hypersurface. To show openess, we define the closed subset

$$Y := \{(c, \xi) \in \mathbb{C}^{\binom{n+d}{d}} \times \mathbb{P}^n \mid \xi \in V_+(F_c) \text{ is singular}\}.$$

Then, $\mathbb{C}^{\binom{n+d}{d}} \setminus S_{\mathbb{A}}$ is the image of $Y$ via the projection $\pi : \mathbb{C}^{\binom{n+d}{d}} \times \mathbb{P}^n \twoheadrightarrow \mathbb{C}^{\binom{n+d}{d}}$, and $\pi$ is a closed map. Hence, $\mathbb{C}^{\binom{n+d}{d}} \setminus S_{\mathbb{A}}$ is closed, too.

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## 15. Aufgabenblatt zum 14.2.2024

**Problem 137.** Let $\Delta \subset M_{\mathbb{R}}$ be a smooth lattice polytope, i.e., its normal fan $\Sigma := \mathcal{N}(\Delta)$ is supposed to be smooth. Denote $\mathbb{C}[\Delta] := \{f \in \mathbb{C}[M] \mid \operatorname{supp}(f) \subseteq \Delta\}$. Each $f \in \mathbb{C}[\Delta]$ gives rise to a subvariety $Z(f) \subseteq T = \operatorname{Spec} \mathbb{C}[M]$. Denote by $\overline{Z}(f)$ the closure of $Z(f)$ in $X := \mathbb{TV}(\Sigma)$.

a) For a given $f \in \mathbb{C}[M]$ and a given chart $\mathbb{TV}(\sigma) \subseteq X$ describe the closure of $Z(f)$ in $\mathbb{TV}(\sigma)$. Does this set equal $\overline{Z}(f) \cap \mathbb{TV}(\sigma)$?

b) Show that the set of $f \in \mathbb{C}[\Delta]$ such that $\overline{Z}(f)$ is smooth forms an open, dense subset of $\mathbb{C}[\Delta]$.

c) Denote by $\mathcal{J}(f) \subseteq \mathcal{O}_X$ the ideal sheaf of $\overline{Z}(f) \subset X$. Under which assumptions do we obtain $\overline{Z}(f) \cong \mathcal{O}_X(-\Delta)$? Is it always true?

d) Show that for smooth, reflexive polyhedra $\Delta$, there is a dense subset of $f \in \mathbb{C}[\Delta]$ such that $\overline{Z}(f)$ is a smooth Calabi-yau variety.

*Solution:* To be used only in true emergency (a) The closure $\overline{Z}(f)$ of $Z(f) \subseteq T$ does not contain any toric prime divisors, i.e., components of $X \setminus T$. Thus, $\overline{Z}(f) \cap \mathbb{TV}(\sigma)$ is the closure of $Z(f)$ in $\mathbb{TV}(\sigma)$.

(b) Either, one argues locally, for each affine chart separately, then we only obtain that the smooth locus is dense. Or, we argue similarily to 136(b).

(c) We need that, for each $a \in \Sigma(1)$, that $\min\langle \operatorname{supp} f, a\rangle = \min\langle \Delta, a\rangle$. This, e.g., satisfied when all vertices of $\Delta$ belong to $\operatorname{supp}(f)$.

(d) This follows from the adjunction formula: $\omega_X \cong \omega_X \otimes \mathcal{J}(f)^\vee \cong \mathcal{O}_X(-\Delta) \otimes \mathcal{O}_X(\Delta)|_{\overline{Z}} = \mathcal{O}_{\overline{Z}}$.

**Problem 138.** Let $\Sigma$ be the fan in $\mathbb{Q}^3$ built from the rays

$$\Sigma(1) = \{e^i, a^i, (-1,-1,-1) \mid i = \mathbb{Z}/3\mathbb{Z}\}$$

(with $e^i$ denoting the canonical basis vectors and $a^i := (1,1,1) + e^i$) and being spanned by the three-dimensional cones $\langle(-1,-1,-1), e^i, e^{i+1}\rangle$, $\langle e^i, e^{i+1}, a^{i+1}\rangle$, $\langle e^i, a^i, a^{i+1}\rangle$, and $\langle a^1, a^2, a^3\rangle$ for $i = \mathbb{Z}/3\mathbb{Z}$. Show that $\Sigma$ is not the normal fan of a polytope, i.e. that $\mathbb{TV}(\Sigma)$ is complete, but not projective.

*Solution:* To be used only in true emergency