Best regards,
Klaus

HUMBOLDT-UNIVERSITÄT ZU BERLIN

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Klaus Altmann

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An algorithm for computing equisingular deformations

by

Klaus Altmann

§1. Introduction.

This paper is a direct continuation of [Al 2]; in particular, we use the same notations. (Note the only difference: The sheaves of differential forms with logarithmic poles are denoted by $\Omega_X\langle D\rangle$ instead of $\Omega(\log D)$.)

(1.1.) In §2 we fix an arbitrary smooth subdivision $\Sigma < \Sigma_0$ and compute the image $\mathrm{Im}\left(ESE_{X_\Sigma}(k[\varepsilon]) \longrightarrow Def_R(k[\varepsilon])\right)$ (cf. Proposition (2.6)).
This together with Theorem [Al 2] (3.4) imply our main result - an algorithm for computing all equisingular first-order deformations in $Def_R(k[\varepsilon])$ (cf. Theorem (4.1)). None of the smooth subdivisions $\Sigma < \Sigma_0$, but only the starting f.r.p.p. decomposition Σ_0 itself is used there, hence, this algorithm seems to be an easy method to determine $\overline{ES}(k[\varepsilon])$ by computers. In particular, for each equation f we can decide if there are equisingular deformations below $\Gamma(f)$ or not.
Finally, an example is given in (4.3).

(1.2) §3 is of purely illustrating character and coinsides partly with §4 of [Al 2]. The great distance between, roughly speaking, "maximal" and "minimal" embedded resolutions (yielding the over-$\Gamma(f)$-deformations or all elements of $\overline{ES}(k[\varepsilon])$, respectively) is subdivided into elementary steps, i.e. single blowing ups of $\mathbb{P}^1_k$-copies. In this way, it is possible to regard the equisingular deformations below $\Gamma(f)$ exactly in the moment of their formation.

§2. Computation of $\operatorname{Im}\left(ESE_X(k[\varepsilon]) \xrightarrow{\gamma} \operatorname{Def}_R(k[\varepsilon])\right)$ (for a fixed embedded resolution X)

For this paragraph we fix an arbitrary smooth f.r.p.p. subdivision $\Sigma < \Sigma_0$ with the corresponding good resolution $\pi: X \longrightarrow \mathbb{A}^3_k$.

(2.1) The connecting morphism of the cohomology sequences of

$$0 \longrightarrow \Theta_X\langle -D-Y\rangle \longrightarrow \Theta_X\langle -D\rangle \longrightarrow \mathcal{N}_{Y|X} \longrightarrow 0$$

yields the following diagram, which may be written in two different versions:

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
\langle \text{monomials} \ge \Gamma(f)\rangle & \twoheadrightarrow & \langle \text{monomials} \ge \Gamma(f)\rangle/(f) & \longrightarrow & ESE_X(k[\varepsilon]) \\
\cap & & \cap & & \downarrow \gamma \\
k[\underline{x}] & \twoheadrightarrow & R & \twoheadrightarrow & \operatorname{Def}_R(k[\varepsilon]) \\
\downarrow & & \downarrow & & \downarrow \\
k[\underline{x}]/\langle \text{monomials} \ge \Gamma(f)\rangle & \xrightarrow{\sim} & R/\langle \text{monomials} \ge \Gamma(f)\rangle & \overset{\psi}{\twoheadrightarrow} & \operatorname{Coker}\gamma \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

and in the cohomological language

$$\begin{array}{ccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
H^0(X, \mathcal{O}_X(Y)) & \twoheadrightarrow & H^0(X, \mathcal{N}_{Y|X}) & \longrightarrow & H^1(X, \Theta_X\langle -D-Y\rangle) \\
\cap & & \cap & & \downarrow \gamma \\
H^0(X\backslash D, \mathcal{O}_X(Y)) & \twoheadrightarrow & H^0(X\backslash D, \mathcal{N}_{Y|X}) & \twoheadrightarrow & H^1(X\backslash D, \Theta_X\langle -D-Y\rangle) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_D(X, \mathcal{O}_X(Y)) & \xrightarrow{\sim} & H^1_D(X, \mathcal{N}_{Y|X}) & \overset{\psi}{\twoheadrightarrow} & H^2_D(X, \Theta_X\langle -D-Y\rangle) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

(The first columns are identified according to [Al 1] (2.2) - we take $k[\underline{x}] = H^0\left(X\backslash D, \mathcal{O}_X(-\sum_{a \in \Sigma(1)} m(a)D_a)\right) \xrightarrow[\sim]{\cdot 1/f} H^0(X\backslash D, \mathcal{O}_X(Y))$ - and $H^1(X, \mathcal{O}_X(Y)) = H^1(X, \mathcal{N}_{Y|X}) = 0$;
for the right hand side we use (2.5)(γ) and (4.2) of [Al 1] - in the latter one the vanishing of $H^2(X, \Theta_X\langle -D-Y\rangle)$ has been proved.)

Definition. For $\xi = \sum_{r \ge 0} \xi_r \cdot x^r \in k[\underline{x}] \cong H^0(X\backslash D, \mathcal{O}_X(Y))$ we denote by $\xi_{<\Gamma(f)}$ the image of ξ in

$$k[\underline{x}]/\langle \text{monomials} \ge \Gamma(f)\rangle = H^1_D(X, \mathcal{O}_X(-\textstyle\sum m(a)D_a)) \cong H^1_D(X, \mathcal{O}_X(Y)) .$$

Taking the canonical section of $k[\underline{x}] \twoheadrightarrow k[\underline{x}]/\langle \text{monomials} \ge \Gamma(f)\rangle$, we get $\xi_{<\Gamma(f)} = \sum_{\substack{r \ge 0 \\ r < \Gamma(f)}} \xi_r \cdot x^r$.

(2.2) **Proposition.** *1) For $i=1,2,3$ let $\varphi_i: H^1_D(X, \mathcal{O}_X(e^i)) \longrightarrow H^1_D(X, \mathcal{O}_X(-\sum m(a)D_a))$ be the multiplication by $x_i \frac{\partial f}{\partial x_i}$. Under the isomorphism*

$$H^1_D(X, \mathcal{O}_X(-\textstyle\sum m(a)D_a)) \xrightarrow[\sim]{\cdot 1/f} H^1_D(X, \mathcal{O}_X(Y)) \xrightarrow{\sim} H^1_D(X, \mathcal{N}_{Y|X}) .$$

then

$$\operatorname{Coker}\gamma = \operatorname{Coker}\left(\bigoplus_{i=1}^{3} \varphi_i\right).$$

2) Let $\xi = \sum_{r \ge 0} \xi_r \cdot x^r \in k[\underline{x}]$ define an element of $\operatorname{Def}_R(k[\varepsilon])$ (the infinitesimal deformation $\hat{f}(x,\varepsilon) = f(x) - \varepsilon\,\xi(x)$). Then this deformation is induced by $ESE_X(k[\varepsilon])$ if and only if

$$\xi_{<\Gamma(f)} \in \operatorname{Im}\left(\bigoplus_{i=1}^{3} \varphi_i\right).$$

Proof. 1) By the second diagram of (2.1) it holds

$$\operatorname{Coker}\gamma = H^1_D(X, \mathcal{N}_{Y|X})/\operatorname{Ker}\psi = \operatorname{Coker}\left(H^1_D(X, \Theta_X\langle -D\rangle) \to H^1_D(X, \mathcal{N}_{Y|X})\right).$$

On the other hand, we can lift the surjection $\Theta_X\langle -D\rangle \twoheadrightarrow \mathcal{N}_{Y|X}$ to the homomorphism $\Theta_X\langle -D\rangle \longrightarrow \mathcal{O}_X(Y)$ given by $\eta \longmapsto \frac{\eta(f)}{f}$

(i) In local coordinates (take the same notations as in the proof of [Al 2](2.5): $f = x^{r^\alpha} \cdot f_\alpha$) we obtain

$$\frac{\eta(f)}{f} = \frac{\eta(f_\alpha)}{f} + \frac{\eta(x^{r^\alpha})}{x^{r^\alpha}} .$$

Since $\eta \in \Theta_X\langle -D\rangle$, the section $\frac{\eta(x^{r^\alpha})}{x^{r^\alpha}}$ is regular on X, and $\frac{\eta(f)}{f}$ is indeed an element of the sheaf $\mathcal{O}_X(Y)$.

(ii) The projections $\Theta_X\langle -D\rangle \twoheadrightarrow \mathcal{N}_{Y|X}$ and $\mathcal{O}_X(Y) \twoheadrightarrow \mathcal{N}_{Y|X}$ are locally given by

$$\eta \longmapsto \left[f_\alpha \in {}^{(f_\alpha)}/_{(f_\alpha^2)} \longmapsto \eta(f_\alpha) \in {}^{\mathcal{O}_X}/_{(f_\alpha)}\right] \quad \text{and}$$

$$a \longmapsto \left[f_\alpha \in {}^{(f_\alpha)}/_{(f_\alpha^2)} \longmapsto a f_\alpha \in {}^{\mathcal{O}_X}/_{(f_\alpha)}\right] \text{, respectively.}$$

Then, the congruence

$$\frac{\eta(f)}{f}\cdot f_\alpha = \eta(f_\alpha) + \frac{\eta(x^{r^\alpha})}{x^{r^\alpha}}\cdot f_\alpha \equiv \eta(f_\alpha) \quad (\text{modulo } f_\alpha)$$

shows that the diagram

$$\begin{array}{ccc} & \nearrow & \mathcal{O}_X(Y) \\ & & \downarrow \\ \Theta_X\langle -D\rangle & \longrightarrow\!\!\!\!\rightarrow & \mathcal{N}_{Y|X} \end{array} \quad \text{commutes.}$$

Since $H^1_D(X, \mathcal{O}_X(Y)) \xrightarrow{\sim} H^1_D(X, \mathcal{N}_{Y|X})$ is an isomorphism, we obtain

$$\operatorname{Coker}\gamma = \operatorname{Coker}\left(H^1_D(X, \Theta_X\langle -D\rangle) \longrightarrow H^1_D(X, \mathcal{O}_X(Y))\right).$$

Finally, the first claim follows by the equation

$$\eta(f) = \sum_{i=1}^{3} \left(x_i \frac{\partial f}{\partial x_i}\right)\cdot \frac{\eta(x_i)}{x_i}$$

and taking the isomorphism

$$\Theta_X\langle -D\rangle \xrightarrow{\sim} \bigoplus_{i=1}^{3} \mathcal{O}_X(e^i)$$

$$\eta \longmapsto \left(\frac{\eta(x_1)}{x_1}, \frac{\eta(x_2)}{x_2}, \frac{\eta(x_3)}{x_3}\right) .$$

2) $\xi \in \underline{k}[\underline{x}] = H^0(X\setminus D, \mathcal{O}_X(-\sum m(a)D_a)) \cong H^0(X\setminus D, \mathcal{O}_X(Y))$ maps onto $0 \in \operatorname{Coker}\gamma$ if and only if

$\xi_{<\Gamma(f)} \in H^1_D(X, \mathcal{O}_X(-\sum m(a)D_a)) \cong H^1_D(X, \mathcal{O}_X(Y))$ vanishes in $\operatorname{Coker}(\bigoplus_{i=1}^{3} \varphi_i)$. □

(2.3) Our next task will be to describe the maps φ_i by the methods of torus embeddings. For this purpose it is usefull to regard the dual version of these maps:

$$\varphi_i^*: H^2(X, \omega_X(\sum m(a) D_a)) \longrightarrow H^2(X, \omega_X(-e^i)) \quad ,$$

and the homomorphisms are still given by multiplication by $x_i \frac{\partial f}{\partial x_i}$.

Now, for $r \in M$ we define the following sets:

$$A_r := \left\{ a \in \Delta \,/\, \langle a,-r\rangle \le -m(a) \right\} = \left\{ a \in \Delta \,/\, \langle a,r\rangle \ge m(a) \right\} ,$$

$$B^\Sigma_{i,t} := \left\{ a \in \Delta \,/\, \langle a,-t\rangle \le \phi_i(a) \right\} \text{ with } \phi_i(a) := \begin{cases} 0 & \text{for } a \in \Sigma^{(1)},\ a \ne e^i \\ 1 & \text{for } a = e^i \end{cases} ,$$

$$H_t := \left\{ a \in \Delta \,/\, \langle a,t\rangle < 0 \right\} .$$

Then, the convex sets $(\Delta \setminus H_t)$ are contained in $B^\Sigma_{i,t}$, and the maps φ_i^* are equal to some homomorphisms

$$\begin{array}{l} \varphi_i^*: \bigoplus_{r\in M} H^1(A_r, k) \longrightarrow \bigoplus_{t\in M} H^1(B^\Sigma_{i,t}, k) \qquad (i=1,2,3) \; . \\ \qquad \Big\| \,(\text{cf.}\,(2.5)) \\ \quad \bigoplus_{\substack{r\ge 0 \\ r<\Gamma(f)}} k\cdot x^{-r} \end{array}$$

(As we are really interested in the dual of, for instance, $H^2(X, \omega_X(\sum m(a)D_a))$, the notations are chosen such that A_r describes the cohomology of the $-r$(th) factor of this sheaf. The relations "≤" or "≥" - instead of the strict ones - in the definitions of A_r and $B^\Sigma_{i,t}$ are induced by taking ω_X(divisor) instead of $\mathcal{O}_X$(divisor).)

But, what does φ_i^* look like? We have to make some general remarks concerning the computation of cohomology on torus embeddings:

(2.4) Denote by $j: T \hookrightarrow X_\Sigma$ a torus embedding in the sense of [Ke].

1) Let $\mathcal{L} \subseteq j_*\mathcal{O}_T = j_*\, k[M]^\sim$ be an M-graded invertible sheaf with order function $\Phi: |\Sigma| \longrightarrow \mathbb{R}$; for $r \in M$ let $A_r := \{a \in \Delta / \langle a,r\rangle < \Phi(a)\}$.

Then, if $\alpha \in \Sigma$ is an arbitrary cone, we obtain

$$\mathcal{L}(r)_{|X_\alpha} = \begin{cases} \underline{\mathbb{C}} & (\forall a \in \alpha: \langle a,r\rangle \ge \Phi(a)) \\ 0 & (\exists a \in \alpha: \langle a,r\rangle < \Phi(a)) \end{cases} ,$$

hence $\mathcal{L}(r)_{|X_\alpha} = H^0(\alpha, \alpha \cap A_r) \otimes \underline{k}$. In particular, the sheaf $\mathcal{L}(r)$ and the pair (Δ, A_r) yield exactly the same Cech complexes.

2) Let $\mathcal{L}^1, \mathcal{L}^2 \subseteq j_*\mathcal{O}_T$ be M-graded invertible sheaves with Φ^1, Φ^2 and A^1_r, A^2_r as before. Assume that there is an $s \in M$ with $x^s \cdot \mathcal{L}^1 \subseteq \mathcal{L}^2$ (equivalent: $\Phi^1 + s \ge \Phi^2$ as functions on Δ).

Then, for each $r \in M$ there is an inclusion $A^2_{r+s} \subseteq A^1_r$, which provides the commutative diagram

$$\begin{array}{ccc} \Gamma(X_\alpha, \mathcal{L}^1(r)) & \xrightarrow{\cdot x^s} & \Gamma(X_\alpha, \mathcal{L}^2(r+s)) \\ \| & & \| \\ H^0(\alpha, \alpha \cap A^1_r) & \hookrightarrow & H^0(\alpha, \alpha \cap A^1_{r+s}) . \end{array}$$

Again by taking Cech cohomology we obtain a description of the multiplication by x^s on the cohomological level:

$$\begin{array}{ccc} H^n(X,\mathcal{L}^1) & \xrightarrow{x^s} & H^n(X,\mathcal{L}^2) \\ \| & & \| \\ \bigoplus_{r\in M} H^n(\Delta, A^1_r) & \xrightarrow{\varphi} & \bigoplus_{r\in M} H^n(\Delta, A^2_r) \end{array}$$

(φ is induced by the inclusion $A^2_{r+s} \subseteq A^1_r$; in particular, φ is homogeneous of degree s.)

3) Let $\mathcal{L}^i, \Phi^i, A^i_r$ (i=1,2) as before, assume that there is a Laurent polynomial $g(\underline{x}) \in k[M]$ with $g(\underline{x})\cdot\mathcal{L}^1 \subseteq \mathcal{L}^2$.

Then, by M-graduation of both sheaves $\mathcal{L}^1$ and $\mathcal{L}^2$, this fact is equivalent to

$$x^s \cdot \mathcal{L}^1 \subseteq \mathcal{L}^2 \quad \text{for all } s \in \operatorname{supp} g .$$

Hence, the method of (2) can be applied to describe the maps

$H^n(X,\mathcal{L}^1) \xrightarrow{\cdot g(x)} H^n(X,\mathcal{L}^2)$

(2.5) The third part of the previous general remark applies exactly to the special maps φ^*_i regarded in 2.3). Denoting by $\Delta^\Sigma_i \subseteq \Delta$ the union of all closed Σ-cones not containing e^i, we obtain the following

Lemma. *1)* $H^1(A_r,k) = \begin{cases} k\cdot x^{-r} & (\text{for } r\geq 0 \text{ and } r<\Gamma(f)) \\ 0 & (\text{otherwise}) \end{cases}$ *, and the perfect pairing with* $H^1_D(X, \mathcal{O}_X(-\sum m(a)D_a)) = \bigoplus_{\substack{r\geq 0 \\ r<\Gamma(f)}} k\cdot x^r$ *is built in the obvious way.*

2) For i=1,2,3 and $t\in M$ the cohomology group $H^1(B^\Sigma_{i,t},k)$ is equal to

(i) $H_0(\Delta^\Sigma_i \cap \mathbb{H}_t)\cdot x^{-t}$ *(for $t_i = -1$ and $t_\nu \geq 0$ for all $\nu \neq i$);*

(ii) $H_0(\Delta^\Sigma_i \cap \mathbb{H}_t)/H_0(\{e^j\})\cdot x^{-t}$ *(for $t_i=-1$, $t_j \leq -1$ ($j\neq i$), and the remaining component is ≥ 0);*

(iii) 0 *(for $t \neq -1$ or $t \leq -(1,1,1)$).*

3) Let $f(\underline{x}) = \sum_{s\in \operatorname{supp} f} \lambda_s \cdot x^s$ be the explicit description of our starting equation. Let r, i and t be such that $H^1(A_r,k)$, $H^1(B^\Sigma_{i,t},k) \neq 0$ (i.e. $r\geq 0$, $r<\Gamma(f)$ and $t_i=-1$, $t \not\leq -(1,1,1)$, respectively).

Then, the x^{-t}-part of $\varphi^*_i(x^{-r})$ is given by

$$s_i \ \lambda_s \cdot \left[H_0(\{a^*\}) \in H_0(\Delta^\Sigma_i \cap \mathbb{H}_t)\right] \quad \textit{with } s:=-t+r \textit{ (because of } (-t)=s+(-r)) \textit{ and } a^* \in \Sigma^{(1)}_0 \textit{ such that } \langle a^*,r\rangle < m(a^*).$$

In particular, this part of $\varphi^*_i(x^{-r})$ vanishes, unless $s \geq \Gamma(f)$.

Proof. 1) $A_r = \Delta \setminus \{a\in\Delta \,/\, \langle a,r\rangle < m(a)\} = \Delta\setminus(\text{convex set})$, and the above conditions for r arise by $r\geq 0$ iff $\partial\Delta \subseteq A_r$ and
$r<\Gamma(f)$ iff $A_r \neq \Delta$.

2) $\Delta\setminus\mathbb{H}_t \subseteq B^\Sigma_{i,t}$, and the only vertex of $\Sigma^{(1)}$ in which both sets can differ is e^i. Hence, the non-vanishing of $H^1(B^\Sigma_{i,t},k)$ implies $e^i \notin \Delta\setminus\mathbb{H}_t$, $e^i \in B^\Sigma_{i,t}$, and we obtain $t_i = \langle e^i,t\rangle = -1$.

Assuming this from now on, we see that $B^\Sigma_{i,t}$ contains exactly the same elements of $\Sigma^{(1)}$ as $\Delta\setminus[\Delta^\Sigma_i \cap \mathbb{H}_t]$. In particular, both subsets of Δ (consisting of open or closed halfspaces in every cone of Σ) are homotopy equivalent and yield the same cohomology. Without loss of generality we take i=1 and consider the above three cases:

(i) $t_2,t_3 \geq 0$: Then, $\partial\Delta \subseteq \Delta\setminus[\Delta^\Sigma_1\cap\mathbb{H}_t]$, and
$H^1(\Delta\setminus[\Delta^\Sigma_1\cap\mathbb{H}_t],k) = H_0(\Delta^\Sigma_1\cap\mathbb{H}_t)$ follows by the Alexander duality

(ii) $t_2 \leq -1$, $t_3 \geq 0$: This means $e^1, e^3 \in (\Delta\setminus[\Delta^\Sigma_1\cap\mathbb{H}_t])$, $e^2 \notin (\Delta\setminus[\Delta^\Sigma_1\cap\mathbb{H}_t])$ and therefore, the connected component C of e^2 in $\Delta^\Sigma_1\cap\mathbb{H}_t$ has no influence on the cohomology:

$$H^1(\Delta\setminus[\Delta^\Sigma_1\cap\mathbb{H}_t],k) = H^1(\Delta\setminus[(\Delta^\Sigma_1\cap\mathbb{H}_t)\setminus C],k) = H_0([\Delta^\Sigma_1\cap\mathbb{H}_t]\setminus C) = H_0(\Delta^\Sigma_1\cap\mathbb{H}_t)/H_0(\{e^2\}).$$

(The middle equality again follows by the Alexander duality.)

(iii) $t_2,t_3 \leq -1$: By $\mathbb{H}_t = \Delta$ we obtain
$\Delta\setminus[\Delta^\Sigma_1\cap\mathbb{H}_t] = \Delta\setminus\Delta^\Sigma_1$,
and this set can be contracted to the point e^1.

3) The linear map $H^1(A_r,k) \to H^1(B^\Sigma_{i,t},k)$ is constructed by the inclusion $B^\Sigma_{i,t} \subseteq A_r$ (cf. (2.4)); in dual terms this means that $H_0(\Delta\setminus A_r) \longrightarrow H_0(\Delta^\Sigma_i\cap\mathbb{H}_t)/\ldots$ is induced by

$(\Delta\setminus A_r) \subseteq (\Delta\setminus B^{\Sigma}_{i,t}) \sim (\Delta^{\Sigma}_i \cap \mathbf{H}_t)$:

Take an element $a^* \in \Sigma_0^{(1)}$ with $\langle a^*, r\rangle < m(a^*)$ (i.e. $a^* \in \Delta\setminus A_r$); assuming $s \geq \Gamma(f)$, we obtain

$$\langle a^*, t\rangle = \langle a^*, r\rangle - \langle a^*, s\rangle > 0 \quad (\text{i.e. } a^* \in \mathbf{H}_t),$$

and x^{-r} maps onto the corresponding connected component in $\Delta^{\Sigma}_i \cap \mathbf{H}_t$ (multiplied by the coefficient of x^s in $x_i \frac{\partial f}{\partial x_i}$). □

(2.6) Now, we are in the position to determine the deformations of $\mathrm{Im}\big(ESE_X(k[\varepsilon]) \xrightarrow{\ \gamma\ } \mathrm{Def}_R(k[\varepsilon])\big)$ exactly:

Definition. For $i=1,2,3$ let $M_i := \{r \in M \,/\, r \geq 0,\ \Gamma(f) - e_i \leq r < \Gamma(f)\}$ ($\{e_1, e_2, e_3\}$ denotes the canonical $\mathbb{Z}$-basis of M);

then, we can choose (and fix) a map

$$a:\ M_i \longrightarrow \Sigma_0^{(1)}$$
$$r \longmapsto a(r) \text{ with } \langle a(r), r\rangle < m\big(a(r)\big).$$

Recall the definitions

$$\mathbf{H}_t := \{a \in \Delta \,/\, \langle a,t\rangle < 0\} \quad (\text{for } t \in M) \quad \text{and}$$
$$\Delta^{\Sigma}_i := \bigcup\{\overline{\alpha} \,/\, \overline{\alpha} \in \Sigma,\ e^i \notin \overline{\alpha}\} \subseteq \Delta.$$

Proposition. *(I) Given the following data*

1) $i \in \{1,2,3\}$,

2) $t \in M$ with: a) $t_i = -1$

b) (i) $t_\nu \geq 0$ (i.e. $e^\nu \notin \mathbf{H}_t$) for all $\nu \neq i$, or

(ii) $t_j \leq -1$ ($i \neq j$) and the remaining component is ≥ 0,

c) there exists an $r \in M_i$ with $r - t \geq \Gamma(f)$ and $\langle a(r), t + e_i\rangle \geq 0$,

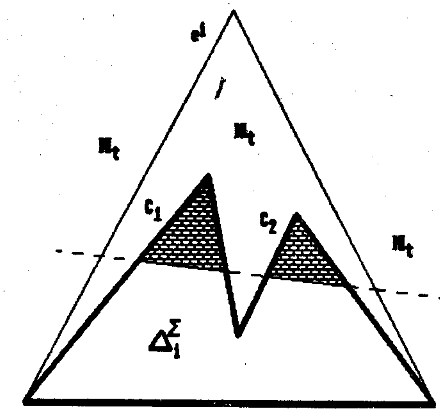
3) a connected component C of $\Delta^{\Sigma}_i \cap \mathbf{H}_t$ not containing any of the vertices e^1, e^2, e^3,

then, the deformation defined by

$$\sum_{\substack{r \in M_i \\ a(r) \in C}} (r_i + 1)\,\lambda_{r-t}\cdot x^r = \Big(x^{t+e_i}\cdot \frac{\partial f}{\partial x_i}\Big)\Big|_{M_i \cap a^{-1}(C)}$$

comes from $ESE_X(k[\varepsilon])$.

(II) $\mathrm{Im}(\gamma) \subseteq \mathrm{Def}_R(k[\varepsilon])$ as a k-vectorspace is spanned by the over-$\Gamma(f)$-deformations and all deformations contructed in the above way.



Proof. By Proposition (2.2), $\mathrm{Im}(\gamma)$ is spanned by the over-$\Gamma(f)$-deformations together with the images of the maps φ_i ($i=1,2,3$). However, in Lemma (2.5)(2) it is shown that the data $\{i,t,C\}$ meeting 1), 2a), 2b) and 3) of the claim form a k-basis of

$$\bigoplus_{i=1}^{3} H^2_D(X, \mathcal{O}_X(e^i)) = \bigoplus_{i=1}^{3}\bigoplus_{t \in M} H^1(B^{\Sigma}_{i,t}, k) \quad (\text{or its k-dual});$$

finally, part (3) of the same Lemma gives

$$\varphi_i(\{i,t,C\})\big|_{k\cdot x^r} = \begin{cases}(r_i+1)\,\lambda_{r-t} & (\text{for } a(r) \in C)\\ 0 & (\text{otherwise})\end{cases}.$$

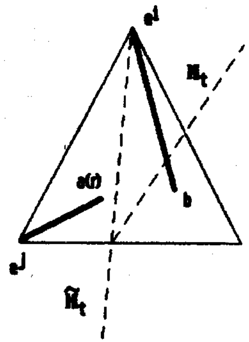
It remains to prove that we are able to restrict ourselves to $r \in M_i$ (instead of $r \geq 0$, $r < \Gamma(f)$) and that the additional assumption 2c) for t can be made:

Let $\{i,t,C\}$ be as before and take an $r \geq 0$, $r < \Gamma(f)$ such that $\varphi_i(\{i,t,C\})\big|_{k\cdot x^r} \neq 0$.

Claim. $\langle a(r), t\rangle \geq -a(r)_i$.

$\langle a(r), t\rangle < -a(r)_i$ would imply that there is an $j \neq i$ with $t_j \leq -1$ (cf. case (ii)), and we would obtain the following situation:

$\tilde{\mathbf{H}}_t := \{a \in \Delta \,/\, \langle a,t\rangle < -a_i\} \subseteq \mathbf{H}_t$ contains $a(r)$ and e^j, but not the vertex e^i. Hence, there is no cone $\overline{be^i} \in \Sigma$, $b \notin \tilde{\mathbf{H}}_t$ ($b \notin \mathbf{H}_t$) meeting $\overline{a(r)e^j}$, and $a(r)$ and e^j must be contained in the same connected component of $\Delta^{\Sigma}_i \cap \mathbf{H}_t$.



Now, $(r-t)\in \operatorname{supp} f$ implies $r-t\geq\Gamma(f)$; in particular, we obtain

$$\langle a(r), r-t\rangle \geq m\big(a(r)\big)$$

and therefore

$$\langle a(r), r\rangle \geq m\big(a(r)\big) - a(r)_i . \qquad \square$$

Remarks. 1) Condition 2c) guarantees that there are only a few (in particular, a finite number of) $t\in M$ fulfilling 2).
2) The above construction is - of course - independent of the choice of the function $a: M_i \longrightarrow \Sigma_0^{(1)}$.

(2.7) In (2.6)-(2.8) of [Al1] we already tried to describe the image of γ. For elements $\xi\in H^1(X,\Theta_X\langle -D-Y\rangle)$ (given explicitly by a 1-cocycle $\{\xi_{\alpha\beta}\}$) the induced deformation $\gamma(\xi)_{<\Gamma(f)}$ was computed directly. Now, we want to give a short dictionary to understand this formulae in the cohomological language used here.

(i) For $i=1,2,3$ we obtain elements $\xi(x_i)\in H^1(X,\mathcal{O}_X(-\sum_{a>0} a_i D_a))$ (given by $\xi_{\alpha\beta}(x_i)$ in [Al1]).

(ii) The exact sequence

$$0\longrightarrow \mathcal{O}_X(-\sum_{a>0} a_i D_a)\longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{\sum a_i D_a}\longrightarrow 0 \quad ,$$

together with $H^1(X,\mathcal{O}_X)=0$, shows that $\xi(x_i)$ can be lifted to an element $b_i\in H^0(X,\mathcal{O}_{\sum a_i D_a})$.

(In [Al1] these sections are given locally by $b_i^\alpha\in\mathcal{O}_X$:

$$\xi_{\alpha\beta}(x_i) = b_i^\beta - b_i^\alpha \quad \text{for every two cones } \alpha,\beta\in\Sigma.)$$

(iii) Multiplying by $\frac{\partial f}{\partial x_i}$ provides a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(-\sum_{a>0} a_i D_a) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{\sum a_i D_a} & \longrightarrow & 0\\
 & & \downarrow\cdot\frac{\partial f}{\partial x_i} & & \downarrow\cdot\frac{\partial f}{\partial x_i} & & \downarrow\cdot\frac{\partial f}{\partial x_i} & & \\
0 & \longrightarrow & \mathcal{O}_X(-\sum m(a) D_a) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{\sum m(a) D_a} & \longrightarrow & 0 \;.
\end{array}$$

Therefore, we obtain $\sum_{i=1}^{3}\frac{\partial f}{\partial x_i} b_i \in H^0(X,\mathcal{O}_{\sum m(a)D_a})$ - still written as a local $\mathcal{O}_X$-section in [Al1].

(iv) Finally, we recall the isomorphism

$$H^0_{(D)}(X,\mathcal{O}_{\sum m(a)D_a}) \xrightarrow{\sim} H^1_D(X,\mathcal{O}_X(-\sum m(a)D_a)) = k[x]/_{\langle \text{monomials} \geq \Gamma(f)\rangle} .$$

§3: Changing the embedded resolution

(3.1) Let $\Sigma<\Sigma_0$ be a smooth subdivision with the following property:

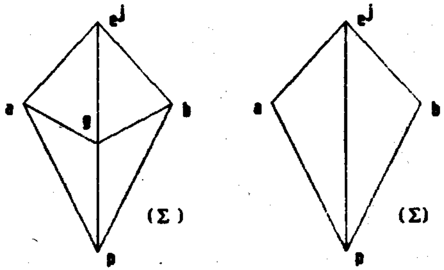
For $i=1,2,3$ the convex hull $\operatorname{conv}(a,b)$ of arbitrary elements $a,b\in\Sigma_0^{(1)}\setminus\{e^i\}$ is contained in Δ_i^Σ. $\quad (*)$

Then, we obtain by [Al1](2.9):

$$\operatorname{Im}\big(ESE_{X_\Sigma}(k[\varepsilon])\longrightarrow Def_R(k[\varepsilon])\big) = \{\text{isomorphism classes of first order "over-}\Gamma(f)\text{-deformations"}\}.$$

(3.2) Embedded resolutions X_Σ meeting the property $(*)$ and the f.r.p.p. decompositions Σ_i defined in [Al2](2.4) represent the two extremal values of $\operatorname{Im}\big(ESE_{X_\Sigma}(k[\varepsilon])\longrightarrow Def_R(k[\varepsilon])\big)$ (equal to the set of all over-$\Gamma(f)$-deformations by (3.1) or to $\overline{ES}(k[\varepsilon])$ by [Al2](3.4), respectively).
It is possible to connect these "maximal" and "minimal" f.r.p.p. decompositions by a chain of elementary subdivisions, and we can try to compare the images of ESE at each step:

(3.3) Definition. Let Σ', Σ be smooth f.r.p.p.decompositions finer than Σ_0. Σ' will be called an elementary subdivision of Σ if it is obtained by barycentrical subdivision of exactly one 2-dimensional cone $\overline{pe^j} \in \Sigma$:



$(a, b, p \in \Sigma^{(1)},$
$g := e^j + p \in \Sigma'^{(1)}).$

(The corresponding proper map $\sigma: X' \longrightarrow X$ is the blowing up of the closed orbit $\overline{\mathrm{orb}\,\overline{pe^j}} \subseteq X$, which is isomorphic to $\mathbb{P}^1_k$.)

(3.4) How do the data of Proposition (4.6)(1) change under elementary subdivisions?

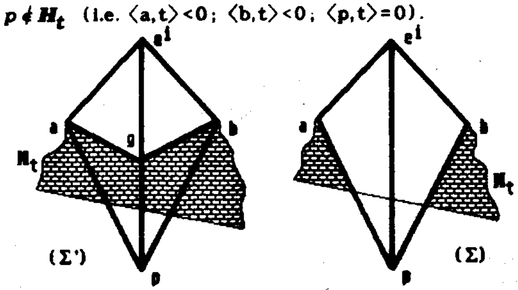
$i, t, \mathbf{H}_t$ are independent of the actual f.r p.p. decomposition;

$\Delta_i^{\Sigma} \subseteq \Delta_i^{\Sigma'}$ really change iff $i = j$, but both sets still contain the same elements of $\Sigma_0^{(1)}$ (all but e^i).

Therefore, the crucial point must be the arrangement of the connected components of $\Delta_i^{\Sigma} \cap \mathbf{H}_t$, i.e. which elements of $\Sigma_0^{(1)}$ are contained in a common one?

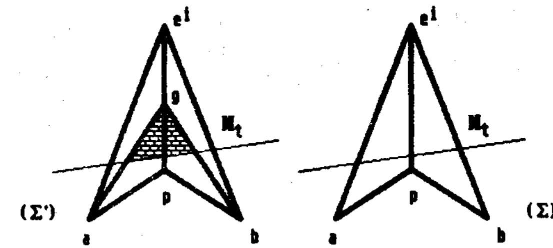
There are only two possibilities for an essential distinction between Σ' and Σ:

1) $a, b, g \in \boldsymbol{H}_t$, $p \notin \boldsymbol{H}_t$ (i.e. $\langle a,t\rangle < 0$; $\langle b,t\rangle < 0$; $\langle p,t\rangle = 0$).



The two connected components of $\Delta_i^{\Sigma} \cap \mathbf{H}_t$ that contain a and b, respectively, are joint in $\Delta_i^{\Sigma'} \cap \mathbf{H}_t$.

2) $a, b, p \notin \boldsymbol{H}_t$, $g \in \boldsymbol{H}_t$ (i.e. $\langle a,t\rangle \geq 0$; $\langle b,t\rangle \geq 0$; $\langle p,t\rangle = 0$).



In $\Delta_i^{\Sigma'} \cap \mathbf{H}_t$, there appears a new component containing g but no element of $\Sigma_0^{(1)}$. Therefore, the image of ESE($k[\varepsilon]$) in $\mathrm{Def}_R(k[\varepsilon])$ will remain unchanged.

(3.5) We define two characteristic integers of the elementary subdivision $\Sigma' < \Sigma$:

$$k := \det(a, b, e^i)$$

$$d := \det(b, a, p) \qquad (a, b, p, e^i \in \Sigma^{(1)} \subseteq \mathbb{Z}^3).$$

By construction, $k \geq 1$ is valid.

Lemma. *1)* $k \cdot p + d \cdot e^i = a + b$

2) Let $Z := \overline{\mathrm{orb}\,\overline{pe^i}} \subseteq X$ (centre of blowing up); then, k and d equal certain intersection numbers on X:

$$(D_p \cdot Z) = -k$$

$$(D_{e^i} \cdot Z) = -d.$$

3) $\mathcal{N}_{Z|X} \simeq \mathcal{O}_Z(-k) \oplus \mathcal{O}_Z(-d)$. *Therefore, the exceptional divisor of the blowing up $\sigma: X' \longrightarrow X$ is isomorphic to the Hirzebruch surface $\mathbb{F}_{k-d}$ over Z.*

Proof 1) By the definitions of k and d and by

$$\det(a, p, e^i) = \det(b, e^i, p) = 1 \quad (\Sigma \text{ is smooth!}),$$

we obtain

$$\det(a+b, e^i, p) = \det(a, b, kp + de^i) = 0.$$

Therefore, the vectors a+b and $kp+de^l$ are contained in $\overline{ab}\cap\overline{pe^l}$, i.e. there exists a $\lambda\in Q$ with $kp+de^l=\lambda(a+b)$.

Finally, we have

$$k = \det(a,kp,e^l) = \det(a,kp+de^l,e^l) = \lambda\cdot\det(a,a+b,e^l) =$$
$$= \lambda\cdot\det(a,b,e^l) = \lambda\cdot k\,.$$

2) For $r\in\mathbb{Z}^3$ the divisors $(x^r)=\sum_{a\in\Sigma(1)}\langle a,r\rangle\cdot D_a$ vanish in Pic X. In particular, we get the following equations between the corresponding intersection numbers:

$$(D_{e^l}\cdot Z)+p_l(D_p\cdot Z)+a_l+b_l=\left([D_{e^l}+p_lD_p+a_lD_a+b_lD_b]\cdot Z\right)=0$$

and

$$p_j(D_p\cdot Z)+a_j+b_j=\left([p_jD_p+a_jD_a+b_jD_b]\cdot Z\right)=0 \quad (\text{for } j\neq l).$$

3) $Z=D_p\cap D_{e^l}$ yields

$$\mathcal{N}_{Z|X}=\mathcal{N}_{Z|D_p}\oplus\mathcal{N}_{Z|D_{e^l}}=\mathcal{O}_Z\left((D_p\cdot Z)\right)\oplus\mathcal{O}_Z\left((D_{e^l}\cdot Z)\right),$$

and the rest then is clear by [Ha](II, § 8). □

(3.6) Let us return to the situation of (3.4):

We had the following conditions for t, which are nessecary for the arrangement of connected components of $\Delta_l^\Sigma\cap\mathbb{H}_t$ to be changed:

$$t_l=\langle e^l,t\rangle=-1\,,$$
$$\langle p,t\rangle=0\,,$$
$$\left[\langle a,t\rangle,\langle b,t\rangle<0 \ \ (\text{cf. }(1)),\ \text{or}\ \langle a,t\rangle,\langle b,t\rangle\geq 0\ \ (\text{cf. }(2))\right].$$

Now, the first part of the previous Lemma yields

$$\langle a,t\rangle+\langle b,t\rangle=\langle kp+de^l,t\rangle=k\cdot\langle p,t\rangle+d\cdot\langle e^l,t\rangle=-d\,,$$

and we have to distinguish between two cases:

Case 1. $d\leq 1$

Only $\langle a,t\rangle,\langle b,t\rangle\geq 0$ can appear, and as already mentioned in (3.4)(2), $ESE_{X'}(k[\varepsilon])$ and $ESE_X(k[\varepsilon])$ induce the same image in $Def_R(k[\varepsilon])$.

Case 2. $d\geq 2$

The only possibility is $\langle a,t\rangle,\langle b,t\rangle<0$, i.e. two connected components C_1 and C_2 of $\Delta_l^\Sigma\cap\mathbb{H}_t$ are joint to a common one, namely C of $\Delta_l^{\Sigma'}\cap\mathbb{H}_t$.

In order to see what happens, let us write down the map

$$\left(\bigoplus_{i=1}^3\varphi_i\right):\ \bigoplus_{i=1}^3 H^1_D(X,\mathcal{O}_X(e^i))\longrightarrow H^1_D(X,\mathcal{O}_X(-\sum m(a)D_a))$$

as a matrix A_Σ:

The columns and rows correspond to the data (i, t, C) and the elements $r\in\bigcup_{i=1}^3 M_i$ (cf. Proposition (2.6)), respectively. Each column represents an equisingular deformation of type (i) or (ii) (cf. (I)(2b) of Proposition (2.6)), and in this way $\mathrm{Im}(\gamma)$ is spanned by all columns of the matrix A_Σ.

Now, joining the components C_1 and C_2 means the construction of a new matrix by

a) summing up the columns (i,t,C_1) and (i,t,C_2) to a common one (i, t, C) (if neither C_1 nor C_2 contain one of the vertices e^1,e^2,e^3), or

b) deleting these colums (otherwise).

(The latter version can only be actual by dealing with type-(ii)-deformations; then one of the triples (i,t,C_ν) did already not appear as a column of the starting matrix, i.e. only one column vanishes really.)

Altogether, for $d\geq 2$ there are exactly (d-1) values of $t\in M$ that imply a changing of the connected components of $\Delta_i^\Sigma\cap\mathbb{H}_t$. Therefore, almost this number of columns must be deleted (maybe after adding some of them to other ones) in order to get the matrix $A_{\Sigma'}$ from A_Σ; we obtain

$$\dim_k\mathrm{Im}(\gamma)-\dim_k\mathrm{Im}(\gamma')=\mathrm{rank}\,A_\Sigma-\mathrm{rank}\,A_{\Sigma'}\leq d-1\,.$$

Remark. The map $\sigma: X'\longrightarrow X$ is the blowing up of an "admissible centre" in the sense of [Kaw]. By Theorem 2 of this paper we obtain

$$(\mathbb{R}^+\sigma_*)(\Theta_{X'}\langle -D'-Y'\rangle)=\Theta_X\langle -D-Y\rangle\langle -Z\rangle$$

which yields the exact sequence

$$\begin{array}{ccccccc} H^0(X,\mathcal{N}_{Z|D_p}) & \longrightarrow H^1(X,\Theta_{X'}\langle -D'-Y'\rangle)\longrightarrow & H^1(X,\Theta_X\langle -D-Y\rangle) & \longrightarrow & H^1(X,\mathcal{N}_{Z|D_p}) & \longrightarrow 0 \\ \|{\scriptstyle (3.5)(2)} & & & & \|{\scriptstyle (3.5)(2)} & \\ H^0(Z,\mathcal{O}_Z(-d)) & & & & H^1(Z,\mathcal{O}_Z(-d)) & . \end{array}$$

Again, we have the above two cases for d:

Case 1. $d\leq 1$

Then $H^1(Z,\mathcal{O}_Z(-d))=0$, and the map $ESE_{X'}(k[\varepsilon])\longrightarrow ESE_X(k[\varepsilon])$ must be surjective.

Case 2. $d \geq 2$

By $H^0(Z, \mathcal{O}_Z(-d)) = 0$, we can compute the difference of the ESE-functors:

$$0 \longrightarrow ESE_{X'}(k[\varepsilon]) \longrightarrow ESE_X(k[\varepsilon]) \longrightarrow k^{d-1} \longrightarrow 0 .$$

But in order to recognize the difference of the images in $Def_R(k[\varepsilon])$, a comparison of the above matrices $A_{\Sigma'}$ and A_Σ will still be necessary.

(3.7) Finally, we want to consider what happens with the matrix A_Σ by not only one single elementary subdivision but by reaching the property (*) of (3.1) at once:

(*) means that all elements of $\Sigma_0^{(1)}$ contained in $\Delta_i^{\Sigma'} \cap \mathbf{H}_t$ even belong to the same connected component. In particular, there are no type-(ii)-deformations ($e^j \in \Delta_i^{\Sigma'} \cap \mathbf{H}_t$!) that are contained in $Im(\gamma')$ - the corresponding columns of A_Σ will be totally deleted by turning to the matrix $A_{\Sigma'}$.

On the other hand, all columns of A_Σ that correspond to a pair (i,t) of type (i) (cf. Proposition (2.6)(I)(2)) will be summed up, thus obtaining only one single column of $A_{\Sigma'}$ that represents the trivial deformation $x^{t+e_i} \cdot \frac{\partial f}{\partial x_i}$.

§4. An algorithm to determine the equisingular deformations below $\Gamma(f)$

(4.1) Analogously to Proposition (2.6) it is possible to compute all deformations of $\overline{ES}(k[\varepsilon]) \subseteq Def_R(k[\varepsilon])$. The corresponding algorithm does not use any of the smooth subdivisions of Σ_0 regarded before, but only the starting f.r.p.p. decomposition Σ_0 itself.

Let $\Delta_i := \bigcup \{ \overline{\alpha} \,/\, \overline{\alpha} \in \Sigma_0,\ e^i \notin \overline{\alpha} \} \subseteq \Delta$ $(i = 1,2,3)$ and take the definition of $M_i \subseteq M$, $a: M_i \longrightarrow \Sigma_0^{(1)}$ and $\mathbf{H}_t$ of (2.6).

Theorem. *(I) Given the following data*

1) $i \in \{1,2,3\}$,

2) $t \in M$ *with: a)* $t_i = -1$

b) (i) $t_\nu \geq 0$ *(i.e.* $e^\nu \notin \mathbf{H}_t$*) for all* $\nu \neq i$ *, or*

(ii) $t_j \leq -1$ $(i \neq j)$ *and the remaining component is* ≥ 0,

c) there exists an $r \in M_i$ *with* $r - t \geq \Gamma(f)$ *and* $\langle a(r), t+e_i \rangle \geq 0$,

3) a connected component C of $\Delta_i \cap \mathbf{H}_t$*, not containing any of the vertices* e^1, e^2, e^3 ;

then, the deformation defined by

$$\sum_{\substack{r \in M_i \\ a(r) \in C}} (r_i + 1)\, \lambda_{r-t} \cdot x^r = \left(x^{t+e_i} \cdot \frac{\partial f}{\partial x_i} \right) \Big|_{M_i \cap a^{-1}(C)}$$

is contained in $\overline{ES}(k[\varepsilon])$.

(II) $ES(k[\varepsilon]) \subseteq Def_R(k[\varepsilon])$ *as a k-vectorspace is spanned by the over-*$\Gamma(f)$*-deformations and all deformations contructed in the above way.*

Proof. Take the three resolutions Σ_ν $(\nu = 1,2,3)$ of [Al 2](2.4). Then, by Theorem [Al 2](3.4) and Proposition (2.6) the above claim were valid if the Δ_i would be replaced by all $\Delta_i^{\Sigma_\nu}$ $(\nu = 1,2,3)$ simultaneously.

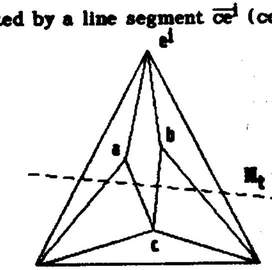
Step 1. Let $i, \nu \in \{1,2,3\}$, $t \in M$ be fixed. By construction it is clear that $\Delta_i \subseteq \Delta_i^{\Sigma_\nu}$, hence $\Delta_i \cap \mathbf{H}_t \subseteq \Delta_i^{\Sigma_\nu} \cap \mathbf{H}_t$.

Now, both sets contain the same elements of $\Sigma_0^{(1)}$, and the connected components of $\Delta_i^{\Sigma_\nu} \cap \mathbf{H}_t$ (restricted to $\Delta_i \cap \mathbf{H}_t$) are built by taking the union of several complete components of $\Delta_i \cap \mathbf{H}_t$.

For the deformations induced by $\Delta_i^{\Sigma_\nu} \cap \mathbf{H}_t$ this means that they split into sums of deformations induced by $\Delta_i \cap \mathbf{H}_t$.

Step 2. The connected components of $\Delta_i \cap \mathbf{H}_t$ *and* $\Delta_i^{\Sigma_i} \cap \mathbf{H}_t$ *correspond to each other and contain the same elements of* $\Sigma_0^{(1)}$*.*

Let $a, b \in \Sigma_0^{(1)} \cap [\Delta_i \cap \mathbf{H}_t]$ be contained in different components of $\Delta_i \cap \mathbf{H}_t$, then they can be separated by a line segment $\overline{ce^i}$ (contained in a cone of Σ_0) with $c \notin \mathbf{H}_t$.



By the construction of Σ_i (cf. [Al 2](2.4)), this f.r.p.p. decomposition contains $\overline{P_i(c)e^i}$ as one cone of the canonical partition of $\overline{ce^i}$. Because of $t_i=-1$, $\langle c,t\rangle \geq 0$ implies $\langle P_i(c),t\rangle \geq 0$, and $\overline{P_i(c)e^i}$ will separate a and b as elements of $\Delta_i^{\Sigma_i} \cap \mathbf{H}_t$. (The opposite direction was already done in step 1.) □

Remark. 1) Similar to (3.7), all type-(i)-deformations in $\overline{ES}(k[\varepsilon])$ consist of pieces of trivial deformations.

2) The k-dimensions of $\overline{ES}(k[\varepsilon])$ and $\overline{ES}(k[\varepsilon])/_{\langle \text{monomials} \geq \Gamma(f)\rangle}$ can be obtained by computing the rank of the following matrix A (cf. (3.6), Case 2):

The *rows* correspond to elements $r \in \bigcup_{i=1}^{3} M_i$,

the *columns* correspond to triples (i,t,C) with (I),(1)-(3) of the above Theorem and

$$a_{r,(i,t,C)} := \begin{cases} (r_i+1)\cdot\lambda_{r-t} & \text{for } a(r)\in C \\ 0 & \text{otherwise} \end{cases}$$

(Of course, this matrix does not depend on the special choice of the function $a: M_i \longrightarrow \Sigma_0^{(1)}$.)

3) Compare with Theorem (5.8) of [Al 1]: If the sets Δ_i are convex, there will be no type-(ii)-deformations, and all deformations of type (i) will be trivial.

(4.2) **Corollary.** *The k-vectorspace* $\overline{ES}(k[\varepsilon])/_{\langle \text{monomials} \geq \Gamma(f)\rangle}$ *and, in particular, the fact whether* $\overline{ES}$ *is exactly the functor of over-*$\Gamma(f)$*-deformations or not, are independent of the coefficients* λ_s *of f with*

$$\langle a,s\rangle \geq m(a) + \max\{a_1,a_2,a_3\} \quad \text{for all } a \in \Sigma_0^{(1)} \setminus \{e^1,e^2,e^3\}.$$

Proof. Let λ_s be a coefficient of f that appears in the matrix A (defined in the previous remark). If

$$a_{r,(i,t,C)} = a_i \cdot \lambda_s \quad (s=r-t),$$

then we take the element $a := a(r) \in \Sigma_0^{(1)} \setminus \{e^1,e^2,e^3\}$, and now we obtain

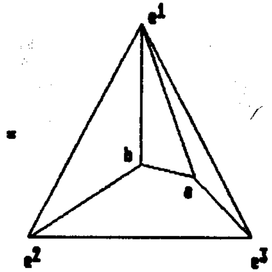
$\langle a,r\rangle < m(a)$ (by definition of $a(r)$),

$\langle a,t\rangle \geq -a_i$ (by (I)(2c) of the Theorem),

hence $\langle a,s\rangle < m(a)+a_i$. □

(4.3) **Example.** Let $f(x,y,z) := x^5+y^6+z^5+y^3z^2$ (cf. [Al 1], §3); we get

$\Sigma_0 =$



$(a=(12,10,15);\ m(a)=60$ and $b=(1,1,1);\ m(b)=5)$.

First, two important properties of Σ_0 become obvious:

1) Δ_1 is convex

2) $\Sigma_0^{(1)} \setminus \{e^1,e^2,e^3\} = \{a,b\}$, and these two elements are connected in Σ_0 directly.

Therefore, all type-(i)-deformations of f have to be trivial, and non-trivial deformations of type (ii) can only be obtained by

$i=2$, and $b,e^3 \in \mathbf{H}_t$ are separated by $\overline{ae^2}$, or

$i=3$, and $a,e^2 \in \mathbf{H}_t$ are separated by $\overline{be^3}$.

Now, we will investigate these two cases. If ever possible we shorten the algorithm of (4.1) by additional restrictions to r and t - decreasing their number is very useful for making the computation by hand.

Case 1. $i=2$

1) We look for all $r \in M_2$ with

$\langle a,r\rangle \geq m(a)$ (otherwise $a=a(r)$ would be possible) and

$m(b)-b_2 \leq \langle b,r\rangle < m(b)$ (because of $b=a(r)$).

It follows that

$12r_1+10r_2+15r_3 \geq 60$ and

$r_1+r_2+r_3=4$, i.e.: $15r_1+15r_2+15r_3=60$,

and the only solution is $r=(0,0,4)$.

2) For $t \in M$ we obtain the following conditions:

$t_1 \geq 0$, $t_2=-1$, $t_3 \leq -1$ (cf. part (I)(2a,b) of Theorem (4.1));

$(0,0,4) - t \geq \Gamma(f)$ (cf. (I)(2c)), hence $t_1 = 0$;

$\langle b,t \rangle \geq -b_2$ (cf. (I)(2c)), hence $t_3 \geq -t_1 = 0$,

which yields a contradiction.

Case 2. $i=3$

1) Again we start with the search for possible $r \in M_3$:

$m(a) - a_3 \leq \langle a,r \rangle < m(a)$ and

$\langle b,r \rangle \geq m(b)$

yield the conditions

$12r_1 + 10r_2 + 15r_3 \leq 59$ and even

$r_1 + r_2 + r_3 = 5$.

2) Conditions for $t \in M$:

$t_1 \geq 0$, $t_2 \leq -1$, $t_3 = -1$ similar to the first case;

$\langle a,t \rangle < 0$, $\langle b,t \rangle \geq 0$ ($a \in H_t$, $b \notin H_t$).

It follows that

$12t_1 + 10t_2 < 15$ but

$t_1 + t_2 \geq 1$ (i.e. $12t_1 + 12t_2 \geq 12$),

hence $2t_2 \geq -2$.

Therefore, we obtain $t_2 = -1$ together with

$12t_1 < 25$ and $t_1 \geq 2$,

i.e. $t = (2,-1,-1)$.

3) Because of

$\langle a,(2,-1,-1) \rangle = 24 - 10 - 15 = -1$,

we obtain a new condition for the elements $r \in M_3$ to represent a non-trivial row of the matrix A:

$r - (2,-1,-1) \geq \Gamma(f)$ implies

$\langle a,r \rangle + 1 \geq m(a)$,

hence $12r_1 + 10r_2 + 15r_3 = 59$.

From this we get the condition

$2r_1 + 5r_3 = 9$ and

$r_1 + r_2 + r_3 = 5$

with the only solution $r = (2,2,1)$.

Altogether we obtain the following description of the matrix A (cf. Remark of (4.1)):

The only column not representing a trivial deformation is given by

$i = 3$, $t = (2,-1,-1)$, C = connected component of $a \in \Delta$;

the only non-vanishing element on this column is

$(r_i+1) \cdot \lambda_{r-t} = 2\lambda_{(0,3,2)}$ (in the row corresponding to $r = (2,2,1)$).

Since $(0,3,2) \in M$ represents a vertex of $\Gamma(f)$, the coefficient $\lambda_{(0,3,2)}$ can never vanish ($\lambda_{(0,3,2)} = 1$ in our special example). Therefore, we have proved

$$\overline{ES}(k[\varepsilon]) / \langle \text{monomials} \geq \Gamma(f) \rangle = k \cdot x^2 y^2 z ,$$

not only for the special equation f, but for all equations having this special Newton boundary.

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