# One parameter families containing three dimensional toric Gorenstein singularities

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## Contents



# 1 Introduction

## 1.1

Let $\sigma$ be a rational, polyhedral cone and $Y = Y_\sigma$ the affine toric variety it defines, which is normal but possibly singular; we study the deformation theory of $Y$. The infinitesimal deformation space $T^1_Y$ is a multigraded vector space, with homogeneous pieces determined by the combinatorial formulas of [Al 1].

If $Y_\sigma$ has only an isolated Gorenstein singularity, we can say even more (cf. [Al 2], [Al 3]): $T^1$ is concentrated in a single multidegree, and the corresponding homogeneous piece has an elementary geometric description in terms of Minkowski summands of a certain lattice polytope. We even obtain the entire versal deformation of $Y_\sigma$ (cf. [Al 4]).

## 1.2

Our first aim in the present paper is to interpret geometrically the formula for $T^1$ for arbitrary toric singularities in every multidegree. We again do this in terms of Minkowski summands of certain polyhedra; however, these polyhedra are not necessarily compact any longer, and their vertices are not necessarily lattice points (cf. 2.6).

In [Al 2] we studied so-called toric deformations. These only exist in negative multidegrees (that is, with multidegrees in $-\sigma^{\vee}$). They are genuine deformations with smooth parameter space, and are characterized by the fact that their total space is again toric. Now, armed with our new description of $T^1_Y$, we describe the Kodaira–Spencer map in these terms in Theorem 3.3. Moreover, in 3.5, we extend the construction of genuine deformations to nonnegative degrees using a partial modification of our singularity $Y_\sigma$. Although the total space is no longer toric, we can still describe it and its Kodaira–Spencer map combinatorially.

## 1.3

After that, we focus on three dimensional Gorenstein toric singularities. As already mentioned, everything is known in the case of isolated singularities. However, as soon as $Y_\sigma$ has a one dimensional singular locus (necessarily of transversal type $\mathrm{A}_k$), the situation changes drastically: $T^1_Y$ spreads in general into infinitely many multidegrees. In 4.3, using our geometric description of the graded pieces of $T^1_Y$, we detect all the nontrivial ones, and determine their dimension (which is usually 1). The easiest example of this kind is the cone over the weighted projective plane $\mathbb{P}(1,2,3)$ (cf. 4.4 and 4.8).

At present it seems impossible to describe the entire versal deformation, which is an infinite dimensional space. However, the infinitesimal deformations corresponding to the one dimensional homogeneous pieces of $T^1_Y$ are unobstructed, and we lift them in 4.5 to genuine one parameter families. Since the corresponding multidegrees are in general nonnegative, this can be done using the construction introduced in 3.5. Our treatment of the cone over $\mathbb{P}(1,2,3)$ continues in this spirit in 4.8.

These one parameter families can be thought of as a kind of skeleton, from which one hopes eventually to build the entire versal deformation. The most important open questions are the following:

1. which sets of one parameter families belong to a common irreducible component of the base space?

2. how can those families be combined to find a general fiber of this component (a smoothing of $Y_\sigma$)?

Answers to these questions would provide important information about three dimensional flips.

# 2 Visualizing $T^1$

## 2.1 Notation

As usual in toric geometry (see [Oda] for a detailed introduction), $N$ and $M$ denote dual lattices (that is, finitely generated, free Abelian groups) with a perfect pairing $\langle\ ,\ \rangle : N \times M \to \mathbb{Z}$, and $N_{\mathbb{R}}$, $M_{\mathbb{R}}$ the corresponding $\mathbb{R}$-vector spaces obtained by extension of scalars. Let $\sigma \subset N_{\mathbb{R}}$ be the polyhedral cone with vertex at 0 spanned by fundamental generators $a^1, \ldots, a^M \in N$. We assume that the $a^i$ are primitive, that is, not proper multiples of other elements of $N$. We write $\sigma = \big\langle a^1, \ldots, a^M \big\rangle$.

The dual cone $\sigma^\vee := \{r \in M_{\mathbb{R}} \mid \langle \sigma, r\rangle \geq 0\}$ is given by the inequalities corresponding to $a^1, \ldots, a^M$. Intersecting $\sigma^\vee$ with the lattice $M$ yields a finitely generated semigroup $\sigma^\vee \cap M$. We denote by $E \subset \sigma^\vee \cap M$ its minimal set of generators, and call it the *Hilbert basis.* Then the affine toric variety $Y_\sigma := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M] \subset \mathbb{C}^E$ has defining equations corresponding to the linear dependence relations among elements of $E$.

## 2.2

Most of the rings and modules relevant to the study of $Y_\sigma$ are $M$-graded (or multigraded); this applies to the modules $T^i_Y$, that describe infinitesimal deformations of $Y_\sigma$ and obstructions to extending them. Let $R \in M$. In [Al 1] and [Al 3], we defined the subsets

$$E_j^R := \big\{ r \in E \mid \big\langle a^j, r \big\rangle < \big\langle a^j, R \big\rangle \big\} \subset E$$

for $j = 1, \ldots, M$. These provide the main tool for building a complex $\operatorname{Span}(E^R)_\bullet$ of free Abelian groups, with

$$\operatorname{Span}(E^R)_{-k} := \bigoplus_{\substack{\tau \text{ a face of } \sigma \\ \text{with } \dim \tau = k}} \operatorname{Span}(E^R_\tau),$$

where

$$E^R_\tau := \bigcap_{a^j \in \tau} E^R_j \quad \text{for faces } \tau < \sigma, \quad \text{but} \quad E^R_0 := \bigcup_{j=1}^{N} E^R_j\,,$$

and where the differentials are the obvious maps.

**Theorem 2.1 (cf. [Al 1], [Al 3])** *The homogeneous piece of $T^1_Y$ in degree $-R$ is given by*

$$T^1_Y(-R) = H^1\Big( \operatorname{Span}(E^R)^*_\bullet \otimes_{\mathbb{Z}} \mathbb{C} \Big).$$

*If we assume in addition that $Y_\sigma$ is smooth in codimension two, the same thing holds for $T^2_Y$:*

$$T^2_Y(-R) = H^2\Big(\operatorname{Span}(E^R)^*_\bullet \otimes_{\mathbb{Z}} \mathbb{C}\Big).$$

In particular, to calculate $T^1_Y(-R)$, we need the vector spaces $\operatorname{Span}_{\mathbb{C}} E^R_j$ and $\operatorname{Span}_{\mathbb{C}} E^R_{jk}$, where $a^j, a^k$ span a two dimensional face of $\sigma$. The first of these is easily determined:

$$\operatorname{Span}_{\mathbb{C}} E^R_j = \begin{cases} 0 & \text{if } \langle a^j, R\rangle \le 0, \\ (a^j)^\perp & \text{if } \langle a^j, R\rangle = 1, \\ M_{\mathbb{C}} & \text{if } \langle a^j, R\rangle \ge 2. \end{cases}$$

The second is always contained in $(\operatorname{Span}_{\mathbb{C}} E^R_j) \cap (\operatorname{Span}_{\mathbb{C}} E^R_k)$ as a subspace of codimension $\le 2$. As the following example shows, its actual size reflects the infinitesimal deformations of the two dimensional cyclic quotient singularity corresponding to the plane cone spanned by $a^j, a^k$. (These singularities are exactly the transversal types of the singularities of $Y_\sigma$ in codimension two.)

**Example 2.2** Write $Y_{n,q}$ for the two dimensional quotient of $\mathbb{C}^2$ by the action of $\mathbb{Z}/n\mathbb{Z}$ via $\left(\begin{smallmatrix} \xi & 0 \\ 0 & \xi^q \end{smallmatrix}\right)$, where $\xi$ is a primitive $n$th root of unity and $q$ is coprime to $n$. Then $Y_{n,q}$ is the toric variety given by the cone $\sigma = \langle (1,0); (-q,n)\rangle \subset \mathbb{R}^2$. The set $E \subset \sigma^\vee \cap \mathbb{Z}^2$ consists of the lattice points $r^0, \dots, r^w$ along the compact faces of the boundary of the convex hull $\operatorname{conv}\big((\sigma^\vee \setminus \{0\}) \cap \mathbb{Z}^2\big)$. There are integers $a_v \ge 2$ such that $r^{v-1} + r^{v+1} = a_v r^v$ for $v = 1, \dots, w-1$. These may be obtained by expanding $n/(n-q)$ as a negative continued fraction (cf. [Oda], §1.6).

Assume $w \ge 2$ and let $a^1 = (1,0)$ and $a^2 = (-q,n)$. Then only two sets $E^R_1$ and $E^R_2$ are involved, and the previous theorem states that

$$T^1_Y(-R) = \left(\frac{\operatorname{Span}_{\mathbb{C}} E^R_1 \cap \operatorname{Span}_{\mathbb{C}} E^R_2}{\operatorname{Span}_{\mathbb{C}}(E^R_1 \cap E^R_2)}\right)^*.$$

The only multidegrees $R \in \mathbb{Z}^2$ contributing to $T^1_Y$ are as follows:

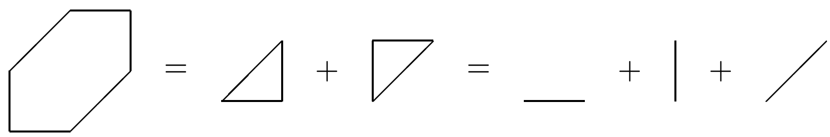
(i) $R = r^1$ (and the case $R = r^{w-1}$ is similar): then $\operatorname{Span}_{\mathbb{C}} E^R_1 = (a^1)^\perp$, $\operatorname{Span}_{\mathbb{C}} E^R_2 = \mathbb{C}^2$ (or $(a^2)^\perp$ if $w = 2$), and $\operatorname{Span}_{\mathbb{C}} E^R_{12} = 0$. Hence, $\dim T^1(-R) = 1$ (or 0 if $w = 2$).

(ii) $R = r^v$ with $2 \le v \le w-2$: $\operatorname{Span}_{\mathbb{C}} E^R_1 = \operatorname{Span}_{\mathbb{C}} E^R_2 = \mathbb{C}^2$, and $\operatorname{Span}_{\mathbb{C}} E^R_{12} = 0$. Thus $\dim T^1(-R) = 2$.

(iii) $R = p \cdot r^v$ with $1 \le v \le w-1$, and $2 \le p < a_v$ for $w \ge 3$; (or $v = 1 = w-1$ with $2 \le p \le a_1$ for $w = 2$): then $\operatorname{Span}_{\mathbb{C}} E^R_1 = \operatorname{Span}_{\mathbb{C}} E^R_2 = \mathbb{C}^2$, and $\operatorname{Span}_{\mathbb{C}} E^R_{12} = \mathbb{C} \cdot R$. In particular, $\dim T^1(-R) = 1$.

## 2.3 Minkowski summands

**Definition 2.3** We define the *Minkowski sum* of two polyhedra $Q', Q'' \subset \mathbb{R}^n$ to be the polyhedron $Q' + Q'' := \{p' + p'' \mid p' \in Q', p'' \in Q''\}$. Obviously, this notion also makes sense for translation classes of polyhedra in arbitrary affine spaces.



Every polyhedron $Q$ decomposes as the Minkowski sum $Q = Q^{\mathrm{c}} + Q^{\infty}$ of a (compact) polytope $Q^{\mathrm{c}}$ and the *cone of unbounded directions* $Q^{\infty}$. The latter is uniquely determined by $Q$, whereas the compact summand is not. However, we can take $Q^{\mathrm{c}}$ to be the minimal one – given as the convex hull of the vertices of $Q$ itself. If $Q$ was already compact, then $Q^{\mathrm{c}} = Q$ and $Q^{\infty} = 0$.

> *A polyhedron $Q'$ is called a* Minkowski summand *of $Q$ if there is a $Q''$ such that $Q = Q' + Q''$ and if, additionally, $(Q')^{\infty} = Q^{\infty}$.*

In particular, a Minkowski summand always has the same cone of unbounded directions as the original polyhedron, whereas its compact edges are a dilation of those of the original polyhedron (the dilation factor 0 is allowed).

## 2.4 Setup

Consider the cone $\sigma \subset N_{\mathbb{R}}$ and fix some element $R \in M$. Then

$$\mathbb{A}(R) := [R = 1] = \{a \in N_{\mathbb{R}} \mid \langle a, R\rangle = 1\} \subset N_{\mathbb{R}}$$

is an affine space; provided that $R$ is primitive, it comes with a lattice $\mathbb{L}(R) := [R = 1] \cap N$. The corresponding vector space is $\mathbb{A}_0(R) := [R = 0]$; this always has the lattice $\mathbb{L}_0(R) := [R = 0] \cap N$. We define the *crosssection* of $\sigma$ in degree $R$ to be the polyhedron

$$Q(R) := \sigma \cap [R = 1] \subset \mathbb{A}(R)$$

(here $Q$ stands for *Querschnitt*). This has cone of unbounded directions $Q(R)^{\infty} = \sigma \cap \mathbb{A}_0(R) \subset N_{\mathbb{R}}$. The compact part $Q(R)^{\mathrm{c}}$ is generated by its vertices $\overline{a}^j := a^j / \langle a^j, R\rangle$ for $j$ satisfying $\langle a^j, R\rangle \geq 1$. A trivial but nevertheless important observation is the following: the vertex $\overline{a}^j$ is a lattice point (that is, $\overline{a}^j \in \mathbb{L}(R)$), if and only if $\langle a^j, R\rangle = 1$.

Fundamental generators of $\sigma$ contained in $R^{\perp}$ can still be "seen" as edges in $Q(R)^{\infty}$, but those with $\langle \bullet, R\rangle < 0$ are "invisible" in $Q(R)$. In particular, we can recover the cone $\sigma$ from $Q(R)$ if and only if $R \in \sigma^{\vee}$.

## 2.5

Write $d^1, \ldots, d^N \in R^\perp \subset N_{\mathbb{R}}$ for the compact edges of $Q(R)$. As in [Al 4], §2, for each compact 2-face $\varepsilon < Q(R)$ we define its *sign vector* $\underline{\varepsilon} \in \{0, \pm 1\}^N$ to be

$$\varepsilon_i := \begin{cases} \pm 1 & \text{if } d^i \text{ is an edge of } \varepsilon, \\ 0 & \text{otherwise,} \end{cases}$$

where the signs are chosen so that the oriented edges $\varepsilon_i \cdot d^i$ fit into a cycle along the boundary of $\varepsilon$. This determines $\underline{\varepsilon}$ up to sign (and either choice will do). In particular, $\sum_i \varepsilon_i d^i = 0$.

**Definition 2.4** For each $R \in M$ we define the vector spaces

$$V(R) := \Big\{(t_1, \ldots, t_N) \;\Big|\; \sum_i t_i \varepsilon_i d^i = 0 \text{ for every compact 2-face } \varepsilon < Q(R)\Big\}$$

$$W(R) := \mathbb{R}^{\#\{\text{vertices of } Q(R) \text{ not in } N\}}.$$

The cone $C(R) := V(R) \cap \mathbb{R}^N_{\geq 0}$ measures the dilation of each compact edge, and therefore parametrizes exactly the Minkowski summands of positive multiples of $Q(R)$. For this reason, we call elements of $V(R)$ *generalized Minkowski summands*; they may have edges of negative length. (See [Al 4], Lemma 2.2 for a discussion of the compact case.) The vector space $W(R)$ provides coordinates $s_j$ for each vertex $\overline{a}^j \in Q(R) \setminus N$, that is, $\langle a^j, R\rangle \geq 2$.

## 2.6

Each compact edge $d^{jk} = \overline{\overline{a}^j \overline{a}^k}$ gives rise to a set of equations $G_{jk}$ relating elements $(\underline{t}, \underline{s}) \in V(R) \oplus W(R)$. These[1] sets are of one of the following three types:

(0) $G_{jk} = \emptyset$;

(1) $G_{jk} = \{s_j - s_k = 0\}$ whenever both coordinates exist in $W(R)$;

(2) $G_{jk} = \{t_{jk} - s_j = 0, t_{jk} - s_k = 0\}$, omitting equations that do not make sense.

[1] The edge corresponds to a codim 2 singular locus of $Y_\sigma$, whose transversal type is the cyclic quotient singularity $\frac{1}{n}(1, q)$ described in Example 2.2. As one can see at the end of 2.7, the choice of $G_{jk}$ is essentially governed by the position of the monomial relative to the classification of Example 2.2 into cases (i), (ii), (iii).

Restricting $V(R) \oplus W(R)$ to the (at most) three coordinates $t_{jk}$, $s_j$, $s_k$, the actual choice of $G_{jk}$ is made such that these equations yield a subspace of dimension $1 + \dim T^1_{\langle a^j, a^k \rangle}(-R)$. Notice that the dimension of $T^1(-R)$ for the two dimensional quotient singularity assigned to the plane cone $\langle a^j, a^k \rangle$ can be obtained from Example 2.2.

**Theorem 2.5** *The infinitesimal deformations of $Y_\sigma$ in degree $-R$ equal*

$$T^1_Y(-R) = \left\{ (\underline{t}, \underline{s}) \in V_{\mathbb{C}}(R) \oplus W_{\mathbb{C}}(R) \;\middle|\; \begin{array}{l} (\underline{t}, \underline{s}) \textit{ satisfies the} \\ \textit{equations } G_{jk} \textit{ of } 2.6 \end{array} \right\} \big/ \mathbb{C} \cdot (\underline{1}, \underline{1}).$$

The vector space $V(R)$ (encoding Minkowski summands) is, in a sense, the main tool to describe infinitesimal deformations. Depending on which of the above Types (0)–(2) the $G_{jk}$ belong to, the elements of $W(R)$ either provide additional parameters, or introduce conditions that exclude Minkowski summands not of some prescribed type.

If $Y$ is smooth in codimension two, then $G_{jk}$ is always of Type (2). In particular, the variables $\underline{s}$ are completely determined by the $\underline{t}$, and we obtain:

**Corollary 2.6** *If $Y$ is smooth in codimension two, then $T^1_Y(-R)$ is contained in $V_{\mathbb{C}}(R)/\mathbb{C} \cdot (\underline{1})$. It is made up of those $\underline{t}$ such that $t_{jk} = t_{kl}$ whenever $d^{jk}$, $d^{kl}$ are compact edges with a common non-lattice vertex $\overline{a}^k$ of $Q(R)$. Thus $T^1_Y(-R)$ equals the set of equivalence classes of those Minkowski summands of $\mathbb{R}_{\geq 0} \cdot Q(R)$ that preserve the stars of non-lattice vertices of $Q(R)$ up to homothety.*

## 2.7

**Proof of Theorem 2.5, Step 1** From Theorem 2.1, we know that $T^1_Y(-R)$ equals the complexified cohomology of the complex

$$N_{\mathbb{R}} \to \bigoplus_j \left(\operatorname{Span}_{\mathbb{R}} E_j^R\right)^* \to \bigoplus_{\langle a^j, a^k \rangle < \sigma} \left(\operatorname{Span}_{\mathbb{R}} E_{jk}^R\right)^*.$$

According to 2.2, an element of $\bigoplus_j \left(\operatorname{Span}_{\mathbb{R}} E_j^R\right)^*$ can be represented by a family of elements

$$\begin{cases} b^j \in N_{\mathbb{R}} & \text{if } \langle a^j, R \rangle \geq 2, \\ b^j \in N_{\mathbb{R}} / \mathbb{R} \cdot a^j & \text{if } \langle a^j, R \rangle = 1. \end{cases}$$

Dividing by the image of $N_{\mathbb{R}}$ means shifting this family by a common vector $b \in N_{\mathbb{R}}$. On the other hand, the family $\{b^j\}$ must map to 0 in the complex;

that is, for each compact edge $\overline{\overline{a}^j, \overline{a}^k} < Q$ the functions $b^j$ and $b^k$ must be equal on $\operatorname{Span}_{\mathbb{R}} E^R_{jk}$. Since

$$(a^j, a^k)^\perp \subset \operatorname{Span}_{\mathbb{R}} E^R_{jk} \subset (\operatorname{Span}_{\mathbb{R}} E^R_j) \cap (\operatorname{Span}_{\mathbb{R}} E^R_k),$$

we immediately obtain the necessary condition $b^j - b^k \in \mathbb{R}a^j + \mathbb{R}a^k$. However, the actual behavior of $\operatorname{Span}_{\mathbb{R}} E^R_{jk}$ will require a closer look in Step 3 below.

**Step 2** We introduce new "coordinates":

$\overline{b}^j := b^j - \langle b^j, R\rangle\, \overline{a}^j \in R^\perp$, which is defined even in the case $\langle a^j, R\rangle = 1$;

$s_j := -\langle b^j, R\rangle$ for $j$ meeting $\langle a^j, R\rangle \geq 2$ (inducing an element of $W(R)$).

The shift of the $b^j$ by an element $b \in N_{\mathbb{R}}$ (that is, $(b^j)' = b^j + b$) appears in these new coordinates as

$$\begin{aligned} (\overline{b}^j)' &= (b^j)' - \langle (b^j)', R\rangle\, \overline{a}^j = b^j + b - \langle b^j, R\rangle\, \overline{a}^j - \langle b, R\rangle\, \overline{a}^j \\ &= \overline{b}^j + b - \langle b, R\rangle\, \overline{a}^j, \\ s'_j &= -\langle (b^j)', R\rangle = s_j - \langle b, R\rangle\,. \end{aligned}$$

In particular, an element $b \in R^\perp$ does not change the $s_j$, but shifts the points $\overline{b}^j$ inside the hyperplane $R^\perp$. Hence, the set of the $\overline{b}^j$ should be considered modulo translation inside $R^\perp$ only. On the other hand, the condition $b^j - b^k \in \mathbb{R}a^j + \mathbb{R}a^k$ changes into $\overline{b}^j - \overline{b}^k \in \mathbb{R}\overline{a}^j + \mathbb{R}\overline{a}^k$ or even $\overline{b}^j - \overline{b}^k \in \mathbb{R}(\overline{a}^j - \overline{a}^k)$ (consider the values of $R$). Hence, the $\overline{b}^j$ form the vertices of a Minkowski summand of $Q(R)$, or at least a generalized Minkowski summand. Modulo translation, this summand is completely described by the dilation factors $t_{jk}$ obtained from

$$\overline{b}^j - \overline{b}^k = t_{jk} \cdot (\overline{a}^j - \overline{a}^k).$$

Now, the remaining part of the action of $b \in N_{\mathbb{R}}$ comes down to an action of $\langle b, R\rangle \in \mathbb{R}$ only:

$$\begin{aligned} t'_{jk} &= t_{jk} - \langle b, R\rangle\,, \quad \text{and} \\ s'_j &= s_j - \langle b, R\rangle\,, \quad \text{as we already know.} \end{aligned}$$

Up to now, we have found that $T^1_Y(-R) \subset V_{\mathbb{C}}(R) \oplus W_{\mathbb{C}}(R)/(\underline{1}, \underline{1})$.

**Step 3** Actually, the elements $b^j$ and $b^k$ must coincide on $\operatorname{Span}_{\mathbb{R}} E^R_{jk}$, which may be larger than just $(a^j, a^k)^\perp$. To measure the difference, consider the quotient space $\operatorname{Span}_{\mathbb{R}} E^R_{jk}/(a^j, a^k)^\perp$ contained in the two dimensional vector space $M_{\mathbb{R}}/(a^j, a^k)^\perp = \operatorname{Span}_{\mathbb{R}}(a^j, a^k)^*$. Since this quotient space coincides with the set $\operatorname{Span}_{\mathbb{R}} E^{\overline{R}}_{jk}$ corresponding to the two dimensional cone $\langle a^j, a^k \rangle \subset \operatorname{Span}_{\mathbb{R}}(a^j, a^k)$, where $\overline{R}$ denotes the image of $R$ in $\operatorname{Span}_{\mathbb{R}}(a^j, a^k)^*$, we may assume that $\sigma = \langle a^1, a^2 \rangle$ (that is, $j = 1$, $k = 2$) represents a two dimensional cyclic quotient singularity. In particular, we only need to discuss the three cases (i)–(iii) of Example 2.2:

In (i) and (ii) we have $\operatorname{Span}_{\mathbb{R}} E^R_{12} = 0$, that is, no additional equation is needed. This means $G_{12} = \emptyset$ is of Type (0) (see 2.6). On the other hand, if $T^1_Y = 0$, then the vector space $\mathbb{R}^3_{(t_{12}, s_1, s_2)}/\mathbb{R} \cdot (\underline{1})$ has to be killed by identifying the three variables $t_{12}$, $s_1$ and $s_2$; we obtain Type (2).

Case (iii) provides $\operatorname{Span}_{\mathbb{R}} E^R_{12} = \mathbb{R} \cdot R$. Hence, as an additional condition we obtain that $b^1$ and $b^2$ have to be equal on $R$. By definition of $s_j$, this means that $s_1 = s_2$, and $G_{12}$ must be of Type (1). $\square$

# 3 Genuine deformations

## 3.1

In [Al 2] we studied the so-called *toric deformations* in a given multidegree $-R \in M$. These are genuine deformations, in the sense that they are defined over smooth parameter spaces; they are characterized by the fact that the total space together with the embedding of the special fiber still belongs to the toric category. Despite the fact they look so special, it seems that toric deformations cover a big part of the versal deformation of $Y_\sigma$. They only exist in negative degrees (that is, $R \in \sigma^\vee \cap M$), but here they form a kind of skeleton. If $Y_\sigma$ is an isolated toric Gorenstein singularity, then toric deformations even provide all irreducible components of the versal deformation (cf. [Al 4]).

After briefly recalling the idea of this construction, we show how the new formula for $T^1_Y$ of Theorem 2.1 can be used to describe the Kodaira–Spencer map of toric deformations. We follow this by the study of nonnegative degrees: if $R \notin \sigma^\vee \cap M$, then we are still able to construct genuine deformations of $Y_\sigma$; however, these are no longer toric.

## 3.2

Let $R \in \sigma^\vee \cap M$. Then, as in [Al 2], §3, an $m$-parameter toric deformation of $Y_\sigma$ in degree $-R$ corresponds to a splitting of $Q(R)$ as a Minkowski sum

$$Q(R) = Q_0 + Q_1 + \cdots + Q_m$$

satisfying the following conditions:

(i) $Q_0 \subset \mathbb{A}(R)$ and $Q_1, \dots, Q_m \in \mathbb{A}_0(R)$ are polyhedra with common cone of unbounded directions $Q(R)^\infty$.

(ii) Each supporting hyperplane $t$ of $Q(R)$ defines a face $F(Q_i, t)$ of each of the polyhedra $Q_0, Q_1, \dots, Q_m$, and the Minkowski sum of these faces equals $F(Q(R), t)$. With at most one exception (which can depend on $t$), these faces contain lattice vertices (points of $N$).

**Remark 3.1** In [Al 2] we distinguished between the case of primitive and nonprimitive elements $R \in M$: if $R$ is a multiple of some element of $M$, then $\mathbb{A}(R)$ does not contain lattice points at all. In particular, condition (ii) just means that $Q_1, \dots, Q_m$ must be lattice polyhedra.

On the other hand, for primitive $R$, the $(m+1)$ summands $Q_i$ appear on an equal footing and may be put into the same space $\mathbb{A}(R)$. Their Minkowski sum must then be interpreted inside this affine space.

**Given a Minkowski decomposition, how to obtain the corresponding toric deformation?**

Defining $\widetilde{N} := N \oplus \mathbb{Z}^m$ (and $\widetilde{M} := M \oplus \mathbb{Z}^m$), we have to embed the summands as $(Q_0, 0)$, $(Q_1, e^1), \dots, (Q_m, e^m)$ into the vector space $\widetilde{N}_{\mathbb{R}}$; here $\{e^1, \dots, e^m\}$ denotes the standard basis of $\mathbb{Z}^m$. Together with $(Q(R)^\infty, 0)$, these polyhedra generate a cone $\widetilde{\sigma} \subset \widetilde{N}$ containing $\sigma$ via the inclusion

$$N \hookrightarrow \widetilde{N} \quad \text{defined by} \quad a \mapsto \big(a; \langle a, R\rangle, \dots, \langle a, R\rangle\big).$$

Actually, $\sigma$ equals $\widetilde{\sigma} \cap N_{\mathbb{R}}$, and we obtain an inclusion $Y_\sigma \hookrightarrow X_{\tilde{\sigma}}$ between the associated toric varieties.

On the other hand, $[R, 0]\colon \widetilde{N} \to \mathbb{Z}$ and $\mathrm{pr}_{\mathbb{Z}^m}\colon \widetilde{N} \to \mathbb{Z}^m$ induce regular functions $f\colon X_{\tilde{\sigma}} \to \mathbb{C}$ and $(f^1, \dots, f^m)\colon X_{\tilde{\sigma}} \to \mathbb{C}^m$, respectively. The resulting map $(f^1 - f, \dots, f^m - f)\colon X_{\tilde{\sigma}} \to \mathbb{C}^m$ is flat and has $Y_\sigma \hookrightarrow X_{\tilde{\sigma}}$ as special fiber.

## 3.3

Let $R \in \sigma^\vee \cap M$ and $Q(R) = Q_0 + \dots + Q_m$ be a decomposition satisfying conditions (i) and (ii) of 3.2. Denote by $(\overline{a}^j)_i$ the vertex of $Q_i$ induced from $\overline{a}^j \in Q(R)$, so that $\overline{a}^j = (\overline{a}^j)_0 + \dots + (\overline{a}^j)_m$.

**Theorem 3.2** *The Kodaira–Spencer map of the corresponding toric deformation $X_{\tilde{\sigma}} \to \mathbb{C}^m$ is the map*

$$\varrho\colon \mathbb{C}^m = T_{\mathbb{C}^m,0} \longrightarrow T^1_Y(-R) \subset V_{\mathbb{C}}(R) \oplus W_{\mathbb{C}}(R)/\mathbb{C}\cdot(\underline{1}, \underline{1})$$

*sending* $e^i$ *to the pair* $[Q_i, \underline{s}^i] \in V(R) \oplus W(R)$ *(for* $i = 1, \ldots, m$*) with*

$$s^i_j := \begin{cases} 0 & \text{if the vertex } (\overline{a}^j)_i \text{ of } Q_i \text{ belongs to the lattice } N, \\ 1 & \text{if } (\overline{a}^j)_i \text{ is not a lattice point.} \end{cases}$$

**Remark 3.3** Setting $e^0 := -(e^1 + \cdots + e^m)$, we obtain $\varrho(e^0) = [Q_0, \underline{s}^0]$ where $\underline{s}^0$ is defined in the same way as $\underline{s}^i$ above.

## 3.4 Proof of Theorem 3.2

We to derive the formula for the Kodaira–Spencer map from the more technical result of [Al 2], Theorem 5.3. If we use in addition [Al 3], Theorem 6.1, this theorem describes $\varrho(e^i) \in T^1_Y(-R) = H^1\big(\operatorname{Span}_{\mathbb{C}}(E^R)^*_\bullet\big)$ as follows:

Let $E = \{r^0, \ldots, r^w\} \subset \sigma^\vee \cap M$. Its elements may be lifted via $\widetilde{M} \twoheadrightarrow M$ to $\tilde{r}^v \in \widetilde{\sigma}^\vee \cap \widetilde{M}$ (for $v = 0, \ldots, w$); denote their $i$th entry of the $\mathbb{Z}^m$-part by $\eta^v_i$. Then, given elements $v^j \in \operatorname{Span} E^R_j$, we may represent them as $v^j = \sum_v q^j_v r^v$ with $q^j \in \mathbb{Z}^{E^R_j}$, and $\varrho(e^i)$ sends $v^j$ to the integer $-\sum_v q^j_v \eta^v_i$. Using the notation of 2.7 for $\varrho(e^i)$, this means that $b^j$ sends elements $r^v \in E^R_j$ onto $-\eta^v_i \in \mathbb{Z}$.

By construction of $\widetilde{\sigma}$, we have the inequalities

$$\big\langle \big((\overline{a}^j)_0, 0\big), \tilde{r}^v \big\rangle \geq 0 \quad \text{and} \quad \big\langle \big((\overline{a}^j)_i, e^i\big), \tilde{r}^v \big\rangle \geq 0 \quad \text{for } i = 1, \ldots, m,$$

which sum up to $\langle \overline{a}^j, r^v \rangle = \big\langle \big(\overline{a}^j, \underline{1}\big), \tilde{r}^v \big\rangle \geq 0$. On the other hand, $r^v \in E^R_j$ is equivalent to $\langle \overline{a}^j, r^v \rangle < 1$. Hence, for $i = 0, \ldots, m$, whenever $(\overline{a}^j)_i \in Q_i$ belongs to the lattice, the corresponding inequality becomes an equality. With at most one exception, this must always happen. Hence for $i = 1, \ldots, m$,

$$\big\langle (\overline{a}^j)_i, r^v \big\rangle + \eta^v_i = \begin{cases} 0 & \text{if } (\overline{a}^j)_i \in N, \\ \langle \overline{a}^j, r^v \rangle & \text{if } (\overline{a}^j)_i \notin N, \end{cases}$$

meaning that $b^j = (\overline{a}^j)_i$ or $b^j = (\overline{a}^j)_i - \overline{a}^j$ respectively. By definition of $\overline{b}^j$ and $s_j$ given in 2.7, we are done. □

## 3.5

We now treat the case of nonnegative degrees; let $R \in M \setminus \sigma^\vee$. It often happens that the easiest way to solve a problem is to change the question until there is no problem left. We can do this here by changing our cone $\sigma$ into some $\tau^R$ such that the degree $-R$ becomes negative. We define

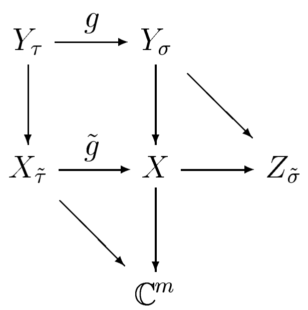
$$\tau := \tau^R := \sigma \cap [R \geq 0], \quad \text{that is,} \quad \tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0} \cdot R.$$

The cone $\tau$ defines an affine toric variety $Y_\tau$. Since $\tau \subset \sigma$, it comes with a map $g\colon Y_\tau \to Y_\sigma$; in other words, $Y_\tau$ is an open part of a modification of $Y_\sigma$. The important observation is

$$\begin{array}{rcccl} \tau \cap [R=0] & = & \sigma \cap [R=0] & = & Q(R)^\infty, \quad \text{and} \\ \tau \cap [R=1] & = & \sigma \cap [R=1] & = & Q(R), \end{array}$$

implying $T^1_{Y_\tau}(-R) = T^1_{Y_\sigma}(-R)$ by Theorem 2.6. Moreover, even the genuine toric deformations $X_{\tilde\tau} \to \mathbb{C}^m$ of $Y_\tau$ carry over to $m$-parameter (nontoric) deformations $X \to \mathbb{C}^m$ of $Y_\sigma$:

**Theorem 3.4** *Each Minkowski decomposition $Q(R) = Q_0 + Q_1 + \cdots + Q_m$ satisfying (i) and (ii) of 3.2 provides an $m$-parameter deformation $X \to \mathbb{C}^m$ of $Y_\sigma$. Via some birational map $\tilde g\colon X_{\tilde\tau} \to X$, it is compatible with the toric deformation $X_{\tilde\tau} \to \mathbb{C}^m$ of $Y_\tau$ presented in 3.2.*




*The total space $X$ is no longer toric, but it sits via birational maps between $X_{\tilde\tau}$ and some affine toric variety $Z_{\tilde\sigma}$ that still contains $Y_\sigma$ as a closed subset.*

## 3.6 Proof

We first construct $\widetilde N$, $\widetilde M$ and $\tilde\tau \subset \widetilde N_{\mathbb{R}}$ by the recipe of 3.2. In particular, $N \subset \widetilde N$, and the projection $\pi\colon \widetilde M \to M$ sends $[r; g_1, \ldots, g_m]$ onto $r + (\sum_i g_i)R$. Defining $\tilde\sigma := \tilde\tau + \sigma$ (hence $\tilde\sigma^\vee = \tilde\tau^\vee \cap \pi^{-1}(\sigma^\vee)$), we obtain the commutative diagram

$$\begin{array}{ccc} \mathbb{C}[\tilde\tau^\vee \cap \widetilde M] & \longleftarrow & \mathbb{C}[\tilde\sigma^\vee \cap \widetilde M] \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{C}[\tau^\vee \cap M] & \longleftarrow & \mathbb{C}[\sigma^\vee \cap M] \end{array}$$

with surjective vertical maps. The canonical elements $e_1, \ldots, e_m \in \mathbb{Z}^m \subset \widetilde M$ together with $[R; 0] \in \widetilde M$ are preimages of $R \in M$. Hence, the corresponding monomials $x^{e_1}, \ldots, x^{e_m}, x^{[R,0]}$ in the semigroup algebra $\mathbb{C}[\tilde\tau^\vee \cap \widetilde M]$ (these were called $f^1, \ldots, f^m, f$ in 3.2) map onto $x^R \in \mathbb{C}[\tau^\vee \cap M]$, which is not regular on $Y_\sigma$. We define $Z_{\tilde{\sigma}}$ as the affine toric variety corresponding to $\tilde{\sigma}$, and $X$ as

$$X := \operatorname{Spec} B, \quad \text{where} \quad B := \mathbb{C}[\tilde{\sigma}^\vee \cap \widetilde{M}][f^1 - f, \ldots, f^m - f] \subset \mathbb{C}[\tilde{\tau}^\vee \cap \widetilde{M}].$$

This means that $X$ is obtained from $X_{\tilde{\tau}}$ by eliminating all the variables except those lifted from $Y_\sigma$ and the deformation parameters themselves. By construction of $B$, the vertical algebra homomorphisms $\pi$ induce a surjection $B \twoheadrightarrow \mathbb{C}[\sigma^\vee \cap M]$.

**Lemma 3.5** *Any element of* $\mathbb{C}[\tilde{\tau}^\vee \cap \widetilde{M}]$ *can be written in a unique way as a sum*

$$\sum_{(v_1,\ldots,v_m)\in\mathbb{N}^m} c_{v_1,\ldots,v_m} \cdot (f^1 - f)^{v_1} \cdots (f^m - f)^{v_m},$$

*with* $c_{v_1,\ldots,v_m} \in \mathbb{C}[\tilde{\tau}^\vee \cap \widetilde{M}]$ *such that* $s - e_i \notin \tilde{\tau}^\vee$ *(for* $i = 1, \ldots, m$*) for any of its monomial terms* $x^s$*. Moreover, these sums belong to the subalgebra* $B$ *if and only if their coefficients* $c_{v_1,\ldots,v_m}$ *do so.*

**Proof** *(a) Existence:* Let $s - e_i \in \tilde{\tau}^\vee$ for some $s, i$. Then, with $s' := s - e_i + [R, 0]$ we obtain

$$x^s = x^{s'} + x^{s-e_i}(x^{e_i} - x^{[R,0]}) = x^{s'} + x^{s-e_i}(f^i - f).$$

Since $e_i = 1$ and $[R, 0] = 0$ if evaluated on $(Q_i, e^i) \subset \tilde{\tau}$, this process eventually stops.

*(b) Membership of* $B$*:* For the previous reduction step, we have to show that if $s \in \mathbb{C}[\tilde{\sigma}^\vee \cap \widetilde{M}]$, then the same holds for $s'$ and $s - e_i$. Since $\pi(s') = \pi(s) \in \sigma^\vee$, this is clear for $s'$. It remains to check that $\pi(s - e_i) \in \sigma^\vee$. Let $a \in \sigma$ be an arbitrary test element; we distinguish two cases:

Case 1: $\langle a, R\rangle \geq 0$. Then $a$ belongs to the subcone $\tau$, and $\pi(s - e_i) \in \tau^\vee$ yields $\langle a, \pi(s - e_i)\rangle \geq 0$.

Case 2: $\langle a, R\rangle \leq 0$. This implies

$$\langle a, \pi(s - e_i)\rangle = \langle a, s\rangle - \langle a, R\rangle \geq \langle a, s\rangle \geq 0.$$

*(c) Uniqueness:* Let $p := \sum c_{v_1,\ldots,v_m} \cdot (f^1 - f)^{v_1} \ldots (f^m - f)^{v_m}$ (satisfying the above conditions) be equal to 0 in $\mathbb{C}[\tilde{\tau}^\vee \cap \widetilde{M}]$. Using the projection $\pi\colon \widetilde{M} \to M$ makes everything $M$-graded. Since the factors $(f^i - f)$ are homogeneous (of degree $R$), we may assume that the same holds for $p$, hence also for its coefficients $c_{v_1,\ldots,v_m}$.

**Claim 3.6** *These coefficients are just monomials.*

Indeed, if $s, s' \in \widetilde{\tau}^\vee$ had the same image under $\pi$, we could assume that some $e_i$-coordinate of $s'$ is smaller than that of $s$. Hence, $s - e_i$ would still be equal to $s$ on $(Q_0, 0)$ and on any $(Q_j, e^j)$ (for $j \neq i$), but even greater than or equal to $s'$ on $(Q_i, e^i)$. This would imply $s - e_i \in \widetilde{\tau}^\vee$, contradicting our assumption for $p$.

Say $c_{v_1,\dots,v_m} = \lambda_{v_1,\dots,v_m} x^\bullet$; we use the projection $\widetilde{M} \to \mathbb{Z}^m$ to take $p$ into the ring $\mathbb{C}[\mathbb{Z}^m] = \mathbb{C}[y_1^{\pm 1}, \dots, y_m^{\pm 1}]$. The elements $x^\bullet$, $f^i$, $f$ map onto $y^\bullet$, $y_i$, and 1, respectively. Hence, $p$ turns into

$$\overline{p} = \sum_{(v_1,\dots,v_m)\in\mathbb{N}^m} \lambda_{v_1,\dots,v_m} \cdot y^\bullet \cdot (y_1 - 1)^{v_1} \cdots (y_m - 1)^{v_m}.$$

By induction through $\mathbb{N}^m$, we obtain that vanishing of $\overline{p}$ implies the vanishing of its coefficients: replace $y_i - 1$ by $z_i$, and take partial derivatives. This proves the claim.

Now, we can easily see that $X \to \mathbb{C}^m$ is flat and has $Y_\sigma$ as special fiber: Lemma 3.5 means that for $k = 0, \dots, m$, we have inclusions

$$B/(f^1 - f, \dots, f^k - f) \hookrightarrow \mathbb{C}[\widetilde{\tau}^\vee \cap \widetilde{M}]/(f^1 - f, \dots, f^k - f).$$

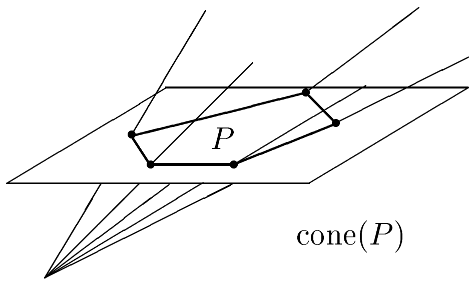
The values $k < m$ yield that $(f^1 - f, \dots, f^m - f)$ forms a regular sequence even in the subring $B$, meaning that $X \to \mathbb{C}^m$ is flat. With $k = m$ we obtain that the surjective map $B/(f^1 - f, \dots, f^m - f) \to \mathbb{C}[\sigma^\vee \cap M]$ is also injective. $\square$

# 4 Three dimensional toric Gorenstein singularities

## 4.1

By [Ish], Theorem 7.7, toric Gorenstein singularities always arise from the following construction: assume given a *lattice polytope* $P \subset \mathbb{R}^n$. We embed the whole space (containing $P$) into the “height one” slice of $N_\mathbb{R} := \mathbb{R}^n \oplus \mathbb{R}$, and consider the cone $\sigma$ generated by $P$; write $M_\mathbb{R} := (\mathbb{R}^n)^* \oplus \mathbb{R}$ for the dual space and $N, M$ for the natural lattices. Our polytope $P$ may be recovered from $\sigma$ as

$$P = Q(R^*) \subset \mathbb{A}(R^*) \quad \text{with } R^* := [\underline{0}, 1] \in M.$$




The fundamental generators $a^1, \ldots, a^M \in \mathbb{L}(R^*)$ of $\sigma$ coincide with the vertices of $P$. (This involves a slight abuse of notation; we use the same symbol $a^j$ for both $a^j \in \mathbb{Z}^n$ and $(a^j, 1) \in M$.)

If $\overline{a^j a^k}$ is an edge of $P$, we denote by $\ell(j,k) \in \mathbb{Z}$ its "length" induced from the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Every edge provides a codimension two singular stratum of $Y_\sigma$ with transversal type $\mathrm{A}_{\ell(j,k)-1}$. In particular, $Y_\sigma$ is smooth in codimension two if and only if every edge of $P$ is primitive, that is, of length $\ell = 1$.

## 4.2

As usual, we fix some element $R \in M$. We know the vector spaces $V(R)$ and $W(R)$ from 2.5; we introduce the subspace

$$V'(R) := \left\{ \underline{t} \in V(R) \;\middle|\; t_{jk} \neq 0 \Rightarrow 1 \leq \langle a^j, R \rangle = \langle a^k, R \rangle \leq \ell(j,k) \right\} \subset V(R).$$

This represents the Minkowski summands of $Q(R)$ that contract to a point any compact edge *not* satisfying $\langle a^j, R \rangle = \langle a^k, R \rangle \leq \ell(j,k)$.

**Theorem 4.1** *For $T^1_Y(-R)$, we distinguish two different types of $R \in M$:*

*(i) If $R \leq 1$ on $P$ (or equivalently $\langle a^j, R \rangle \leq 1$ for $j = 1, \ldots, M$), then $T^1_Y(-R) = V_{\mathbb{C}}(R)/(\underline{1})$. Moreover, concerning Minkowski summands, we may replace the polyhedron $Q(R)$ by its compact part $P \cap [R = 1]$ (a face of $P$).*

*(ii) If $R$ does not satisfy the previous condition, then $T^1_Y(-R) = V'(R)$.*

**Proof** The first case follows from Theorem 2.6 simply because $W(R) = 0$. For (ii), we assume that there are vertices $a^j$ contained in the affine halfspace $[R \geq 2]$. These vertices can be connected to one another inside this halfspace via paths along edges of $P$.

The two dimensional cyclic quotient singularities corresponding to the edges $\overline{a^j a^k}$ of $P$ are themselves Gorenstein. In the language of Example 2.2, this means that $w = 2$, and we obtain

$$\dim T^1_{\langle a^j, a^k \rangle}(-R) = \begin{cases} 1 & \text{if } \langle a^j, R\rangle = \langle a^k, R\rangle = 2, \dots, \ell(j,k) \\ & \quad \text{(case (iii) in 2.2)}, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $T^1_{\langle a^j, a^k \rangle}(-R)$ cannot be two dimensional, and (in the notation of 2.6) the equations $s_j - s_k = 0$ belong to $G_{jk}$ whenever $\langle a^j, R\rangle, \langle a^k, R\rangle \geq 2$. This means that for elements of

$$T^1_Y \subset \big(V_{\mathbb{C}}(R) \oplus W_{\mathbb{C}}(R)\big) / \mathbb{C} \cdot (\underline{1}, \underline{1}),$$

all entries of the $W_{\mathbb{C}}(R)$-part must be equal, or even zero, after dividing by $\mathbb{C} \cdot (\underline{1}, \underline{1})$. Moreover, if not both $\langle a^j, R\rangle$ and $\langle a^k, R\rangle$ equal one, the vanishing of $T^1_{\langle a^j, a^k \rangle}(-R)$ implies that $G_{jk}$ also contains the equation $t_{jk} - s_\bullet = 0$. □

**Corollary 4.2** *For toric Gorenstein singularities, the condition* 3.2, *(ii) that allow us to build genuine deformations simplifies:* $Q_1, \dots, Q_m$ *just have to be lattice polyhedra.*

**Proof** If $R \leq 1$ on $P$, then $Q(R)$ is itself a lattice polyhedron. Hence, condition (ii) automatically comes down to this simpler form.

In the second case, $T^1_Y(-R)$ involves some $W(R)$ part. On the one hand, via the Kodaira–Spencer map, it indicates which vertices of which polyhedron $Q_i$ belong to the lattice. On the other, we have observed in the previous proof that the entries of $W(R)$ are all equal. This exactly implies our claim. □

## 4.3

To treat three dimensional toric Gorenstein singularities, we now focus on *plane lattice polygons* $P \subset \mathbb{R}^2$. The vertices $a^1, \dots, a^M$ are arranged in a cycle. For $j \in \mathbb{Z}/M\mathbb{Z}$, we write $d^j := a^{j+1} - a^j \in \mathbb{L}_0(R^*)$ for the edge from $a^j$ to $a^{j+1}$ (see 2.4), and $\ell(j) := \ell(j, j+1)$ for its length.

Let $s^1, \dots, s^M$ be the fundamental generators of the dual cone $\sigma^\vee$, labelled so that $\sigma \cap (s^j)^\perp$ equals the face spanned by $a^j, a^{j+1} \in \sigma$. In particular, skipping the last coordinate of $s^j$ yields the (primitive) inner normal vector at the edge $d^j$ of $P$.

**Remark 4.3** For the convenience of those who prefer to live in $M$ rather than $N$, we show how to see the integers $\ell(j)$ in the dual world: choose a fundamental generator $s^j$ and denote by $r, r' \in M$ the elements of the Hilbert basis closest to $s^j$ along the two adjacent faces of $\sigma^\vee$, respectively (see Figure 1 in 4.7). Then, $\{R^*, s^j\}$ together with either $r$ or $r'$ form a basis of the lattice $M$, and $(r + r') - \ell(j)R^*$ is a multiple of $s^j$.

In the very special case of plane lattice polygons (that is, three dimensional toric Gorenstein singularities), we can describe $T^1_Y$ and the genuine deformations (for fixed $R \in M$) explicitly. First, we can easily spot the degrees carrying infinitesimal deformations:

**Theorem 4.4** *In general (with the exceptions (4–5)), $T^1_Y(-R)$ is only nontrivial in the following cases:*

*(1) $R = R^*$ with $\dim T^1_Y(-R) = M - 3$;*

*(2) $R = qR^*$ (for $q \geq 2$) with $\dim T^1_Y(-R) = \max\big\{0, \#\{j \mid q \leq \ell(j)\} - 2\big\}$;*

*(3) $R = qR^* - ps^j$ with $2 \leq q \leq \ell(j)$ and $p \in \mathbb{Z}$ sufficiently large such that $R \notin \operatorname{int}(\sigma^\vee)$. In this case, $T^1_Y(-R)$ is one dimensional.*

*Additional degrees exist only in the following two (overlapping) exceptional cases:*

*(4) $P$ contains a pair of parallel edges $d^j$, $d^k$, both longer than every other edge. Then $\dim T^1_Y(-qR^*) = 1$ for $q$ in the range*

$$\max\big\{\ell(l) \;\big|\; l \neq j, k\big\} < q \leq \min\big\{\ell(j), \ell(k)\big\}.$$

*(5) $P$ contains a pair of parallel edges $d^j$, $d^k$ with distance $d$ ($d := \big\langle a^j, s^k\big\rangle = \big\langle a^k, s^j\big\rangle$). If $\ell(k) > d \geq \max\big\{\ell(l) \;\big|\; l \neq j, k\big\}$, then $\dim T^1_Y(-R) = 1$ for $R = qR^* + ps^j$ with $1 \leq q \leq \ell(j)$ and $1 \leq p \leq \big(\ell(k) - q\big)/d$.*

The cases (1), (2), (4), and (5) yield at most finitely many (negative) degrees in $T^1_Y$. Type (3) consists of $\ell(j) - 1$ infinite series corresponding to any vertex $a^j \in P$; these series contain only nonnegative degrees, except possibly for the leading elements ($R$ might sit on $\partial\sigma^\vee$).

**Proof** These assertions are straightforward consequences of Theorem 4.1, so that the following brief remark should be sufficient: the condition $\langle a^j, R\rangle = \langle a^{j+1}, R\rangle$ means $d^j \in R^\perp$. Moreover, if $R \notin \mathbb{Z} \cdot R^*$, there is at most one edge (or a pair of parallel edges) having this property. $\square$

## 4.4

**Example 4.5** A typical example of a nonisolated, three dimensional toric Gorenstein singularity is the cone over the weighted projective space $\mathbb{P}(1,2,3)$. We use it to illustrate our calculations of $T^1$ as well as the forthcoming construction of genuine one parameter families. $P$ has the vertices $(-1,-1)$, $(2,-1)$, $(-1,1)$, so that $\sigma$ is the cone generated by

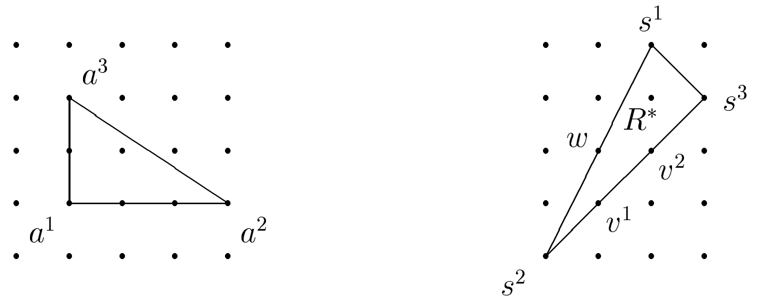
$$a^1 = (-1,-1;1), \quad a^2 = (2,-1;1), \quad a^3 = (-1,1;1).$$

Since our singularity is a cone over a projective variety, $\sigma^\vee$ also appears as a cone over some lattice polygon. In this example, $\sigma$ and $\sigma^\vee$ happen to be isomorphic. We obtain

$$\sigma^\vee = \langle s^1, s^2, s^3 \rangle \quad \text{with} \quad s^1 = [0,1;1], s^2 = [-2,-3;1], s^3 = [1,0;1].$$

The Hilbert basis $E \subset \sigma^\vee \cap \mathbb{Z}^3$ consists of these three fundamental generators together with

$$R^* = [0,0;1], \quad v^1 = [-1,-2;1], \quad v^2 = [0,-1;1], \quad w = [-1,-1;1].$$



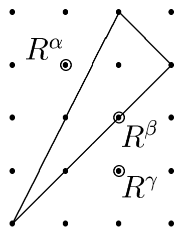

$\sigma = \text{cone}(P)$ $\qquad\qquad$ $\sigma^\vee$

In particular, $Y_\sigma$ has embedding dimension 7. The edges of $P$ have length $\ell(1) = 3$, $\ell(2) = 1$, and $\ell(3) = 2$. Thus $Y_\sigma$ has one dimensional singularities of transversal type $A_2$ and $A_1$. According to Theorem 4.4, $Y_\sigma$ admits only infinitesimal deformations of type (3). Their degrees come in three series:

($\alpha$) $2R^* - p_\alpha s^3$ with $p_\alpha \geq 1$. Even the leading element $R^\alpha = [-1,0,1]$ is not contained in $\sigma^\vee$.

($\beta$) $2R^* - p_\beta s^1$ with $p_\beta \geq 1$. The leading element equals $R^\beta = v^2 = [0,-1,1]$ and sits on the boundary of $\sigma^\vee$.

($\gamma$) $3R^* - p_\gamma s^1$ with $p_\gamma \geq 2$. The leading element is $R^\gamma = [0,-2,1] \notin \sigma^\vee$.




$\sigma^\vee \subset M_{\mathbb{R}}$

## 4.5

Each degree belonging to type (3) (that is, $R = qR^* - ps^j$ with $2 \leq q \leq \ell(j)$) provides an infinitesimal deformation. To show that they are unobstructed by describing how they lift to genuine one parameter deformations should be no problem: just split the polygon $Q(R)$ into a Minkowski sum satisfying conditions (i) and (ii) of 3.2, then construct $\widetilde{\tau}$, $\widetilde{\sigma}$ and $(f^1 - f)$ as in 3.2 and 3.5.

However, we prefer to present the result for our special case all at once using new coordinates. Let $P \subset \mathbb{A}(R^*) = \mathbb{R}^2 \times \{1\} \subset \mathbb{R}^3 = N_{\mathbb{R}}$ be a lattice polygon as in 4.3, and $R = qR^* - ps^j$ as just mentioned. Then $\sigma, \tau \subset N_{\mathbb{R}}$ are the cones over $P$ and $P \cap [R \geq 0]$ respectively, and the one parameter family in degree $-R$ is obtained as follows:

**Proposition 4.6** *The cone $\widetilde{\tau} \subset N_{\mathbb{R}} \oplus \mathbb{R} = \mathbb{R}^4$ is generated by the elements:*

*(i)* $(a, 0) - \langle a, R\rangle\, (\underline{0}, 1)$ *as* $a \in P \cap [R \geq 0]$ *runs through the vertices from the* $R^\perp$*-line to* $a^j$*,*

*(ii)* $(a, 0) - \langle a, R\rangle\, (d^j/\ell(j), 1)$ *as* $a \in P \cap [R \geq 0]$ *runs from* $a^{j+1}$ *back up to the line* $R^\perp$*, and*

*(iii)* $(\underline{0}, 1)$ *and* $(d^j/\ell(j), 1)$*.*

*The vector space $N_{\mathbb{R}}$ containing $\sigma$ sits in $N_{\mathbb{R}} \oplus \mathbb{R}$ as $N_{\mathbb{R}} \times \{0\}$. Via this embedding, we have as usual $\widetilde{\sigma} = \widetilde{\tau} + \sigma$. The monomials $f$ and $f^1$ are given by their exponents $[R, 0]$ and $[R, 1] \in M \oplus \mathbb{Z}$ respectively.*

Geometrically, one can think of $\widetilde{\tau}$ as generated by the interval $I$ with vertices as in (iii) and by the polygon $P'$ obtained as follows: "squeeze" $P \cap [R \geq 0]$ along $R^\perp$ by a cone with base $q/\ell(j) \cdot \overline{a^j a^{j+1}}$ and some vertex on the line $R^\perp$; take $-\langle \bullet, R\rangle$ as an additional, fourth coordinate. Then, $[R^*, 0]$ is still 1 on $P'$ and equals 0 on $I$. Moreover, $[R, 0]$ vanishes on $I$ and on the $R^\perp$-edge of $P'$; $[R, 1]$ vanishes on the whole of $P'$.

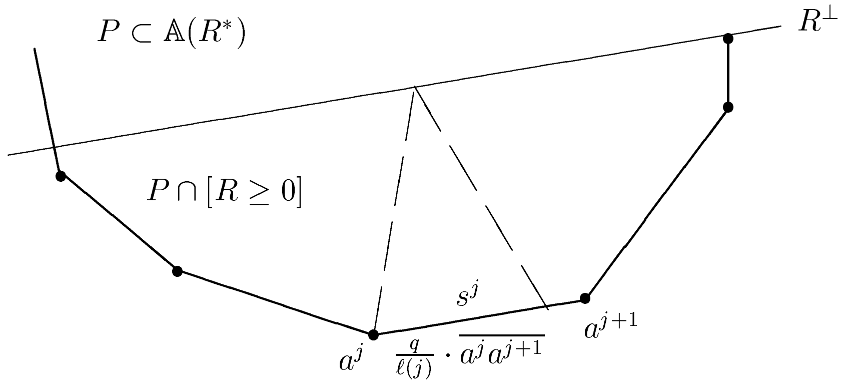



Figure 1: Parallel lines $R^\perp$ and $a^j a^{j+1}$ at distance $q/p$ apart in $\mathbb{A}(R^*)$

**Proof** We change coordinates. If $g := \gcd(p, q)$ denotes the "length" of $R$, then we can find an $s \in M$ such that $\{s, R/g\}$ forms a basis of $M \cap (d^j)^\perp$. Adding some $r \in M$ with $\langle d^j/\ell(j), r\rangle = 1$ (the $r$ of Remark 4.3 will do) yields a $\mathbb{Z}$-basis for the whole lattice $M$. We consider the following commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow[\sim]{(s,\,r,\,R/g)} & \mathbb{Z}^3 \\ {\scriptstyle (\mathrm{id},0)}\downarrow & & \downarrow{\scriptstyle (\mathrm{id},\, g\cdot \mathrm{pr}_3)} \\ N \oplus \mathbb{Z} & \xrightarrow[\sim]{([s,0],\,[r,0],\,[R/g,0],\,[R,1])} & \mathbb{Z}^3 \oplus \mathbb{Z} \end{array}$$

The left hand side contains the data relevant for our proposition. Carrying them over to the right yields:

- $[0, 0, g] \in (\mathbb{Z}^3)^*$ as the image of $R$;
- $[0, 0, g, 0], [0, 0, 0, 1] \in (\mathbb{Z}^4)^*$ as the images of $[R, 0]$ and $[R, 1]$ respectively;
- $\tau$ becomes a cone with affine crosssection

$$Q([0, 0, g]) = \\ \operatorname{conv}\Big( \big( \langle a, s\rangle \,/\, \langle a, R\rangle \,;\, \langle a, r\rangle \,/\, \langle a, R\rangle \,;\, 1/g \big) \,\Big|\, a \in P \cap [R \geq 0] \Big);$$

- $I$ changes into the unit interval $(Q_1, 1)$ reaching from $(0, 0, 0, 1)$ to $(0, 1, 0, 1)$;

- finally, cone$(P')$ maps onto the cone spanned by the convex hull $(Q_0, 0)$ of the points $\big(\langle a,s\rangle \,/\, \langle a,R\rangle\,;\langle a,r\rangle \,/\, \langle a,R\rangle\,;1/g;0\big)$ for $a \in P \cap [R \geq 0]$ on the $a^j$-side and $\big(\,\langle a,s\rangle \,/\, \langle a,R\rangle\,;\langle a,r\rangle \,/\, \langle a,R\rangle - 1;1/g;0\big)$ for $a$ on the $a^{j+1}$-side respectively.

Since $Q([0,0,g])$ equals the Minkowski sum of the interval $Q_1 \subset \mathbb{A}_0([0,0,g])$ and the polygon $Q_0 \subset \mathbb{A}([0,0,g])$, we are done by 3.2. □

## 4.6

To see how the original equations of the singularity $Y_\sigma$ are perturbed, it is useful to study first the dual cones $\widetilde{\tau}^\vee$ or $\widetilde{\sigma}^\vee = \widetilde{\tau}^\vee \cap \pi^{-1}(\sigma^\vee)$:

**Proposition 4.7** *If $s \in \sigma^\vee \cap M$, the element of $M \oplus \mathbb{Z}$ given by*

$$S := \begin{cases} [s,0] & \text{if } \langle d^j, s\rangle \geq 0 \\ [s, -\langle d^j/\ell(j), s\rangle] & \text{if } \langle d^j, s\rangle \leq 0 \end{cases}$$

*is a lift of $s$ to $\widetilde{\sigma}^\vee \cap (M \oplus \mathbb{Z})$. (Notice that it does not depend on $p, q$, but only on $j$.) Moreover, if $s^v$ runs through the edges of $P \cap [R \geq 0]$, the elements $S^v$ together with $[R,0]$ and $[R,1]$ form the fundamental generators of $\widetilde{\tau}^\vee$.*

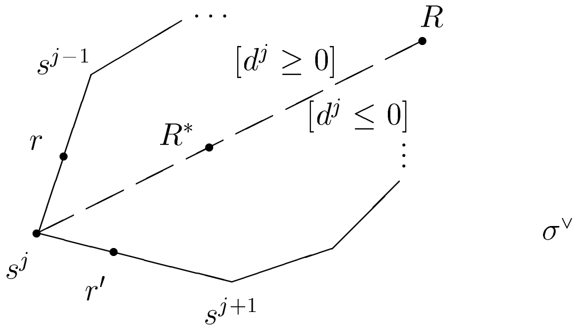
**Proof** Since we know $\widetilde{\tau}$ from the previous proposition, the calculations are straightforward and we omit them. □

## 4.7

Recall from 2.1 that $E$ denotes the minimal set generating the semigroup $\sigma^\vee \cap M$. Any $s \in E$ has an associated variable $z_s$, and $Y_\sigma \subset \mathbb{C}^E$ is given by binomial equations arising from linear relations among elements of $E$. Everything will be clear by considering an example: a linear relation such as $s^1 + 2s^3 = s^2 + s^4$ transforms into $z_1 z_3^2 = z_2 z_4$.

The fact that $\sigma$ defines a Gorenstein variety (that is, $\sigma$ is a cone over a lattice polytope) implies that $E$ consists only of $R^*$ and elements of $\partial\sigma^\vee$ including the fundamental generators $s^v$. If $E \cap \partial\sigma^\vee$ is ordered clockwise, then any two adjacent elements together with $R^*$ form a $\mathbb{Z}$-basis of the three dimensional lattice $M$.

In particular, any three consecutive elements of $E \cap \partial\sigma^\vee$ provide a unique linear relation among them and $R^*$. (We have already met this fact in Remark 4.3, where $r, s^j, r'$ were consecutive elements.) The resulting "boundary" equations do not generate the ideal of $Y_\sigma \subset \mathbb{C}^E$. Nevertheless, to describe a deformation of $Y_\sigma$, it is sufficient to know only how this set of equations is perturbed. Moreover, if one has to avoid boundary equations "overlapping" a certain spot on $\partial\sigma^\vee$, then it will even be possible to drop up to two of them from the list.




**Theorem 4.8** *The one parameter deformation of $Y_\sigma$ in degree $-(qR^* - ps^j)$ is completely determined by the following perturbations:*

*(i) (Boundary) equations involving only variables induced from $[d^j \geq 0] \subset \sigma^\vee$ remain unchanged. The same statement holds for $[d^j \leq 0]$.*

*(ii) The boundary equation $z_r z_{r'} - z_{R^*}^{\ell(j)} z_{s^j}^k = 0$ corresponding to the triple $\{r, s^j, r'\}$ is perturbed to $\big(z_r z_{r'} - z_{R^*}^{\ell(j)} z_{s^j}^k\big) - t z_{R^*}^{\ell(j)-q} z_{s^j}^{k+p} = 0$. Divide everything by $z_{s^j}^k$ if $k < 0$.*

**Proof** Restricted to either $[d^j \geq 0]$ or $[d^j \leq 0]$, the map $s \mapsto S$ lifting elements of $E$ to $\widetilde{\sigma} \cap (M \oplus \mathbb{Z})$ is linear. Hence, any linear relation remains true, and part (i) is proved.

For the second part, we consider the boundary relation $r + r' = \ell(j)R^* + ks^j$ with a suitable $k \in \mathbb{Z}$. By Proposition 4.6, the summands involved lift to the elements $[r, 0]$, $[r', 1]$, $[R^*, 0]$, and $[s^j, 0]$ respectively. In particular, the relation breaks down and has to be replaced by

$$\begin{aligned} [r,0] + [r',1] &= [R,1] + \big(\ell(j) - q\big)[R^*,0] + (k+p)[s^j,0], \quad \text{and} \\ \ell(j)[R^*,0] + k[s^j,0] &= [R,0] + \big(\ell(j) - q\big)[R^*,0] + (k+p)[s^j,0]. \end{aligned}$$

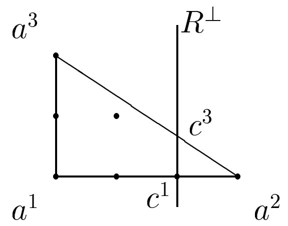
The monomials corresponding to $[R, 1]$ and $[R, 0]$ are $f^1$ and $f$ respectively. They are *not* regular on the total space $X$, but their difference $t := f^1 - f$ is. Hence, the difference of the monomial versions of both equations yields the result.

Finally, we should remark that (i) and (ii) cover all boundary equations except those overlapping the intersection of $\partial\sigma^\vee$ with $\overline{R^*R}$. $\square$

## 4.8

We return to Example 4.4 and discuss the one parameter deformations in degree $-R^\alpha$, $-R^\beta$, and $-R^\gamma$ respectively:

**Case** $\alpha$**:** $R^\alpha = [-1,0,1] = 2R^* - s^3$ means $j=3$, $q=\ell(3)=2$, and $p=1$. Hence, the line $R^\perp$ has distance $q/p=2$ from its parallel through $a^3$ and $a^1$. In particular, $\tau = \langle a^1, c^1, c^3, a^3\rangle$ with $c^1=(1,-1,1)$ and $c^3=(3,-1,3)$.




$\tau \subset \sigma$

We construct the generators of $\widetilde{\tau}$ by the recipe of Proposition 4.6: $a^3$ treated via (i) and $a^1$ treated via (ii) yield the same element $A := (-1,1,1,-2)$; from the $R^\perp$-line we obtain $C^1 := (1,-1,1,0)$ and $C^3 := (3,-1,3,0)$; finally (iii) provides $X := (0,0,0,1)$ and $Y := (0,-1,0,1)$. Hence, $\widetilde{\tau}$ is the cone over the pyramid with plane base $XYC^1C^3$ and $A$ as top. (The relation between the vertices of the quadrangle equals $3C^1 + 2X = C^3 + 2Y$.) Moreover, $\widetilde{\sigma}$ equals $\widetilde{\sigma} = \widetilde{\tau} + \mathbb{R}_{\geq 0} a^2$ with $a^2 := (a^2, 0)$. Since $A + 2X + 2a^2 = C^3$ and $A + 2Y + 2a^2 = 3C^1$, $\widetilde{\sigma}$ is a simplex generated by $A$, $X$, $Y$, and $a^2$.

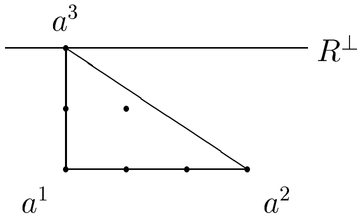
Denoting the variables assigned to $s^1, s^2, s^3, R^*, v^1, v^2, w \in E \subset \sigma^\vee \cap M$ by $Z_1, Z_2, Z_3, U, V_1, V_2, W$ respectively, there are six boundary equations:

$$Z_3 W Z_1 - U^3 = Z_1 Z_2 - W^2 = 0,$$
$$WV_1 - UZ_2 = Z_2V_2 - V_1^2 = V_1Z_3 - V_2^2 = V_2Z_1 - U^2 = 0.$$

Only the latter four are covered by Theorem 4.8. They are perturbed to

$$WV_1 - UZ_2 = Z_2V_2 - V_1^2 = V_1Z_3 - V_2^2 = V_2Z_1 - U^2 - t_\alpha Z_3 = 0.$$

**Case** $\beta$**:** $R^\beta = [0,-1,1] = 2R^* - s^1$ means $j=1$, $\ell(1)=3$, $q=2$, and $p=1$. Hence, $R^\perp$ still has distance 2, but now from the line $a^1a^2$.



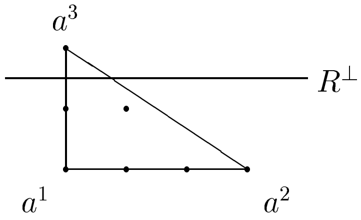

$\tau = \sigma$

We obtain

$$\widetilde{\tau} = \langle(-1,-1,1,-2);(0,-1,1,-2);(-1,1,1,0);(0,0,0,1);(1,0,0,1)\rangle\,.$$

The boundary equation $Z_3WZ_1 - U^3 = 0$ corresponds to Theorem 4.8, (ii); it is perturbed to $Z_3WZ_1 - U^3 - t_\beta UZ_1 = 0$.

**Case $\gamma$:** $R^\gamma = [0,-2,1] = 3R^* - 2s^1$ means $j = 1$, $q = \ell(1) = 3$, and $p = 2$.




$\tau \subset \sigma$

Here we have

$$\widetilde{\tau} = \langle(-1,-1,1,-3);(-2,1,2,0);(-1,2,4,0);(0,0,0,1);(1,0,0,1)\rangle\,,$$

and the previous boundary equation provides $Z_3WZ_1 - U^3 - t_\gamma Z_1^2 = 0$.

# 5 Work in progress and open problems

## 5.1

For a fixed $j$, denote by $t_{q,p}$ the parameter corresponding to $R = qR^* - ps^j$ and take them as coefficients of formal power series $t_q(z) := \sum_{p\gg 0} t_{q,p} z^p$ for $2 \le q \le \ell(j)$.

**Conjecture 5.1** *The one parameter families corresponding to a fixed $j$ fit into a common huge family defined over a smooth, infinite dimensional parameter space. As in Theorem 4.8, the boundary equations sitting completely inside $[d^j \ge 0]$ or $[d^j \le 0]$ remain unchanged. The equation $z_r z_{r'} - z_{R^*}^{\ell(j)} z_{s^j}^k = 0$ turns into $\big(z_r z_{r'} - z_{R^*}^{\ell(j)} z_{s^j}^k\big) - \sum_{2\le q\le \ell(j)} t_q(z_{s^j})\, z_{R^*}^{\ell(j)-q} z_{s^j}^k = 0$.*

If this conjecture is true, then it should be possible to prove it directly by lifting equations and relations, without using Minkowski decompositions.

| | Equations | Perturbations |
|---|---|---|
| (0) | $Z_3WZ_1 - U^3$ | $-(t_\alpha + Z_3\xi_\alpha)UZ_3 - (t_\beta + Z_1\xi_\beta)UZ_1 - (t_\gamma + Z_1\xi_\gamma)Z_1^2$ |
| (1) | $Z_1Z_2 - W^2$ | $-t_\alpha(t_\alpha + Z_3\xi_\alpha)\xi_\beta - t_\alpha V_1 - \xi_\alpha\xi_\beta(V_2 + t_\beta)Z_1$ $-\xi_\alpha(V_2 + t_\beta)^2$ |
| (2) | $WV_1 - UZ_2$ | $-(t_\alpha + Z_3\xi_\alpha)t_\alpha\xi_\gamma - \xi_\alpha(t_\gamma + Z_1\xi_\gamma)(V_2 + t_\beta)$ |
| (3) | $Z_2V_2 - V_1^2$ | $-(t_\alpha + Z_3\xi_\alpha)\xi_\beta V_1 - t_\alpha\xi_\gamma W - \xi_\alpha\xi_\beta(t_\gamma + Z_1\xi_\gamma)U$ $-\xi_\alpha(t_\gamma + Z_1\xi_\gamma)^2$ |
| (4) | $V_1Z_3 - V_2^2$ | $+(t_\alpha + Z_3\xi_\alpha)\xi_\beta Z_3 - (t_\beta + Z_1\xi_\beta)V_2 - (t_\gamma + Z_1\xi_\gamma)U$ |
| (5) | $V_2Z_1 - U^2$ | $-(t_\alpha + Z_3\xi_\alpha)Z_3$ |
| (6) | $V_2W - UV_1$ | $-(t_\alpha + Z_3\xi_\alpha)(t_\gamma + Z_1\xi_\gamma + U\xi_\beta)$ |
| (7) | $Z_3W - UV_2$ | $-(t_\beta + Z_1\xi_\beta)U - (t_\gamma + Z_1\xi_\gamma)Z_1$ |
| (8) | $Z_2Z_3 - V_1V_2$ | $-t_\alpha\xi_\gamma U - (t_\beta + Z_1\xi_\beta)V_1 - (t_\gamma + Z_1\xi_\gamma)W$ |
| (9) | $Z_1V_1 - UW$ | $-(t_\alpha + Z_3\xi_\alpha)(V_2 + t_\beta)$ |

Table 5.1: Deformation of the cone over $\mathbb{P}(1,2,3)$. The base space is defined by $t_\alpha t_\beta = t_\alpha t_\gamma = 0$.

## 5.2

Jan Stevens has used the computer algebra system Macauly to compute the versal deformation of our example Cone($\mathbb{P}(1,2,3)$). In the list of perturbed equations given in Table 5.1, we denote the deformation parameters by $t_\alpha, t_\beta, t_\gamma, \xi_\alpha, \xi_\beta$ and $\xi_\gamma$. The first ones are the same as used in 4.8 covering $R^\alpha, R^\beta, R^\gamma$, respectively. The latter three parameters are formal power series $\xi_\alpha = \xi_\alpha(Z_3)$, $\xi_\beta = \xi_\beta(Z_1)$, and $\xi_\gamma = \xi_\gamma(Z_1)$; their coefficients, including that of $Z^0$ are the actual deformation parameters. Up to shift of degrees, the notation is similar to that of the previous conjecture; here we have $\deg_M \xi_\alpha = R^\alpha - s^3$, $\deg_M \xi_\beta = R^\beta - s^1$, and $\deg_M \xi_\gamma = R^\gamma - s^1$, respectively.

Equation (0) is not needed as a generator of the ideal of $Y_\sigma \subset \mathbb{C}^7$. It was put into the list since (0)–(5) are the boundary equations mentioned in 4.8. I'm afraid that the plus sign in (4) seems to be no mistake since, for instance, the original relation between equations (4), (6), and (7) lifts to $U \cdot \boxed{4} + Z_3 \cdot \boxed{6} - V_2 \cdot \boxed{7} - (t_\gamma + Z_1\xi_\gamma) \cdot \boxed{5} = 0$.

The versal base space is given by $t_\alpha t_\beta = t_\alpha t_\gamma = 0$, that is, it consists of two (infinite dimensional) smooth components. Restricting the deformation to the subspaces $[t_\alpha = \xi_\alpha = 0]$ and $[t_\beta = t_\gamma = \xi_\beta = \xi_\gamma = 0]$ yields two smooth families corresponding to $j = 1$ and $j = 3$, respectively. They equal exactly the deformations predicted in 5.1 and contain the 3 one parameter families shown in 4.8.

## 5.3

Let us focus on the smooth two parameter family involving only $t_\alpha$ and $\xi_\beta := \xi_\beta(0)$, setting the remaining parameters to zero. From the above list, only the following perturbation terms survive:

| | | | |
|---|---|---|---|
| (0) | $\big(Z_3WZ_1 - U^3\big) - t_\alpha UZ_3 - \xi_\beta UZ_1^2$ | (5) | $\big(V_2Z_1 - U^2\big) - t_\alpha Z_3$ |
| (1) | $\big(Z_1Z_2 - W^2\big) - t_\alpha^2\xi_\beta - t_\alpha V_1$ | (6) | $\big(V_2W - UV_1\big) - t_\alpha\xi_\beta U$ |
| (2) | $\big(WV_1 - UZ_2\big)$ | (7) | $\big(Z_3W - UV_2\big) - \xi_\beta Z_1U$ |
| (3) | $\big(Z_2V_2 - V_1^2\big) - t_\alpha\xi_\beta V_1$ | (8) | $\big(Z_2Z_3 - V_1V_2\big) - \xi_\beta Z_1V_1$ |
| (4) | $\big(V_1Z_3 - V_2^2\big) + t_\alpha\xi_\beta Z_3 - \xi_\beta Z_1V_2$ | (9) | $\big(Z_1V_1 - UW\big) - t_\alpha V_2$ |

Equation (3) turns into a binomial by restricting to the subfamily $[\xi_\gamma = 0]$ or $[t_\alpha = 0]$. This makes it difficult to hope that our two parameter family has a total space which is at least close to being toric (for instance, by some partial modifications).

## 5.4

Nevertheless, we try to obtain those "mixed" deformations involving degrees $-(qR^* - ps^j)$ with more than one $j$ by the methods developed so far. We begin with the $t_\alpha$-family introduced in 4.8 and try to deform varieties close to its total space. We recall the relevant cones:

$\sigma = \langle a^1, a^2, a^3\rangle$ and $\tau = \langle a^1, c^1, c^3, a^3\rangle$, with

$$\begin{array}{l} a^1 = (-1,-1,1),\\ a^2 = (2,-1,1),\\ a^3 = (-1,1,1)\end{array} \quad\text{and}\quad \begin{array}{l} c^1 = (1,-1,1),\\ c^3 = (3,-1,3);\end{array}$$

$\widetilde{\sigma} = \langle A, X, Y, a^2\rangle$ and $\widetilde{\tau} = \langle A, C^1, C^3, X, Y\rangle$, with

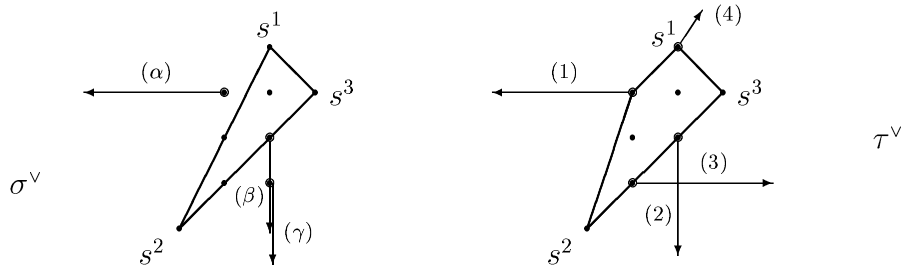
$$\begin{array}{l} A = (-1,1,1,-2),\\ C^1 = (1,-1,1,0),\\ C^3 = (3,-1,3,0)\end{array} \quad\text{and}\quad \begin{array}{l} X = (0,0,0,1),\\ Y = (0,-1,0,1),\\ a^2 = (a^2,0).\end{array}$$

The two dimensional faces ("edges") of the simplicial cone $\widetilde{\sigma}$ are smooth. Hence, by 2.6 the variety $Z_{\widetilde{\sigma}}$ is rigid.

That means we have to focus on infinitesimal deformations of $X_{\widetilde{\tau}}$. We begin at the level of special fibers with a quick view of $T^1_\tau$ in comparison with $T^1_\sigma$: again from 2.6, we see that the vector space $T^1_\tau(-R)$ is nontrivial only for $R = [0,-1,0]$ and for the four series

$$R^{(1)}_\lambda = R^\alpha - \lambda s^3; \quad R^{(2)}_\lambda = R^\beta - \lambda s^1; \quad R^{(3)}_\lambda = v^1 - \lambda R^\alpha; \quad R^{(4)}_\lambda = s^1 - \lambda s^2$$

with $\lambda \geq 0$.




$T^1(-R)$ is always one dimensional, except that $\dim T^1_\tau(-[0,-2,0]) = 2$; here the series (2) and (3) meet (with $\lambda = 1$). In particular, we have lost the $\gamma$-deformations in $Y_\tau$. They are not compatible with the partial modification $Y_\tau \to Y_\sigma$.

As already mentioned in 4.8, $\tilde{\tau}$ is the cone over the pyramid with plane base $XYC^1C^3$ and $A$ as its top. The edges of the triangular face $AC^1C^3$ correspond to $\mathrm{A}_1$ singularities; the remaining edges are smooth. The Hilbert basis $E$ of $\sigma^\vee \cap M \subset \tau^\vee \cap M$ lifts uniquely to $\tilde{\tau}^\vee \cap \mathbb{Z}^4$; together with $[R^\alpha, 0]$ and $[R^\alpha, 1]$ we obtain the Hilbert basis of the latter semigroup:

$$
\begin{array}{lll}
S^1 = [0,1,1,1] & V^1 = [-1,-2,1,0] & R^* = [0,0,1,0] \\
S^2 = [-2,-3,1,0] & V^2 = [0,-1,1,0] & [R^\alpha,0] = [-1,0,1,0] \\
S^3 = [1,0,1,0] & W = [-1,-1,1,0] & [R^\alpha,1] = [-1,0,1,1]
\end{array}
$$

Using our formula of 2.6 for $T^1$ again, we see that $T^1_{\tilde{\tau}}(-R)$ is at most one dimensional. $X_{\tilde{\tau}}$ admits infinitesimal deformations in the following degrees:

(a) $R^{(a)}_\lambda = [-1,-2,1,-1] - \lambda[R^\alpha, 1]$ for $\lambda \geq 1$.

Here, $R$ equals 2 on the vertices $A$, $C^1$, $C^3$, and it is nonpositive otherwise. In particular, $Q(R) = \tilde{\tau} \cap [R = 1]$ is a triangle, and we obtain $V(R) = W(R) = \mathbb{R}$. Projecting $\mathbb{Z}^4 \to M$ maps this series onto $R^{(3)}_\lambda$ $(\lambda \geq 1)$ which was mentioned above.

(b) $R^{(b)}_{\lambda,\mu} = [-1,-2,1,0] - \lambda[R^\alpha, 0] - \mu[R^\alpha, 1]$ for $\lambda, \mu \geq 0$.

The compact part of $Q(R)$ equals the interval reaching from $C^1/2$ to $C^3/2$. Again, this series projects onto the third one for $\tau$.

(c) $R^{(c)}_{\lambda,\mu} = [0,-1,1,-1] - \lambda S^1 - \mu[R^\alpha, 1]$ for $\lambda \geq 2$; $\mu \geq 0$.

$Q(R)$ now has $\overline{AC^1}/2$ as compact part. Mapping the degrees into $M$ yields the $\beta$-series $R^{(2)}_\lambda$ with $\lambda \geq 2$.

(d) $R^{(d)}_{\lambda,\mu} = [0,1,1,0] - \lambda S^2 - \mu[R^\alpha, 1]$ for $\lambda, \mu \geq 0$.

This final series corresponds to the edge $\overline{AC^3}$, and it maps onto $R^{(4)}_\lambda$ $(\lambda \geq 0)$.

## 5.5

From the previous section we know that there is no infinitesimal deformation either of $X_{\tilde{\tau}}$ or $Z_{\tilde{\sigma}}$ in any lift of degree $-R_1^{(2)}$. In particular, our attempt to reconstruct the $(t_\alpha, \xi_\beta)$-deformation of 5.3 has failed.

However, since we are just interested in mixing $\alpha$- and $\beta$-deformations, we may try degree $T := R_{2,0}^{(c)} = [0, -3, -1, -3]$ from the series (c) above. It is a lift of $R_2^{(2)} \in M$ corresponding to the deformation parameter arising as the coefficient of $Z$ in the formal power series $\xi_\beta(Z)$. By abuse of notation, we will call this parameter $\xi_\beta$ again. The corresponding family may be obtained from 5.3 via the substitution $\xi_\beta := \xi_\beta Z_1$.

Now we may construct the genuine deformation of $X_{\tilde{\tau}}$ occurring in degree $-T$. The usual approach (splitting $Q(R)$ into a Minkowski sum) yields the following result:

- On the partial modification of the new total space we have variables $Z_1, Z_2, Z_3, U, V_1, V_2, W$ arising as lifts from $\sigma^\vee$. Moreover, we call $R_0, R_1$, $T_0, T_1$ the variables corresponding to the lifts of $[R^\alpha, 0]$, $[R^\alpha, 1]$ or to the two lifts of $T$, respectively. In particular, $t_\alpha = R_1 - R_0$ and $\xi_\beta = T_1 - T_0$ are the deformation parameters.

- The modified total space corresponds to a cone

$$\Delta = \langle B^1, B^2, B^3, C^3, Y, Z \rangle,$$

  where $B^1, B^2, B^3$ have arisen from the Minkowski splitting $\overline{AC^1} = \{B^1\} + \overline{B^2B^3}$ inside the affine plane $[T = 1]$. The variables mentioned above correspond to the equally named points of $\Delta^\vee \cap \mathbb{Z}^5$. The Hilbert basis consists of $\{Z_1, Z_2, Z_3, V_1, V_2, R_0, R_1, T_0, T_1\}$ and an additional element $S$ (with $U = S + Z_1$, $W = S + R_1$). It might be useful to know $\Delta$ and its dual cone in detail. Therefore, we write down the following table indicating the pairing between the mutually dual cones:

| | $Z_1$ | $Z_2$ | $Z_3$ | $U$ | $V_1$ | $V_2$ | $W$ | $R_0$ | $R_1$ | $T_0$ | $T_1$ | $S$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $B^1$ | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 1 |
| $B^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
| $B^3$ | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 |
| $C^3$ | 2 | 0 | 6 | 3 | 2 | 4 | 1 | 0 | 0 | 0 | 0 | 1 |
| $Y$ | 0 | 3 | 0 | 0 | 2 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| $Z$ | 2 | 0 | 0 | 3 | 0 | 0 | 3 | 6 | 2 | 0 | 0 | 0 |

- Proceeding as in the proof of Theorem 4.8, the ten original equations defining $Y_\sigma$ (cf. the second column of Table 5.1) induce binomial equations for the modified total space:

  (0) and (5) provide $WZ_1 - R_1U = U^2 - Z_1^3T_0 = R_0Z_3 - Z_1^3T_1 = V_2Z_1 - R_1Z_3 = 0$;

  (1) provides $Z_1Z_2 - R_1V_1 = W^2 - Z_1R_1^2T_0 = R_0V_1 - Z_1R_1^2T_1 = 0$;

  (9) provides $Z_1V_1 - V_2R_1 = UW - T_0R_1Z_1^2 = V_2R_0 - T_1R_1Z_1^2 = 0$;

  the equations (2), (3), (4), (6), (7), and (8) remain unchanged.

- To describe the deformation of $X_{\tilde{\tau}}$ we have to eliminate $T_0$ and $T_1$; just their difference $\xi_\gamma = T_1 - T_0$ may occur. There is nothing to do with equations (2), (3), (4), (6), (7), (8); for the remaining ones we obtain

$$
\begin{array}{ll}
(0;5) & WZ_1 - R_1U = U^2 - R_0Z_3 + \xi_\beta Z_1^3 = V_2Z_1 - R_1Z_3 = 0 \\
(1) & Z_1Z_2 - R_1V_1 = W^2 - R_0V_1 + \xi_\beta R_1^2 Z_1 \\
(9) & Z_1V_1 - V_2R_1 = UW - V_2R_0 + \xi_\beta R_1 Z_1^2.
\end{array}
$$

- Finally, we should remember that we are eventually interested in deformations of $X$ rather than $X_{\tilde{\tau}}$. Hence, we have to do the same job again with $t_\alpha = R_1 - R_0$. The best result for equations (0), (1), (5), and (9) is

$$
\begin{array}{ll}
(0) & (Z_3WZ_1 - U^3) - U(t_\alpha Z_3 + \xi_\beta Z_1^3) \\
(1) & (Z_1Z_2 - W^2) - t_\alpha V_1 - \xi_\beta R_1^2 Z_1 \\
(5) & (V_2Z_1 - U^2) - t_\alpha Z_3 - \xi_\beta Z_1^3 \\
(9) & (Z_1V_1 - UW) - t_\alpha V_2 - \xi_\beta R_1 Z_1^2.
\end{array}
$$

  In particular, we have not been able to eliminate $R_1$ from equations (1) and (9). It indicates that our deformation of $X_{\tilde{\tau}}$ *does not blow down* onto the $X$ level.

## 5.6

We have used $\tau \subset \sigma$ to obtain a partial modification $Y_\tau \to Y_\sigma$. Deformations of $Y_\sigma$ have been obtained by blowing down deformations of $Y_\tau$.

Is it possible, or even better, to consider the whole modification $Y_\Sigma \to Y_\sigma$ instead of the open chart only? $\Sigma$ is the fan consisting of the two cones $\tau = \sigma \cap [R \geq 0]$ and $\tau' := \sigma \cap [R \leq 0]$. Apart from the fact that this approach looks more natural, there might be another advantage: the lack of certain degrees providing infinitesimal deformations of $Y_\tau$ may be overcome by blowing down local trivial deformations of $Y_\Sigma$.

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