



The graph of monomial ideals[☆]

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Received 12 September 2002

Available online 7 April 2005

Dedicated to Wolmer Vasconcelos on the occasion of his 65th birthday

Abstract

There is a natural infinite graph whose vertices are the monomial ideals in a polynomial ring $\mathbb{K}[x_1, \dots, x_n]$. The definition involves Gröbner bases or the action of the algebraic torus $(\mathbb{K}^*)^n$. We present algorithms for computing the (affine schemes representing) edges in this graph. We study the induced subgraphs on multigraded Hilbert schemes and on square-free monomial ideals. In the latter case, the edges correspond to generalized bistellar flips.

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MSC: 13P10; 14B07

1. Edge ideals

The most important tool for computing with ideals in a polynomial ring $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \dots, x_n]$ over a field \mathbb{K} is the theory of Gröbner bases. It furnishes degenerations of arbitrary ideals in $\mathbb{K}[\mathbf{x}]$ to monomial ideals along one-parameter subgroups of $(\mathbb{K}^*)^n$; see [2, Section 15.8]. Monomial ideals are combinatorial objects. They represent the most special points in the “world of ideals”. The following adjacency relation among monomial ideals extracts the combinatorial essence of Gröbner degenerations.

[☆] Partially supported by NSF grant DMS-0200729.

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Definition. We define the infinite graph of monomial ideals $\mathcal{G} = \mathcal{G}_{n, \mathbb{K}}$ as follows. The vertices of \mathcal{G} are the monomial ideals in $\mathbb{K}[\mathbf{x}]$, and two monomial ideals M_1, M_2 are connected by an edge if there exists an ideal I in $\mathbb{K}[\mathbf{x}]$ such that the set of all initial monomial ideals of I , with respect to all term orders, is precisely $\{M_1, M_2\}$.

First examples of interesting finite subgraphs can be obtained by restricting to artinian ideals of a fixed colength r . We consider the induced subgraph on the set

$$\mathcal{G}^r := \mathcal{G}_{n, \mathbb{K}}^r := \{M \subseteq \mathbb{K}[\mathbf{x}] \text{ monomial ideal} : \dim_{\mathbb{K}} \mathbb{K}[\mathbf{x}]/M = r\}.$$

Proposition 1. The finite graphs \mathcal{G}^r are connected components of the graph \mathcal{G} .

Proof. Since Gröbner degenerations preserve the colength of an ideal, the graph \mathcal{G}^r is a union of connected components of \mathcal{G} . Hence it suffices to show that \mathcal{G}^r is connected. One can connect two vertices of \mathcal{G}^r , i.e., two monomial ideals $M_1, M_2 \subseteq \mathbb{K}[\mathbf{x}]$ of the same colength, by a sequence of “moving single boxes” in their socles. Hence, we may assume that the vector spaces $M_i/(M_1 \cap M_2)$ are one-dimensional, generated by single monomials m_i . But then, the ideal

$$I := (M_1 \cap M_2) + \langle m_1 - m_2 \rangle$$

provides an edge connecting M_1 and M_2 inside \mathcal{G}^r . \square

The monomial ideals of colength r in $\mathbb{K}[x, y]$ are in bijection with the partitions of the integer r . We computed $\mathcal{G}_{2, \mathbb{K}}^r$, the graph of partitions, up to $r = 13$, using the algorithm in Section 2. Here is a small example. The graph $\mathcal{G}_{2, \mathbb{K}}^4$ consists of five vertices and eight edges, and it equals the cone of the vertex $(2, 2)$ over the 4-cycle

$$(1, 1, 1, 1) \longleftrightarrow (2, 1, 1) \longleftrightarrow (3, 1) \longleftrightarrow (4) \longleftrightarrow (1, 1, 1, 1). \tag{1}$$

We conjecture that $\mathcal{G}_{2, \mathbb{K}}^r$ is independent of the field \mathbb{K} , for all r , but we are still lacking a combinatorial rule for deciding when two partitions form an edge.

Remark 2. Not all connected components of the graph \mathcal{G} are finite. For instance, the induced subgraph on the principal ideals is an infinite connected component.

Let us now take a closer look at the ideals that define the edges in \mathcal{G} . Since monomial ideals are homogeneous with respect to the \mathbb{Z}^n -grading of $\mathbb{K}[\mathbf{x}]$, one expects that edges arise from ideals I which admit an $(n - 1)$ -dimensional grading.

Definition. An ideal $I \subseteq \mathbb{K}[\mathbf{x}]$ is an edge providing ideal if the set of initial monomial ideals $\text{in}_{\prec}(I)$, as \prec ranges over all term orders on $\mathbb{K}[\mathbf{x}]$, has cardinality two. We call I an edge ideal if there exists $c \in \mathbb{Z}^n$ with both positive and negative coordinates such that I is homogeneous with respect to the induced $(\mathbb{Z}^n/\mathbb{Z}c)$ -grading of $\mathbb{K}[\mathbf{x}]$.

Proposition 3. Every edge ideal is an edge providing ideal. Given any edge providing ideal I , there is only one non-monomial ideal \tilde{I} among its initial ideals $\text{in}_w(I)$, $w \in \mathbb{N}^n$. Moreover, \tilde{I} is an edge ideal connecting the same vertices as I does.

Proof. The first statement holds because generators of edge ideals have the form $\lambda_0 \mathbf{x}^u + \lambda_1 \mathbf{x}^{u+c} + \dots + \lambda_r \mathbf{x}^{u+rc}$. Hence, there are only two equivalence classes of term orders, given by $c < 0$ and $c > 0$. For the second statement note that the *Gröbner fan* of I is a regular polyhedral subdivision of $\mathbb{R}_{\geq 0}^n$ which has exactly two maximal cones. Their intersection is an $(n - 1)$ -dimensional cone C . The unique (up to scaling) vector c perpendicular to C has both positive and negative coordinates. Fix a vector w in the relative interior of C . Then $\tilde{I} := \text{in}_w(I)$ is $\mathbb{Z}^n / \mathbb{Z}c$ -homogeneous and has the same two initial monomial ideals as I does. \square

Here is an example to illustrate this for $n = 2$. The ideal $I = \langle x^4 + x^2y + y^2 + x + y + 1 \rangle$ is edge providing. The unique edge ideal is $\tilde{I} = \text{in}_{(1,2)}(I) = \langle x^4 + x^2y + y^2 \rangle$. A general reference for undefined terminology and notation used in this paper is the book by Sturmfels [10].

2. Computing the graph

We fix a primitive vector $c \in \mathbb{Z}^n$ with $c_i > 0$ and $c_j < 0$ for some $i, j \in \{1, 2, \dots, n\}$. Here *primitive* means that the greatest common divisor of c_1, c_2, \dots, c_n is one.

Lemma 4. *For any monomial ideal M in $\mathbb{K}[\mathbf{x}]$, there exists an affine scheme $\Omega_c(M)$ which parametrizes all $(\mathbb{Z}^n / \mathbb{Z}c)$ -homogeneous ideals I with $\text{in}_{c < 0}(I) = M$.*

Proof. For any minimal generator \mathbf{x}^u of M let r_u be the largest integer such that $u + r_u c$ is non-negative. Introduce unknown coefficients $\lambda_{u,1}, \dots, \lambda_{u,r_u}$ and form

$$\underline{\mathbf{x}}^u + \lambda_{u,1} \mathbf{x}^{u+c} + \lambda_{u,2} \mathbf{x}^{u+2c} + \dots + \lambda_{u,r_u} \mathbf{x}^{u+r_u c}. \quad (2)$$

The ideal I generated by the polynomials (2) satisfies $\text{in}_{c < 0}(I) = M$ if and only if they form a Gröbner basis with the underlined leading terms. By Buchberger's criterion, this means that all S -pairs reduce to zero, giving an explicit system of polynomial equations in terms of the $\lambda_{u,i}$. On the other hand, we would like the coordinates $\lambda_{u,i}$ to be uniquely determined from I . This is the case if we require that (2) describes a *reduced* Gröbner basis, imposing $\lambda_{u,i} = 0$ whenever $\mathbf{x}^{u+ic} \in M$. \square

We call $\Omega_c(M)$ the *Schubert scheme* of M in direction c . In the case when M is generated by a subset of the variables then $\Omega_c(M)$ is a Schubert cell in the Grassmannian. If M_1, M_2 are two monomial ideals, then the scheme-theoretic intersection

$$\Omega_c(M_1, M_2) := \Omega_c(M_1) \cap \Omega_{-c}(M_2)$$

parametrizes all c -edge ideals between M_1 and M_2 .

Algorithm 5. (Input: c, M_1, M_2 . Output: $\Omega_c(M_1, M_2)$).

Step 1: Construct the affine scheme $\Omega_c(M_1)$ using the procedure in the proof above. Using S -pair reduction, one obtains a set of polynomials in variables $\lambda_{u,i}$, and the universal c -edge ideal over the base $\Omega_c(M_1)$ is described by the polynomials (2).

Step 2: Construct the affine scheme $\Omega_{-c}(M_2)$ as in Step 1. This gives a set of polynomials in some other variables $\tilde{\lambda}_{\bar{u},i}$ representing the universal c -edge ideal.

Step 3: Form additional joint equations in both sets of variables $\lambda_{u,i}$ and $\tilde{\lambda}_{\bar{u},i}$ which express the requirement that the universal ideal over $\Omega_c(M_1)$ coincides with the universal ideal over $\Omega_{-c}(M_2)$. This is done by reducing the polynomials (2) of Step 1 modulo those of Step 2 and reading off the coefficients with respect to \mathbf{x} .

Let us demonstrate how Algorithm 5 works for a small example.

Example 6. Let $M_1 = \langle x^6, x^2y, y^2 \rangle$, $M_2 = \langle x^2, xy^2, y^6 \rangle$ and $c = (1, -1)$. In Step 1 we introduce three indeterminates a_1, a_2, a_3 . The ideals in $\Omega_c(M_1)$ are of the form

$$\langle \underline{x^6}, \underline{x^2y} + a_1x^3, \underline{y^2} + a_2xy + a_3x^2 \rangle. \tag{3}$$

These polynomials are a Gröbner basis with underlined leading terms if and only if

$$a_1^2 - a_1a_2 + a_3 = 0. \tag{4}$$

In Step 2 we similarly compute the affine scheme $\Omega_{-c}(M_2)$ to be the hypersurface

$$b_3^2 - b_1b_3 + b_2 = 0, \tag{5}$$

carrying the universal ideal

$$\langle \underline{x^2} + b_1xy + b_2y^2, \underline{xy^2} + b_3y^3, \underline{y^6} \rangle. \tag{6}$$

Finally, in Step 3 we enforce the condition that the ideals in (3) and (6) are equal, given that (4) and (5) hold. This is done by reducing the generators of (3) modulo the Gröbner basis (6) and collecting coefficients in the normal forms. We obtain

$$\{a_1 - a_3b_1 + a_3b_3, a_2 - a_3b_1, a_3b_2 - 1, b_2 - b_1b_3 + b_3^2\}. \tag{7}$$

Example 7. The Schubert schemes $\Omega_c(M_i)$ in the previous example are reduced and irreducible. However, this is not true in general. For instance, for $M = \langle x^6, y^5, z^9, y^3z^5, x^4y^3z^2, x^3y^2z^4, x^2y^4z^3 \rangle$ we obtain $\Omega_{(-3,0,1)}(M) \simeq \text{Spec} \mathbb{K}[\varepsilon]/(\varepsilon^2)$.

Our next result will imply that the lower index “ c ” can be dropped from $\Omega_c(M_1, M_2)$.

Theorem 8. *Given any two monomial ideals M_1, M_2 in $\mathbb{K}[\mathbf{x}]$, there is at most one direction $c \in \mathbb{Z}^n$ such that the scheme $\Omega_c(M_1, M_2)$ is non-empty. Moreover, if $\Omega_c(M_1, M_2) \neq \emptyset$, then M_1, M_2 have equal Hilbert functions with respect to an induced $(\mathbb{Z}^n / \mathbb{Z}c')$ -grading if and only if $c' = \pm c$.*

The proof of Theorem 8 will be given in the next section. If M_1 and M_2 are connected by an edge in our graph \mathcal{G} , then c is uniquely determined, and we simply write

$$\Omega(M_1, M_2) := \Omega_c(M_1, M_2)$$

for the scheme which parameterizes all edge ideals between M_1 and M_2 . If M_1 and M_2 are not connected by an edge in \mathcal{G} then $\Omega(M_1, M_2)$ denotes the empty set. Hence the following algorithm can be used to determine the adjacency relation in \mathcal{G} .

Algorithm 9. (Input: M_1, M_2 . Output: $\Omega(M_1, M_2)$).

Step 1: Compute the \mathbb{N}^n -graded Hilbert series $H(M_i; \mathbf{x})$ of the two given monomial ideals as rational functions, i.e., find the numerator polynomials K_1 and K_2 of

$$H(M_i; \mathbf{x}) = \frac{K_i(x_1, \dots, x_n)}{(1-x_1)(1-x_2)\cdots(1-x_n)}.$$

Step 2: Factor the polynomial $K_1(\mathbf{x}) - K_2(\mathbf{x})$ into irreducible factors. Output $\Omega_c(M_1, M_2) = \emptyset$, unless there is, up to sign, a unique primitive vector $c \in \mathbb{Z}^n$ which has positive and negative coordinates such that the binomial $\mathbf{x}^{c^+} - \mathbf{x}^{c^-}$ appears as a factor.

Step 3: Run Algorithm 5 for the vector $\pm c$ found in Step 2, and output the affine scheme $\Omega(M_1, M_2) = \Omega_c(M_1, M_2)$. (It is still possible that this scheme is empty.)

The correctness of Algorithm 9 follows directly from Theorem 8. An improvement to Step 1 in this algorithm in the context of an ambient multigrading will be discussed in the next section.

As an example consider the two ideals in Example 6. In step 1 we compute

$$\begin{aligned} K_1(x, y) &= 1 - x^6 - x^2y - y^2 + x^6y + x^2y^2, \\ K_2(x, y) &= 1 - x^2 - xy^2 - y^6 + x^2y^2 + xy^6. \end{aligned}$$

The difference $K_1(x, y) - K_2(x, y)$ of these numerator polynomials factors as

$$\begin{aligned} (x-y)(y-1)(x-1)(x^4 + x^3y + x^2y^2 + xy^3 + y^4 + x^3 + x^2y + xy^2 + y^3 + x^2 \\ + xy + y^2 + x + y). \end{aligned}$$

The only binomial factor with both terms non-constant is $x - y$, and we conclude that $\Omega(M_1, M_2)$ equals $\Omega_{(1,-1)}(M_1, M_2)$, the affine scheme described by (7).

3. Multigraded Hilbert schemes

We consider an arbitrary grading of the polynomial ring $\mathbb{K}[\mathbf{x}]$. It is given by an epimorphism of abelian groups $\deg : \mathbb{Z}^n \rightarrow A$. For any function $h : A \rightarrow \mathbb{N}$, the *multigraded Hilbert scheme* Hilb_h parametrizes all homogeneous ideals I such that $\mathbb{K}[\mathbf{x}]/I$ has Hilbert function h . This scheme was introduced in [4]. Multigraded Hilbert schemes provide a natural setting for studying finite subgraphs of $\mathcal{G} = \mathcal{G}_{n, \mathbb{K}}$.

Definition. A multigraded Hilbert scheme Hilb_h has the *induced subgraph property* if any two monomial ideals $M_1, M_2 \in \text{Hilb}_h$ which are connected in $\mathcal{G}_{n, \mathbb{K}}$ can also be connected via an edge ideal I which lies in the same Hilbert scheme Hilb_h .

The induced subgraph property holds for the Hilbert scheme of points, where $A = \{0\}$ is the zero group, by our discussion in Section 1. However, it fails in general.

Example 10. Consider the “super-grading” of $\mathbb{K}[x, y]$ given by $\deg : \mathbb{Z}^2 \rightarrow \mathbb{Z}/2\mathbb{Z}$, $(r, s) \mapsto r + s$, and define $h : \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{N}$ by $h(0) = h(1) = 2$. The two ideals $M_1 = \langle x^4, y \rangle$ and $M_2 = \langle x^2, y^2 \rangle$ are points in Hilb_h . They are connected in \mathcal{G} as was seen in (1). Algorithm 9 finds that the edge ideals are $\langle x^2 + \alpha y, y^2 \rangle$ for any $\alpha \in \mathbb{K}^*$. None of the edge ideals is homogeneous in the given grading. We conclude that the Hilbert scheme Hilb_h does not have the induced subgraph property.

Definition. A grading of $\mathbb{K}[\mathbf{x}]$ is called *positive* if only the constants have degree 0. This implies that the grading group A is torsion-free, i.e., $A \cong \mathbb{Z}^q$ for some q .

A torsion-free grading $\deg : \mathbb{Z}^n \rightarrow \mathbb{Z}^q$ is positive if and only if $\mathbb{N}^n \cap \ker(\deg) = 0$ if and only if the *fibers* $\mathbb{N}^n \cap \deg^{-1}(a)$ are finite if and only if the polyhedra $\mathbb{R}_{\geq 0}^n \cap \deg_{\mathbb{R}}^{-1}(a)$ are compact. Under these circumstances, our graphs behave nicely.

Theorem 11. Let $\deg : \mathbb{Z}^n \rightarrow \mathbb{Z}^q$ be a positive grading and $h : \mathbb{Z}^q \rightarrow \mathbb{N}$ any function. Then the multigraded Hilbert scheme Hilb_h has the induced subgraph property.

We will derive this theorem from the following lemma.

Lemma 12. Let $\deg : \mathbb{Z}^n \rightarrow \mathbb{Z}^q$ be a positive grading and $M_1, M_2 \subset \mathbb{K}[\mathbf{x}]$ monomial ideals with the same Hilbert function. Then $\Omega_c(M_1, M_2) \neq \emptyset$ implies $\deg(c) = 0$.

Proof of Theorem 11. Let M_1 and M_2 be monomial ideals in Hilb_h and I an edge ideal in $\Omega(M_1, M_2)$. Lemma 12 implies that I is homogeneous with respect to the given positive grading \deg . Since M_1 and M_2 are initial ideals of I , all three ideals have the same Hilbert function, and hence I is a point in Hilb_h as desired. \square

Proof of Lemma 12. Let $I \in \Omega_c(M_1, M_2)$, $M_1 = \text{in}_{c < 0}(I)$ and $M_2 = \text{in}_{c > 0}(I)$; denote by h their Hilbert function. The edge ideal I is generated by $\mathbb{Z}^n/\mathbb{Z}c$ -homogeneous polynomials of the form

$$\mathbf{x}^u + \lambda_1 \mathbf{x}^{u+c} + \lambda_2 \mathbf{x}^{u+2c} + \dots + \lambda_r \mathbf{x}^{u+rc} \quad (\lambda_r \neq 0). \tag{8}$$

We shall abuse the symbols M_1, M_2, I to also denote the set of exponents of the monomials in that ideal. For instance, from (8) we infer $u \in M_1$ and $u + rc \in M_2$. We also have the following obvious inclusions among finite sets of monomials:

$$I \cap \deg^{-1}(a) \subseteq M_i \cap \deg^{-1}(a) \quad \text{for } i = 1, 2 \text{ and } a \in \mathbb{Z}^q. \tag{9}$$

Our strategy is this: we first prove Lemma 12 for one-dimensional gradings.

Step 1: $q = 1$. Assume that $c \notin \ker(\deg)$. We claim that

$$M_1 \cap \deg^{-1}(a) = M_2 \cap \deg^{-1}(a) = I \cap \deg^{-1}(a) \quad \text{for all } a \in \mathbb{Z}^q. \tag{10}$$

This implies $M_1 = M_2$, a contradiction which will establish Lemma 12 for $q = 1$.

We may assume $\deg(\mathbb{N}^n) \subseteq \mathbb{N}$ and $\deg(c) < 0$. Since the case $a \leq 0$ is void, we shall prove (10) for positive integers a by induction. Suppose the two equalities hold for all

$a < a_0$. Consider any element $u \in M_1 \cap \text{deg}^{-1}(a_0)$ and a corresponding polynomial $f = \mathbf{x}^u + \lambda_1 \mathbf{x}^{u+c} + \dots + \lambda_r \mathbf{x}^{u+rc} \in I$ with $\lambda_r \neq 0$ and minimal $r \geq 0$. If $r = 0$, then $\mathbf{x}^u \in I$. If $r > 0$, then $u + rc \in M_2$ with $\text{deg}(u + rc) = a_0 + r \cdot \text{deg}(c) < a_0$. This implies $u + rc \in I$ by the induction hypothesis. But then f can be shortened, and we obtain a contradiction. It follows that $M_1 \cap \text{deg}^{-1}(a_0) \subseteq I \cap \text{deg}^{-1}(a_0)$ and, via (9) and $(M_1 \cap \text{deg}^{-1}(a_0)) = h(a_0) = (M_2 \cap \text{deg}^{-1}(a_0))$, that Claim (10) holds true.

Step 2: $q \geq 2$. Consider the polyhedral cone $\sigma := \text{deg}_{\mathbb{R}}(\mathbb{R}_{\geq 0}^n)$ in \mathbb{R}^q . Since $\mathbb{N}^n \cap \ker(\text{deg}) = 0$, the cone σ is pointed which means that the dual cone σ^\vee is full-dimensional. For a linear map $\ell : \mathbb{Z}^q \rightarrow \mathbb{Z}$ the following statements are equivalent:

$$\begin{aligned} \mathbb{N}^n \cap \ker(\ell \circ \text{deg}) = 0 &\iff \mathbb{N}^n \cap \text{deg}^{-1}(\ell^{-1}(a)) \text{ are finite for all } a \in \mathbb{Z} \\ &\iff \text{deg}(\mathbb{N}^n) \cap \ell^{-1}(a) = \sigma \cap \ell^{-1}(a) \text{ are finite} \\ &\iff \sigma \cap (\ker \ell) = 0 \\ &\iff \ell \in (\text{int } \sigma^\vee) \cup (-\text{int } \sigma^\vee). \end{aligned}$$

Fix a basis \mathcal{B} of $(\mathbb{R}^q)^*$ consisting of linear forms ℓ which satisfy this condition. For each $\ell \in \mathcal{B}$, we apply Step 1 to the one-dimensional grading $(\ell \circ \text{deg}) : \mathbb{Z}^n \rightarrow \mathbb{Z}^q \rightarrow \mathbb{Z}$, and we conclude that c lies in $\ker(\ell \circ \text{deg})$. Therefore, $c \in \bigcap_{\ell \in \mathcal{B}} \ker(\ell \circ \text{deg}) = \ker(\text{deg})$, since $\mathbb{R}\mathcal{B} = (\mathbb{R}^q)^*$. This finishes the proof of Lemma 12 and of Theorem 11. \square

Suppose that M_1 and M_2 are monomial ideals on a multigraded Hilbert scheme Hilb_h . For $a \in A$ we denote by $P_a(M_i) \in \mathbb{N}^n$ the sum of all vectors $u \in \mathbb{N}^n$ such that $\mathbf{x}^u \notin M_i$ and $\text{deg}(u) = a$. Here the number of summands is $h(a)$, the value of the Hilbert function at a .

Lemma 13. *Let $M_1, M_2 \in \text{Hilb}_h, \Omega_c(M_1, M_2) \neq \emptyset$, and $\text{deg}(c) = 0$. If M_1, M_2 differ in a degree $a \in A$, then $P_a(M_1) - P_a(M_2)$ is a positive integer multiple of c .*

Proof. Let $I \in \Omega_c(M_1, M_2)$. We may assume that deg equals the c -grading $\mathbb{Z}^n \rightarrow \mathbb{Z}^n/\mathbb{Z}c$. For a degree $a \in \mathbb{Z}^n/\mathbb{Z}c$ we denote by I_a and $(M_i)_a$ the homogeneous parts of the corresponding ideals. Let \mathcal{L} be a finite set of polynomials such that $(M_1)_a$ and $(M_2)_a$ are contained in $\text{in}_{c<0}(\mathcal{L})$ and $\text{in}_{c>0}(\mathcal{L})$, respectively. For an element $\lambda_0 \mathbf{x}^u + \dots + \lambda_r \mathbf{x}^{u+rc} \in \mathcal{L}$ with $\lambda_0, \lambda_r \neq 0$ we call r its length. The *total length* of \mathcal{L} is the sum of the lengths of all polynomials in \mathcal{L} . Now, whenever there are two elements $f, g \in \mathcal{L}$ having the same highest or the same lowest monomial, then we can reduce the total length of \mathcal{L} without loosing $(M_1)_a \subseteq \text{in}_{c<0} \mathcal{L}$ and $(M_2)_a \subseteq \text{in}_{c>0} \mathcal{L}$. Just replace $\{f, g\}$ by the shorter polynomial among them and $f - g$. Iterating this several times, we arrive at a set \mathcal{L} none of whose polynomials have common ends. The set \mathcal{L} provides a bijection $(M_1)_a \xrightarrow{\sim} (M_2)_a$ via $\text{in}_{c<0}(f) \mapsto \text{in}_{c>0}(f)$. \square

We are now prepared to tie up some loose ends from the last section. Let us first reexamine the process of finding the correct direction c in Algorithm 9. Factoring the numerator difference of the Hilbert series can be replaced by the following procedure.

Algorithm 14. (Input: $M_1, M_2 \in \text{Hilb}_h$ with respect to a positive grading or $A=0$. Output: $\Omega(M_1, M_2)$).

Step 1: Pick a degree $a \in A$ in which the monomial ideals M_1 and M_2 are different. Compute the vectors $P_a(M_1)$ and $P_a(M_2)$.

Step 2: If $P_a(M_1) = P_a(M_2)$ then stop and output the empty set. Otherwise let c be the primitive vector in direction $P_a(M_1) - P_a(M_2)$.

Step 3: Using Algorithm 5, compute and output $\Omega_c(M_1, M_2)$.

Finally, it is time to present the

Proof of Theorem 8. Suppose that $\Omega_c(M_1, M_2)$ and $\Omega_{c'}(M_1, M_2)$ are both non-empty, where c and c' are primitive vectors in \mathbb{Z}^n which have positive and negative coordinates. The group $A := \mathbb{Z}^n / \mathbb{Z}c' \simeq \mathbb{Z}^{n-1}$ is torsion-free and the canonical map $\text{deg} : \mathbb{Z}^n \rightarrow A$ is a positive grading. Applying Lemma 12 to this grading, we find that $c = \pm c'$. Finally, Lemma 13 excludes $c' = -c$. \square

One important question regarding Hilbert schemes is under which circumstances Hilb_h is connected. While classical Hilbert schemes are known to be connected [8], Santos [9] recently constructed a disconnected multigraded Hilbert scheme. The graph introduced in this paper provides a tool for studying connectivity questions.

Definition. For a subscheme $\mathcal{H} \subseteq \text{Hilb}_h$, we denote by $\mathcal{G}(\mathcal{H}) \subseteq \mathcal{G}$ the subgraph with vertices and edges built from monomial and edge ideals in \mathcal{H} . In particular, the induced subgraph property means that $\mathcal{G}(\text{Hilb}_h)$ is an induced subgraph of \mathcal{G} .

Lemma 15. Let $\text{deg} : \mathbb{Z}^n \rightarrow \mathbb{Z}^q$ be a positive grading of $\mathbb{K}[\mathbf{x}]$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. If \mathcal{H} is an irreducible component of Hilb_h then the graph $\mathcal{G}(\mathcal{H})$ is connected.

Proof. The positive grading implies that Hilb_h is a projective scheme [4, Corollary 1.2]. Hence \mathcal{H} is irreducible and projective. The algebraic torus $(\mathbb{K}^*)^n$ acts on \mathcal{H} with finitely many fixed points (the monomial ideals). Consider any two monomial ideals M_1, M_2 which lie in \mathcal{H} . Then $\{M_1, M_2\}$ is an edge in $\mathcal{G}(\mathcal{H})$ if and only if M_1 and M_2 are in the closure of a one-dimensional torus orbit on \mathcal{H} . The irreducible variety \mathcal{H} contains a connected projective curve C , not necessarily irreducible, which lies in \mathcal{H} and contains both points M_1 and M_2 . We can degenerate the curve C by a generic one-parameter subgroup of $(\mathbb{K}^*)^n$ to a curve C' which is $(\mathbb{K}^*)^n$ -invariant. This can be done, for instance, by a Gröbner basis computation in the homogeneous coordinates of the projective variety \mathcal{H} . The degenerate curve C' still contains M_1 and M_2 , it is connected (since, by Stein Factorization, flat degenerations of connected projective schemes are connected; see e.g., Exercise III/11.4 in [5]), and it is set-theoretically a union of closures of one-dimensional torus orbit on \mathcal{H} . Hence M_1 can be connected to M_2 by a sequence of edges in $\mathcal{G}(\mathcal{H})$. \square

Corollary 16. For positive gradings with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, the multigraded Hilbert scheme Hilb_h is connected if and only if the graph $\mathcal{G}(\text{Hilb}_h)$ is connected.

Proof. The if direction always holds even if the grading is not positive and Hilb_h is not compact. Indeed, if I_1 and I_2 are arbitrary ideals in Hilb_h then we can connect them to

their initial ideals $\text{in}_{<}(I_1)$ and $\text{in}_{<}(I_2)$ under some term order $<$. Connecting these two monomial ideals along the graph $\mathcal{G}(\text{Hilb}_h)$ establishes a path in Hilb_h which connects I_1 and I_2 . For the only-if direction we use Lemma 15. Suppose Hilb_h is connected. Then the graph of irreducible components is connected, where two components are connected by an edge in this graph if and only if they intersect. On the other hand, with Hilb_h , all its irreducible components are torus invariant. Hence, by Gröbner degenerations, every non-empty intersection of irreducible components of Hilb_h contains at least one monomial ideal. Using Lemma 15, we can then connect any two monomial ideals $M_1, M_2 \in \text{Hilb}_h$ by a sequence of edges in $\mathcal{G}(\text{Hilb}_h)$. \square

We do not know at present whether Lemma 15 and Corollary 16 remain valid if the grading is not positive. Corollary 16 had been proved previously by Maclagan and Thomas for the special case of *toric Hilbert schemes* [6]. Here “toric” means that h is the characteristic function of $\deg(\mathbb{N}^n)$. The disconnected example in [9] is a toric Hilbert scheme. It was constructed using methods from polyhedral geometry.

4. Simplicial complexes

Every class of monomial ideals determines an induced subgraph of \mathcal{G} . In this section we study the induced finite subgraph on square-free monomial ideals in $\mathbb{K}[\mathbf{x}]$. These ideals correspond to simplicial complexes on $[n] := \{1, 2, \dots, n\}$. We write Δ_{n-1} for the full simplex on $[n]$. Faces of Δ_{n-1} are subsets of $[n]$, and they are identified with their incidence vectors in $\{0, 1\}^n$. Fix an arbitrary simplicial complex $X \subset \Delta_{n-1}$. Its *Stanley–Reisner ideal* and its *Stanley–Reisner ring* are

$$M_X := \langle \mathbf{x}^u : u \in \Delta_{n-1} \setminus X \rangle \subseteq \mathbb{K}[\mathbf{x}] \quad \text{and} \quad A_X := \mathbb{K}[\mathbf{x}]/M_X.$$

The A_X -module $\text{Hom}_{\mathbb{K}[\mathbf{x}]}(M_X, A_X)$ describes the *infinitesimal deformations* of A_X . It is \mathbb{Z}^n -graded. Elements λ of degree c in $\text{Hom}_{\mathbb{K}[\mathbf{x}]}(M_X, A_X)$ look like $\mathbf{x}^u \mapsto \lambda(u) \mathbf{x}^{u+c}$, where λ ranges over a subspace of the vector space of maps $\Delta_{n-1} \setminus X \rightarrow \mathbb{K}$, cf. [1]. The equations defining this subspace include $\lambda(u) = 0$ whenever $u + c \notin \mathbb{N}^n$. For any $c \in \mathbb{Z}^n$ and any $\lambda \in \text{Hom}_{\mathbb{K}[\mathbf{x}]}(M_X, A_X)_c$, we define an ideal as follows:

$$I_\lambda := \langle \mathbf{x}^u + \lambda(u) \mathbf{x}^{u+c} : u \in \Delta_{n-1} \setminus X \rangle.$$

If $\mathbf{x}^{u+c} \in M_X$, then the value $\lambda(u)$ does not matter neither for $\lambda \in \text{Hom}_{\mathbb{K}[\mathbf{x}]}(M_X, A_X)$, nor for I_λ . We will set $\lambda(u) := 0$ in this case, cf. the end of the proof of Lemma 4.

Theorem 17. *Let $c \in \mathbb{Z}^n$ be a vector with both positive and negative coordinates.*

(a) *The map $\text{Hom}_{\mathbb{K}[\mathbf{x}]}(M_X, A_X)_c \rightarrow \Omega_c(M_X)$, $\lambda \mapsto I_\lambda$ is an isomorphism of schemes over \mathbb{K} . In particular, the Schubert scheme $\Omega_c(M_X)$ is an affine space.*

(b) *The monomial c -neighbors of M_X in \mathcal{G} come from $\text{Hom}_{\mathbb{K}[\mathbf{x}]}(M_X, A_X)_c$ via*

$$M'_X(\lambda) := \langle \mathbf{x}^u : u \in \Delta_{n-1} \setminus X, \lambda(u) = 0 \rangle + \langle \mathbf{x}^{v+c} : v \in \Delta_{n-1} \setminus X, \lambda(v) \neq 0 \rangle.$$

Proof. (a) Each pair $(\mathbf{x}^u, \mathbf{x}^v)$ of minimal M_X -generators provides a condition on both sides, in addition to the previously mentioned vanishing of certain λ -coordinates. The condition

is gotten via the linearity of $\lambda \in \text{Hom}$, on the one hand, and via the S-polynomials, on the other. In both cases, one obtains that $\lambda(u) = \lambda(v)$ whenever $\mathbf{x}^{(u \cup v)+c} \notin M_X$. In particular, these equations are linear.

(b) We must show that the generators $\mathbf{x}^u + \lambda(u)\mathbf{x}^{u+c}$ with $u \in \Delta_{n-1} \setminus X$ form a Gröbner basis of I_λ also for the term order $c > 0$. Let $\mathbf{x}^v = \text{in}_{c>0}(f)$ be the initial term of some element $f \in I_\lambda$. We must show that \mathbf{x}^v is a multiple of the $(c > 0)$ -leading term of some $\mathbf{x}^u + \lambda(u)\mathbf{x}^{u+c}$. After reducing f to normal form with respect to the generators, only two cases remain. Either f is a binomial or a monomial.

Case 1: f equals $\mathbf{x}^{v-c-u}(\mathbf{x}^u + \lambda(u)\mathbf{x}^{u+c})$ with $\lambda(u) \neq 0$. Then \mathbf{x}^v is divisible by $\mathbf{x}^{u+c} = \text{in}_{c>0}(\mathbf{x}^u + \lambda(u)\mathbf{x}^{u+c})$, and we are done.

Case 2: f equals \mathbf{x}^v , i.e., $\mathbf{x}^v \in I_\lambda$. For $w \in \mathbb{N}^n$ let $\bar{w} = \{i : w_i \neq 0\}$ denote its support. Then $\bar{w}_1 + \bar{w}_2 = \overline{\bar{w}_1 \cup \bar{w}_2}$, and $\bar{w} = w$ for elements $w \in \Delta_{n-1}$. The ideal M_X being square-free, we have $\mathbf{x}^w \in M_X$ if and only if $\mathbf{x}^{\bar{w}} \in M_X$. In particular, since $\mathbf{x}^v \in \text{in}_{c<0}(I_\lambda) = M_X$, we have $\mathbf{x}^{\bar{v}} \in M_X$ and $\bar{v} \in \Delta_{n-1} \setminus X$. It suffices to show that $\lambda(\bar{v}) = 0$. Suppose $\lambda(\bar{v}) \neq 0$. Then $v + c \geq \bar{v} + c \geq 0$. Now, $\lambda(\bar{v})\mathbf{x}^{v+c} = \mathbf{x}^{v-\bar{v}}(\mathbf{x}^{\bar{v}} + \lambda(\bar{v})\mathbf{x}^{\bar{v}+c}) - \mathbf{x}^v$ implies that $\mathbf{x}^{v+c} \in I_\lambda$, hence $\bar{v} + c \in \Delta_{n-1} \setminus X$. Setting $w_1 := \bar{v}$ and $w_2 := \bar{v} + c$, we find $\overline{(w_1 \cup w_2) + c} = \bar{v} + c \in X$ (since $\lambda(\bar{v}) \neq 0$), i.e., $\mathbf{x}^{(w_1 \cup w_2)+c} \notin M_X$. The equations mentioned in (a) imply $\lambda(\bar{v}) = \lambda(\bar{v} + c)$. We can now replace by v by $v + c$ and run the same argument again. After iterating this step finitely many times, the hypothesis $v + c \geq 0$ will no longer hold, so that $\lambda(\bar{v} + c) = 0$ and hence $\lambda(\bar{v}) = 0$. This completes the proof. \square

To make the previous theorem more useful, we shall apply the description of the vector spaces $\text{Hom}_{\mathbb{K}[\mathbf{x}]}(M_X, A_X)_c$ given by Altmann and Christophersen in [1].

Notation. For a subset $N \subseteq X$, we denote by $\langle N \rangle$ the union of all relatively open simplices $|f|$, $f \in N$, in the geometric realization $|X|$. For $c \in \mathbb{Z}^n$ with non-trivial positive and negative parts c^+ and c^- , we denote by $a, b \subseteq [n]$ their respective supports, and

$$N_c := \{f \in X : a \subseteq f, f \cap b = \emptyset, f \cup b \notin X\},$$

$$\tilde{N}_c := \{f \in N_c : f \cup b' \notin X \text{ for some proper subset } b' \text{ of } b\}.$$

The following results are proved in [1]. If $c_i \leq -2$ for some i then $\text{Hom}_{\mathbb{K}[\mathbf{x}]}(M_X, A_X)_c$ vanishes. If not, i.e., if $c^- = b$, let N^1, \dots, N^m be the subsets of N_c which correspond to those connected components of $\langle N_c \rangle$ that do not touch \tilde{N}_c . There is an isomorphism $\mathbb{K}^m \xrightarrow{\sim} \text{Hom}_{\mathbb{K}[\mathbf{x}]}(M_X, A_X)_c$. It sends $(\lambda_1, \dots, \lambda_m)$ to the map $\lambda : \Delta_{n-1} \setminus X \rightarrow \mathbb{K}$ defined as $\lambda(u) := \lambda_i$ if $(u \cup a) \setminus b \in N^i$ and $\lambda(u) := 0$ otherwise. Theorem 17(a) implies that $\Omega_c(M_X)$ is trivial unless $a = \text{supp}(c^+)$ is a face of X ($a \notin X \Rightarrow N_c = \emptyset$).

Suppose $a = \text{supp}(c^+) \in X$ and $c^- = b$ and fix N^1, \dots, N^m as above. Then each non-empty subset $\{i_1, \dots, i_\ell\} \subseteq \{1, \dots, m\}$ determines a monomial ideal as follows:

$$M'_X = M'_X(i_1, \dots, i_\ell) = \langle \mathbf{x}^u : u \in \Delta_{n-1} \setminus X, (u \cup a) \setminus b \notin N^{i_1} \cup \dots \cup N^{i_\ell} \rangle$$

$$+ \langle \mathbf{x}^{v+c} : v \in \Delta_{n-1} \setminus X, (v \cup a) \setminus b \in N^{i_1} \cup \dots \cup N^{i_\ell} \rangle.$$

These $2^m - 1$ ideals are generally not distinct. However, if $(a \cup b) \notin X$, then $\langle N_c \rangle$ is connected, hence $m = 1$ for $\tilde{N}_c = \emptyset$ and $m = 0$ for $\tilde{N}_c \neq \emptyset$. Theorem 17 and the results quoted from [1] imply

Corollary 18. *The ideals M'_X are all the neighbors of M_X in \mathcal{G} in direction c .*

We next identify the square-free monomial ideals among the neighbors M'_X of M_X . From the generators \mathbf{x}^{v+c} we see that M'_X is not square-free unless $c_i \leq 1$ for all i . Hence from now on we assume that $c^+ = a$ and $c^- = b$ are non-empty disjoint subsets of $[n]$. We further suppose that $\{N^1, \dots, N^\ell\}$ is a non-empty subset of the connected components of $\langle N_c \rangle$ that do not touch \tilde{N}_c . With these data (giving rise to an M'_X) we associate the distinguished subcomplex

$$F := \{f \setminus a : f \in N^1 \cup \dots \cup N^\ell\} \subseteq 2^{[n] \setminus (a \cup b)}.$$

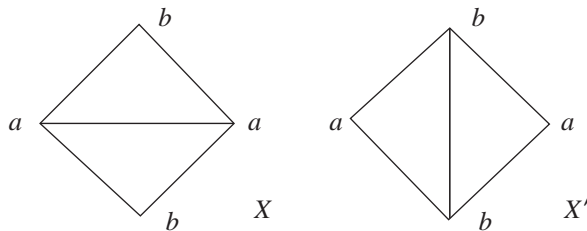
Notation. Let $\mathbf{a} := \{f : f \subseteq a\}$ be the full simplex on a , $\partial(\mathbf{a}) := \{f : f \subsetneq a\}$, and similarly define \mathbf{b} , $\partial(\mathbf{b})$ from b . If Y and Z are subcomplexes (or just subsets) of X on disjoint sets of vertices, then their *join* is the simplicial complex $Y * Z := \{f \cup g : f \in Y, g \in Z\}$. In particular, $\{a\} * F = N^1 \cup \dots \cup N^\ell$, and it is straightforward to check that the triple join $\mathbf{a} * F * \partial(\mathbf{b})$ is a subcomplex of X .

Theorem 19. *The monomial ideal M'_X is square-free if and only if $(\mathbf{a} * F * \mathbf{b}) \cap X = \mathbf{a} * F * \partial(\mathbf{b})$ if and only if $\mathbf{a} * F * \{b\}$ is disjoint from X if and only if $X \cap (F * \{b\}) = \emptyset$. If this holds then the neighboring simplicial complex X' with $M_{X'} = M'_X$ is given by*

$$X' = (X \setminus (\mathbf{a} * F * \partial(\mathbf{b}))) \cup (\partial(\mathbf{a}) * F * \mathbf{b}). \tag{11}$$

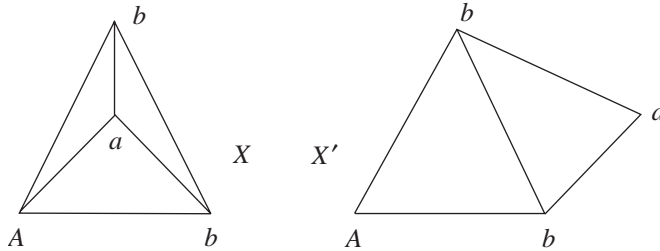
Theorem 19 describes all the edges $\{X, X'\}$ in the *graph of simplicial complexes*, that is, the subgraph of \mathcal{G} induced on square-free monomial ideals. The transition from X to X' generalizes the familiar notion of *bistellar flips*. They correspond to the case $\ell = m = 1$ and $b \notin X$. Here the condition in the first sentence of Theorem 19 is automatically satisfied, meaning that the c -neighbor M'_X of M_X is square-free. These bistellar flips are a standard tool for locally altering combinatorial manifolds (see e.g. [11]) or triangulations of point configurations (see e.g. [6]).

Example 20. Let a be an edge in a triangulated manifold X of dimension two. If $a \cup b$ supports the two triangles meeting along a , then $N_c = \{a\}$, $\tilde{N}_c = \emptyset$, i.e., $\ell = m = 1$, $N^1 = N_c$, and $F = \{\emptyset\}$. We are in the $b \notin X$ case, and $\mathbf{a} * F * \partial(\mathbf{b})$ consists of the two triangles and their faces.



From X , we remove $\{a\} * F * \partial(\mathbf{b})$, i.e., the two triangles and their common edge. They are replaced, in X' , by the two triangles $\partial(\mathbf{a}) * F * \{b\}$ with common edge b .

Example 21. We still consider a triangulated surface. Let a be a trivalent vertex being adjacent to an edge b and a third vertex A . In particular, $a \cup b \in X$.



Here $N_c = \{\overline{Aa}\}$, $\tilde{N}_c = \emptyset$, $\ell = m = 1$, $N^1 = N_c$, $F = \{A\}$. One obtains X' from X by removing the edge \overline{Aa} together with the adjacent triangles and, afterwards, adding the triangle formed by edge b and vertex A . The new complex X' is no longer part of a triangulation of a two-dimensional manifold, since the edge b is incident to three triangles in X' . The geometric realization $|X'|$ looks like $|X|$ plus the additional triangle $\Delta(bba)$ sticking out of it.

Example 22. Finally, we would like to show that flipping backwards Example 21 gives an instance with $m = 2$, i.e., with more than one neighbor in a fixed tangent direction c . Let X consist of three triangles $\Delta(AST)$, $\Delta(BST)$, $\Delta(CST)$ sharing the common edge $a := \overline{ST}$. With $b := \{B\}$, we obtain $N_c = \{\Delta(AST), \Delta(CST)\}$ and $\tilde{N}_c = \emptyset$. Since \overline{ST} does not belong to N_c , this yields $m = 2$, $N^1 = \{\Delta(AST)\}$, and $N^2 = \{\Delta(CST)\}$. Now, choosing F among $\{A\}$, $\{B\}$, or $\{A, B\}$, we have three possibilities to construct neighbors of X . In each case, the square free condition of Theorem 19 is satisfied, and we obtain the following results for X' :

- (A) $X' = \{\Delta(BTA), \Delta(BST), \Delta(BAS), \Delta(CST)\} \cup \{\text{faces}\}$; this is like the X from Example 21.
- (B) $X' = \{\Delta(BTC), \Delta(BST), \Delta(BCS), \Delta(AST)\} \cup \{\text{faces}\}$; this looks similar as the previous complex—but now the other triangle $\Delta(CST)$ has been subdivided.
- (AB) $X' = \{\Delta(BTA), \Delta(BST), \Delta(BAS), \Delta(BTC), \Delta(BCS)\} \cup \{\text{faces}\}$, i.e., both triangles $\Delta(AST)$ and $\Delta(CST)$ are subdivided by the same inner vertex B .

Proof of Theorem 19. Recall from Corollary 18 that M'_X has two types of generators. The first were \mathbf{x}^u with $u \notin X$ and $(u \cup a) \setminus b \notin N^1 \cup \dots \cup N^\ell$, the second were \mathbf{x}^{v+c} with $v \notin X$ and $(v \cup a) \setminus b \in N^1 \cup \dots \cup N^\ell$. Hence, M'_X is square-free if and only if every non-reduced generator $v + c$ of the latter type finds some reduced generator g with $g \leq v + c$. Since this implies $g \subseteq (v \cup a) \setminus b \in N^1 \cup \dots \cup N^\ell \subseteq N_c \subseteq X$, the generator g cannot be of type one. If there is a type two generator $g = u + c \leq v + c$, then, since $u \cap a = \emptyset$, we obtain $u \subseteq v \setminus a$, and $u \notin X$ implies $v \setminus a \notin X$. On the other hand, if $v \setminus a \notin X$, then $u := v \setminus a$ indeed does the job. We conclude that M'_X is square-free if and only if there is no $v \in \Delta_{n-1} \setminus X$ with $(v \cup a) \setminus b \in N^1 \cup \dots \cup N^\ell$ and $v \setminus a \in X$. This condition is equivalent to the one stated in Theorem 19. To see this take $f := v \setminus (a \cup b) \in F$ or $v := f \cup (a \cup b)$.

Now, let us assume that this square-free condition is satisfied and take Eq. (11) of the theorem as a definition of some subset $X' \subset \Delta_{n-1}$. We first show that X' is indeed a simplicial complex. Afterwards, we will check that $M_{X'} = M'_X$.

Step 1. We claim the following: Let f' be a subset of f which lies in $2^{[n] \setminus (a \cup b)}$. Then, $f' \in F$ and $(a \cup f) \in X$ if and only if $f \in F$ and $(a \cup f' \cup b) \notin X$.

If F was referring to N_c instead to its subset $N^1 \cup \dots \cup N^\ell$, the claim would follow directly from the definition. On the other hand, if both $a \cup f'$ and $a \cup f$ are in N_c , then the whole flag in between belongs to N_c and, moreover, to the same connected component of $\langle N_c \rangle$. In particular, $f' \in F$ if and only if $f \in F$.

Step 2. $X' = X \setminus \{[a] * F * \partial(\mathbf{b})\} \cup \{\partial(\mathbf{a}) * F * \{b\}\}$ is a simplicial complex:

First, we check that $\{a\} * F * \partial(\mathbf{b})$ is, inside X , closed under enlargement. Let $(a \cup f' \cup b'') \subseteq (a \cup f \cup b') \in X$ with $f' \in F$, $b'' \subsetneq b$, $f \in 2^{[n] \setminus (a \cup b)}$, and $b' \supseteq b''$. Then, since $(a \cup f) \in X$, we may use Step 1 to obtain $f \in F$. Moreover, since $X \cap [a * F * \{b\}] = \emptyset$, the set b'' cannot equal b .

Now, we check the subsets of the elements of $X' \setminus X$. Take, w.l.o.g., $g := (a'' \cup f' \cup b) \subseteq (a' \cup f \cup b)$ with $a'' \subseteq a' \subsetneq a$ and $f \in F$. If $(a \cup f' \cup b) \in X$, then $g \in X$, and we are done. If not, then Step 1 implies that $f' \in F$, hence $g \in [\partial(\mathbf{a}) * F * \{b\}]$.

Step 3. $M'_X = M_{X'}$: Translating the square-free M'_X generators into the $(\mathbf{a} * F * \mathbf{b})$ -language, we obtain as exponents $(a' \cup f \cup b') \notin X$ with $f \notin F$ for those of the first type and $(a \cup f)$ with $f \in F$ for those of the second. (The condition $(f \cup b) \notin X$ follows automatically from $X \cap [a * F * \{b\}] = \emptyset$.) While the type two generators are the minimal elements of $\{[a] * F * \partial(\mathbf{b})\} \subseteq X \setminus X'$, the type one generators do neither belong to X , nor to the part being changed during the transition to X' . Hence, it remains to consider $g = (a' \cup f \cup b')$ belonging to neither X , nor $[\partial(\mathbf{a}) * F * \{b\}]$ and to show that $\mathbf{x}^g \in M'_X$. If $f \notin F$, then g is obviously a generator of type one. Assuming $f \in F$, then there are two possibilities: If $a' = a$, then $g \supseteq (a \cup f)$, i.e., a type two generator takes care. If $a' \subsetneq a$, then $b' \subsetneq b$, hence $(a \cup f) \cup b' \notin X$ implies $(a \cup f) \in \tilde{N}_c$, and we obtain a contradiction to $f \in F$. \square

5. The next steps

The following list of open problems arises naturally from our investigations.

Problem. Does the graph $\mathcal{G}_{n, \mathbb{K}}$ depend on the field \mathbb{K} ? For instance, is $\mathcal{G}_{n, \mathbb{R}} = \mathcal{G}_{n, \mathbb{C}}$? For \mathbb{K} algebraically closed, does $\mathcal{G}_{n, \mathbb{K}}$ depend on the characteristic of \mathbb{K} ?

While Theorem 19 shows that the graph of simplicial complexes is independent of the field \mathbb{K} , the following example suggests that the answer might be “yes”, anyway. Let $n = 2$, $M_1 = \langle x^4, y^2 \rangle$, and $M_2 = \langle x^2, y^4 \rangle$. The edge ideals connecting M_1 and M_2 have the form

$$I = \langle x^4, y^2 + a_1yx + a_2x^2 \rangle = \langle x^2 + b_1yx + b_2y^2, y^4 \rangle,$$

where a_1, a_2, b_1, b_2 are scalars in \mathbb{K} satisfying $a_2b_2 - 1 = a_1 - a_2b_1 = b_1^3 - 2b_1b_2 = 0$. These three equations define a scheme which is the reduced union of two irreducible components if $\text{char } \mathbb{K} \neq 2$, and which is non-reduced but irreducible if $\text{char } \mathbb{K} = 2$.

Problem. Find a purely combinatorial description for the graph of partitions. What is the exact relationship with the directed graphs studied by Evain in [3]?

Problem. Does the induced subgraph property hold for all toric Hilbert schemes ?

Problem. Do there exist monomial ideals $M_1, M_2 \subseteq \mathbb{K}[\mathbf{x}]$ having the same Hilbert function with respect to two different gradings $\mathbb{Z}^n / \mathbb{Z}c$ and $\mathbb{Z}^n / \mathbb{Z}c'$? Or is this rather common provided that $\Omega(M_1, M_2) = \emptyset$?

Our results in Sections 2 and 3 provide a method for constructing the graph of a multi-graded Hilbert scheme Hilb_h , provided the following problem has been solved.

Problem. Develop a practical algorithm for computing all monomial ideals in Hilb_h .

Peeva and Stillman [7] recently proved a connectivity theorem for Hilbert schemes over the exterior algebra, using methods similar to those in Section 4. Monomial ideals in the exterior algebra being square-free, the following question arises.

Problem. Given any grading and Hilbert function on the exterior algebra, is its flip graph the same as the subgraph induced from our graph of simplicial complexes?

It is natural to wonder about the topological significance of these flips.

Problem. Which topological invariants remain unchained by the generalized flips of Theorem 19. How can one decide, by means of a practical algorithm, whether two given simplicial complexes can be connected by a chain of those flips?

Acknowledgement

We would like to thank Rekha Thomas for carefully reading a manuscript version of this paper.

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