problem 70.** a) Let $E = V(y^2 - x^3 + x) \subseteq \mathbb{A}^2$, and denote by $\overline{E} \subseteq \mathbb{P}^2$ the projective closure obtained by homogenizing the equation. Show that $\overline{E} \setminus E$ consist of a single point $P$.

b) Denote by $(\mathbb{A}^2)' \subseteq \mathbb{P}^2$ one of the standard charts containing $P$. Describe the affine coordinate ring of $\overline{E} \cap (\mathbb{A}^2)'$.

*Solution:* (a) $\overline{E} = V(y^2z - x^3 + xz^2) \Rightarrow P = (0:1:0)$. This is the only point in $\overline{E}$ with $z = 0$.

(b) In the affine $(x,z)$-coordinates (with $y = 1$) $\overline{E}$ is given by the affine equation $z = x^3 - xz^2$, and $P$ corresponds to the origin. Thus, the affine coordinate ring of this chart equals $k[x,z]/(x^3 - xz^2 - z)$.

**Problem 71.** a) The map $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ sending $(c_0, \ldots, c_n) \mapsto (c_0 : \ldots : c_n)$ looks locally like $\pi^{-1}\big(D_+(z_i)\big) = D(z_i) \to D_+(z_i)$. Within the Spec language, this could be understood as

$$k[\mathbf{z}]_{(z_i)} \hookrightarrow k[\mathbf{z}]_{z_i} = k[\mathbf{z}]_{(z_i)}[z_i, z_i^{-1}] = k[\mathbf{z}]_{(z_i)} \otimes_k k[z_i, z_i^{-1}].$$

Geometrically, this means that $D(z_i) \cong D_+(z_i) \times k^*$. Show this directly at the level of (closed) points.

b) Let $v_2 : \mathbb{P}^1 \to \mathbb{P}^2$ be the Veronese embedding $v_2 : (z_0 : z_1) \mapsto (w_0 : w_1 : w_2) := (z_0^2 : z_0z_1 : z_1^2)$. Describe the ring homomorphism corresponding to the restriction $v_2|_{D_+(z_0)} : D_+(z_0) \to D_+(w_0)$.

*Solution:* (a) We just assume that $i = 0$ for a better layout. Then,

$$\begin{aligned} D_+(z_0) &= \{(c_0 : c_1 : \ldots : c_n) \mid c_0 \neq 0\} &= \{(1 : c'_1 : \ldots : c'_n) \mid c'_i \in k\} \\ & &= \{(c'_1, \ldots, c'_n) \mid c'_i \in k\} \end{aligned}$$

with $c_i = c_0 \cdot c'_i$. On the other hand,

$$D(z_0) = \{(c_0, c_1, \ldots, c_n) \mid c_0 \neq 0\} = \{c_0 \cdot (1, c'_1, \ldots, c'_n) \mid c'_i \in k, c_0 \in k^*\}.$$

(b) $k[\frac{w_1}{w_0}, \frac{w_2}{w_0}] \to k[\frac{z_1}{z_0}]$, $\frac{w_1}{w_0} \mapsto \frac{z_1}{z_0}$, $\frac{w_2}{w_0} \mapsto (\frac{z_1}{z_0})^2$.

**Problem 72.** a) Let $J \subseteq k[\mathbf{z}] := k[z_0, \ldots, z_n]$ be an ideal. Show that $(J : (\mathbf{z})^\infty)$ is the largest ideal $J'$ containing $J$ but still satisfying $J'_{z_i} = J_{z_i}$ for all $i = 0, \ldots, n$.

b) Let $J \subseteq k[\mathbf{z}] := k[z_0, \ldots, z_n]$ be a *homogeneous* ideal. Show that $(J : (\mathbf{z})^\infty)$ is the largest homogeneous ideal $J'$ containing $J$ but still satisfying $J'_{(z_i)} = J_{(z_i)}$ for all $i = 0, \ldots, n$ where these expressions mean the homogeneous localizations.

*Solution:* (a) For a given $J \subseteq k[\mathbf{z}]$ we know that $\{f \in k[\mathbf{z}] \mid f/1 \in J_{z_i}\} = (J : (z_i)^\infty)$. On the other hand, we know that the left hand side is the largest extension of $J$ which does not change the localization in $z_i$.
Afterwards, one should use that $\bigcap_{i=0}^n (J : (z_i)^\infty) = (J : (\mathbf{z})^\infty)$.

(b) If $J$ is homogeneous, then $(J : (\mathbf{z})^\infty)$ is homogeneous, too. Moreover, if $J' \supseteq J$ is another homogeneous ideal, then we have $J'_{(z_i)} = J_{(z_i)}$ if and only if $J'_{z_i} = J_{z_i}$. While ($\Leftarrow$) is clear (homogeneous localizations are just the subsets of degree 0), the implication ($\Rightarrow$) can be obtained as follows:
If $f/z_i^k \in J'_{z_i}$ with $f \in J'$, then we decompose $f = \sum_d f_d$ by its degrees. In particular, $f_d \in J'$ for all $d$. Hence, $f_d/z_i^d \in J'_{(z_i)} = J_{(z_i)} \subseteq J_{z_i}$. Thus, $f_d/1 \in J_{z_i}$, i.e., $f/1$ and $f/z_i^k$ belong to $J_{z_i}$, too.

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## 5. Aufgabenblatt zum 22.5.2023

**Problem 73.** For an ideal $I \subseteq k[\mathbf{x}]$ with $\mathbf{x} = (x_1, \ldots, x_n)$ denote by $I^h := (f^h \mid f \in I) \subseteq k[\mathbf{z}]$ with $\mathbf{z} = (z_0, \ldots, z_n)$ and $x_i = z_i/z_0$ its homogenization. On the contrary, for a homogeneous ideal $J \subseteq k[\mathbf{z}]$ we denote by $J^0 \subseteq k[\mathbf{x}]$ its dehomogenization obtained by $z_0 \mapsto 1$ and $z_i \mapsto x_i$ for $i \geq 1$. It equals the homogenous localization $J_{(z_0)}$. Eventually, we denote by $V_{\mathbb{P}}(J) \subseteq \mathbb{P}^n$ and $V_{\mathbb{A}}(I) \subseteq \mathbb{A}^n = D_+(z_0) \subset \mathbb{P}^n$ the respective vanishing loci.

a) Recall that $V_{\mathbb{A}}(J^0) = V_{\mathbb{P}}(J) \cap D_+(z_0)$ inside $\mathbb{A}^n = D_+(z_0)$. Assume that $k = \overline{k}$, and use the Hilbert Nullstellensatz to show that then $V_{\mathbb{P}}(I^h) = \overline{V_{\mathbb{A}}(I)}$ inside $\mathbb{P}^n_k$.

b) Show by presenting a suitable example that the equality of (a) fails for $k = \mathbb{R}$.

c) In Subsection (11.2) we had considered $\mathbb{A}' := \mathbb{A}^{n+1}$ instead of $\mathbb{P} := \mathbb{P}^n$. In particular, we denote $V_{\mathbb{A}'}(J) \subseteq \mathbb{A}'$ for the affine subsets induced by homogeneous ideals $J \subseteq k[\mathbf{z}]$. Comparing both situations via $\pi : \mathbb{A}' \setminus 0 \to \mathbb{P}$ we have now open subsets $D(z_0) \subset \mathbb{A}'$ and $D_+(z_0) \subset \mathbb{P}$ with $D(z_0) = \pi^{-1}\big(D_+(z_0)\big)$, see Problem 71.
We have seen in Subsection (16.6) that $V_{\mathbb{A}'}(J) \cap D(z_0) = \pi^{-1}\big(V_{\mathbb{A}}(J^0)\big)$ with $V_{\mathbb{A}}(J^0) \subseteq \mathbb{A} = D_+(z_0) \subset \mathbb{P}$. Or, with other symbols, and $V_{\mathbb{A}'}(I^h) \cap D(z_0) = \pi^{-1}\big(V_{\mathbb{A}}(I)\big)$. Using this, we have got in Subsection (11.2) that $V_{\mathbb{A}'}(I^h) = \overline{V_{\mathbb{A}'}(I^h) \cap D(z_0)}$ inside $\mathbb{A}'$. Now, use this to derive $V_{\mathbb{P}}(I^h) = \overline{V_{\mathbb{P}}(I^h) \cap D_+(z_0)}$ inside $\mathbb{P}$.

*Solution:* (a) Let $g(\mathbf{z}) = 0$ on $V_{\mathbb{A}}(I) \subseteq D_+(z_0) \subseteq \mathbb{P}^n$. Then, on the one hand, $g^0(\mathbf{x}) = 0$ on $V_{\mathbb{A}}(I) \subseteq \mathring{A}^n$, i.e. there is an $N$ with $(g^0)^N \in I$, hence $((g^0)^h)^N = ((g^0)^N)^h \in I^h$. On the other, $g = z_0^e \cdot (g^0)^h$, thus $g^N = z_0^{eN} \cdot ((g^0)^h)^N \in I^h$. So it follows that $g(\mathbf{z}) = 0$ on $V_{\mathbb{P}}(I^h)$, too.
(b) For $k = \mathbb{R}$ consider the example $I = (x_1^2 + (x_2 - x_3)^2 + 1)$. While $V_{\mathbb{A}}(I) = \emptyset$, we have $I^h = (z_1^2 + (z_2 - z_3)^2 + z_0^2)$ with $(0:0:1:1) \in V_{\mathbb{P}}(I^h)$.
(c) We have to show that there is no homogeneous ideal $J \subseteq k[\mathbf{z}]$ such that

$$V_{\mathbb{P}}(I^h) \cap D_+(z_0) \subseteq V_{\mathbb{P}}(J) \subsetneq V_{\mathbb{P}}(I^h).$$

However, from this we may apply $\pi^{-1}$ (and add $0 \in \mathbb{A}'$ at the last to gadgets) to obtain

$$V_{\mathbb{A}'}(I^h) \cap D(z_0) \subseteq V_{\mathbb{A}'}(J) \subsetneq V_{\mathbb{A}'}(I^h).$$

And this is a contradiction to the statements before.

**Problem 74.** a) Let $H$ denote the hexagon with the vertices $v_1 = [0,0]$, $v_2 = [1,0]$, $v_3 = [2,1]$, $v_4 = [2,2]$, $v_5 = [1,2]$, $v_6 = [0,1]$. Describe the corresponding embedding $\mathbb{P}(H) \hookrightarrow \mathbb{P}^6$ by giving some homogeneous equations by hand and, afterwards, “all” homogeneous equations by using Singular or Macauly2.

b) If $\Delta_1, \Delta_2 \subseteq M_{\mathbb{Q}}$ are lattice polyhedra, then we define their *Minkowski sum* as $\Delta_1 + \Delta_2 := \{a + b \mid a \in \Delta_1,\, b \in \Delta_2\}$. Show that this is again a lattice polyhedron and that its vertices are sums of the vertices of $\Delta_1$ and $\Delta_2$, respectively. Does every such sum provide a vertex of $\Delta_1 + \Delta_2$?

c) Calculate the Minkowski sum $\Delta$ of the triangles $\Delta_1 = \operatorname{conv}\{[0,0];\ [1,0];\ [1,1]\}$ and $\Delta_2 = \operatorname{conv}\{[0,0];\ [1,1];\ [0,1]\}$. Can you find a Minkowski decomposition of $\Delta$ into one-dimensional summands?

d) Show that there is always a regular map $\mathbb{P}(\Delta_1 + \Delta_2) \to \mathbb{P}(\Delta_1 \times \Delta_2)$. Describe this map explicitely for $\Delta_1 = \Delta_2 = [0,1] \subseteq \mathbb{Q}^1$.

e) Show how the two-dimensional $\mathbb{P}(H)$ of part (a) becomes a closed subset of both $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Describe its equations in both instances.

*Solution:* (a) There is the additional lattice point $v_0 = [1,1]$. If $z_i$ denotes the homogeneous coordinate associate to $v_i$, then, e.g., $z_1 z_0 - z_2 z_6$ or $z_1 z_4 - z_0^2$ are equations.
(b) $\Delta = H$ from (a). In the decomposition (c) $[0,0] + [1,1] = [1,1]$ is not a vertex.
(c) $H = \operatorname{conv}\{[0,0];\ [1,0]\} + \operatorname{conv}\{[0,0];\ [0,1]\} + \operatorname{conv}\{[0,0];\ [1,1]\}$.
(d) $\mathbf{z} \mapsto \mathbf{w}$ with $w_{a,b} := z_{a+b}$. If all $\mathbf{w}$-coordinates were vanishing, then all $\mathbf{z}$-coordinates vanish, too. For the example we use first the Veronese and Segre embedding: $\mathbb{P}^1 = \nu_2(\mathbb{P}^1) \subseteq \mathbb{P}^2 \to \mathbb{P}^3$ $(z_0 : z_1) \mapsto (z_0^2 : z_0 z_1 : z_1^2)$ and $(w_0 : w_1 : w_2) \mapsto (w_0 : w_1 : w_1 : w_2)$, hence altogether $(z_0 : z_1) \mapsto (z_0^2 : z_0 z_1 : z_0 z_1 : z_1^2)$. Alternatively, this could be understood as the diagonal map $\mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$.
(e) Denote the coordinates of the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ by $(w_0 : \ldots : w_8) = (x_0 y_0 : x_0 y_1 : \ldots : x_2 y_1 : x_2 y_2)$. Hence, $(z_0 : \ldots : z_6) \mapsto (z_1 : z_0 : z_6 : z_2 : z_3 : z_0 : z_0 : z_4 : z_5)$. The equations are $w_1 = w_5 = w_6$, i.e., $x_0 y_1 = x_1 y_2 = x_2 y_0$. They are bilinear, i.e., linear in the $\mathbb{P}^2$-arguments $(x_0, x_1, x_2)$ and in $(y_0, y_1, y_2)$.
Similarly, for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, we get just a single equation like $x_0 y_0 z_1 = x_1 y_1 z_0$ which is again multilinear – namely linear in all of the three $\mathbb{P}^1$-arguments $(x_0, x_1)$, $(y_0, y_1)$, and $(z_0, z_1)$.

**Problem 75.** Let $\Delta \subseteq M_{\mathbb{R}}$ be a lattice polytope and $\Sigma = \mathcal{N}(\Delta)$ the associated (inner) normal fan in the dual space $N_{\mathbb{R}}$. Denote by $n : \mathbb{TV}(\Sigma) \to \mathbb{P}(\Delta) \subseteq \mathbb{P}^{\#(\Delta \cap M)-1}$ the map being glued from the following local pieces:

For each $\sigma \in \Sigma$ there is some (maybe non-unique) $w = w(\sigma) \in \Delta \cap M$ such that $\min\langle \Delta, a\rangle = \langle w, a\rangle$ for all $a \in \sigma$. On the other hand, $w \in \Delta \cap M$ gives rise to a homogeneous coordinate $z_w$ of $\mathbb{P}^{\#(\Delta \cap M)-1}$; denote $U_w := D_+(z_w)$. Now, there is an inclusion $(\Delta - w) \subseteq \sigma^\vee$.

a) Check this inclusion.

b) Derive a morphism between the affine varieties $p_\sigma : \mathbb{TV}(\sigma) \to U_w \cap \mathbb{P}(\Delta)$ with $w = w(\sigma)$.

c) Under which conditions do we have $\mathbb{R}_{\geq 0} \cdot (\Delta - w) = \sigma^\vee$? And, if both cones are different, what is the true relation between them (or, between their respective duals)?
Maybe, instead of checking this formally, it would be more helpful to explain and digest this via pictures and examples. At least as a first step.

d) Show that these local maps glue, i.e., that for faces $\tau \leq \sigma$ the restiction $p_{\sigma|\mathbb{TV}(\tau)}$ equals $p_\tau$ when considered as maps towards $\mathbb{P}(\Delta)$.

e) Assuming equality in (c), show that $p_\sigma : \mathbb{TV}(\sigma) \to U_w \cap \mathbb{P}(\Delta)$ is an isomorphism after replacing $\Delta$ by some dilation $\Delta_N := N \cdot \Delta$ with $N \gg 0$. Can you give an exact condition for the minimal possible $\Delta_N$?

*Solution:* (a) If $a \in \sigma$, then $\langle \Delta, a \rangle \geq \langle w, a \rangle$, i.e., $\langle \Delta - w, a \rangle \geq 0$. Thus, $\Delta - w \subseteq \sigma^\vee$.

(b) The affine coordinate ring of $U_w$ is the homogenous localization of $k[\Delta] := k[\mathbb{N} \cdot (\Delta \cap M, 1)]$ by $z_w = \chi^{[w,1]}$. This ring equals $k[(\Delta \cap M) - w] \subseteq \sigma^\vee \cap M]$.

(c) The elements of $\sigma$ attain their minimum on $\Delta$ in a face $F(\sigma) \leq \Delta$. Then

$$\sigma^\vee = \mathbb{R}_{\geq 0} \cdot (\Delta - F),$$

and the latter equals $\mathbb{R}_{\geq 0} \cdot (\Delta - w)$ if and only if $w \in \operatorname{int} F$ is a point of the relative interior of $F$. In general, $\mathbb{R}_{\geq 0} \cdot (\Delta - F)^\vee$ is just a face of $\mathbb{R}_{\geq 0} \cdot (\Delta - w)^\vee$.

(e) $p_\sigma$ is an isomorphism for all $\sigma$ iff for all vertices $w \in \Delta_N$ (that is, the corresponding $\sigma$ are full-dimensional cones) the set $\Delta_N - w$ contains the Hilbert basis of the cone $\mathbb{R}_{\geq 0} \cdot (\Delta_N - w)$. And, actually, this cone does not depend on $N$ at all. Only the finite set $\Delta_N - w$ does.

**Problem 76.** A lattice polyhedron $\Delta \subseteq M_\mathbb{Q}$ is called *normal* if $d\Delta \cap M = d(\Delta \cap M)$ where the latter means the set of all sums obtained by exactly $d$ summands from $\Delta \cap M$.

a) Show that $\nabla := \operatorname{conv}\{[000], [100], [010], [112], [113]\}$ is not normal – namely, $[d-1, 1, 1] \in dP \cap M$, but not in $d(P \cap M)$ for any $d \geq 2$.

b) Let $\Delta \subseteq M_\mathbb{Q}$ be a lattice polytope and $d \in \mathbb{N}$. Show that the homogeneous coordinates $z_v$ of $\mathbb{P}(\Delta)$ corresponding to a vertex $v \in \Delta$ cannot simultaneously vanish. Use this to construct the natural map $\varphi : \mathbb{P}(d\Delta) \to \mathbb{P}(\Delta)$.

c) If $\Delta$ is additionally normal, then define the $d$-th Veronese map $\nu_d : \mathbb{P}(\Delta) \to \mathbb{P}(d\Delta)$ and check that it is the inverse of $\varphi$ of (b).

d) Show that (two-dimensional) lattice polygons are always normal.

*Solution:* (a) $[d-1, 1, 1] = \frac{1}{2}\,[1, 1, 2] + (d - \frac{3}{2})\,[1.0.0] + \frac{1}{2}\,[0, 1, 0] + \frac{1}{2}\,[0, 0, 0]$. Actually, the point $[1, 1, 3]$ was not needed at all – it is just there to ensure that the set $(M \cap \nabla)$ spans the lattice $M = \mathbb{Z}^3$, which makes the example nicer.

(b) Let $w \in \Delta \cap M$ be an arbitrary lattice point inside $\Delta$. Then, there is a non-negative, rational linear combination $w = \sum \lambda_v v$ with $\sum_v = 1$ such that the sum is taken over all vertices of $\Delta$. Multiplying with the common denominator, we obtain $k \cdot w = \sum_v k_v \cdot v$ with $k, k_v \in \mathbb{N}$. This translates into the equation $z_w^k - \prod_v z_v^{k_v}$ for the variety $\mathbb{P}(\Delta)$. Thus, if all "vertex coordinates" $z_v$ vanish, then $z_w$ will vanish, too.

Now, on the open subset $U_v \subseteq \mathbb{P}(d\Delta)$ defined by $z_{dv} \neq 0$, we may define $\mathbb{P}(d\Delta) \supseteq U_v \stackrel{\varphi}{\to} \mathbb{P}(\Delta)$ via $\varphi(c)_w := c_{(d-1)v+w}$ for any $w \in \Delta \cap M$. This guarantees that $\varphi(c)_v \neq 0$.

(c) We simply define $\nu_d : \mathbb{P}(\Delta) \to \mathbb{P}(d\Delta)$ by $\nu_d(c)_w := c_{v_1} \cdot \ldots \cdot c_{v_d}$ where $w \in d\Delta \cap M$ and $v_i \in \Delta \cap M$ with $\sum_{i=1}^d v_i = w$. The existence of those lattice points $v_i$ follows from the normality of $\Delta$, and the independence of $\nu_d$ on their choice is a consequence of the equations of $\mathbb{P}(\Delta)$: Any two choices leed to an affine relation among the points of $\Delta \cap M$. Finally, one checks that the image satisfies the equations of $\mathbb{P}(d\Delta)$.

(d) Polygons $\Delta$ can be decomposed into elementary triangles. And those are normal.

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## 6. Aufgabenblatt zum 29.5.2023

**Problem 77.** Show that the local blowing up map $\varphi : k[x,y] \hookrightarrow k[x, \frac{y}{x}]$ is not flat – do this by calculating all modules $\mathrm{Tor}_i^{k[x,y]}(k[x,\frac{y}{x}], k)$ with $k = k[x,y]/(x,y)$. What is the geometric meaning of $\mathrm{Tor}_0^{k[x,y]}(k[x,\frac{y}{x}], k) = k[x,\frac{y}{x}] \otimes_{k[x,y]} k = k[x,\frac{y}{x}]/(x,y) = k[\frac{y}{x}]$? Likewise, with $t = \frac{y}{x}$, the map $\varphi$ can be denoted by $k[x,xt] \hookrightarrow k[x,t]$.

Having done this – can you find now an injection $M \hookrightarrow N$ of $k[x,y]$-modules that does not stay injective after being tensorized with $k[x,\frac{y}{x}]$?

*Solution:* Resolving $k$, the Koszul complex $0 \to k[x,xt] \to k[x,xt]^2 \to k[x,xt] \to 0$ involves the maps $(xt,-x)^T$ and $(x,xt)$. Tensorizing with $k[x,t]$ changes the complex into $0 \to k[x,t] \to k[x,t]^2 \to k[x,t] \to 0$, but keeps the maps. Thus, $\mathrm{Tor}_1 = \mathrm{Tor}_0 = k[t]$. The latter equality reflects the open, affine piece of the exceptional divisor $E = \mathbb{P}^1$ (since the tensor product with $k$ calculates the fiber over 0).

Just use $M := (x,y)$ and $N := k[x,y]$ – then one has the exact sequence $0 \to M \to N \to k \to 0$.

**Problem 78.** a) Let $\ell \subset \mathbb{A}^2_k$ be a line through the origin. Describe both, the total and the strict transform $\pi^{-1}(\ell)$ and $\pi^{\#}(\ell)$ inside the blowing up $\widetilde{\mathbb{A}}^2_k$. Try both the ("naive") geometric description and the algebraic one via the covering with affine charts.

b) Use the example of blowing up the origin $f = \pi : \widetilde{\mathbb{A}}^2_k \to \mathbb{A}^2_k$ to show that it might happen that $\overline{f^{-1}(y)} \neq f^{-1}(\overline{y})$ (for points $y \in \mathbb{A}^2_k$). Is, however, one of the two sides always contained in the other?

*Solution:* (a) $\pi^{-1}(\ell \setminus 0) = \{\ell\} \times (\ell \setminus 0) \subset \mathbb{P}^1 \times (\mathbb{A}^2 \setminus 0)$. Thus $\pi^{\#}(\ell) = \overline{\pi^{-1}(\ell \setminus 0)} = \{\ell\} \times \ell \subset \widetilde{\mathbb{A}}^2_k \subset \mathbb{P}^1 \times \mathbb{A}^2$. On the other hand, $\pi^{-1}(\ell) = (\{\ell\} \times \ell) \cup (\mathbb{P}^1 \times \{0\})$ where the latter component equals $E$.

*Algebraic description*: The restriction of $\pi$ to one chart corresponds to $k[x,y] \hookrightarrow k[x,t]$ with $t = y/x$. The equation of $\ell$ is some linear form $\ell(x,y) = \ell_1 x + \ell_2 y$ where not both $\ell_1, \ell_2$ vanish. Now, the total transform is given by $\ell(x,tx) = \ell_1 x + \ell_2 tx = x \cdot (\ell_1 + \ell_2 t)$. Here, $V(x)$ equals $E$ (or, better, the part from $E$ being contained in our chart, that is, one point of $E = \mathbb{P}^1$ is missing), and $V(\ell_1 + \ell_2 t)$ is the strict transform (within our chart).

It is interesting to observe that, restricted to our chart, $[x = 0]$ equals $E$, but $\pi^{\#}(\ell)$ equals $[t = -\ell_1/\ell_2 \text{constant}]$. That is, both lines are transversal to each other. However, if $\ell_2 = 0$, then the strict transform is empty (in our chart), i.e., it is entirely contained in the other chart. And exactly this happes to our chart for $\ell_1 = 0$. If $\ell_1, \ell_2 \neq 0$, then the strict transform is contained in both charts, i.e., in their intersection as well.

(b) We always have that $\overline{f^{-1}(y)} \subseteq f^{-1}(\overline{y})$. However, you might take $y := \eta_\ell$, i.e. the generic point of a line $\ell \subseteq \mathbb{A}^2_k$ containing 0. Then, $\pi^{-1}(\eta_\ell)$ is the generic point of the strict transform $\pi^\#(\ell)$ and $\overline{\pi^{-1}(\eta_\ell)}$ is the entire strict transform. On the other hand, $\overline{\eta_\ell} = \ell$, and its pre-image is the total transform $\pi^{-1}(\eta)$.

**Problem 79.** Let $I \subseteq A$ be an ideal in a ring $A$.

a) Show that $\pi : \operatorname{Proj} \oplus_{d\geq 0} I^d \to \operatorname{Spec} A$ is an isomorphism outside $V(I)$.

b) Assume that $I = (f)$ with a non-zero divisor $f \in A$. Show that the blowing up of $I$ is an isomorphism everywhere.

*Solution:* (a) We show that $\pi$ is an isomorphism outside $V(f) \supseteq V(I)$ (with $f \in I$). Since $V(I) = \bigcap_{f\in I} V(f)$, this will prove the claim.
We localize $A \to A_f$, i.e. w.l.o.g. we may assume that $f \in A^*$. But then, $I = (1)$, and $\oplus_{d\geq 0} I^d = A[t]$. On the other hand, $\operatorname{Proj} A[t] = D_+(t) = \operatorname{Spec} A[t]_{(t)} = \operatorname{Spec} A$.
Alternatively, one might look at $I = (f_1, \ldots, f_n)$. Then, the blowing up is locally given by $A[\frac{f_1}{f_i}, \ldots, \frac{f_n}{f_i}]$. However, the homomorphism $A \to A[\frac{f_1}{f_i}, \ldots, \frac{f_n}{f_i}]$ becomes an isomorphism after localizing with $f_i$.
(b) There is only one chart, namely $A \to A[\frac{f}{f}] = A$.

**Problem 80.** Let $\Sigma$ be the two-dimensional fan in $\mathbb{Q}^2$ that is spanned by the six rays

$$a^0 = (1,0),\ b^2 = (1,1),\ a^1 = (0,1),\ b^0 = (-1,0),\ a^2 = (-1,-1),\ b^1 = (0,-1),$$

i.e. it consists of six two-dimensional cones, six rays, and the origin.

a) Show that the three fans induce two different morphisms $\varphi_a : \mathbb{TV}(\Sigma) \to \mathbb{P}^2$ and $\varphi_b : \mathbb{TV}(\Sigma) \to \mathbb{P}^2$. Can you comment the relation between, e.g., $\varphi_a$ and the blowing up of $0 \in \mathbb{A}^2$?

b) Show that $\varphi_a$ is birational, i.e. it provides an isomorphism between certain non-empty, open (hence dense) subsets $U \subseteq \mathbb{TV}(\Sigma)$ and $V \subseteq \mathbb{P}^2$. Can you spot those $U, V$ (as large as possible) explicitely?

c) Describe the rational (i.e., not everywhere defined) map $(\varphi_b \circ \varphi_a^{-1}) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ by explicit coordinates. In which points is this map not defined?

*Solution:* (a) Denote by $\Sigma_a$ and $\Sigma_b$ the two-dimensional fans spanned by $\{a^0, a^1, a^2\}$ and $\{b^0, b^1, b^2\}$, i.e. arising from $\Sigma$ by removing the $b$-rays and the $a$-rays, respectively. Then, both $\Sigma_a$ and $\Sigma_b$ are fans describing the variety $\mathbb{P}^2$, and the fact that $\Sigma$ is a subdivision of these fans means that we have toric maps $\varphi_a : \mathbb{TV}(\Sigma) \to \mathbb{TV}(\Sigma_a) \xrightarrow{\sim} \mathbb{P}^2$ and $\varphi_b : \mathbb{TV}(\Sigma) \to \mathbb{TV}(\Sigma_b) \xrightarrow{\sim} \mathbb{P}^2$.
Over each of the three charts of $\mathbb{P}^2$, the morphism $\varphi_a$ is the blowing up of its origin. Thus, altogether, $\varphi_a$ is the blowing up of $\mathbb{P}^2$ in three points. Alternatively, this can be visualized as cutting the corners off the triangle representing $\mathbb{P}^2 = \mathbb{P}(\triangle)$. This yields a hexagon, and $\Sigma$ equals the normal fan of this hexagon.

(b) Denote by $\Sigma'$ the one-dimensional fan in $\mathbb{Q}^2$ that consists of the rays $\{a^0, a^1, a^2\}$ (and the origin). The toric variety $\mathbb{TV}(\Sigma')$ becomes then an open subset in both $\mathbb{TV}(\Sigma)$ and $\mathbb{TV}(\Sigma_a)$.

(c) One of the possible identifications $(\mathbb{C}^*)^2 = \mathbb{TV}(0) \subseteq \mathbb{TV}(\Sigma_a) \overset{\sim}{\to} \mathbb{P}^2$ mentioned in (a) works as $(x, y) \mapsto (1 : x : y)$. Doing the same with $\Sigma_b$, we obtain $(x, y) \mapsto (1 : \frac{1}{x} : \frac{1}{y})$. The composition yields $(1 : x : y) \mapsto (1 : \frac{1}{x} : \frac{1}{y})$. In homogenous coordinates, this yields the so-called CREMONA transformation $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $(x : y : z) \mapsto (\frac{1}{x} : \frac{1}{y} : \frac{1}{z}) = (yz : zx : xy)$. It is undefined in the three points $(1 : 0 : 0)$, $(0 : 1 : 0)$, and $(0 : 0 : 1)$.

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## 7. Aufgabenblatt zum 5.6.2023

**Problem 81.** Let $I \subseteq A$ be an ideal in a ring $A$. Then, we denote by

$$\widetilde{X} := \operatorname{Proj} \oplus_{d\geq 0} I^d \cdot t^d \ \stackrel{\pi}{\to}\ \operatorname{Spec} A =: X$$

the blowing up of $X$ in in $Z := V(I) = \operatorname{Spec} A/I$, cf. Problem 79. If $J \subseteq I$, then $Y := \operatorname{Spec} A/J$ is sandwiched between $Z$ and $X$, i.e., $Z \subseteq Y \subseteq X$. We define the strict transform of $Y$ as

$$\pi^{\#}(Y) := \overline{\pi^{-1}\big(Y\big) \setminus \pi^{-1}(Z)}$$

where $E = \pi^{-1}(Z)$ was the so-called exceptional divisor in $\widetilde{X}$. Show that $\pi^{\#}(Y)$ equals (is isomorphic) to the blowing up of $Y$ in $Z$.

*Solution:* We consider the situation via the local charts. If $I = (f_1, \ldots, f_n)$, then the $k$-th chart of $\widetilde{X}$ is $\operatorname{Spec} A[\underline{f}/f_k] \to \operatorname{Spec} A$ where $A[\underline{f}/f_k]$ should be understood as a subring of the localization $A_{f_k}$, i.e., $f_k$ turns into a non-zero divisor (being a unit in the ambient ring $A_{f_k}$).

The strict transform $\pi^{\#}(Y)$ is given by the ideal $(J : f_k^\infty) := (J : (f_k)^\infty)$ in $A[\underline{f}/f_k]$, i.e., we are supposed to show that the kernel of

$$p_k : A[\underline{f}/f_k] \to (A/J)[\underline{f}/f_k]$$

equals exactly this ideal. Moreover, one has to check that these local isomorphisms glue along the intersections of the charts, i.e., that all these homomorphisms fit with further localizations $A \to A_{f_k} \to A_{f_k f_l}$. But this is straightforward.

Obviously, the homomorphisms $p_k$ are surjective. To consider their kernels, it is convenient to switch into the level of their ambient rings, i.e., to investigate the extension of $p_k$

$$p_k : A_{f_k} \to (A/J)_{f_k} = A_{f_k}/(J \cdot A_{f_k}).$$

Here, the kernel is obvious. Thus, it remains to understand the following situation: Let $J \subseteq A$ be an ideal, $f \in A$ an element, and $B \subseteq A_f$ an $A$-subalgebra. What is $(J \cdot A_f) \cap B$? It is easy to check that

$$(J \cdot A_f) \cap B = \big((J \cdot B) : f^\infty\big).$$

**Problem 82.** Recall Problems 31, 32, 35 from Algebra I; they deal with the notion of direct limits. You can find them with their proposed solutions earlier in this text. I have, additionally, added them (without solutions) at the end of this sheet.

**Problem 83.** Let $\mathcal{O}$ be a presheaf of rings, let $\mathcal{F}, \mathcal{G}$ be presheaves of abelian groups or, for the second problem, of $\mathcal{O}$-modules on a topological space $X$. Let $p \in X$. Show that there are natural isomorphisms $\varphi : (\mathcal{F} \oplus \mathcal{G})_p \stackrel{\sim}{\to} \mathcal{F}_p \oplus \mathcal{G}_p$ and $\psi : (\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})_p \stackrel{\sim}{\to} \mathcal{F}_p \otimes_{\mathcal{O}_p} \mathcal{G}_p$.

*Solution:* The canonical map $\psi : (\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})_p \to \mathcal{F}_p \otimes_{\mathcal{O},p} \mathcal{G}_p$ is clearly surjective. Moreover, if $\psi(\sum_i a_p^i \otimes b_p^i) = 0$ in $\mathcal{F}_p \otimes_{\mathcal{O},p} \mathcal{G}_p$, then this is witnessed by a finite sum of bilinearity relations. All participating elements can then be lifted on a common open neighborhood $U \ni p$.

**Problem 84.** Let $f : \mathcal{F} \to \mathcal{G}$ be a map of sheaves of abelian groups. Show that the map $f$ is

a) zero (i.e. $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is zero for all open $U \subseteq X$), or

b) injective (i.e. $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open $U \subseteq X$), or

c) an isomorphism

if and only if for all $p \in X$ the corresponding maps $f_p : \mathcal{F}_p \to \mathcal{G}_p$ are zero, injective, or an isomorphism, respectively.

*Solution:* (c) Assuming isomorphisms $f_p : \mathcal{F}_p \to \mathcal{G}_p$, it remains to show that $\mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for each open $U \subseteq X$. Let $t \in \mathcal{G}(U)$. For every $p \in X$ there is an $s_p \overset{f}{\mapsto} t_p$. Choose a neighborhood $p \in U(p) \subseteq U$ such that there is a section $s(p) \in \Gamma(U(p), \mathcal{F})$ with $s(p)_p = s_p$. In particular, $f(s(p)) \in \Gamma(U(p), \mathcal{G})$ with $f(s(p))_p = t_p$. Thus, there is a, maybe smaller, $p \in V(p) \subseteq U(p)$ such that $f(s(p)) = t$ after being restricted to $V(p)$, i.e. such that $f(s(p)|_{V(p)}) = t|_{V(p)}$. In particular, locally on a (the) open covering $\{V(p)\}$ of $U$, we have found pre-images of $t$.
Now, since $f$ is injective by (b), all pre-images of $t$ (if they exist) are uniquely determined on each level, i.e. on each open subset of $U$. In particular, $s(p)|_{V(p)\cap V(q)} = s(q)|_{V(p)\cap V(q)}$, since both sides are pre-images of $t|_{V(p)\cap V(q)}$. Thus, they glue, i.e. there is an $s \in \Gamma(U, \mathcal{F})$ such that $s|_{V(p)} = s(p)|_{V(p)}$. Finally, we see that $f(s) = t$, since both sides coincide when restricted to $V(p)$ for each $p \in U$.

*Here are the old problems from Algebra I dealing with direct limits:*

**Problem 31.** a) Let $I := (I, \leq)$ be a poset. It turns into a category via objects $:= I$ and $\mathrm{Hom}_I(a,b) := \begin{cases} \{(a,b)\} & \text{if } a \leq b, \\ \emptyset & \text{otherwise} \end{cases}$. A "*directed system* on $I$ with values in a category $\mathcal{C}$" is a (covariant) functor $I \to \mathcal{C}$; the "direct limit" $\varinjlim X_i$ of such a system $X = (X_i \mid i \in I)$ is defined via the following universal property: $\mathrm{Hom}_{\mathcal{C}}(\varinjlim X_i, Z) = \{\varphi \in \prod_i \mathrm{Hom}(X_i, Z) \mid i \leq j \Rightarrow \varphi_i = \varphi_j \circ [X_i \to X_j]\}$. In particular, there are canonical maps $X_j \to \varinjlim X_i$ (as the image of $\mathrm{id} \in \mathrm{Hom}_{\mathcal{C}}(\varinjlim X_i, \varinjlim X_i)$). Translate the notion of the direct limit into that of an initial object in some category.

b) What is $\varinjlim X_i$ if $I$ contains a maximum? What is $\varinjlim X_i$ if all elements of $I$ are mutually non-comparable, i.e. if $i \leq j \Leftrightarrow i = j$?

c) Let $\mathcal{C} = \mathcal{M}od_R$ be the category of modules over some ring $R$. For an element $m_j \in M_j$ we will use the same symbol $m_j$ for its canonical image in $M := \oplus_{i \in I} M_i$, too. Using this notation, show that $\varinjlim M_i = M/N$ where the submodule $N \subseteq M$ is generated by all differences $m_j - \varphi_{jk}(m_j)$ with $m_j \in M_j$, $j \leq k$, and $\varphi_{jk} : M_j \to M_k$ being the associated $R$-linear map.

d) Assume $(I, \leq)$ to be *filtered*, i.e. for $i, j \in I$ there is always a $k = k(i,j) \in I$ with $i, j \leq k$. If $\mathcal{C} = \mathcal{M}od_R$, then $\varinjlim M_i = \coprod_i M_i/\sim$, where $\coprod$ means the disjoint union (as sets) and "$\sim$" is the equivalence relation generated by $[\varphi_{ij}(m_i) \sim m_i$ for $i \leq j]$ (with $\varphi_{ij} : M_i \to M_j$). (*Hint:* First, define an $R$-module structure of the right hand side. Then check that an element $x \in M_i$ turns into $0 \in \varinjlim M_i$ if and only if there is a $j \geq i$ with $\varphi_{ij}(x) = 0 \in M_j$.)

*Solution:* (a) For a fixed directed system $X = (X_i \mid i \in I)$ which includes compatible maps $\psi_{ij} : X_i \to X_j$ for $i \leq j$, we define the category

$$\mathcal{C}^X := \big\{(Z, \varphi_i \mid i \in I) \,\big|\, Z \in \mathcal{C}, \varphi_i = \varphi_j \circ \psi_{ij}\big\}$$

with the obvious morphisms. Then, $\widetilde{Z} := \varinjlim X_i$ together with the maps $\widetilde{\varphi}_j : X_j \to \varinjlim X_i$ is the initial object of the category $\mathcal{C}^X$.

(b) $\varinjlim X_i = X_{\max I}$ and $\varinjlim X_i =$ coproduct (being the direct sum in $\mathcal{M}od_R$ and the disjoint union in $\mathcal{S}et$).

(c) For a directed system $(M_i \mid i \in I)$ (including compatible maps $\phi_{ij} : X_i \to X_j$ for $i \leq j$), we define $M := \oplus_{i \in I} M_i$ and $N$ as in the problem. Then, we have natural maps $\iota_j : M_j \hookrightarrow M \twoheadrightarrow M/N$. The quotient construction with $N$ ensures $\iota_k = \iota_j \circ \varphi_{jk}$ for $j \leq k$. Finally, the universal property follows directly from this construction: If we have compatible $R$-linear maps $f_j : M_j \to L$, then we obtain, e.g. by the universal property of the direct sum, a map $M \to L$. And the compatibilities among the maps $f_i$ ensures that $N$ is sent to 0 via this map.

(d) Denote $C := \coprod_{i \in I} M_i$. Then, if $m_i \in M_i \subseteq C$ is a representative of $\overline{m_i} \in C/\sim$ and $r \in R$, then it is clear how to obtain $r \cdot \overline{m_i} := \overline{rm_i}$. Moreover, this construction

is compatible with $\varphi_{ij}(m_i) \sim m_i$ – just because $varphi_{ij}$ is linear.
More intersting is the addition – this is exactly the part where the filtering becomes essential: If we were supposed to add $\overline{m_i}$ and $\overline{m_j}$, then we may choose a $k = k(i,j)$ with $i, j \leq k$. But then, by the definition of $\sim$, we obtain $\overline{m_i} = \overline{\varphi_{ik}(m_i)}$ and $\overline{m_j} = \overline{\varphi_{jk}(m_j)}$, i.e. both summands are represented by elements in $M_k$. There, we can add them, and this solves the problem.

**Problem 32.** a) Let $P \in \operatorname{Spec} R$ be a prime ideal and $M$ an $R$-module. Show that the localisation $M_P$ is the direct limes of modules $M_f$ with distinguished elements $f \in R$. What is the associated poset $(I, \leq)$? Is it filtered?

b) Show that infinite direct sums are filtered direct limits of finite direct sums. What is the underlying poset?

*Solution:* (a) The underlying poset is $I := R \setminus P$ with $f \leq g$ being defined via the relation $f|g$. If this is the case, we have natural maps $R_f \to R_g$. This poset is filtered because of $k(f,g) := fg$. Now, we can check the universal property for the compatible system of maps $\{M_f \to M_P \mid f \in I\}$.

(b) Let $\Lambda$ be a set, and we consider $R$-modules $M_\lambda$ for $\lambda \in \Lambda$. The basic poset $I$ is defined as

$$I := \{S \subseteq \Lambda \mid \#S < \infty\} \subseteq 2^\Lambda$$

with the inclusion relation. For each $S \in I$ we define $M_S := \oplus_{\lambda \in S} M_\lambda$ which induces natural embeddings $M_S \hookrightarrow M_{S'}$ whenever $S \subseteq S'$. They are compatible with the overall embedding $M_S \hookrightarrow \oplus_{\lambda \in \Lambda} M_\lambda$.

**Problem 35.** In the category of directed systems of $R$-modules on a poset $I := (I, \leq)$ (cf. Problem 31) define kernels, images, and cokernels in a natural way, e.g.

$$\ker\big(\varphi : (M_i \mid i \in I) \to (N_i \mid i \in I)\big) := \big(\ker[\varphi_i : M_i \to N_i] \mid i \in I\big).$$

This leads to the notion of exact sequences of directed systems.

a) Show that $\varinjlim$ is right exact (by constructing a right adjoint functor).

b) Show that *filtered* direct limits with values in $\mathcal{M}od_R$ are even exact.

c) Consider the set $I := \{m, a, b\}$ with $m < a$ and $m < b$. Show that the direct limit over this $I$ (even with values in $\mathcal{M}od_R$) is not left exact.

*Solution:* (a) The right adjoint functor is $Z \mapsto$ [constant system $Z$]. Indeed, the universal property of the direct limit says $\operatorname{Hom}(\varinjlim_i M_i, N) = \operatorname{Hom}(\{M_i\}, N) = \operatorname{Hom}(\{M_i\}, \{N_i := N\})$.

(c) Consider $(0, M, M) \hookrightarrow (M, M, M)$. The direct limits of both systems are $M \oplus M$ and $M^3/\sim$ with $(m, 0, 0) \sim (0, m, 0)$ and $(m, 0, 0) \sim (0, 0, m)$, i.e. the latter becomes isomorphic to $M$. The map $\varinjlim(0, M, M) \to \varinjlim(M, M, M)$ becomes the addition $M \oplus M \to M$. It is not injective at all.

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## 8. Aufgabenblatt zum 12.6.2023

**Problem 85.** In class we had defined the so-called constant presheaf $\mathcal{F} := \underline{A}^{\rm pre}$ via $\mathcal{F}(U) := A$. Afterwards, assuming that $X$ is a locally connected topological space, we define another presheaf $\mathcal{G}$ via $\mathcal{G}(U) := A^{\pi_0(U)}$. Show that $\mathcal{G}$ is actually a sheaf, namely $\mathcal{G} = \mathcal{F}^a$. It is called the “constant sheaf” $\mathcal{G} = \underline{A}$.

*Solution:* (i) *$\mathcal{G}$ is a sheaf.* Assume that $U = \bigcup_i U_i$ and $s \in A^{\pi_0(U)}$, i.e. $s : \pi_0(U) \to A$. If $s_i : \pi_0(U_i) \to \pi_0(U) \to A$ vanishes for all $i$, then $s$ vanishes, too: If $C \subseteq U$ is a connected component, then we choose an index $i$ with $U_i \cap C \neq \emptyset$. Hence, there is a connected component $C_i$ of $U_i$ intersecting $C$, hence being contained in $C$. Thus, via the map $\pi_0(U_i) \to \pi_0(U)$, $C_i$ maps to $C$, and we obtain $C \mapsto 0$.

On the other hand, if $s_i : \pi_0(U_i) \to A$ are compatible maps, then we start again with some $C \in \pi_0(U)$, choose some $C_i$, and define $s(C) := s_i(C_i)$. It remains to show that different choices $C_i \in \pi_0(U_i)$ and $C'_j \in \pi_0(U_j)$ (even $i = j$ is a non-trivial case) lead to $s_i(C_i) = s_j(C'_j)$. This can be done by creating a chain of mutually overlapping $C_v \in \pi_0(U_v)$. However, the following alternative point of view simplifies the situation a lot:

Assigning to $A$ the discrete topology, the set $A^{\pi_0(U)} = \mathrm{Maps}(\pi_0(U), A)$ can be identified with the set of all *continous* maps $U \to A$. And now, the glueing procedure is straightforward.

(ii) *$\mathcal{G}$ equals $\mathcal{F}^a$.* There is an obvious map of presheaves $\mathcal{F} \to \mathcal{G}$; for each open $U \subseteq X$ it becomes $A \to A^{\pi_0(U)}$ sending $a \in A$ to the constant map $\pi_0(U) \stackrel{a}{\to} A$ or $U \stackrel{a}{\to} A$. For each $P \in X$, the associated map $A = \mathcal{F}_P \to \mathcal{G}_P$ on stalks is an isomorphism; its inverse $\mathcal{G}_P \to A$ is $f \mapsto f(P)$.

**Problem 86.** Let $A$ be a ring and $X := \operatorname{Spec} A$. Show that the functor $M \mapsto \widetilde{M}$ from the category of $A$-modules into the category of $\mathcal{O}_X$-modules is fully faithful, i.e. that it induces isomorphisms on the sets $\mathrm{Hom}(\bullet, \bullet)$.

*Solution:* $\widetilde{M} \to \widetilde{N}$ is equivalent to a system of compatible $A_f$-linear maps $M_f \to N_f$ with $f \in A$.

**Problem 87.** Let $S$ be a graded ring and $f \in S_1$. Show that, for every $k \in \mathbb{Z}$, the $S_{(f)}$- modules $S_{(f)}$ and $S(k)_{(f)}$ are isomorphic where $S(k)$ denotes the degree shift by $k$. Find a counterexample for $\deg f = 2$.

*Solution:* The map $S_{(f)} \to S(k)_{(f)}$ with $s/f^\ell \mapsto s/f^{\ell-k}$ defines the desired isomorphism.
For a counterexample, take $S := k[x, y, f]$ with $\deg x = \deg y = 1$ and $\deg f = 2$. Then $S_{(f)} = k[x^2/f,\ xy/f,\ y^2/f] = k[A, B, C]/(AC - B^2)$ and, as a module, $S_{(f)}$ is (of course) generated by the single element 1. On the other hand, $S(1)_{(f)}$ is generated by $x$ and $y$. There is no way to do it with just one element.

Or, alternatively, take $S = k[f]$ with $\deg f = 2$. Then $S_{(f)} = k$. On the other hand, $S(1)_{(f)} = 0$.

**Problem 88.** Define $\mathcal{F}$ as the sheaf of regular sections of the map $h : \widetilde{\mathbb{A}^n} \to \mathbb{P}^{n-1}$ arising from blowing up of the origin $\pi : \widetilde{\mathbb{A}^n} \to \mathbb{A}^n$. Recall that such a section $s$ assignes to each $\ell \in \mathbb{P}^{n-1}$ a point $c \in \ell \subseteq \mathbb{A}^n$.

On the other hand, we define $\mathcal{G} := \widetilde{S(-1)}$ with $S := k[\mathbf{z}] := k[z_1, \ldots, z_n]$. It is a sheaf of $\mathcal{O}_{\mathbb{P}^{n-1}}$-modules where $\mathcal{O}_{\mathbb{P}^{n-1}} = \widetilde{S}$.

a) Show that the sheaves $\mathcal{F}$ and $\mathcal{G}$ are isomorphic by investigating (and glueing) their local pieces on the open subsets $D_+(z_i)$. The sheaf $\mathcal{F} = \mathcal{G}$ is usually called $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$.

b) What is its global sections? That is, determine $\Gamma\big(\mathbb{P}^{n-1}, \mathcal{O}(-1)\big) = \mathcal{O}(-1)(\mathbb{P}^{n-1})$.

*Solution:* (a) *Local Sections* of $h$: Locally, on $D_+(z_i)$, the map $h : \widetilde{\mathbb{A}^n} \to \mathbb{P}^{n-1}$ corresponds to the embedding $\iota_i : k[\mathbf{z}/z_i] \hookrightarrow k[\mathbf{z}, \mathbf{z}/z_i] = k[z_i, \mathbf{z}/z_i]$. Thus, over $D_+(z_i)$, a regular section $s_i$ corresponds to a $k$-algebra-homomorphism

$$s_i^* : k[z_i, \mathbf{z}/z_i] \to k[\mathbf{z}/z_i]$$

satisfying $s_i^* \circ \iota_i = \mathrm{id}_{k[\mathbf{z}/z_i]}$. That is, $s_i^*$ just corresponds to the choice of an element $s_i^*(z_i) \in k[\mathbf{z}/z_i]$. In other words, $\mathcal{F}\big(D_+(z_i)\big) \cong k[\mathbf{z}/z_i]$. So far, this results coincides with the sheaf $\mathcal{O} = \widetilde{S}$ and, actually, with all sheaves $\mathcal{O}(\ell) = \widetilde{S(\ell)}$ for $\ell \in \mathbb{Z}$, cf. Problem 87.

*Glueing these sections:* Since $z_j = z_j/z_i \cdot z_i$, two sections $s_i$ and $s_j$ coincide on $D_+(z_i z_j)$ iff

$$s_j^*(z_j) = s_i^*(z_j) = s_i^*(z_j/z_i \cdot z_i) = z_j/z_i \cdot s_i^*(z_i).$$

This can be written in a more symmetric way as

$$s_j^*(z_j)/z_j = s_i^*(z_i)/z_i \quad \text{inside } k[z_1^{\pm 1}, \ldots, z_n^{\pm 1}] \subset k(\mathbf{z}).$$

Thus, $is_i$ is better encoded by $s_i^*(z_i)/z_i \in z_i^{-1} k[\mathbf{z}/z_i] = k[\mathbf{z}](-1)_{(z_i)}$ instead of $s_i^*(z_i) \in k[\mathbf{z}/z_i]$. It leads to the fact that the identification

$$\mathcal{F}\big(D_+(z_i)\big) \xrightarrow{\sim} k[\mathbf{z}](-1)_{(z_i)} = \mathcal{G}\big(D_+(z_i)\big)$$

is compatible with changing the charts $D_+(z_i) \supset D_+(z_i z_j) \subset D_+(z_j)$. That is, it yields a global isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{G}$.

(b) The global sections are the intersection

$$\Gamma(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(-1)) = \bigcap_{i=1}^{n} k[\mathbf{z}](-1)_{(z_i)} = 0.$$

This is analogous to (16.8) with $\mathcal{O}_{\mathbb{P}^{n-1}} = \mathcal{O}_{\mathbb{P}^{n-1}}(0)$. Similarly, $z_\nu \in \bigcap_{i=1}^n k[\mathbf{z}](1)_{(z_i)}$, i.e. these elements of $S = k[\mathbf{z}]$ become global sections of the sheaf $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$. While $S$ does not collect functions, this provides an alternative interpretation of its elements. And, interpreting the sheaf $\mathcal{O}(1)$ as the sheaf of sections (in its original meaning) of a bundle over $\mathbb{P}^{n-1}$, this interpretation is even a geometric one.

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## 9. Aufgabenblatt zum 19.6.2023

**Problem 89.** a) Let $A$ be a ring and $f, g \in A$ with $D(f) \subseteq D(g)$ within $\operatorname{Spec} A$. Show that $g \in A_f^*$.
(*Hint:* Reduce the problem w.l.o.g. to the case $f = 1$.)

b) Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a *non-local* homomorphism of local rings. Show that, for every $S$-module $N$, the modules $\operatorname{Tor}_i^R(N, R/\mathfrak{m})$ vanish for every $i \in \mathbb{Z}$.

*Solution:* (a) We replace $A$ by $A_f$ – then the relation $D(f) \subseteq D(g)$ remains true, but $D(f) = \operatorname{Spec} A_f$ becomes the whole space. Thus, the problem can, w.l.o.g., be reduced to the following one: Let $g \in A$ such that $D(g) = \operatorname{Spec} A$. Show that $g \in A^*$. But this is clear – if $g \notin A^*$, then there was a prime $P \in \operatorname{Spec} A$ containing $g$, i.e., $P \notin D(g)$.

(b) If $\varphi(\mathfrak{m}) \not\subseteq \mathfrak{n}$, then there is an $r \in \mathfrak{m}$ which becomes a unit in $S$. Thus, while $N \xrightarrow{\cdot r} N$ is an isomorphism, the map $R/\mathfrak{m} \xrightarrow{\cdot r} R/\mathfrak{m}$ is zero. In particular, if $T := \operatorname{Tor}_i^R(N, R/\mathfrak{m})$, then the bifunctoriality of Tor implies that the endomorphism $T \xrightarrow{\cdot r} T$ is an isomorphism and, at the same time, equal to 0.

**Problem 90.** Let $X = [0,1] \subset \mathbb{R}$ with the classical, i.e., Euclidean topology.

a) We define $\mathcal{F}$ as the so-called *skyscraper sheaf* on $0 \in X$ (with value $\mathbb{Z}$): For each open $U \subseteq X$ we define

$$\mathcal{F}(U) := \begin{cases} \mathbb{Z} & \text{if } 0 \in U \\ 0 & \text{otherwise} \end{cases}$$

with the canonical restriction maps (always $\mathrm{id}_{\mathbb{Z}}$, whenever this makes sense). Show that $\mathcal{F}$ is really a sheaf and calculate all its stalks.

b) Let $\mathcal{G} = \underline{\mathbb{Z}}$ be the constant sheaf (the sheafification of the constant pre-sheaf). Then, show that $\mathcal{H}om(\mathcal{F}, \mathcal{G})_0 = 0$. Compare this with $\operatorname{Hom}(\mathcal{F}_0, \mathcal{G}_0)$.

*Solution:* (a) $\mathcal{F}$ is obviously a presheaf. It stalks are $\mathcal{F}_0 = \mathbb{Z}$ and $\mathcal{F}_c = 0$ for $c \neq 0$. Thus, to check the sheaf property, one could either check the axioms, or one calculates $\mathcal{F}^a$ – which is easy because only one single stalk is non-trival – and observes that $\mathcal{F}^a = \mathcal{F}$.

Alternatively, we consider the closed embedding $i : 0 \hookrightarrow X$ and realize that $\mathcal{F} = i_*(\underline{\mathbb{Z}})$ where $\underline{\mathbb{Z}} = \mathbb{Z}$ is the constant (pre-)sheaf on the point 0.

(b) Obviously, $\operatorname{Hom}(\mathcal{F}_0, \mathcal{G}_0) = \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ (at least if we consider $\mathbb{Z}$-linear maps). In particular, it is non-zero.

On the other hand, let $f \in \mathcal{H}om(\mathcal{F}, \mathcal{G})_0$ and let this be represented by a map $f : \mathcal{F}_{|U} \to \mathcal{G}_{|U}$ with a connected open $U \ni 0$, e.g., some interval $U = [0, t)$. We take

$V := U \setminus \{0\}$ and consider the restriction diagram

$$\begin{array}{ccc} \mathbb{Z} = \mathcal{F}(U) & \xrightarrow{f} & \mathcal{G}(U) = \mathbb{Z} \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ 0 = \mathcal{F}(V) & \xrightarrow{0} & \mathcal{G}(V) = \mathbb{Z}. \end{array}$$

It implies that $f = 0$.

**Problem 91.** Let $f : X \to Y$ and $g : Y \to Z$ be continous maps between topological spaces. Show that for a sheaf $\mathcal{H}$ on $Z$ we have that $(gf)^{-1}(\mathcal{H}) = f^{-1}g^{-1}\mathcal{H}$.

*Solution:* It suffices to look at the case of pre-sheaves. Then, for an open $U \subset X$ we have that

$$\begin{aligned} (f^{-1}g^{-1}\mathcal{H})(U) &= \varinjlim_{V \supseteq f(U)} (g^{-1}\mathcal{H})(V) = \varinjlim_{V \supseteq f(U)} \varinjlim_{W \supseteq g(V)} \mathcal{H}(W) \\ &= \varinjlim_{W \supseteq gf(U)} \mathcal{H}(W). \end{aligned}$$

Both directed systems for $W$ coincide – one can see this better by rewriting the conditions into $U \subseteq f^{-1}V$, $V \subseteq g^{-1}W$, and $U \subseteq f^{-1}g^{-1}W$. The $U$ is given, both is conditions for $W$ – and the $V$ does not matter at all. It is just supposed to exist – but for this one might take $V := g^{-1}W$.

**Problem 92.** a) Let $\mathcal{R}$ be a sheaf of rings on some space $X$. A sheaf $\mathcal{F}$ of $\mathcal{R}$-modules is called locally free if there is an open covering $X = \bigcup_i U_i$ such that all restrictions $\mathcal{F}|_{U_i}$ are isomorphic to direct sums of copies of $\mathcal{R}|_{U_i}$. Show that tensorizing with locally free sheaves is an exact functor.

b) Let $S = \mathbb{C}[z_0, z_1]$ and take $X := \operatorname{Proj} S$. It becomes a locally ringed space via $\mathcal{O}_X := \widetilde{S}$. Show that the sheaves of $\mathcal{O}_X$-modules $\mathcal{O}_X(k) := \widetilde{S(k)}$ (for $k \in \mathbb{Z}$) are locally free.

c) Show that $\mathcal{O}_X$ and $\mathcal{O}_X(-1)$ are not isomorphic to each other.

d) Show that $\mathcal{O}_X(k) \otimes_{\mathcal{O}_X} \mathcal{O}_X(k') \cong \mathcal{O}_X(k + k')$.

e) Show that $\mathcal{O}_X(k) \cong \mathcal{O}_X(k') \Leftrightarrow k = k'$.

*Solution:* (a) The exactness can be checked locally, e.g., on the stalks – but there, the functor is $\otimes_{\mathcal{R}_P} \mathcal{R}_P^{\oplus I}$, hence exact.

(b) On $D_+(z_i)$ we know that $S(k)_{(z_i)} = z_i^k \cdot S_{(z_i)}$, cf. Problem 87. This shows that, locally, $\mathcal{O}_X(k)$ is free of rank 1.

(c) See Problem 88: We know that $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = \mathbb{C}$, but $\Gamma(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-1)) = 0$.

(d) Locally on $D_+(z_i)$ we know that $(z_i^k \cdot S_{(z_i)}) \otimes_{S_{(z_i)}} (z_i^{k'} \cdot S_{(z_i)}) \xrightarrow{\sim} z_i^{k+k'} \cdot S_{(z_i)}$. Since this map is simply given by the multiplication within $\operatorname{Quot} S = \mathbb{C}(z_0, z_1)$, it does not depend on the choice of $i$, i.e., it glues foor different indiced $i, j$.
To mention a counter example, i.e., a situation where this argument does not work: The isomorphisms $(\cdot z_i^k) : S_{(z_i)} \xrightarrow{\sim} z_i^k \cdot S_{(z_i)}$ depend on $i$. That is, when you replace $i$ by $j$, then both maps induce different maps $S_{(z_i z_j)} \to z_i^k \cdot S_{(z_i z_j)} = z_j^k \cdot S_{(z_i z_j)}$ on the set $D_+(z_i z_j) = D_+(z_i) \cap D_+(z_j)$.

(e) If $k > k'$, then any isomorphism $f : \mathcal{O}_X(k) \xrightarrow{\sim} \mathcal{O}_X(k')$ would induce an isomorphism $f \otimes \mathrm{id}_{\mathcal{O}(-k'-1)} : \mathcal{O}_X(k-k'-1) \xrightarrow{\sim} \mathcal{O}_X(-1)$. But, by arguments like that from (c), this cannot exist.

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## 10. Aufgabenblatt zum 26.6.2023

**Problem 93.** a) Let $\mathcal{F} := \underline{A}$ be the constant sheaf on $U := (-2,0) \cup (0,2) \subseteq \mathbb{R}$; denote by $j : U \hookrightarrow \mathbb{R}$ the natural embedding. What are the germs of $j_*\mathcal{F}$ in the points 0, 1, 2, and 3, respectively?

b) I mentioned in class, as a general philosophy, that in a series of constructions with sheaves it suffices to sheafify only once at the end. Recall this philosophy in the construction of, say $(\mathcal{F} \otimes \mathcal{G}) \otimes \mathcal{H}$.

Demonstrate that this philosophy does not work in the context of (a) – if we start with $\mathcal{F} := \underline{A}^{\text{pre}}$, show that $(j_*\mathcal{F}^a)^a \neq (j_*\mathcal{F})^a$. Why does the usual stalk argument does not work anymore?

c) Let $j : U \to X$ be an open embedding and $\mathcal{F}, \mathcal{G}$ be sheaves on $U$ and $X$, respectively. Are there natural maps between $\mathcal{F}$ and $(j_*\mathcal{F})|_U$ or between $\mathcal{G}$ and $j_*(\mathcal{G}|_U)$?

d) Let $i : K \to X$ be a closed embedding of topological spaces, i.e. $K \subseteq X$ is a closed subset with the induced toplogy. If $\mathcal{F}$ is a sheaf on $K$, then calculate the stalks of $i_*\mathcal{F}$ in terms of those of $\mathcal{F}$.

e) Let $f : X \to Y$ be a closed (and continous) map, i.e. images of closed subsets are closed in $Y$. Show that $(f_*\mathcal{F})_y = \varinjlim_{U \supseteq f^{-1}y} \mathcal{F}(U)$ for sheaves $\mathcal{F}|X$ and $y \in Y$.

*Solution:* (a) $A \oplus A$, $A$, $A$, and 0.

(b) $\otimes$ commutes with stalks; for $j_*$ this does not even make sense. In this particular example, we consider the stalks at 0: $(j_*\mathcal{F}^a)^a_0 = (j_*\mathcal{F}^a)_0 = A \oplus A$, but $(j_*\mathcal{F})^a_0 = (j_*\mathcal{F})_0 = A$.

(c) First, we even have $\mathcal{F} = (j_*\mathcal{F})|_U$. For $\mathcal{G}$, we consider open $V \subseteq X \Rightarrow \varrho_{V,U\cap V} : \mathcal{G}(V) \to \mathcal{G}(U \cap V) = (j_*(\mathcal{G}|_U))(V)$ gives $\mathcal{G} \to j_*(\mathcal{G}|_U)$.

(d) First, if $p \in X \setminus K$, then $(i_*\mathcal{F})|_{X\setminus K} = 0$ implies that $(i_*\mathcal{F})_p = 0$. On the other hand, if $p \in K$, then $(i_*\mathcal{F})_p = \mathcal{F}_p$: Both sides are direct limits over the sections $\mathcal{F}(W)$ with $W \subseteq K$ being open subsets containing $p$ – but for the left hand side we have the additional property that $W$ has to be of the form $W = U \cap K$ for an open subset $U \subseteq X$. But this is not a restriction at all (every $W$ has this property), both limits coincide.

(e) The universal property of the direct limes gives a natural map

$$(f_*\mathcal{F})_y = \varinjlim_{f^{-1}V \supseteq f^{-1}y} \mathcal{F}(f^{-1}V) \to \varinjlim_{U \supseteq f^{-1}y} \mathcal{F}(U)$$

(with $U \subseteq X$ and $V \subseteq Y$). The closeness assumption of $f$ ensures that for all $U \supseteq f^{-1}y$ there is a $V \ni y$ with $U \supseteq f^{-1}V \supseteq f^{-1}y$: Just take $V := Y \setminus f(X \setminus U)$.

**Problem 94.** a) Let $\varphi : A \to B$ be a ring homomorphism and denote by $f : \operatorname{Spec} B \to \operatorname{Spec} A$ the associated map between the associated affine schemes. Assume that $M$ and $N$ are $A$- and $B$-modules, respectively. For the corresponding sheaves show that

$$f^*\widetilde{M} = \widetilde{M \otimes_A B}$$

on $\operatorname{Spec} B$ and

$$f_*\widetilde{N} = \widetilde{N_A}$$

on $\operatorname{Spec} A$.

b) Let $j : D(a) \hookrightarrow \operatorname{Spec} A$ be the "nice" open embedding obtained for an $a \in A$. Does (a) say something about $\widetilde{M}|_{D(a)}$?

*Solution:* (a) We have for all $a \in A$ the following chain of equalities: $(f_*\widetilde{N})(D(a)) = \widetilde{N}\big(D(\varphi(a))\big) = N_{\varphi(a)} = (N_A)_a$. They are compatible with the inclusions $D(aa') \subseteq D(a)$, hence provide an isomorphism of sheaves.
There are natural maps $M \to \Gamma(\operatorname{Spec} B, f^{-1}\widetilde{M})$ and $M \otimes_A B \to \Gamma(\operatorname{Spec} B, f^*\widetilde{M})$. The latter induces $\mathcal{O}_B$-linear sheaf homomorphism $\widetilde{M \otimes_A B} \to f^*\widetilde{M}$. On the level of stalks this yields the natural maps $(M \otimes_A B)_Q \to M_{\varphi^{-1}(Q)} \otimes_{A_{\varphi^{-1}(Q)}} B_Q$ for $Q \in \operatorname{Spec} B$. However, these maps are clearly isomorphisms.

(b) $\widetilde{M}|_{D(a)} = j^*\widetilde{M} = \widetilde{M_a}$ on $D(a) = \operatorname{Spec} A_a$.

**Problem 95.** a) Denote by $R^*$ the group of units in a ring $R$. Similarly, if $\mathcal{O}_X$ is a sheaf of rings on $X$, then we define $\mathcal{O}_X^*(U) := \mathcal{O}_X(U)^*$. Show that this defines a sheaf of abelian groups. Moreover, for a section $s \in \Gamma(U, \mathcal{O}_X)$ it satisfies $s \in \Gamma(U, \mathcal{O}_X^*) \Leftrightarrow s_P \in \mathcal{O}_{X,P}^*$ for all $P \in U$.

b) We call locally free sheaves of rank one on a ringed space $(X, \mathcal{O}_X)$ *invertible sheaves*. Let $L$ be such an invertible sheaf, i.e. there is an open cover $\{U_i \subseteq X\}_{i\in I}$ with isomorphisms $\varphi_i : L|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$. Show that the composition maps $\varphi_j \circ \varphi_i^{-1}$ are given by elements $h_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$ having the property $h_{ji} = h_{ij}^{-1}$ and $h_{ij} \cdot h_{jk} \cdot h_{ki} = 1$ on $U_i \cap U_j \cap U_k$ ("cocycle condition").

c) How do the $h_{ij}$ change if the isomorphisms $\varphi_i$ are altered?

d) Show that $L \cong \mathcal{O}_X \Leftrightarrow$ there are elements $g_i \in \Gamma(U_i, \mathcal{O}_X^*)$ such that $h_{ij} = g_i \cdot g_j^{-1}$ ("$h_{\bullet\bullet}$ is a coboundary").

e) How does one obtain the cocycle $\{H_{ij}\}$ for the sheaves $L \otimes L'$ and $L^\vee := \mathcal{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)$ out of the cocycles $\{h_{ij}\}$ and $\{h'_{ij}\}$ of $L$ and $L'$, respectively?

f) let $\{U_i \subseteq X\}_{i\in I}$ be an open cover, and let $h_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$ be elements satisfying the cocycle condition. Show that there is an, up to isomorphism unique, invertible sheaf $L$ on $X$ inducing the given $h_{ij}$ via (b).

g) How do the cocycles of the sheaves $\mathcal{O}_{\mathbb{P}^n}(\ell)$ on $\mathbb{P}^n$ look like? (Consider at least $\ell = -1, 0, 1$ and $n = 1$ or $n = 2$.)

h) Show that the set of isomorphism classes of invertible sheves forms a group; it is called the "*Picard group*" $\operatorname{Pic} X$. Note that the group operation is $\otimes$ and that the inverse of $L$ is given by $L^\vee := \mathcal{H}om_{\mathcal{O}_X}(L, \mathcal{O}_X)$.

*Solution:* (f) Existence part: $L(U) := \{s \in \prod_i \mathcal{O}_{U_i}(U \cap U_i) \mid s_i = h_{ij}s_j\}$.

**Problem 96.** a) Let $\varphi : A \to B$ be an injective ring homomorphism. Show (without using (b)) that $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is dominant, i.e., that $f(\operatorname{Spec} B)$ is dense in $\operatorname{Spec} A$, i.e., that its closure equals the whole $\operatorname{Spec} A$.
(Hint: A set $X \subseteq \operatorname{Spec} A$ is contained in a proper closed subset $F \subsetneq \operatorname{Spec} A$ iff there is a non-empty (nice?) open subset $U \subseteq \operatorname{Spec} A$ being disjoint to $X$.)

b) Let $\varphi : A \to B$ be a ring homomorphism leading to $f : \operatorname{Spec} B \to \operatorname{Spec} A$. Assume that $f(\operatorname{Spec} B) \subseteq V(I)$ for some ideal $I \subseteq A$. Show that then exists another ideal $I' \subseteq I$ with $V(I') = V(I)$ such that $\varphi$ factorizes via $A/I'$. Moreover, give an example where $I' = I$ cannot be achieved.

*Solution:* (a) Assume that $D(a) \subseteq \operatorname{Spec} A$ is non-empty (i.e., $a \notin \sqrt{0}$) but disjoint to $f(\operatorname{Spec} B)$. This means that

$$D(\varphi(a)) = f^{-1}D(a) = \emptyset, \text{ i.e., } \varphi(a) \in \sqrt{0} \subseteq B.$$

However, under an injective ring homomorphism, non-nilpotent elements cannot become nilpotent.

(b) Let $J := \ker \varphi$. Then $\varphi : A/J \hookrightarrow B$ leads to a dominant map $f : \operatorname{Spec} B \to \operatorname{Spec} A/J$, i.e. $V(I) \supseteq \overline{f(\operatorname{Spec} B)} = V(J)$. Now, we set $I' := I \cap J$. On the one hand, we have the chain of maps

$$A \twoheadrightarrow A/I' \twoheadrightarrow A/J \hookrightarrow B,$$

and on the other, we know that $V(I') = V(I) \cup V(J) = V(I)$.

The counter example was already mentioned in class:

$$X := \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \hookrightarrow \operatorname{Spec} k[\varepsilon]$$

sends this single point into $V(\varepsilon) \subset \operatorname{Spec} k[\varepsilon]$. However, the map does not factor via $X_{\text{red}} = \operatorname{Spec} k[\varepsilon]/(\varepsilon)$.

For the glueing of schemes, please have a look at Problem [Hart, II/2.12].

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VL "Algebra II"
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## 11. Aufgabenblatt zum 3.7.2023

**Problem 97.** Let $f : \operatorname{Spec} B \to \operatorname{Spec} A$ be a morphism that is induced from a ring homomorphism $\varphi : A \to B$. Assume that there is an open covering $\operatorname{Spec} A = \bigcup_i U_i$ such that all maps $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ are closed embeddings, i.e., locally (on the target) of the form $\operatorname{Spec} A_i/J_i \hookrightarrow \operatorname{Spec} A_i$. Show that this implies that $\varphi : A \to B$ is surjective. (I.e. $f$ is a closed embedding "on the direct way", namely not just using some open covering.)

*Solution:* Refining the open covering $\{U_i\}$ leads to the w.l.o.g. assumption $U_i = \operatorname{Spec} A_{f_i}$. Thus, $f|_{f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$ is associated to the ring homomorphism $A_{f_i} \to B_{\varphi(f_i)}$. Now, we may use Proposition 4 in (2.6).

**Problem 98.** a) Let $X$ be a scheme with an open, affine cover $\{U_i = \operatorname{Spec} A_i\}$. Show that the affine schemes $\operatorname{Spec}(A_i)_{\mathrm{red}}$ (with $(A_i)_{\mathrm{red}} := A_i/\sqrt{0}$) can be glued to become a reduced scheme $X_{\mathrm{red}}$. Are there maps between $X$ and $X_{\mathrm{red}}$? Are they finite? How do they look like on the topological level?

b) Let $X$ be an irreducible scheme. Show that there is a unique "generic point" $\eta \in X$, i.e. it is characterized by the property $\overline{\eta} = X$. How can one obtain open affine subsets $\operatorname{Spec} A \subseteq X$ containing $\eta$?

c) Let $X$ be an integral (irreducible and reduced) scheme. Show that $\mathcal{O}_{X.\eta}$ is a field (the "function field" of $X$). How does it look like for $X = \mathbb{A}^2_{\mathbb{C}}$, or $X = \mathbb{P}^2_{\mathbb{C}}$?

*Solution:* a) $A_f/\sqrt{(0)} = (A/\sqrt{0})f$. $X_{\mathrm{red}} \hookrightarrow X$ is a closed embedding, in particular finite, and it is an isomorphism on the underlying topological spaces.

b) W.l.o.g. $X$ is reduced. Then, if $\operatorname{Spec} A \subseteq X$ is an open subset, $A$ is an integral domain, and the point $\eta := (0) \in \operatorname{Spec} A \subseteq X$ does not depend on $A$. The stalk is $\mathcal{O}_{X,\eta} = \operatorname{Quot}(A)$. For instance, if $X = \mathbb{A}^2_{\mathbb{C}}$, or $X = \mathbb{P}^2_{\mathbb{C}}$, then $\mathcal{O}_{X,\eta} = \mathbb{C}(x, y)$.

**Problem 99.** Let $f : X \to \operatorname{Spec} B$ be a morphism of schemes. If $X = \bigcup_{i \in I} \operatorname{Spec} A_i$, then we had defined in class the scheme theoretic image of $f$ as $\operatorname{Spec} B/J$ with $J := \bigcap_i \ker(B \to A_i) \subseteq B$. It was the the smallest closed subscheme of $\operatorname{Spec} B$ such that $f$ factorizes through.

a) Assume that the index set $I$ is finite. Show that $V(J) = \overline{f(X)}$.

b) $f$ corresponds to a ring homomorphism $f^* : B \to \Gamma(X, \mathcal{O}_X)$. What is the relation between $\ker(f^*)$ and the ideal $J$ from (a)?

*Solution:* (a) While "$\supseteq$" is clear, we have to show that $f$ is dominant whenever $J = 0$. This is the global version of Problem 96. We can prove it similar: If $D(b) \neq \emptyset$ was disjoint to $f(X)$, then $b \in B$ is not nilpotent, but all $\varphi_i(b) \in A_i$ become nilpotent via $\varphi_i : B \to A_i$. Thus, there is a common $n$ such that $\varphi_i(b^n) = 0 \in A_i$, hence $b^n = 0$.

Alternatively, one can check that $Z := \operatorname{Spec} \prod_{i \in I} A_i = \sqcup_{i \in I} \operatorname{Spec} A_i$. There is a canonical, surjective map $\pi : Z \twoheadrightarrow X$, and $f$ and $f \circ \pi$ have the same image. On the other hand, $Z$ is affine, i.e., Problem 96 applies directly.

b) The restriction morphisms $\Gamma(X, \mathcal{O}_X) \to \Gamma(\operatorname{Spec} A_i, \mathcal{O}_X) = A_i$ induce a map $\Gamma(X, \mathcal{O}_X) \to \prod_{i \in I} A_i$ which is injective (by the sheaf axioms). Thus, $J = \ker(f^*)$.

**Problem 100.** Let $X$ be a scheme and $x \in X$ be a closed point. This gives rise to the local ring $A := \mathcal{O}_{X,x}$. We denote its maximal ideal by $\mathfrak{m} \subset A$. We call $T_x^* X := \mathfrak{m}/\mathfrak{m}^2$ the cotangent space of $X$ in $x$.

a) Show that an open embedding $U \hookrightarrow X$ and a closed embedding $Z \hookrightarrow X$ give rise to isomorphisms $T_x^* X \xrightarrow{\sim} T_x^* U$ and surjections $T_x^* X \twoheadrightarrow T_x^* Z$, respectively (if $x \in U$ and $x \in Z$).

b) Determine these maps explicitely for the origin $x = (0, 0)$ with respect to the closed embeddings
(i) $Z_1 = \operatorname{Spec} \mathbb{C}[x, y]/(y) \hookrightarrow \operatorname{Spec} \mathbb{C}[x, y]$,
(ii) $Z_2 = \operatorname{Spec} \mathbb{C}[x, y]/(y^2) \hookrightarrow \operatorname{Spec} \mathbb{C}[x, y]$, and
(iii) $Z_3 = \operatorname{Spec} \mathbb{C}[x, y]/(y^2 - x^3) \hookrightarrow \operatorname{Spec} \mathbb{C}[x, y]$.
Moreover, draw a rough picture of the three situations (i)-(iii).

c) Choose some concrete tangent vector $0 \neq t \in T_{(0,0)} Z_3$ and describe the associated morphism $\operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2 \to Z_3$.

*Solution:* (a) For a local embedding $U \hookrightarrow X$, the local rings $\mathcal{O}_{U,x}$ and $\mathcal{O}_{X,x}$ coincide. The closed embedding $Z \hookrightarrow X$ locally looks like $\operatorname{Spec} A/J \hookrightarrow \operatorname{Spec} A$. The point $x \in Z$ corresponds to some maximal ideal $P \subset A$ above $J$. Hence, $\mathcal{O}_{X,x} = A_P$ and $\mathcal{O}_{Z,x} = (A/J)_P = A_P/(J \cdot A_P)$. The maximal ideal is induced by $P$ in both cases. Thus,

$$\mathfrak{m}_{X,x}/\mathfrak{m}_{X,x}^2 = PA_P/P^2A_P = P/P^2 \otimes_A A_P = P/P^2$$

and

$$\mathfrak{m}_{Z,x}/\mathfrak{m}_{Z,x}^2 = (P/J)/(P/J)^2 = P/(P^2 + J).$$

(b) We obtain $T_{(0,0)}^* \mathbb{C}^2 = (x, y)/(x, y)^2 = \mathbb{C}x \oplus \mathbb{C}y$. Similarily,

$$T_{(0,0)}^* Z_1 = (x, y)/(x^2, xy, y^2; y) = (x, y)/(x^2, y) = \mathbb{C}x,$$

and

$$T_{(0,0)}^* Z_2 = (x, y)/(x^2, xy, y^2; y^2) = T_{(0,0)}^* \mathbb{C}^2 = \mathbb{C}x \oplus \mathbb{C}y,$$

and

$$T_{(0,0)}^* Z_3 = (x, y)/\big(x^2, xy, y^2; (y^2 - x^3)\big) = T_{(0,0)}^* \mathbb{C}^2 = \mathbb{C}x \oplus \mathbb{C}y,$$

(c) We have got $T_{(0,0)}^* Z_3 = T_{(0,0)}^* \mathbb{C}^2 = \mathbb{C}^2$, and we choose $t = (2, 1) \in T_{(0,0)} Z_3 = \Big((x, y)/(x, y)^2\Big)^*$, i.e., $t : (x, y)/(x, y)^2 \to \mathbb{C}$ with $x \mapsto 2$ and $y \mapsto 1$.

The associated morphism $t : \operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2 \to Z_3$ corresponds to the ring homomorphism $\mathbb{C}[x, y]/(y^2 - x^3) \to \mathbb{C}[\varepsilon]/\varepsilon^2$ sending $x \mapsto 2\varepsilon$ and $y \mapsto \varepsilon$. The reason for having no “non-$\varepsilon$-term”, i.e., $x \mapsto (0 + 2\varepsilon)$, is that the point of interest is $(0, 0)$, i.e., having

0 as both coordinates.

If we were looking at $T_{(1,1)}\mathbb{C}^2$ instead (with the same tangent vector $(2,1)$), then we would have obtained $x \mapsto 1+2\varepsilon$ and $y \mapsto 1+\varepsilon$ instead. Why didn't I take $T_{(1,1)}Z_3$? While $(1,1) \in Z_3$, the tangent vector $(2,1)$ does not belong to $T_{(1,1)}Z_3$. The latter can be calculated as follows: Substitute $a := x-1$ and $b := y-1$, then

$$\mathbb{C}[x,y]/(y^2 - x^3) = \mathbb{C}[a,b]/(b^2 + 2b - a^3 - 3a^2 - 3a),$$

and we look at the point $(a,b) = (0,0)$. The cotangent space is $(a,b)/(a^2, ab, b^2, 2b - 3a)$, and the vector $(2,1)$ does not define a correct linear map

$$(a,b)/(a^2, ab, b^2, 2b - 3a) \to \mathbb{C}, \quad a \mapsto 2,\ b \mapsto 1.$$

On the other hand, it does not make sense to compare the tangent spaces of some $X$ within different points, anyway. We have no parallel transport.

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## 12. Aufgabenblatt zum 10.7.2023

**Problem 101.** Let $F$ be a locally free sheaf on an integral, i.e. irreducible and reduced scheme $X$. Show that, for open subsets $U \subseteq X$, the restriction map $\Gamma(X, F) \to \Gamma(U, F)$ is injective.
Give counter examples for the cases when one of the assumptions is violated.

*Solution:* W.l.o.g. $X = \operatorname{Spec} A$ with $A$ being a domain and $F = \mathcal{O}_X$. Now, all localizations are injective.

Examples for violated assumptions: (i) $X = \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[x]$ and $F = \widetilde{M}$ with $M = \mathbb{C}[x]/(x) = \mathbb{C}$ yields the 0-map to $U := X \setminus \{0\}$, and (ii) $X = \operatorname{Spec}(\mathbb{C} \times \mathbb{C})$ is the disjoint union of two points. Even $F = \mathcal{O}_X$ violates the claim now.

**Problem 102.** a) Show directly that the diagonal $\Delta : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}} \times_{\mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$ is a closed embedding. What is the homogeneous ideal of $\Delta(\mathbb{P}^1_{\mathbb{C}}) \subseteq \mathbb{P}^3_{\mathcal{C}}$ after additionally using the Segre embedding? Do you see the Veronese embedding within this picture?
b) Let $X := \mathbb{A}^1_{\mathbb{C}} \cup \mathbb{A}^1_{\mathbb{C}}$ glued along the common $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$. Show directly that there are affine open $U_1, U_2 \subseteq X$ such that either $U_1 \cap U_2$ is not affine or that $U_1 \cap U_2 = U$ is affine with $U_i = \operatorname{Spec} A_i$ and $U = \operatorname{Spec} B$ such that $A_1 \otimes_{\mathbb{C}} A_2 \to B$ is not surjective.
c) In the situation of (b) show that $\Delta(X) \subseteq X \times_{\operatorname{Spec} \mathbb{C}} X$ is not a closed subset.

*Solution:* (a) The Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$, $(x_0 : x_1), (y_0 : y_1) \mapsto (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1)$ corresponds to the homogeneous coordinate rings $\mathbb{C}[z_0, \ldots, z_3] \to$ $\to \mathbb{C}[z_0, \ldots, z_3]/(z_0 z_3 - z_1 z_2)$. The diagonal $\Delta : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$ maps $(w_0 : w_1) \mapsto (w_0 : w_1), (w_0 : w_1) \mapsto (w_0^2 : w_0 w_1 : w_0 w_1 : w_1^2)$. In particular, It is obtained from the homogeneous ring homomorphism

$$\mathbb{C}[z_0, \ldots, z_3] \to \mathbb{C}[w_0, w_1], \quad z_0 \mapsto w_0^2,\ z_1, z_2 \mapsto w_0 w_1,\ z_3 \mapsto w_1^2$$

factoring through $\mathbb{C}[z_0, \ldots, z_3]/(z_0 z_3 - z_1 z_2)$. It is a surjection not onto $\mathbb{C}[w_0, w_1]$, but onto its even part. In the projective situation, this is sufficient for providing a closed embedding. The kernel is generated by $(z_1 - z_2,\ z_0 z_3 - z_1 z_2)$. Thus, the embedding $\Delta : \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ factors via the second Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$, followed by a linear embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^3$.

(b) In $X := U_1 \cup U_2$ (with $U_i = \mathbb{A}^1$ for $i = 1, 2$) the intersection of $U_1$ and $U_2$ is, via definition, $U_{12} = \mathbb{A} \setminus \{0\}$. The restriction maps $\varrho_i$ on the coordinate rings are both the localization maps $\varrho_i : \mathbb{C}[x] \to \mathbb{C}[x]_x$. The two copies of $\mathbb{C}[x]$ do not generate the larger ring $\mathbb{C}[x]_x$.

(c) Just looking at the closed points, $X$ consists of $\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ and two points $0_A$ and $0_B$. In $X \times X$, all 4 pairs $(0_A, 0_A)$, $(0_A, 0_B)$, $(0_B, 0_A)$, $(0_B, 0_B)$ belong to the closure of $\Delta(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}) = \{(t, t) \mid t \neq 0\}$. However, $\Delta(\mathbb{A}^1_{\mathbb{C}})$ contains only two of them, namely $(0_A, 0_A)$ and $(0_B, 0_B)$.

**Problem 103.** a) Show that $d$-dimensional $k$-varieties (with a perfect field $k$) are birational equivalent to hypersurfaces in $\mathbb{P}^{d+1}$.
(*Hint:* Use the theorem of the primitive element.)

b) Let $f, g \in k[x]$ be two different polynomials with simple roots. Construct a hypersurface of $\mathbb{C}^2$ that is birational equivalent to $V(y^2 - f(x),\, z^2 - g(x)) \subseteq \mathbb{C}^3$.

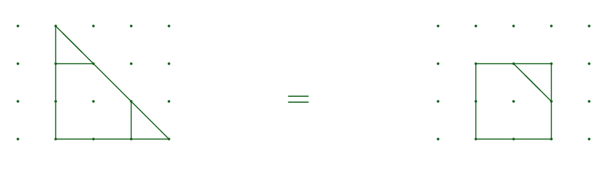
*Solution:* (a) $k =$ perfect $\Rightarrow$ for each field extension $K = k(\alpha_1, \ldots, \alpha_m) \supseteq k$ there is an $e \subseteq \{\alpha_1, \ldots, \alpha_m\}$ with $K \supseteq k(e) \supseteq k$ (separable|transzendent), cf. [ZS, ch. II, Th 30+31, S.104]. "Satz vom primitiven Element" $\Rightarrow$ $d$-dimensional $k$-varieties are birational equivalent to hypersurfaces in $\mathbb{P}^{d+1}$.
(b) Let $\pm y$ and $\pm z$ be the respective roots of the minimal polynomials $m_y(t) = t^2 - f(x)$ and $m_z(t) = t^2 - g(x)$ over $k(x)$. Theorem of the primitive element (actually, its proof) $\rightsquigarrow$ every $\gamma := y + cz$ with $c \in k(x)$ such that $y + cz \neq (-y) + c(-z)$, i.e., $c \neq -y/z$ generates the extension field $K := k(x)(y, z)$ over $k(x)$.
With $c := 1$, i.e., $\gamma := y + z$, we obtain $\big(\gamma^2 - (f+g)\big)^2 = 4fg$. This leads to the hypersurface equation $\gamma^4 - 2(f+g)\gamma^2 + (f-g)^2 = 0$.

**Problem 104.** Assume that the ring $A$ is factorial. Show that this implies $\operatorname{Pic}(\operatorname{Spec} A) = 0$, i.e. every invertible sheaf on $\operatorname{Spec} A$ is isomorphic to $\mathcal{O}_{\operatorname{Spec} A}$.
(*Hint:* For invertible sheaves $\mathcal{L}$ one is supposed to use the cocycle description on an open covering $\{D(g_i)\}$ with $\mathcal{L}|_{D(g_i)} \cong \mathcal{O}_{D(g_i)}$, cf. Problem 95. Via induction by the overall number of prime factors of the $g_i$, one can reduce the claim to the special case that all elements $g_i$ are prime. Now, using again Problem 95, one can attain that $h_{ij} \in A^*$ for all $i, j$.)

*Solution:* Let $p$ be a prime divisor of $g_1 \cdots g_N$ – via induction by the number of prime divisors of $g_1 \cdots g_N$ we may assume that $\mathcal{L}$ is trivial on $D(p) = \operatorname{Spec} A_p$. On the other hand, (prime divisors of $g_1 \cdots g_N$) $\supseteq (g_1, \ldots, g_N) = (1)$. Thus, we may suppose that all $g_i$ are prime.
Now, every $h_{ij} \in A^*_{g_i g_j}$ (using the notation of Problem 95) can be expressed as $h_{ij} = u_{ij} \cdot g_i^{e_i}/g_j^{e_j}$ with $u_{ij} \in A^*$. The elements $u_{ij}$'s do still satisfy the cocycle condition. Hence, we can represent them as $u_{ij} = u_{i0}/u_{j0}$.

**Problem 105.** Show (by using the toric language via polytopes in $M_{\mathbb{R}}$) that the blowing up of $\mathbb{P}^2$ in two points is isomorphic to the blowing up of $\mathbb{P}^1 \times \mathbb{P}^1$ in one single point.

*Solution:*



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## 1. Aufgabenblatt zum 25.10.2024

**Problem 106.** Let $f : X \to Y$ be a morphism of schemes, and let $F|Y$ be an $\mathcal{O}_Y$-module.

a) Show that there is a natural $\Gamma(Y, \mathcal{O}_Y)$-linear map $f^* : \Gamma(Y, F) \to \Gamma(X, f^*F)$.

b) A subset $S \subseteq \Gamma(Y, F)$ is said to "generate $F$" if $S$ generates all stalks $F_y$ as $\mathcal{O}_{Y,y}$-modules. Show that this implies that $f^*(S) \subseteq \Gamma(X, f^*F)$ generates $f^*F$.

c) Prove the so-called projection formula: Let $E$ be an $\mathcal{O}_X$-module and suppose that $F$ is a locally free sheaf on $Y$. Then, $f_*(E \otimes_{\mathcal{O}_X} f^*F) = f_*E \otimes_{\mathcal{O}_Y} F$.

*Solution:* (a) $\Gamma(Y, F) \to F(Y) \otimes_{\mathcal{O}_Y(Y)} \mathcal{O}_X(X) \to \Gamma(X, f^{*\text{prä}}F) \to \Gamma(X, f^*F)$. Alternatively, one may use $f^* \dashv f_*$: $\Gamma(Y, F) \to \Gamma(Y, f_*f^*F) = \Gamma(X, f^*F)$.

(b) For every $x \in X$ we have a commutative diagram

$$\begin{array}{ccc} \Gamma(Y,F) & \longrightarrow & \Gamma(X, f^*F) & & \\ \downarrow & & \downarrow & & \\ F_{f(x)} & \longrightarrow & (f^*F)_x & = & F_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}. \end{array}$$

(c) First, we have a natural map $f_*E \otimes_{\mathcal{O}_Y} F \to f_*E \otimes_{\mathcal{O}_Y} f_*f^*F \to f_*(E \otimes_{\mathcal{O}_X} f^*F)$. The isomorphism property can be checked locally – in particular, we may assume that $F = \mathcal{O}_Y$. But then, the map turns into $\mathrm{id} : f_*E \to f_*E$.

**Problem 107.** Let $E$ be a locally free $\mathcal{O}_X$-module of rank $r$ on a scheme $X$, i.e., there exists an affine, open covering $\{U_i\}_{i \in I}$ of $X$ together with isomorphisms $\phi_i : E|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}^r$.

a) Show that $E^\vee := \mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X)$ is locally free of rank $r$, too. Moreover, it satisfies $E^{\vee\vee} = E$.

b) Analogously to the same construction on modules over rings, we define

$$(\mathrm{Sym}^d E)(U \subseteq X) := \mathrm{Sym}^d E(U).$$

Thus, we obtain via $\mathcal{A} := \oplus_{d \geq 0} \mathrm{Sym}^d E$ a ring sheaf on $X$. How does $\mathcal{A}$ look like for the special case $E = \mathcal{O}_X \cdot s_1 \oplus \ldots \oplus \mathcal{O}_X \cdot s_r$ being a free $\mathcal{O}_X$-module?

c) Let $\pi : \mathrm{Spec}_X \mathcal{A} \to X$ be the gluing of the schemes and morphisms $\mathrm{Spec}\, \mathcal{A}(U_i) \to U_i = \mathrm{Spec}\, B_i$ where $\{U_i\}_{i \in I}$ is like in (a). Show that $\pi$ is a *vector bundle*, i.e., it is *locally* isomorphic to $X \times_{\mathrm{Spec}\, \mathbb{Z}} \mathbb{A}^r_{\mathbb{Z}} \to X$, an the transition maps $U_i \times \mathbb{A}^r \hookleftarrow \pi^{-1}(U_i \cap U_j) \hookrightarrow U_j \times \mathbb{A}^r$ are linear in the fibers (on $U_i \cap U_j$).

d) The sets of sections of $\pi$ – in the original meaning of this word, i.e., $s_U : U \to \pi^{-1}(U)$ with $\pi \circ s_U = \mathrm{id}_U$) form a sheaf of $\mathcal{O}_X$-modules on $X$. Accordingly, we denote $\mathrm{Spec}_X \mathcal{A}$ as $\mathbb{A}$(name of this sheaf).

e) For $X = \mathbb{P}^1_k$ and $E = \mathcal{O}_{\mathbb{P}^1}(\ell)$ describe $\pi : \mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell)) \to \mathbb{P}^1$ in the toric language, i.e., via fans. Can you spot the toric among the global sections of $\pi$ (again, in the original, literal meaning of the word)?

(*Hint:* For the bundle $\mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(-1)) \to \mathbb{P}^1$ we do already know the result – it has to be the blowing up $\widetilde{\mathbb{A}}^2 \to \mathbb{P}^1$.)

*Solution:* (a) The first question is local, i.e., it follows from $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X^r, \mathcal{O}_X) \cong \mathcal{O}_X^r$. For the second, one starts with the natural $\mathcal{O}_X$-module homomorphism

$$E \to \mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{H}om_{\mathcal{O}_X}(E, \mathcal{O}_X), \mathcal{O}_X\big)$$

and checks locally that it is an isomorphism.

(b) In the special case of $E = \mathcal{O}_X \cdot s_1 \oplus \ldots \oplus \mathcal{O}_X \cdot s_r$ we obtain $\mathcal{A} = \mathbb{C}[s_1, \ldots, s_r] \otimes_{\mathbb{C}} \mathcal{O}_X$.

(c) The local triviality follows from (b). Moreover, if $s_1, \ldots, s_r$ and $t_1, \ldots, t_r$ are bases for $E$ on $U_i$ and $U_j$, respectively, then they are related by a regular, i.e., invertible, $(r \times r)$-base change matrix with entries in $\mathcal{O}_{U_i \cap U_j}$. Regularity is equivalent to the determinant being contained in $\mathcal{O}^*_{U_i \cap U_j}$.

(d) The sheaf of sections is isomorphic to $E^\vee$; hence $\operatorname{Spec}_X \mathcal{A} =: \mathbb{A}(E^\vee)$. The reason for this is the fact that sections $s_U : \operatorname{Spec} B = U \to \pi^{-1}(U) = \operatorname{Spec} \operatorname{Sym}^\bullet\big(E(U)\big)$ correspond to $B$-algebra homomorphisms $s_U^* : \operatorname{Sym}^\bullet\big(E(U)\big) \to B$ – recall that $E(U) \cong B^r$ is a $B$-module. Those maps are completely determined by their behavior on degree 1, i.e., by the $B$-module homomorphisms $(s_U^*)_1 : E(U) \to B$. Thus, sections $s$ correspond to $\mathcal{O}_X$-linear maps $s_1^* : E \to \mathcal{O}_X$.

(e) The result depends on the choice of coordinates on $\mathbb{P}^1$. One version of the desired description is the following: The fan of $\mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell))$ is spanned from the two maximal cones

$$\sigma_0 := \langle (1,0), (0,1) \rangle \quad \text{and} \quad \sigma_\infty := \langle (0,1), (-1,\ell) \rangle.$$

This reflects the fact that it was glued from two affine pieces over $U_0, U_\infty \subset \mathbb{P}^1$. The map $\pi : \mathbb{A}(\mathcal{O}_{\mathbb{P}^1}(\ell)) \to \mathbb{P}^1$ is given by the first projection $\operatorname{pr}_1 : \mathbb{Z}^2 \to \mathbb{Z}$. The embeddings $\mathbb{Z} \hookrightarrow \mathbb{Z}^2$, $1 \mapsto (1,i)$ mit $0 \le i \le -\ell$ display the toric among all sections of $\pi$.

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## 2. Aufgabenblatt zum 1.11.2023

**Problem 108.** a) Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module on an integral scheme $X$. Show that $\mathcal{F} \otimes_{\mathcal{O}_X} K(X) = \mathcal{F}_\eta$ where $\eta \in X$ denotes the generic point and both $K(X)$ and $\mathcal{F}_\eta$ mean the constant sheafs with these values.

b) Give an example for a non-coherent subsheaf $\mathcal{F} \subseteq K(X)$ where the claim of Part(a) fails.

*Solution:* (a) Recall that $K(X) = \mathcal{O}_{X,\eta} = \widetilde{\mathrm{Quot}(A)}$. Moreover, we have a canonical map $\mathcal{F} \otimes_{\mathcal{O}_X} K(X) \to \mathcal{F}_\eta$ which arises from the restriction maps $\mathcal{F}(U) \to \mathcal{F}_\eta$ and the fact that $\mathcal{F}_\eta$ is a $\mathcal{O}_{X,\eta}$-module, i.e., a $K(X)$-vector space.

Let us check this map locally: If $\mathcal{F} = \widetilde{M}$ on $\operatorname{Spec} A$, then

$$\mathcal{F} \otimes_{\mathcal{O}_X} K(X) = \widetilde{M} \otimes_{\widetilde{A}} \widetilde{\mathrm{Quot}(A)} = M \otimes_A \widetilde{\mathrm{Quot}(A)} = \widetilde{M_{(0)}}.$$

Note that $M_{(0)}$ does not denote any homogeneous localization (which would not make any sense at all) – but it is the localization by the ideal $(0)$ which equals the stalk $\mathcal{F}_\eta$. Moreover, since the $A$-module $\mathcal{F}_\eta$ is already a $\mathrm{Quot}(A)$-vector space, none of the localizations will change this module. Hence, the associated sheaf is constant.

(b) Take $X = \operatorname{Spec} A$ with $A$ being a DVR, e.g., $A = k[x]_{(x)}$. Its spectrum consists of $\eta = (0)$ and the maximal ideal $\mathfrak{m} = (x)$. Let $\mathcal{F} \subset K(\operatorname{Spec} A) = k(x)$ be the sheaf defined as $\mathcal{F}(\{\eta\}) = k(x)$ and $\mathcal{F}(X) = 0$. Then, $\mathcal{F} \otimes k(x) = \mathcal{F} \neq k(x)$.

This $\mathcal{F}$ is, by the way, an example of a non-coherent ideal sheaf.

**Problem 109.** a) Let $\sigma \subseteq N_{\mathbb{Q}}$ be a polyhedral cone with $N \cong \mathbb{Z}^n$; let $\tau \leq \sigma$ be a face. Show that by

$$k[\sigma^\vee \cap M] \to k[\sigma^\vee \cap \tau^\perp \cap M], \quad x^r \mapsto \begin{cases} x^r & \text{if } r \in \tau^\perp \\ 0 & \text{otherwise} \end{cases}$$

we obtain a closed embedding $\mathbb{TV}(\overline{\sigma}, N/\operatorname{span}(\tau)) \hookrightarrow \mathbb{TV}(\sigma, N)$ where $\overline{\sigma}$ denotes the image of $\sigma$ via $N_{\mathbb{R}} \to N_{\mathbb{R}}/\operatorname{span}(\tau)$. If $\sigma = \tau$, then $\mathbb{TV}(\overline{\tau}, N/\operatorname{span}(\tau)) = \mathbb{TV}(0, N/\operatorname{span}(\tau))$ equals the subtorus $\operatorname{orb}(\tau) := \operatorname{Spec} k[\tau^\perp \cap M]$ of the torus $T = \operatorname{Spec} k[M]$.

b) Let $\Sigma$ be a fan in $N_{\mathbb{Q}}$; let $\tau \in \Sigma$. Show that all $\mathbb{TV}(\overline{\sigma}, N/\operatorname{span}(\tau))$ glue to a closed subvariety of $\mathbb{TV}(\Sigma, N)$. (How to handle those cones $\sigma$ that do *not* contain $\tau$ as a face?) This variety will be denoted by $\overline{\operatorname{orb}}(\tau)$. As $\mathbb{TV}(\overline{\sigma}, N/\operatorname{span}(\tau)) \subseteq \mathbb{TV}(\sigma, N)$ is the closure of $\operatorname{orb}(\tau)$ in $\mathbb{TV}(\sigma, N)$, the above $\overline{\operatorname{orb}}(\tau)$ is the closure of $\operatorname{orb}(\tau)$ in $\mathbb{TV}(\Sigma, N)$.

c) $\overline{\operatorname{orb}}(\tau)$ is toric, too. What is the associated torus? How does the associated fan look like? What is the dimension of $\overline{\operatorname{orb}}(\tau)$? What is $\overline{\operatorname{orb}}(\tau) \cap \overline{\operatorname{orb}}(\tau')$?

d) Let $\Delta \subseteq M_{\mathbb{Q}}$ be a lattice polytope – it induces a morphism $\mathbb{TV}(\mathcal{N}(\Delta), N) \to$

$\mathbb{P}(\Delta) \subseteq \mathbb{P}^{\#(\Delta\cap M)}$ which is an isomorphism whenever $\Delta$ is sufficiently large. For a face $F \leq \Delta$ check that $\mathbb{P}(F) = \mathbb{P}(\Delta) \cap V(z_r \mid r \notin F)$ ($z_r$ denotes the homogeneous coordinate associated to the lattice point $r \in \Delta \cap M$) coincides with some $\overline{\text{orb}}(\tau)$, i.e., the question is – what is $\tau$?.

*Solution:* (a) Since $\tau$ is a face, the $k$-linear map between the two semigroup algebras is multiplicatively.

(b) If $\tau \subseteq \sigma_1, \sigma_2$, then $\tau \subseteq \sigma_1 \cap \sigma_2$ – and all inclusions among cones of a fan are automatiacally face relations. Then, the surjections

$$k[\sigma_i^\vee \cap M] \to k[\sigma_i^\vee \cap \tau^\perp \cap M]$$

from (a) with $i = 1, 2$ are compatible with the ambient

$$k[(\sigma_1 \cap \sigma_2)^\vee \cap M] \to k[(\sigma_1 \cap \sigma_2)^\vee \cap \tau^\perp \cap M]$$

obtained from $(\sigma_1 \cap \sigma_2)^\vee = \sigma_1^\vee + \sigma_2^\vee$. Whenever $\sigma \in \Sigma$ does not contain $\tau$, then $\text{orb}(\tau)$ is disjoint to $\mathbb{TV}(\tau, N)$.
Finally, being the closure (of some subset $\text{orb}(\tau)$) can be checked locally.

(c) The torus is $\text{orb}(\tau) = \operatorname{Spec} k[\tau^\perp \cap M]$. The fan is $\overline{\Sigma} := \{\overline{\sigma} \mid \sigma \in \Sigma,\ \sigma \supseteq \tau\}$. The dimension of $\text{orb}(\tau)$ is that of $\tau^\perp$, i.e., it equals $\text{rank}(N) - \dim \tau$.

(d) $\tau \in \mathcal{N}(\Delta)$ equals $\mathcal{N}(F, \Delta)$ – that is, it is built from all $a \in N_\mathbb{R}$ such that $\min\langle \Delta, a\rangle = \langle F, a\rangle$ (impling that $a \in N_\mathbb{R}$ is constant on $F \subseteq M_\mathbb{R}$).

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## 3. Aufgabenblatt zum 8.11.2023

**Problem 110.** Let $f : (N, \Sigma) \to (N', \Sigma')$; this gives rise to a morphism $F := \mathbb{TV}(f) : \mathbb{TV}(N, \Sigma) \to \mathbb{TV}(N', \Sigma')$. If $\sigma \in \Sigma$, which orbit $\operatorname{orb}(\sigma' \in \Sigma')$ does contain $F(\operatorname{orb}(\sigma))$?

*Solution:* $\sigma' \in \Sigma'$ is the smallest cone such that $f(\sigma) \subseteq \sigma'$.

**Problem 111.** Let $C \subseteq \mathbb{R}^d$ be a convex, closed subset. We call a subset $F \subseteq C$ to be

a) a *supported (or "exposed") face* if there is some non-trivial $h \in (\mathbb{R}^d)^*$ such that $h$ is constant on $F$ and $h(c) > h(F)$ for all $c \in C \setminus F$. In other words, $F = \{c \in C \mid h(c) = \min h(C)\}$,

b) an *extremal face* if it is convex and any $a, b \in C$ with $\frac{a+b}{2} \in F$ enforce that $a, b \in F$, or

c) a *strongly extremal face* if it is convex and any $a, b \in C$ with $\lambda\, a + (1 - \lambda)\, b \in F$ for some $\lambda \in (0, 1)$ enforce that $a, b \in F$.

i) Show that (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c).

ii) If $C$ is polyhedral, i.e., the convex hull of a finite set, then (b) implies (a), too.

*Solution:* (i) The implications (a) $\Rightarrow$ (c) $\Rightarrow$ (b) are trivial. For the equivalence of (b) and (c) assume that $a, b \in C$ with $c := \lambda\, a + (1-\lambda)\, b \in F$. If $\lambda > \frac{1}{2}$, then $c$ is closer to $a$ as to $b$; actually, $c$ is the center of the segment $\overline{ab'}$ with $b' = c + (c - a) = 2c - a$. In particular, we obtain $a, b' \in F$. Hence, by convexity, $\overline{ab'} \subseteq F$, and we can continue with a new $c := b'$.

(ii) Let $F \subseteq C$ be a convex subset satisfying (c). There must be a (unique) minimal (closed, supported/exposed) face $f \leq C$ containing $F$ – and we are going to show that $F = f$. We may, w.o.l.g., assume that $f = C$. This implies (by convexity of $F$) that $F$ contains an interior point $p$ of $C$. Now, if $q \in C$ is arbitrary, there is a $q' \in C$ such that $p$ is an interior point of the line segment $\overline{qq'}$. Hence, $q \in F$.

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## 4. Aufgabenblatt zum 15.11.2023

**Problem 112.** Let $\widetilde{X} = \operatorname{Proj} \oplus_{d\geq 0} I^d \stackrel{\pi}{\to} \operatorname{Spec} A = X$ be the blowing up of $X$ in the ideal $I \subseteq A$. Then, the so-called exceptional divisor $E = \pi^{-1}(V(I)) \subseteq \widetilde{X}$ is given by the ideal sheaf $\pi^{-1}\widetilde{I} \cdot \mathcal{O}_{\widetilde{X}} \subseteq \mathcal{O}_{\widetilde{X}}$. Show that this is isomorphic to one of the sheaves $\mathcal{O}_{\widetilde{X}}(\ell)$. What is $\ell$?

*Solution:* Writing $I = (g_1, \ldots, g_n) \subseteq A$, we can describe $\pi^{-1}\widetilde{I} \cdot \mathcal{O}_{\widetilde{X}}$ locally on $D_+(g_i) = \operatorname{Spec} A[\mathbf{g}/g_i]$ by the ideal $(g_i)$. The sheaves $g_i \cdot \mathcal{O}_{D_+(g_i)}$ glue to $\mathcal{O}_{\widetilde{X}}(1)$.
Note that this is in contrast to the ideal sheaves of $V(F_d) \subset \mathbb{P}^n$ where $F_d$ is a homogeneous polynomial of degree $d$. These ideal sheaves had been $\mathcal{O}(-d)$.

**Problem 113.** Consider the affine toric variety $X = \mathbb{TV}(\langle(1,0), (1,2)\rangle; \mathbb{Z}^2) = \operatorname{Spec} R$ with $R = \mathbb{C}[x,y,z]/(xz - y^2)$ and the point $p = (1,1,1)$ corresponding to the ideal $\mathfrak{m} = (x-1, y-1, z-1) \subset R$.

a) Calculate the $\mathbb{C}$-vector space $\mathfrak{m}/\mathfrak{m}^2$ and conclude that $R_\mathfrak{m}$ is a regular local ring. Does it make a difference if $\mathfrak{m}$ is understood as an ideal of $R$ or of $R_\mathfrak{m}$?

b) Show explicitely that $\mathfrak{m}$, understood as an ideal in $R_\mathfrak{m}$, can be generated by two elements. Does this still hold for $\mathfrak{m}$ considered as an ideal in the original $R$?

*Solution:* (a) Let us denote $\mathfrak{m}$ correctly by $\mathfrak{m}_\mathfrak{m}$ when understood as an ideal of $R_\mathfrak{m}$. Then, $\mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2 = (\mathfrak{m}/\mathfrak{m}^2) \otimes_R R_\mathfrak{m}$. However, since $\mathfrak{m}/\mathfrak{m}^2$ is an $R = \mathfrak{m}$-module, i.e., since $\mathfrak{m}$ annihilates $\mathfrak{m}/\mathfrak{m}^2$, all elements from $R \setminus \mathfrak{m}$ act as units on this module. Hence, the localization $\otimes_R R_\mathfrak{m}$ is an isomorphism on this module. An alternative way to obtain this is

$$\mathfrak{m}_\mathfrak{m}/\mathfrak{m}_\mathfrak{m}^2 = (\mathfrak{m}/\mathfrak{m}^2) \otimes_R R_\mathfrak{m} = \mathfrak{m} \otimes_R (R/\mathfrak{m} \otimes_R R_\mathfrak{m}) = \mathfrak{m} \otimes_R R/\mathfrak{m} = \mathfrak{m}_\mathfrak{m}^2.$$

Now, to calculate this vector space, we obtain

$$\mathfrak{m}/\mathfrak{m}^2 = (x-1, y-1, z-1) \big/ (x-1, y-1, z-1)^2 + (xz - y^2)$$

or, after substituting $a = x-1$, $b = y-1$, $c = z-1$,

$$\mathfrak{m}/\mathfrak{m}^2 = (a,b,c)\big/(a,b,c)^2 + (ac - b^2 + a - 2b + c) = (a,b,c)\big/(a,b,c)^2 + (a - 2b + c).$$

This equals the vector space $(\mathbb{C}a \oplus \mathbb{C}b \oplus \mathbb{C}c)/\mathbb{C}\cdot(a-2b+c)$ which has $\mathbb{C}$-dimension 2. Since this coincides with the Krull dimension of of $X$, we obtain that $p$ is a smooth point.

(b) Any of the three generators $x-1$, $y-1$, or $z-1$ can be expressed by the remaining two. For instance, $y^2 - 1 = xz - 1$ leads to

$$(y-1) = \tfrac{1}{y+1}(xz-1) = \tfrac{1}{y+1}\big((x-1)(z-1) + (x-1) + (z-1)\big),$$

where $(y+1)$ is allowed to become a denominator since it is a unit in $R_\mathfrak{m}$.
If we consider $R$ instead of $R_\mathfrak{m}$, then we have still $\mathfrak{m} = (x-1, y-1) = (y-1, z-1)$; the

former follows from $R/(x-1, y-1) = \mathbb{C}[x, y, z]/\big(xz-y^2, x-1, y-1\big) = \mathbb{C}[z]/(z-1)$. On the other hand, since $R/(x-1, z-1) = \mathbb{C}[x, y, z]/\big(xz-y^2, x-1, z-1\big) = \mathbb{C}[z]/(y^2-1)$, we obtain $\mathfrak{m} \neq (x-1, z-1)$.

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VL "Algebra III"
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# 5. Aufgabenblatt zum 22.11.2023

**Problem 114.** We have seen in class that $D = \sum_i \lambda_i\, p_i \in \operatorname{Div} \mathbb{P}^1_{\mathbb{C}}$ (with $\lambda_i \in \mathbb{Z}$ and closed points $p_i \in \mathbb{P}^1_{\mathbb{C}}$) is a principal divisor $\Leftrightarrow \deg D := \sum_i \lambda_i = 0$.

a) If we replace $\mathbb{P}^1_{\mathbb{C}}$ by the affine line $\mathbb{A}^1_{\mathbb{C}}$ – how does this change the above claim? Which of the two implications does survive?

b) Let $I := V(z_0^2 + z_1^2) \in \mathbb{P}^1_{\mathbb{R}}$. For which $\lambda \in \mathbb{Z}$ is $D(\lambda) = 1 \cdot [0] + 1 \cdot [\infty] - \lambda \cdot I$ a principal divisor in $\mathbb{P}^1_{\mathbb{R}}$? (Here, we used the notation $0 := (1:0) \in \mathbb{P}^1_{\mathbb{R}}$ and $\infty := (0:1) \in \mathbb{P}^1_{\mathbb{R}}$.) Is there a general concept so that this becomes compatible with (a)?

*Solution:* a) In $\mathbb{A}^1_{\mathbb{C}}$, every divisor is a principal divisor: Prime divisors are $c = V(x-c)$, and they equal $\operatorname{div}(x-c)$. In particular, $\operatorname{Cl} \mathbb{A}^1_{\mathbb{C}} = 0$, and the map $\deg : \operatorname{Div} \mathbb{A}^1_{\mathbb{C}} \to \mathbb{Z}$ does not factor via $\operatorname{Cl} \mathbb{A}^1_{\mathbb{C}}$.

b) $f := x_0 x_1/(x_0^2 + x_1^2)$ shows that $\lambda = 1$ provides a principal divisor. Over $\mathbb{R}$, one should define the degree of a divisor as $\deg(\sum_i \lambda_i p_i) := \sum_i \lambda_i \deg p_i$ with $\deg p := [K(p) : \mathbb{R}]$. Recall that $K(p)$ denotes the residue field of $p$, i.e., $K(p) = \mathcal{O}_{\mathbb{P}^1_{\mathbb{R}},p}/\mathfrak{m}_{\mathbb{P}^1_{\mathbb{R}},p}$.

**Problem 115.** Let $C := \operatorname{Spec} \mathbb{C}[x,y]/(y^2 - x^3) \subset \mathbb{A}^2$. Denote $p := (x.y) \in C$, i.e., $p$ is the origin of $\mathbb{A}^2$. While $p$ is a smooth point of $\mathbb{A}^2$, it is a singular point of the curve $C$. Thus, the local ring $\mathcal{O}_{C,p}$ is a domain, but it is not normal.

a) Determine $\operatorname{ord}_p(x)$ and $\operatorname{ord}_p(y)$ by using the more general definition of the order in non-normal 1-codimensional points.

b) What are the principal divisors $\operatorname{div}(y/x)$ and $\operatorname{div}(x/y)$ and which of the two rational functions $y/x$ or $x/y$ belong to $\mathcal{O}_{C,p}$ or to $\mathbb{C}[x,y]/(y^2 - x^3)$?

c) Answer (b) for the squares of $y/x$ and $x/y$.

*Solution:* (a) $\mathcal{O}_{C,p}/(x) = \big(\mathbb{C}[x,y]/(y^2 - x^3)\big)_{(x,y)}/(x) = \mathbb{C}[x,y]/(x,y^2) = \mathbb{C}[y]/(y^2)$ and $\mathcal{O}_{C,p}/(y) = \big(\mathbb{C}[x,y]/(y^2 - x^3)\big)_{(x,y)}/(y) = \mathbb{C}[x,y]/(x^3,y) = \mathbb{C}[x]/(x^3)$ show that $\operatorname{ord}_p(x) = 2$ and $\operatorname{ord}_p(y) = 3$.

(b) $\operatorname{ord}_p(y/x) = 3 - 2 = 1$ and $\operatorname{ord}_p(x/y) = 2 - 3 = -1$. Since the numerators and denominators $x$ and $y$ are non-zero on every $c \in C \setminus \{p\}$, we obtain $\operatorname{ord}_c(y/x) = \operatorname{ord}_c(x/y) = 0$. Hence,

$$\operatorname{div}(y/x) = p \quad \text{and} \quad \operatorname{div}(x/y) = -p.$$

Since $\operatorname{ord}_p(x/y) < 0$, the rational function $x/y$ cannot belong to $\mathcal{O}_{C,p} \supseteq \mathbb{C}[x,y]/(y^2 - x^3)$. For $t := y/x$ we know that $\operatorname{ord}_p(t) = 1 \geq 0$. If $\mathcal{O}_{C,p}$ were normal, this would have implied $t \in \mathcal{O}_{C,p}$. Similarily had $\operatorname{div}(t) \geq 0$ implied that $t \in \mathbb{C}[x,y]/(y^2 - x^3)$. However, we have $\mathbb{C}[x,y]/(y^2 - x^3) = \mathbb{C}[t^2, t^3]$, which shows the missing of $t$. Moreover, the situation does not improve by localizing in the ideal $(t^2, t^3)$.

(c) As we just have seen, $t^2 \in \mathbb{C}[t^2, t^3] = \mathbb{C}[x, y]/(y^2 - x^3)$. Actually, we even have $t^2 = x$.

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## 6. Aufgabenblatt zum 29.11.2023

**Problem 116.** Let $E := V(y^2z - x^3 + xz^2) \subseteq \mathbb{P}^2_{\mathbb{C}}$; it is the usually first example of a smooth elliptic curve.

a) Show that for two (closed, maybe assume distinct) points $p, q \in E$ the line $\overline{pq}$ intersects $E$ in exactly one further point $\ell(p, q)$.

b) Show that the divisor $D := [p] + [q] - [r] - [s]$ (with closed points $p, q, r, s \in E$) is a principal one if $\ell(p, q) = \ell(r, s)$.

c) Consider the map $\Phi : E \to \mathrm{Cl}_0(E) := \ker(\deg) \subseteq \mathrm{Cl}(E)$, $p \mapsto [p] - [(0 : 1 : 0)]$. For points $p, q \in E$ find a third one $r \in E$ such that $\Phi(p) + \Phi(q) = \Phi(r)$.

*Solution:* (a) Let $L = L(x, y, z) = ax + by + c$ be the affine equation (with $z = 1$) for the line $\overline{pq}$; assume, w.l.o.g., $b \neq 0$. Then, substituting $y = -a/bx - c/b$, the affine $E$-equation $y^2 = x^3 - x = x(x^2 - 1)$ becomes an $x$-polynomial of degree 3. Besides $x(p)$ and $x(q)$ it has exactly one further root.

(b) Let $L_{p,q}, L_{r,s} \in \mathbb{C}[x, y, z]$ the homogeneous linear equations of the projective lines $\overline{pq}$ and $\overline{rs}$, respectively. Then, $f := L_{p,q}/L_{r,s} \in K(E)$ is a rational function with $\mathrm{div}(f) = [p] + [q] - [r] - [s]$.

(c) The equation $\Phi(p) + \Phi(q) = \Phi(r)$ means $D := [p] + [q] - [r] - [(0 : 1 : 0)] = 0$ in $\mathrm{Cl}(E)$, i.e., $D$ is supposed to be principal. Via (b), we can obtain this when $\ell(p, q) = \ell(r, (0 : 1 : 0))$.

Let $r \in E$ be the point obtained by reflecting $\ell(p, q)$ at the $x$-axis, i.e., switching the sign of the $y$-coordinate, Then, the line connecting $r$ and $(0 : 1 : 0)$ passes through $\ell(p, q)$, too.

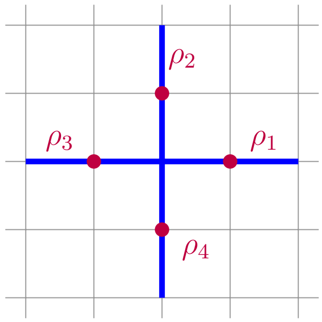
**Problem 117.** a) Denote $X := \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ and let $Y := (\mathbb{P}^2_{\mathbb{C}})'$ be the blow up of $\mathbb{P}^2_{\mathbb{C}}$ in some closed point. Understand $X$ and $Y$ as toric varieties and determine the class groups $\mathrm{Cl}(X)$ and $\mathrm{Cl}(Y)$ together with the class maps assigning each (toric) divisor its class.

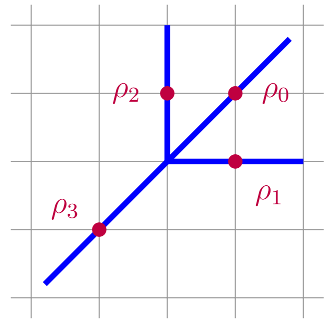
b) In $X$ we have divisors $\mathbb{P}^1_{\mathbb{C}} \times \{q\}$ and $\{p\} \times \mathbb{P}^1_{\mathbb{C}}$. Which of them are toric divisors, i.e., of the form $\overline{\mathrm{orb}}(\rho)$ for $\rho \in \Sigma(1)$?

c) In $Y$ we have the exceptional divisor $E$ and the strict and total transforms of lines $\ell \subset \mathbb{P}^2$. Which of them are toric? What are their corresponding rays?

d) Determine and draw the classes of the divisors from (b) and (c) within $\mathrm{Cl}(X)$ and $\mathrm{Cl}(Y)$ you have got from (a).

*Solution:* (a) The fans of $X$ and $Y$ are $\Sigma_X$ and $\Sigma_Y$,




and the associated maps $\mathbb{Z}^4 \to N = \mathbb{Z}^2$ are given by the matrices

$$A_X = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad A_Y = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & -1 \end{pmatrix}$$

respectively. We denote the kernels of these maps by $K_{X/Y} : \mathbb{Z}^2 \hookrightarrow \mathbb{Z}^4$; these are, e.g., the transposes of the following matrices:

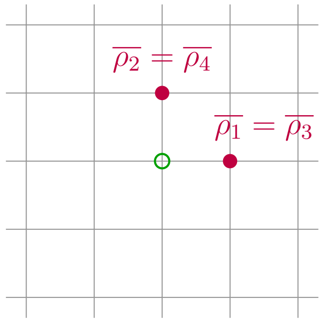
$$K_X^t = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad K_Y^t = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

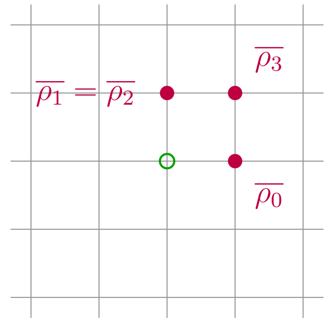
Thus, both class groups $\mathrm{Cl}(X)$ and $\mathrm{Cl}(Y)$ are isomorphic to $\mathbb{Z}^2$, and the matrices $K_X^t$ and $K_Y^t$ understood as linear maps $\mathbb{Z}^4 \twoheadrightarrow \mathbb{Z}^2$ describe $\mathbb{Z}^{\Sigma(1)} \twoheadrightarrow \mathrm{Cl}$.

(b) Only the divisors $\overline{\mathrm{orb}}(\rho_1) = \mathbb{P}^1_{\mathbb{C}} \times \{0\}$, $\overline{\mathrm{orb}}(\rho_3) = \mathbb{P}^1_{\mathbb{C}} \times \{\infty\}$, $\overline{\mathrm{orb}}(\rho_2) = \{0\} \times \mathbb{P}^1_{\mathbb{C}}$, and $\overline{\mathrm{orb}}(\rho_4) = \{\infty\} \times \mathbb{P}^1_{\mathbb{C}}$ are toric.

(c) The exceptional divisor is $E = \overline{\mathrm{orb}}(\rho_0)$. Moreover, the strict and total transforms of the lines $\ell$ are toric if and only $\ell$ itself was. Thus, we do have $\ell_i := \overline{\mathrm{orb}}(\varrho_i) = V(z_i)$ whith $i = 1, 2, 3$ where $z_1, z_2, z_3$ denote the homogeneous coordinates of $\mathbb{P}^2$, and $\varrho_i$ denote the three rays of the $\mathbb{P}^2$-fan. They equal $\rho_i$ before $\rho_0$ was introduced.
The "new" rays $\rho_i$ in $\Sigma_Y$ (with $i = 1, 2, 3$) represent the prime divisors $\pi^{\#}(\ell_i)$ being the strict transforms of the three lines. Since $\ell_3$ is disjoint to the point $p = (0 : 0 : 1) \in \mathbb{P}^2$ which was blown up, it is not effected from the map $\pi : (\mathbb{P}^2)' \twoheadrightarrow \mathbb{P}^2$. That is, $\pi^{\#}(\ell_3) = \pi^*(\ell_3)$ equals the total transform as well.
On the other hand, the closed subschemes $\pi^*(\ell_i)$ for $i = 1, 2$ contain $E$; they are not irreducible, hence no prime divisor. Actually, they are no divisors at all (only prime divisors are schemes). However, we can associate a reducible divisor to it, namely $\pi^{\#}(\ell_i) + E$.

(d) We draw $\mathrm{Cl} = \mathbb{Z}^2$. In both pictures, the origin is in the center. The class $\overline{\rho_i}$ of the divisor $\overline{\mathrm{orb}}(\rho_i)$ can be found via the $i$-th column of the matrices $K_X^t$ and $K_Y^t$, respectively.



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## 7. Aufgabenblatt zum 6.12.2023

**Problem 118.** Let $X$ be a normal $k$-variety ($\overline{k} = k$) with function field $K := K(X) \supseteq k$.

a) Show that elements $f \in K^* \setminus k$ correspond to dominant rational maps $f : X \dashrightarrow \mathbb{P}^1_k$.

b) Let $U \subseteq X$ be an open subset such that $\operatorname{div}(f)_{|U} \geq 0$. Show that $f : X \dashrightarrow \mathbb{P}^1_k$ is then truely defined on $U$. What about $V \subseteq X$ with $(-\operatorname{div}(f))_{|V} \geq 0$? Conclude that $f : X \dashrightarrow \mathbb{P}^1_k$ is always defined on $X \setminus Z$ with some closed $Z \subseteq X$ satisfying $\operatorname{codim}_X(Z) \geq 2$. In particular, this leads to a regular $f : X \to \mathbb{P}^1_k$, whenever $X$ is a curve.

c) Give two examples where one cannot extend $f : X \dashrightarrow \mathbb{P}^1_k$ to a globally defined $X \to \mathbb{P}^1_k$. One with $X$ being a non-normal curve, the second with $X$ being a normal surface.

*Solution:* (a) $f \in K^*$ induces $k[t] \to K$ via $t \mapsto f$. Since $f \notin k$ and $\overline{k} = k$, the element $f$ is transcendental over $k$, hence this map is injective – inducing an embedding $K(\mathbb{P}^1) = k(t) \hookrightarrow K$.

(b) Assume that $U = \operatorname{Spec} A$. Since $\operatorname{ord}_P(f) \geq 0$ for all $P \in \operatorname{Spec} A$ of height one, we obtain that $f \in A$. Hence, $k(t) \to K(X) = \operatorname{Quot}(A)$ with $t \mapsto f$ factors via $k[t] \to A$. Second, for $V \subseteq X$ this works similarily; we just look at $k[\frac{1}{t}] \subset k(t) \to K(X)$. Finally, we use that different prime divisors intersect in closed subsets with codimension at least 2. In particular, if $X$ is a curve, then each point has an open neighborhood satisfying either $\operatorname{div}(f) \geq 0$ or $\operatorname{div}(f) \leq 0$.

(c) First, one can see how normality is important: $\mathbb{P}^2 \supseteq V(y^2z - x^3) \dashrightarrow \mathbb{P}^1$ given by $f := y/x$. Then, we consider the same $f := y/x$ as a rational function $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$. It becomes the linear projection $(x : y : z) \mapsto (x : y)$ out of $(0 : 0 : 1)$ – and it cannot be extended to the whole $\mathbb{P}^2$. To obtain a decent map one is either forced to remove this point, i.e., considering $\mathbb{P}^2 \setminus \{((0 : 0 : 1))\} \to \mathbb{P}^1$, or one replaces $\mathbb{P}^2$ by the blowing up $\mathbb{F}_1 := \widetilde{\mathbb{P}^2}$ of $\mathbb{P}^2$ in $(0 : 0 : 1)$.
Recall from class that this can be observeed within the toric language, too.

**Problem 119.** If $D$ is a divisor on some $X$, then $x \in X$ is called a base point of $D$ if it is contained in the support of all $D' \in |D| := \{D' \geq 0 \mid D' \sim D\}$. We denote by $\operatorname{Bp}(D)$ the set of all these points; it is called the base locus of $D$.

a) Is this notion depending on $D$ or on its class $\overline{D} \in \operatorname{Cl}(X)$?

b) Let $X := \widetilde{\mathbb{A}^2}$ be the blowing up of $\mathbb{A}^2$ in the origin and denote by $E \subset X$ the exceptional (prime) divisor. Draw this situation via some fan and identify the ray corresponding to $E$. Denote $j : T \hookrightarrow X$ the open embedding. What are the local equations of $E$?

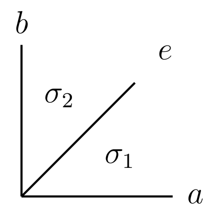
c) Draw the two cones in $M_{\mathbb{R}}$ that represent the monomial sections of $\mathcal{O}(E) \subseteq$

$j_*\mathcal{O}_T = \mathbb{C}[M]$ on the two affine charts of $X$. (*Hint:* These regions are cones with vertex corresponding to the inverse of the local equation of $E$.) Determine $\Gamma(X, \mathcal{O}(E))$ as the intersection of these two regions. What are the base points of $E$?

d) Draw the two cones for $\mathcal{O}(-E)$ and determine $\Gamma(X, \mathcal{O}(-E))$ as the intersection of these two regions. For each vertex of this region (representing some global section of $\mathcal{O}(E)$) determine the associated effective divisor being equivalent to $-E$. What is their intersection? What is $\mathrm{Bp}(-E)$?

*Solution:* (a) The linear system $|D|$ depends only on the class of $D$ – it consists of all effective divisors within this class.

(b) The fan in $N_{\mathbb{R}} = \mathbb{R}^2$ looks as follows:




The central ray $e$ corresponds to the divisor $E$. The horizontal and vertical rays $a$ and $b$ encode the strict transforms of the prime divisors $V(x), V(y) \subset \mathbb{A}^2$, respectively.

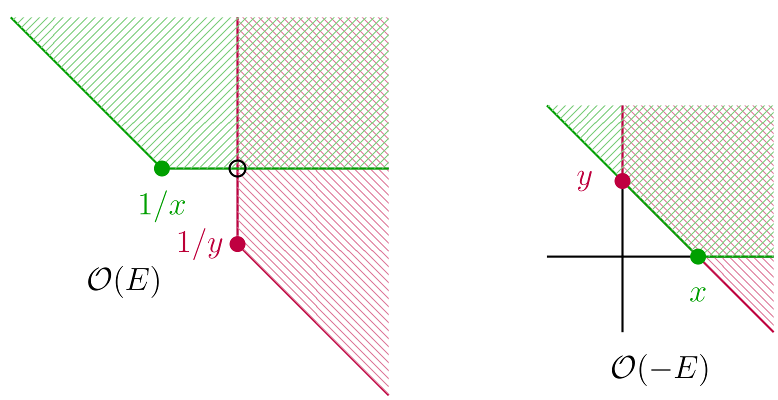
(c) The dual cones of the $\sigma_i$ are generated by

$$\sigma_1^\vee = \langle [[0,1],\ [1,-1]\rangle \quad \text{and} \quad \sigma_2^\vee = \langle [[1,0],\ [-1,1]\rangle.$$

The associated charts are represented by

$$\mathbb{C}[\sigma_1^\vee \cap M] = \mathbb{C}[y, x/y] \quad \text{and} \quad \mathbb{C}[\sigma_2^\vee \cap M] = \mathbb{C}[x, y/x],$$

respectively. The equations of $E$ are $y \in \mathbb{C}[\sigma_1^\vee \cap M]$ and $x \in \mathbb{C}[\sigma_2^\vee \cap M]$. The local generators of the sheaf are their inverses, i.e., $1/y$ and $1/x$, respectively. See the left hand side of the following figure:




The intersection of both regions gives $\Gamma(X, \mathcal{O}(E)) = \mathbb{C}[x,y] = \Gamma(\mathbb{A}^2, \mathcal{O}) = \Gamma(\widetilde{\mathbb{A}^2}, \mathcal{O})$. The origin 0 corresponds to the global section $1 = \chi^0$, and this provides the effective diviasor $E = E + \mathrm{div}(1)$. All other divisors inside $|E|$ are obtained by adding further effective divisors to $E$, i.e., $E$ is always contained in it. Thus, $\mathrm{Bp}(E) = E$.

(d) The drawing is done in the right hand side of the previous figure. We consider the vertex $[1,0]$ encoding $x \in \Gamma(X, \mathcal{O}(-E))$. We obtain

$$\operatorname{div}(x) = 1 \cdot \overline{\operatorname{orb}}(a) + 1 \cdot \overline{\operatorname{orb}}(e),$$

thus $(-E) + \operatorname{div}(x) = \overline{\operatorname{orb}}(a)$. Similarily, $(-E) + \operatorname{div}(y) = \overline{\operatorname{orb}}(b)$. In particular, these two effective divisors are disjoint (look at the orbits they are containing). Thus, $\operatorname{Bp}(-E) = \emptyset$, i.e., $(-E)$ is basepoint free.

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## 8. Aufgabenblatt zum 13.12.2023

**Problem 120.** Let $X$ be a normal variety over $\mathbb{C}$. In class we have associated to every coherent subsheaf $\mathcal{J} \subset K(X)$ (i.e., fractional ideal sheaf) a divisor $\operatorname{div}(\mathcal{J}) := \sum_P \operatorname{ord}_P(\mathcal{J}) \cdot P$ where the sum runs over all prime divisors $P \subset X$. Moreover, for any given divisor $D$ on $X$ we have seen that $\operatorname{div}(\mathcal{O}_X(-D)) = D$.

a) Show that, for any $\mathcal{J}$, we have $\operatorname{div}(\mathcal{J}^\vee) = -\operatorname{div}(\mathcal{J})$.

b) Starting with $\mathcal{J}$, show that $\mathcal{O}_X\big(-\operatorname{div}(\mathcal{J})\big) = \mathcal{J}^{\vee\vee}$.

c) Note that, for effective divisors $D$, the associated $\mathcal{O}_X(-D)$ is a true ideal sheaf in $\mathcal{O}_X$, i.e., it defines a closed subscheme which we will denote by $Z(D) \subset X$. If $D$ was a prime divisor, then $Z(D) = D$, and we will use this equality as a notation for general effective divisors, too. Thus, we obtain an exact sequence $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ with $\mathcal{O}_D := \iota_* \mathcal{O}_D$ where $\iota : D \hookrightarrow X$ denotes the closed embedding.

*Solution:* (a) We just need to control the local multiplicities in a prime divisor $P$, i.e., in a prime ideal $P \subset A$ (locally $X = \operatorname{Spec} A$), i.e., in the maximal ideal $P \subset A_P$ where the latter is a DVR with $P = (t)$. Now, $\mathcal{J}$ becomes a principal ideal $\mathcal{J} \cdot A_P = (f) = (t^k)$, and then $\mathcal{J}^\vee \cdot A_P = (1/f) = (t^{-k})$.

Note that we have never needed to assume that $\mathcal{J}$ is reflexive. However, $\mathcal{J}^\vee$ is.

(b) We work in a local setup. If $f \in \mathcal{J}$ and $P$ is a prime divisor, then $\operatorname{ord}_P(f) \geq \operatorname{ord}_P(\mathcal{J})$. Thus, $\operatorname{div}(f) - \operatorname{div}(\mathcal{J}) \geq 0$, i.e., in other words, $f \in \mathcal{O}_X\big(-\operatorname{div}(\mathcal{J})\big)$, and we have shown that $\mathcal{J} \subseteq \mathcal{O}_X\big(-\operatorname{div}(\mathcal{J})\big)$. Since the latter sheaf is reflexive, we obtain that it contains $\mathcal{J}^{\vee\vee}$, too.

For the other inclusion, we use the previous one for $\mathcal{J}^\vee$, i.e., we know that $\mathcal{J}^\vee \subseteq \mathcal{O}_X\big(-\operatorname{div}(\mathcal{J}^\vee)\big) = \mathcal{O}_X\big(\operatorname{div}(\mathcal{J})\big)$. Dualizing this, we obtain $\mathcal{O}_X\big(-\operatorname{div}(\mathcal{J})\big) = \mathcal{O}_X\big(\operatorname{div}(\mathcal{J})\big)^\vee \subseteq \mathcal{J}^{\vee\vee}$.

**Problem 121.** Let $A = \mathbb{C}[x,y,z]/(xz - y^2)$ and recall that it is an affine toric variety where the dual cone $\sigma^\vee \subset M_\mathbb{R} = \mathbb{R}^2$ is generated by the lattice points $A = [1,0]$, $B = [1,1]$, and $C = [1,2]$. Then, $x = \chi^A$, $y = \chi^B$, and $z = \chi^C$.

Denote by $P$ and $Q$ the two toric prime divisors given by the two rays of $\sigma$, say $P = V(x,y)$ and $Q = (y,z)$. In particular, $(x,y) = \mathcal{O}_A(-P)$ and $(y,z) = \mathcal{O}_A(-Q)$. Both ideals are reflexive. Finally, recall that $2P = \operatorname{div}(x)$ is a principal divisor.

Compare the ideal $(x)$ with $(x,y)^2 = (x^2, xy, y^2)$. Draw the lattice points corresponding to the monomials of $A$ and both ideals as figures in $\mathbb{Z}^2$. Which of the ideals $(x)$ or $(x^2, xy, y^2)$ is reflexive? What is their reflexive hulls (meaning the double duals) – can one observe this in the picture?

*Solution:* $(x)$ is a free $A$-module, hence reflexive. It equals $\mathcal{O}_A(-2P)$. On the other hand, $(x^2, xy, y^2) = \mathcal{O}_A(-P) \cdot \mathcal{O}_A(-P)$, hence $(x) = (x^2, xy, y^2)^{\vee\vee}$. While

$(x^2, xy, y^2) = (x^2, xy, xz) \subseteq (x)$, we know that $(x) \not\subseteq (x^2, xy, xz)$, hence we need to understand the other inclusion $(x) \subseteq (x^2, xy, xz)^{\vee\vee}$.

Since this is equivalent to $(x^2, xy, xz)^\vee \subseteq (x)^\vee$, we start with an $A$-linear $\phi : (x^2, xy, xz) \to A$ (which is just the multiplication with an element $\phi \in Q(A)$) and it remains to show that $\phi(x) = \phi \cdot x \in A$. Since, for any monomial (fractional) ideal $I$ also $\operatorname{Hom}(I, A)$ is generated by monomials, we may assume that $\phi$ is a monomial, i.e., $\phi = \chi^m$ for some $m \in \mathbb{Z}^2$. Now, we know that $m+2A$, $m+(A+B)$, $m+2B \in \sigma^\vee$, and we have to show that $m + A \in \sigma^\vee$. This follows immediately from studying the picture of $\sigma^\vee$ and the positions of $A, B, C$.

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## 9. Aufgabenblatt zum 20.12.2023

**Problem 122.** Let $\mathcal{L}$ be an invertible sheaf on a $k$-variety $X$.

a) Show that there is a natural map $\varphi_{\mathcal{L}} : \Gamma(X, \mathcal{L}) \otimes_k \mathcal{L}^{-1} \to \mathcal{O}_X$. The image of $\varphi$ is an ideal sheaf defining a closed subscheme $\mathcal{B}(\mathcal{L}) \subseteq X$ (called the "base locus" of $\mathcal{L}$).

b) Show that $\mathcal{L}$ is globally generated iff the base locus is empty.

c) For every $s \in \Gamma(X, \mathcal{L})$ there is an effective divisor $D(s) := V(s) \subseteq X$. Show that (the underlying topological space of) $\mathcal{B}(\mathcal{L})$ coincides with the intersection of the (supports of) the divisors $D(s)$.

d) Show that $\mathcal{B}(\mathcal{L}^k) \supseteq \mathcal{B}(\mathcal{L}^{kl})$ at the level of topological spaces. The intersection $\bigcap_{k\geq 1} \mathcal{B}(\mathcal{L}^k)$ is called the stable base locus of $\mathcal{L}$. Does it come with a natural scheme structure?

e) Do the global sections $\{z_0^2 + z_1^2, z_0^2 + z_2^2\}$ generate the sheaf $\mathcal{O}_{\mathbb{P}^2_{\mathbb{R}}}(2)$?

*Solution:* (a) $\varphi_{\mathcal{L}}$ is obtained by applying $(\otimes_{\mathcal{O}_X} \mathcal{L}^{-1})$ to the natural map $\Gamma(X, \mathcal{L}) \otimes_k \mathcal{O}_X \to \mathcal{L}$.

(b) The points where $\Gamma(X, \mathcal{L}) \otimes_k \mathcal{O}_X \to \mathcal{L}$ (or $\varphi_{\mathcal{L}}$) is surjective are exactly those points where $\mathcal{L}$ is globally generated.

(c) A point $P \in X$ is contained in $\operatorname{supp} D(s)$ iff $\varphi_{\mathcal{L}}(s) = 0$ in $\mathcal{O}_{X,P}/\mathfrak{m}_{X,P}$. Hence, $P \in \bigcap_{s\in\Gamma(\mathcal{L})} \operatorname{supp}(D_s) \Leftrightarrow \operatorname{im}(\varphi_{\mathcal{L}})_P \subseteq \mathfrak{m}_{X,P}$, i.e. iff $P \in \mathcal{B}(\mathcal{L})$.

(d) One has to argue set-theoretically. Hence, the base loci do not stabilize schem scheoretical.

(e) In the chart $D_+(z_0)$, the sections generate the ideal $(1 + y_1^2, 1 + y_2^2) \neq (1)$ with $y_i := z_i/z_0$.

**Problem 123.** Let $\mathcal{F}$ be a coherent sheaf on some scheme $X$. The fact that $\mathcal{F}$ is generated by finitely many global sections is equivalent to the existence of a sheaf homomorphism $f : \mathcal{O}_X^n \to \mathcal{F}$ such that (a) $f$ is surjective, or (b) $\Gamma(X, f)$ is surjective? What is the right answer – (a) or (b)? Give a proof of your answer and a counterexample for the wrong one: Is the condition to strong or to weak?

*Solution:* (a) is true. The condition (b) is neither to strong, nor to weak. It just says that $\Gamma(X, \mathcal{F})$ is a finitely generated $\Gamma(X, \mathcal{O}_X)$-module. Example: On $\mathbb{P}^1$ is $\mathcal{O} \xrightarrow{0} \mathcal{O}(-1)$ surjective on the global sections.

**Problem 124.** a) Let $F_d \in \mathbb{C}[x, y, z]_d$ be a (non-zero) homogeneous polynomial of degree $d$. It defines the curve $C := V_+(F_d) \subset \mathbb{P}^2$, and we denote by $\iota : C \hookrightarrow \mathbb{P}^2$ the closed embedding.

Assume that $H \subseteq \mathbb{P}^2$ is a hyperplane, i.e., a line, that is not contained in $C$ and call $P_1, \ldots, P_d$ the (maybe partially coinciding) points of $C \cap H$. That is, we understand the points of $C \cap H$ with multiplicities.

Show that $\mathcal{O}_C(1) := \iota^*\mathcal{O}_{\mathbb{P}_2}(1)$ is isomorphic to $\mathcal{O}_C(P_1 + \dots P_d)$.

b) Apply (a) to the following situation: $F_3(x,y,z) = y^2z - x(x^2 - z^2)$ and $H = V_+(z)$.

*Solution:* (a) Let $L(x,y,z)$ be the linear equation defining $H$. Since $\mathcal{O}_{\mathbb{P}_2}(1) \cong \mathcal{O}_{\mathbb{P}_2}(H)$, we may investigate $\iota^*\mathcal{O}_{\mathbb{P}_2}(H)$ instead of $\iota^*\mathcal{O}_{\mathbb{P}_2}(1)$. If $p \in C \subset H$, then we denote by $\ell, f \in \mathcal{O}_{\mathbb{P}^2,p}$ the equations of $H$ and $C$ in $p$, respectively. That is, they are induced from $L$ and $F_d$, respectively. Since $p \in C$, we know that $f \in \mathfrak{m}_p \subset \mathcal{O}_{\mathbb{P}^2,p}$. On the other hand, $\ell$ might be in $\mathfrak{m}_p$, or it is a unit in $\mathcal{O}_{\mathbb{P}^2,p}$ – just depending on the fact whether $p \in H$ or not.

The stalk $\mathcal{O}_{\mathbb{P}_2}(H)_p \subseteq K(\mathbb{P}^2) = \mathbb{C}(x,y,z)$ equals $\frac{1}{\ell} \cdot \mathcal{O}_{\mathbb{P}^2,p}$. Pulling $\mathcal{O}_{\mathbb{P}_2}(H)$ back via $\iota^*$ means to consider the surjective map $\iota^* : \mathcal{O}_{\mathbb{P}^2,p} \twoheadrightarrow \mathcal{O}_{C,p}$. The sheaf $\iota^*\mathcal{O}_{\mathbb{P}_2}(H)$ is then given in $p$ via its stalk $\iota^*\mathcal{O}_{\mathbb{P}_2}(H)_p = \frac{1}{\iota^*(\ell)}\mathcal{O}_{C,p} \subset K(C)$. On the other hand, $\iota^*(\ell) \in K(C)$ is a rational function such that $\mathrm{ord}_p(\iota^*(\ell))$ equals the multiplicity of $H \cap C$ in $p$.

To say the full and formally correct truth, we *define* the multiplicity of the intersection points $P_i$ in $C \cap H$ exactly via this order.

(b) $P_1 = P_2 = P_3 = (0:1:0)$.

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## 10. Aufgabenblatt zum 10.1.2024

**Problem 125.** Let $\Delta \subsetneq M_{\mathbb{Q}}$ be a lattice polyhedron and $\Sigma$ be a subdivision of the normal fan $\mathcal{N}(\Delta)$. For each top-dimensional cone $\sigma$, we denote by $\Delta(\sigma)$ the corresponding vertex of $\Delta$. In class we had defined the invertible sheaf $\mathcal{O}_{\mathbb{TV}(\Sigma)}(\Delta)$ by glueing $\mathbf{x}^{\Delta(\sigma)} \cdot \mathcal{O}_{\mathbb{TV}(\sigma)} \subseteq k[M]$ from the open charts $\mathbb{TV}(\sigma) \subseteq \mathbb{TV}(\Sigma) =: X$.

a) Check the technical details of this construction.

b) Could we have, instead, defined $\mathcal{O}_{\mathbb{TV}(\Sigma)}(\Delta) := \sum_{\sigma\in\Sigma} \mathbf{x}^{\Delta(\sigma)} \cdot \mathcal{O}_{\mathbb{TV}(\Sigma)}$?

c) Can one see directly that $\dim_k \Gamma(X, \mathcal{O}(\Delta))$ is finite-dimensional for (bounded) polytopes $\Delta$?

d) The just constructed $\mathcal{O}_{\mathbb{TV}(\Sigma)}(\Delta)$ is a subsheaf of $k[M]$ (meaning of $j_*\mathcal{O}_T \subset K(X)$). Hence, it gives rise to a Cartier divisor $D(\Delta)$ with $\mathcal{O}_{\mathbb{TV}(\Sigma)}(D(\Delta)) = \mathcal{O}_{\mathbb{TV}(\Sigma)}(\Delta)$. Write $D(\Delta)$ as a Weil divisor, i.e., as an explicit sum of prime divisors $\overline{\mathrm{orb}}(\rho)$ with $\rho \in \Sigma(1)$ with integer coefficients.

e) Assume that $\Sigma = \mathcal{N}(\Delta)$. Repeat (d), but use $\overline{\mathrm{orb}}(F)$ with facets $F \leq \Delta$ instead of $\overline{\mathrm{orb}}(\rho)$ with $\rho \in \Sigma(1)$.

f) What is the condition (in terms of $\Delta$) for $D(\Delta) \geq 0$?

*Solution:* (a) If $\sigma' \in \Sigma$ is a (maximal) cone of $\Sigma$, then for any $a \in \sigma'$ we know that $\langle \Delta(\sigma'), a\rangle = \min\langle \Delta, a\rangle \leq \langle \Delta(\sigma), a\rangle$. In particular, $\Delta(\sigma) - \Delta(\sigma') \in (\sigma')^\vee \cap M$, hence $\mathbf{x}^{\Delta(\sigma)} = \mathbf{x}^{\Delta(\sigma)-\Delta(\sigma')} \cdot \mathbf{x}^{\Delta(\sigma')} \in k[(\sigma')^\vee \cap M] \cdot \mathbf{x}^{\Delta(\sigma')} = \mathbf{x}^{\Delta(\sigma')} \cdot \mathcal{O}_{\mathbb{TV}(\sigma')}$ is a section. In particular, $\mathbf{x}^{\Delta(\sigma)} \cdot \mathcal{O}_{\mathbb{TV}(\sigma')} \subseteq \mathbf{x}^{\Delta(\sigma')} \cdot \mathcal{O}_{\mathbb{TV}(\sigma')}$, hence $\mathbf{x}^{\Delta(\sigma)} \cdot \mathcal{O}_{\mathbb{TV}(\sigma\cap\sigma')} = \mathbf{x}^{\Delta(\sigma')} \cdot \mathcal{O}_{\mathbb{TV}(\sigma\cap\sigma')}$.

(b) Moreover, since $\sigma'$ in (a) was arbitrary, this shows that $\mathbf{x}^{\Delta(\sigma)}$ is a global section of $\mathcal{O}_{\mathbb{TV}(\Sigma)}(\Delta)$. Since these global sections contain all local generators, they generate $\mathcal{O}_{\mathbb{TV}(\Sigma)}(\Delta)$. Hence, the formula of (b) is true – and may be used as definition, too.

(c) We have seen that $\Delta \cap M$ forms a basis of $\Gamma(\mathbb{TV}(\Sigma), \mathcal{O}(\Delta)$. Whenever the polyhedron $\Delta$ is compact, then it has only finitely many lattice points.

(d) On $\mathbb{TV}(\sigma)$ we have $D(\Delta) = \mathrm{div}(\mathbf{x}^{-\Delta(\sigma)})$, i.e., it equals $-\sum_{\rho\in\sigma(1)} \langle \Delta(\sigma), \rho\rangle \cdot \overline{\mathrm{orb}}(\rho)$. On the other hand, the coefficients can be written as $\min\langle \Delta, \rho\rangle$, thus we can summarize our information to

$$D(\Delta) = -\sum_{\rho\in\Sigma(1)} \min\langle \Delta, \rho\rangle \cdot \overline{\mathrm{orb}}(\rho).$$

(e) Alternatively, we may replace each ray $\rho$ by its associated facet $F \leq \Delta$, writing $\overline{\mathrm{orb}}(\rho) = \overline{\mathrm{orb}}(F)$. Then, we obtain

$$D(\Delta) = -\sum_{F\leq\Delta} d(F, 0) \cdot \overline{\mathrm{orb}}(F)$$

where $d(F,0)$ denotes the lattice distance of 0 from the affine hyperplane spanned by $F$ – and the sign is set positive iff 0 is on the same side as $\Delta$.

(f) $D(\Delta)$ is effective if and only if $\Delta$ contains the origin.

**Problem 126.** b) Let $\Sigma$ be the complete fan in $\mathbb{Q}^2$ that is spanned by the rays $(1,0)$, $(0,1)$, and $\pm(1,1)$. Consider on $X := \mathbb{TV}(\Sigma)$ the ideal sheaves $I_+$, $I_-$ of the closed orbits $\overline{\mathrm{orb}}(\pm(1,1))$, respectively. Show that both sheaves are invertible and describe (draw) for the four sheaves $I_\pm^{\pm 1}$ the local generators. What are their global sections? Which of these four sheaves are globally generated? How does the corresponding morphism look like?

a) Do the same with the subfan $\mathcal{F}$ of $\Sigma$ that is generated by the rays $(1,0)$, $(0,1)$, and $(1,1)$. Consider here the ideal sheaf of $\overline{\mathrm{orb}}(1,1) \subseteq \widetilde{\mathbb{A}^2} = \mathbb{TV}(\mathcal{F})$ and its inverse.

*Solution:* Let $\sigma = \langle (1,0),\, (1,1)\rangle$. Then, in the chart

$$\mathbb{TV}(\sigma) = \operatorname{Spec} k\big[\langle [0,1],\, [1,-1]\rangle \cap \mathbb{Z}^2\big] \subseteq \widetilde{\mathbb{A}^2} \subseteq X$$

the sheaves $I_+ = \mathcal{O}(-E)$ and $I_+^{-1} = \mathcal{O}(E)$ are generated by the monomials $\mathbf{x}^{[0,1]}$ and $\mathbf{x}^{[0,-1]}$, respectively. Hence, $\mathcal{O}(-E)$ is globally generated with $\Delta = \mathrm{conv}\{[0,1],\, [1,0]\} + \mathbb{Q}^2_{\geq 0}$, but $\mathcal{O}(E)$ is not. Nevertheless, $\Gamma(\widetilde{\mathbb{A}^2}, \mathcal{O}(E)) \xleftarrow{\sim} \Gamma(\widetilde{\mathbb{A}^2}, \mathcal{O}) \,\widehat{=}\, \mathbb{Q}^2_{\geq 0}$.

In $X$ there is also the chart $\mathbb{TV}(\tau)$ with $\tau = \langle (1,0),\; (-1,-1)\rangle$. There, $\mathcal{O}(\pm E)$ is trivial, i.e. generated by 1. Thus, $\Gamma(X, \mathcal{O}(-E)) = 0$ (in particular, $-E$ is no longer globally generated) and $\Gamma(X, \mathcal{O}(-E)) = k$. On the other hand, $H := \overline{\mathrm{orb}}(-(1,1))$ gives rise to a globally generated sheaf $I_-^{-1} = \mathcal{O}_X(H)$. The corresponding polyhedron has the vertices $[0,0]$, $[1,0]$, $[0,1]$. The corresponding morphism is the blowing up $X \to \mathbb{P}^2$ contracting $E$.

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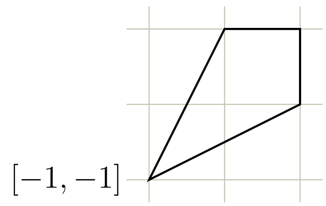
## 11. Aufgabenblatt zum 17.1.2024

**Problem 127.** Let $\Delta \subset \mathbb{R}^2$ be the quadrangle with the vertices $v_1 = [1,0]$, $v_2 = [1,1]$, $v_3 = [0,1]$, $v_4 = [-1,-1]$.




a) Show that $\Delta$ cannot be written as a Minkowski sum $\Delta = \nabla^1 + \nabla^2$ with lattice polygons $0 \neq \nabla^i \subset \mathbb{R}^2$ $(i = 1, 2)$.

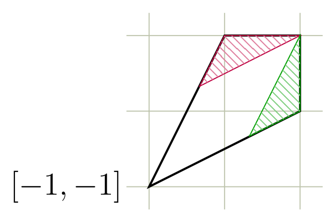
b) Give an example of a decomposition $\ell \cdot \Delta = \nabla^1 + \nabla^2$ with $\ell \in \mathbb{N}$ and $\nabla^i \subset \mathbb{R}^2$ being two lattice triangles $(i = 1, 2)$.

c) Construct the polyhedral cone $C(\Delta)$ (of Minkowski summands of $\Delta$) and explain how the subsemigroup of lattice points within $C(\Delta)$ reflects (a) and (b).

*Solution:* (a) There is the decomposition

$$\Delta = \operatorname{conv}\{[1,1],\ [0,1],\ [-\tfrac{1}{3}, \tfrac{1}{3}]\} + \operatorname{conv}\{[0,-1],\ [0,0],\ [-\tfrac{2}{3}, -\tfrac{4}{3}]\}$$

which is, up to opposite shifts of the two summands, unique. In the figure below,




the red triangle displays the first summand, but the green one is just an integral shift of the second one. Since the summands are non-lattice triangles, we are done.

(b) Multiplying the decomposition of (a) with 3 gives the result.

(c) We have four edges

$$d^1 = \overrightarrow{v_1 v_2} = [0,1],\ \ d^2 = \overrightarrow{v_2 v_3} = [-1,0],\ \ d^3 = \overrightarrow{v_3 v_4} = [-1,-2],\ \ d^4 = \overrightarrow{v_4 v_1} = [2,1],$$

i.e., $C(\Delta) \subset \mathbb{R}^4_{\geq 0} = \{(t_1, t_2, t_3, t_4) \mid t_i \geq 0\}$ is obtained from the two linear closing conditions

$$t_1[0,1] + t_2[-1,0] + t_3[-1,-2] + t_4[2,1] = [0,0].$$

That is, $2t_4 = t_2 + t_3$ and $2t_3 = t_1 + t_4$. Using just the coordinates $(t_3, t_4)$, the non-negaitivity conditions yield

$$t_3, t_4 \geq 0, \quad \text{and} \quad 2t_3 - t_4 = t_1 \geq 0,\ 2t_4 - t_3 = t_2 \geq 0.$$

Hence, $C(\Delta)^\vee = \langle [2,-1];\ [-1,2] \rangle$, yielding $C(\Delta) = \langle (1,2),\ (2,1) \rangle$.

The point $(1,1) \in C(\Delta)$ stands for $(t_1, t_2, t_3, t_4) = (1,1,1,1)$ (i.e., no edge dilation at all); it corresponds to the original $\Delta$. The generator $(1,2)$ means $(t_1, t_2, t_3, t_4) = (0,3,1,2)$; the zero-entry encodes the disappearence of an edge – leading to a triangle. The second generator is $(2,1)$, i.e., $(t_1, t_2, t_3, t_4) = (3,0,2,1)$. Their sum is $(3,3)$, i.e., $3 \cdot \Delta$. The original $\Delta$, however, corresponds to $(1,1) \in C(Q)$ and cannot be written a sum of two integral points on the rays of $C(\Delta)$. Instead, we may write $(1,1) = \frac{1}{3} \cdot (1,2) + \frac{1}{3} \cdot (2,1)$. This reflects the non-integral decomposition we had started with.

**Problem 128.** a) Let $0 \to A \to B \to C \to 0$ be an exact sequence among free $R$-modules of ranks $a$, $b$, and $c$, respectively. In particular, we have $b = a + c$. For any $n \in \mathbb{N}$ we define a decreasing filtration $F^\bullet(\Lambda^n B)$ as follows:

$$F^k(\Lambda^n B) := \langle a_1 \wedge \ldots \wedge a_k \wedge b_{k+1} \wedge \ldots \wedge b_n \mid a_i \in A, b_j \in B \rangle.$$

In particular, $F^0(\Lambda^n B) = \Lambda^n B$ and $F^n(\Lambda^n B) = \Lambda^n A$ and $F^{n+1} = 0$.

Show that there are natural isomorphisms $F^k/F^{k+1} \cong (\Lambda^k A) \otimes_R (\Lambda^{n-k} C)$. That means that they should not depend on the choice of special bases, they should commute with localizations of $R$ and, hence, yield a corresponding result for locally free $\mathcal{O}_X$-modules on some ringed space $(X, \mathcal{O}_X)$.

b) Consider the special case of $n = b$.

*Solution:* (a) We will define an $R$-linear map

$$\Phi_k : (\Lambda^k A) \otimes_R (\Lambda^{n-k} C) \to F^k(\Lambda^n B)/F^{k+1}(\Lambda^n B);$$

there seems to be no good way to define a natural inverse. Thus, aiming at $\Phi_k$, we set

$$(a_1 \wedge \ldots \wedge a_k) \otimes (c_{k+1} \wedge \ldots \wedge c_n) \mapsto (a_1 \wedge \ldots \wedge a_k) \wedge (b_{k+1} \wedge \ldots \wedge b_n)$$

where $b_j \in B$ are some preimages of $c_j \in C$. First, this assignment is well-defined: If we replace some $b_j$ by another $b'_j$ representing $c_j$, then $a_j := b'_j - b_j \in A$, and the RHS is contained in $F^{k+1}(\Lambda^n B)$.

Second, the assignment is multilinear and alternating in both factors. Hence, $\Phi_k$ does indeed define an $R$-linear map as being announced. Moreover, it is obviously surjective. To check injectivity one just chooses compatible bases of $A, B, C$ which, in particular, fixes some splitting of the given exact sequence.

While the last step does leave the canonical setup, we should emphasize that the isomorphism $\Phi_k$ had been defined in a natural way. Thus, it is compatible with localizations and glues to the setup of locally free sheaves.

(b) If $n = b$, then asking for $k \le a$ and $(n-k) \le c$ ensuring that $(\Lambda^k A) \otimes_R (\Lambda^{n-k} C) \neq 0$ implies that $k = a$ and $n - k = c$. In particular, we obtain that

$$F^k(\Lambda^b B)/F^{k+1}(\Lambda^b B) = \begin{cases} (\Lambda^a A) \otimes_R (\Lambda^c C) & \text{if } k = a \\ 0 & \text{otherwise.} \end{cases}$$

Hence, since the filtration consists of a single jump only, $\Lambda^b B = (\Lambda^a A) \otimes_R (\Lambda^c C)$.

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## 12. Aufgabenblatt zum 24.1.2024

**Problem 129.** Let $\Delta \subseteq M_{\mathbb{R}} \cong \mathbb{R}^d$ be a $d$-dimensional lattice polytope and define $\Sigma := \mathcal{N}(\Delta)$. Assume that $\Sigma$ is smooth and denote $X = \mathbb{TV}(\Sigma) = \mathbb{P}(\Delta)$.

a) Show that an invertible line bundle $\mathcal{L}$ on $X$ is basepoint free (i.e., globally generated) if and only if for each $\sigma \in \Sigma$ there is an effective divisor $D$ with $\operatorname{supp}(D) \cap \mathbb{TV}(\sigma) = \emptyset$ and $\mathcal{O}_X(D) \cong \mathcal{L}$.

b) Denote by $\pi : \mathbb{Z}^{\Sigma(1)} \twoheadrightarrow \operatorname{Cl}(X)$ the canonical map assigning to each toric divisor its class in $\operatorname{Cl}(X) = \operatorname{Pic}(X)$. It gives rise to $\pi_{\mathbb{R}} : \mathbb{R}^{\Sigma(1)} \twoheadrightarrow \operatorname{Cl}(X) \otimes \mathbb{R}$. Recall that $\pi_{\mathbb{R}}\big(\mathbb{R}^{\Sigma(1)}_{\geq 0}\big)$ is the so-called effective or pseudo-effective cone $\operatorname{Eff}(X)$. Show that

$$\bigcup_{\sigma \in \Sigma(d)} \pi_{\mathbb{R}}\big(\mathbb{R}^{\Sigma(1)\setminus\sigma(1)}_{\geq 0}\big)$$

equals the so-calles nef cone $\operatorname{Nef}(X)$ being the closure of all semi-ample or basepoint free divisors.

c) Let $\Sigma_a \subseteq \mathbb{R}^2$ $(a \in \mathbb{Z}_{\geq 1})$ be the complete fan spanned by the rays $\Sigma_a(1) = \{(0,-1),\ (1,0),\ (0,1),\ (-1,a)\}$. The corresponding toric variety $F_a := \mathbb{TV}(\Sigma_a)$ is called the $a$-th Hirzebruch surface. Determine the group $\operatorname{Cl}(F_a) = \operatorname{Pic}(F_a)$ and the cones $\operatorname{Nef}(F_a), \operatorname{Eff}(F_a) \subseteq \operatorname{Cl}(F_a)_{\mathbb{R}}$.

*Solution:* (a) The part $(\Leftarrow)$ is clear. On the other hand, if $\mathcal{L}$ is globally generated, then there is a lattice polytope $\nabla \in C(\Delta)$, i.e., $\sigma$ is a refinement of $\mathcal{N}(\nabla)$, such that $\mathcal{L} \cong \mathcal{O}_X(\nabla)$. Now, for any given $\sigma \in \Sigma$, we may translate $\nabla$ such that the corresponding vertex sits in the origin, i.e., $\nabla(\sigma) = 0$. Then, the associated Weil divisor $\operatorname{div}(\nabla)$ is toric, effective, and none of the prime divisors $\overline{\operatorname{orb}}(\rho)$ with $\rho \in \sigma(1)$ occurs. That is, all prime divisors showing up in $\operatorname{div}(\nabla)$ are disjoint to $\mathbb{TV}(\sigma)$.

(b) This follows straight away from (a).

(c) The fan $\Sigma_a$ is obtained from $\begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & a \end{pmatrix} : \mathbb{Z}^4 \to \mathbb{Z}^2$; the Gale transform into the $M$-level yields $\begin{pmatrix} 1 & 0 & 1 & 0 \\ a & 1 & 0 & 1 \end{pmatrix} : \mathbb{Z}^4 \to \mathbb{Z}^2 = \operatorname{Cl}(F_a) = \operatorname{Pic}(F_a)$. In particular, we can observe the fiber class $e_2 = [D_2] = [D_4]$. Moreover, $\operatorname{Eff}(F_a) = \langle e_1, e_2 \rangle = \langle [D_3], [D_2] \rangle$ and

$$\operatorname{Nef}(F_a) = \langle e_1 + ae_2,\ e_2 \rangle \cap \langle e_2, e_1 \rangle \cap \langle e_1, e_2 \rangle \cap \langle e_2, e_1 + ae_2 \rangle = \langle e_1 + ae_2,\ e_2 \rangle.$$

Some additional comments: We study the divisors $D_1$, $D_2$, $D_3$, and $D_1 + D_2$. The associated polyhedra $\Delta(\bullet)$ are (the subscripts indicate the local generator status)

$$\Delta(D_1) = \operatorname{conv}\{[0,0]_{23,34},\ [a,1]_{41},\ [0,1]_{12}\}, \quad \Delta(D_2) = \operatorname{conv}\{[-1,0]_{12,23},\ [1,0]_{34,41}\}$$

and

$$\Delta(D_3) = \{[0,0]_{12,41}\}, \quad \Delta(D_1+D_2) = \operatorname{conv}\{[-1,1]_{12},\ [-1,0]_{23},\ [0,0]_{34},\ [a,1]_{41}\}.$$

One sees immediately that $[D_1+D_2]$ im $\operatorname{Amp}(F_a)$ is very ample. Moreover, the divisors $D_1$ and $D_2$ are both globally generated and yield the contractions to $\mathbb{P}(a,1,1)$ and $\mathbb{P}^1$, respectively. On the other hand, the single global section of $\mathcal{O}(D_3)$ generates the sheaf only in two charts; the remaining local generators are $[0,-1]$ and $[-a,-1]$ for the cones 23 and 34, respectively. The exceptional divisor is its own base locus, and the corresponding rational map sends $F_a \setminus D_3$ to a point.

**Problem 130.** Let $A \to B$ be an algebra, denote $I := \ker(B \otimes_A B \twoheadrightarrow B)$. and consider $B \otimes_A B$ (and thus $I$) as $B$-modules via the multiplication on the left hand factors.

a) Show that $D : B \to I/I^2$, $b \mapsto b \otimes 1 - 1 \otimes b$ is an $A$-derivation.

b) Show that the induced $B$-linear map $\Omega_{B|A} \to I/I^2$ is an isomorphism.

*Solution:* (a) The key equation is $D(bc) = bD(c)+cD(b)-(b\otimes 1-1\otimes b)\cdot(c\otimes 1-1\otimes c)$.

(b) Denote $\Phi : \Omega_{B|A} \to I/I^2$; for every $B$-module $M$ it induces, via $\operatorname{Hom}_B(\bullet, M)$, the $B$-linear map $\Phi_M : \operatorname{Hom}_B(I/I^2, M) \to \operatorname{Der}_A(B, M)$ sending $\varphi \mapsto \varphi \circ D$.

The elements $D(b) = b \otimes 1 - 1 \otimes b$ generate $I$: Indeed, $cD(b) = (bc) \otimes 1 - c \otimes b$, hence modulo those element, we can modify any $\sum_i b_i \otimes c_i \in I$ to $\sum_i (b_i c_i) \otimes 1$. On the other hand, the membership with $I$ means $\sum_i b_i c_i = 0$.

Thus, $\Phi_M$ is injective. For the surjectivity of $\Phi_M$, assume that $f : B \to M$ is an $A$-derivation. We define $F : B \otimes_A B \to M$ via the $A$-bilinear map $(b, c) \mapsto b \cdot f(c)$. This map is even $B$-linear (recall that the $B$-action on the source happens via the first factor). Eventually, we consider the restriction $F|_I$ and it remains to check that $F|_{I^2} = 0$. This follows from the derivation properties of $f$ and the key equation from Part (a).

**Problem 131.** Show directly that $\Omega_E = \mathcal{O}_E$ for smooth $E = \overline{V(y^2 - f_3(x))} \subseteq \mathbb{P}^2$, i.e. $f_3 \in \mathbb{C}[x]$ is a polynomial of degree three without double zeros, e.g. $f_3(x) = x^3 - x$.

*Solution:* Since $y^2 = f(x) = x^3 - x$, we obtain that $dy/f'(x) = dx/(2y)$, hence this expression is a generator of $\Omega$ on the chart $E \cap \mathbb{C}^2$. The other chart covering $(0:1:0)$ has coordinates $a = x/y$ and $c = 1/y$. Hence $dx/y = c \cdot d(a/c) = da - a/c \cdot dc$. The equation $a^3 = ac^2 + c$ yields $(3a^2 - c^2)da = (2ac+1)dc$, thus $(2ac+1)(da - a/c \cdot dc) = (2ac+1) \cdot da - a/c\,(3a^2 - c^2) \cdot da = -2\,da$, i.e. $dx/(2y) = (2ac+1)^{-1} da$ is also a generator of $\Omega$ in $(0:1:0)$.

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## 13. Aufgabenblatt zum 31.1.2024

**Problem 132.** In comparison to the (smooth) elliptic curve $V(y^2z - x(x^2 - z^2)$ wie consider now the singular case: Let $E := V(y^2z - x^3) \subseteq \mathbb{P}^2_{\mathbb{C}}$. Show that $\Omega_E$ is not locally free. What about the sheaf $\mathcal{H}om_{\mathcal{O}_E}(\Omega_E, \mathcal{O}_E)$? Calculate all groups $\operatorname{Ext}^i_{\mathcal{O}_{E,p}}(\Omega_{E,p}, \mathcal{O}_{E,p})$.

*Solution:* For our task, only the affine chart $E = V(y^2 - x^3) \subseteq \mathbb{A}^2_{\mathbb{C}}$ is interesting – it contains the only singular point $p = (0:0:1)$. Denote $R := \mathbb{C}[x,y]/(y^2 - x^3) = \mathbb{C}[t^2, t^3]$; by the usual exact sequence for $\mathbb{C} \to k[x,y] \to R$ we obtain the $R$-module $\Omega_E$ as a cokernel $0 \longrightarrow R \xrightarrow{(-3x^2, 2y)^T} R^2 \longrightarrow \Omega_E \longrightarrow 0$, Note that the injectivity of the map $R \to R^2$ does not come from the general statement; one just checks it directly. Applying $\operatorname{Hom}_R(\bullet, R)$ yields

$$0 \longrightarrow \operatorname{Hom}_R(\Omega_E, R) \longrightarrow R^2 \xrightarrow{(-3x^2, 2y)} R \longrightarrow \operatorname{Ext}^1_R(\Omega_E, R) \longrightarrow 0.$$

This exact sequence may be localized to give information about the points $p \in \operatorname{Spec} R$ as well. It follows that $\operatorname{Hom}_R(\Omega_E, R) \cong R$, i.e. $\mathcal{H}om_{\mathcal{O}_E}(\Omega_E, \mathcal{O}_E)$ is locally free. On the other hand, replacing $R$ by $R_p$ with $p \in \operatorname{Spec} R \subseteq E$, we obtain $\operatorname{Ext}^1_{R_p}(\Omega_E, R_p) = 0$ unless $p = (0,0)$. For $p = (0,0)$ (being represented by the ideal $(x,y) \subset R$) we obtain $\operatorname{Ext}^1 = R_{(x,y)}/(x^2, y) \cong \mathbb{C}^2$. In particular, $\Omega_{E,0}$ is not free.

**Problem 133.** Let $k = \mathbb{F}_3(u)$. We define $C$ as the affine curve $C := V(y^2 - x^3 - u) \subseteq \mathbb{A}^2_k$. Show that the prime ideal $P := (y)$ is a closed point $P \in C$. Moreover, check that the local ring $\mathcal{O}_{C,P}$ is regular, but that $\Omega_C$ is not even free at this point. What happens, if we consider the same $C$ over $\overline{k}$ instead?

*Solution:* Denote $A := k[x,y]/(y^2 - x^3 - u)$. Then $A/P = k[x]/(x^3 + u) = k(\sqrt[3]{u})$ is a field. Since $P$ is principal, the one-dimensional local ring $A_P$ is regular. On the other hand, $\Omega_{C,P} = (A\,dx \oplus A\,dy/2y\,dy)_P = A_P\,dx \oplus A_P/(y)\,dy$, i.e. this module does even contain torsion elements.

In $\overline{k}$ there is an element $v$ with $v^3 + u = 0$. The ideal $(y)$ is not prime anymore, but it can be replaced by $P := (y, x - v)$. But then, this new $P$ is not principal anymore (not even in $A_P$).

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VL “Algebra III”
FU Berlin, Winter 2023/24

## 14. Aufgabenblatt zum 7.2.2024

**Problem 134.** Which of the two-dimensional cyclic quotient singularities $X_{n,q} = \frac{1}{n}(1,q) = \mathbb{TV}(\sigma)$ with $\sigma = \langle(1,0),\ (-q,n)\rangle$ is Gorenstein?

*Solution:* Just the $A_n$-singularities, i.e. those with $q=-1$, i.e. the matrix describing the $(\mathbb{Z}/n\mathbb{Z})$-action has $\det = 1$.

**Problem 135.** Recall that a lattice polytope is called reflexive if it contains 0 as an interior point, and all facets have lattice distance 1 from the origin. In particular, 0 is the only interior lattice point.

a) Draw 5 examples of reflexive polygons, i.e., 2-dimensional gadgets.

b) Find an example of a non-reflexive lattice polytope with exactly one interior lattice point.

*Solution:* (a) double square, triple triangle, the hexagon, and their duals (only the hexagon is self-dual).

(b) I expect that this could be found in [CLS]. Anyway, it has to be of dimension $\geq 3$.

**Problem 136.** a) For fixed $n,d\in\mathbb{N}$, the space $\mathbb{C}^{\binom{n+d}{d}}$ parametrizes all polynomials $f\in\mathbb{C}[x_1,\ldots,x_n]$ of degree $\deg(f)\leq d$. Denote by $S_{\mathbb{A}}\subseteq\mathbb{C}^{\binom{n+d}{d}}$ the locus of all $f\in\mathbb{C}[\underline{x}]$ such that $V(f)\subseteq\mathbb{C}^{n+1}$ is smooth. Is $S_{\mathbb{A}}$ non-empty/open/dense in $\mathbb{C}^{\binom{n+d}{d}}$?

b) The space $\mathbb{C}^{\binom{n+d}{d}}$, or better $\mathbb{P}^{\binom{n+d}{d}-1}$ parametrizes all homogeneous polynomials $F\in\mathbb{C}[z_0,\ldots,z_n]$ of degree $d$. Denote by $S_{\mathbb{P}}\subseteq\mathbb{C}^{\binom{n+d}{d}}$ the locus of all homogeneous $F\in\mathbb{C}[\underline{z}]$ of degree $d$ such that $V_+(F)\subseteq\mathbb{P}^n$ is smooth. Is $S_{\mathbb{P}}$ non-empty/open/dense in $\mathbb{C}^{\binom{n+d}{d}}$?

*Solution:* (a) For each $n,d$, we certainly have $f=1\in S_{\mathbb{A}}$. Or, less strange, if $d\geq 1$, then we might take $f=x_1$. Anyway, this yields that $S_{\mathbb{A}}\neq\emptyset$.

The set $S_{\mathbb{A}}$ is *not open*: For $n=1$ ($x:=x_1$) and $d=2$, each $c\in\mathbb{C}^*$ gives rise to $(\frac{1}{c^2},\frac{2}{c},1)\in\mathbb{C}^3$ encoding $f_c(x)=\frac{1}{c^2}x^2+\frac{2}{c}x+1=(\frac{x}{c}+1)^2=\frac{1}{c^2}\cdot(x+c)^2$. Since this has a double root, $V(f_c)$ is not smooth, i.e., $f_c\in\mathbb{C}^3\setminus S_{\mathbb{A}}$. On the other hand, if $c\to\infty$, then $f_c\to 1$. However, $f_\infty=1\in S_{\mathbb{A}}$.

Instead, we could have looked at $g_c := \frac{x}{c^2}\cdot(x+c)^2$ as well. It corresponds to $(\frac{1}{c^2},\frac{2}{c},1,0)\in\mathbb{C}^4$, and the limit becomes $g_\infty=x\in S_{\mathbb{A}}$.

Thus, the usual method for showing density, namely by proving openess, fails. Instead, we obtain density as a consequence from the density in (b).

(b) At least when $\operatorname{char} k \not| d$, e.g., when $k=\mathbb{C}$, then $f=z_0^d+\ldots+z_n^d$ yields a smooth hypersurface. To show openess, we define the closed subset

$$Y := \{(c, \xi) \in \mathbb{C}^{\binom{n+d}{d}} \times \mathbb{P}^n \mid \xi \in V_+(F_c) \text{ is singular}\}.$$

Then, $\mathbb{C}^{\binom{n+d}{d}} \setminus S_{\mathbb{A}}$ is the image of $Y$ via the projection $\pi : \mathbb{C}^{\binom{n+d}{d}} \times \mathbb{P}^n \twoheadrightarrow \mathbb{C}^{\binom{n+d}{d}}$, and $\pi$ is a closed map. Hence, $\mathbb{C}^{\binom{n+d}{d}} \setminus S_{\mathbb{A}}$ is closed, too.

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# 15. Aufgabenblatt zum 14.2.2024

**Problem 137.** Let $\Delta \subset M_{\mathbb{R}}$ be a smooth lattice polytope, i.e., its normal fan $\Sigma := \mathcal{N}(\Delta)$ is supposed to be smooth. Denote $\mathbb{C}[\Delta] := \{f \in \mathbb{C}[M] \mid \operatorname{supp}(f) \subseteq \Delta\}$. Each $f \in \mathbb{C}[\Delta]$ gives rise to a subvariety $Z(f) \subseteq T = \operatorname{Spec}\mathbb{C}[M]$. Denote by $\overline{Z}(f)$ the closure of $Z(f)$ in $X := \mathbb{TV}(\Sigma)$.

a) For a given $f \in \mathbb{C}[M]$ and a given chart $\mathbb{TV}(\sigma) \subseteq X$ describe the closure of $Z(f)$ in $\mathbb{TV}(\sigma)$. Does this set equal $\overline{Z}(f) \cap \mathbb{TV}(\sigma)$?

b) Show that the set of $f \in \mathbb{C}[\Delta]$ such that $\overline{Z}(f)$ is smooth forms an open, dense subset of $\mathbb{C}[\Delta]$.

c) Denote by $\mathcal{J}(f) \subseteq \mathcal{O}_X$ the ideal sheaf of $\overline{Z}(f) \subset X$. Under which assumptions do we obtain $\overline{Z}(f) \cong \mathcal{O}_X(-\Delta)$? Is it always true?

d) Show that for smooth, reflexive polyhedra $\Delta$, there is a dense subset of $f \in \mathbb{C}[\Delta]$ such that $\overline{Z}(f)$ is a smooth Calabi-yau variety.

**Problem 138.** Let $\Sigma$ be the fan in $\mathbb{Q}^3$ built from the rays

$$\Sigma(1) = \{e^i, a^i, (-1,-1,-1) \mid i = \mathbb{Z}/3\mathbb{Z}\}$$

(with $e^i$ denoting the canonical basis vectors and $a^i := (1,1,1) + e^i$) and being spanned by the three-dimensional cones $\langle(-1,-1,-1), e^i, e^{i+1}\rangle$, $\langle e^i, e^{i+1}, a^{i+1}\rangle$, $\langle e^i, a^i, a^{i+1}\rangle$, and $\langle a^1, a^2, a^3\rangle$ for $i = \mathbb{Z}/3\mathbb{Z}$. Show that $\Sigma$ is not the normal fan of a polytope, i.e. that $\mathbb{TV}(\Sigma)$ is complete, but not projective.