Deformations of Toric Singularities

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In the present paper we investigate the deformation theory of toric singularities (i.e. of those singularities occurring in toric varieties). The best results are obtained for isolated toric Gorenstein singularities: We describe their mini-versal (or semi-universal) deformation in several ways (by a combinatorial construction, by equations, and by interpreting its irreducible components). See section (1.2) for a more detailed survey.

My investigations on this subject started during a one-year stay at the university of Kaiserslautern (1990/91). I would like to thank Duco van Straten, Gert-Martin Greuel, and Theo de Jong for turning my interest toward deformation theory and for many fruitful discussions and hints (and the DFG for financial support).

At the same time I was introduced by Kurt Behnke, Jan Christophersen, and Jan Stevens to the latest developments of deforming two-dimensional quotient singularities. This led to the definition of so-called toric deformations and my first attempts of classifying them. Then, during a short stay (in 1992) at Cornell University, discussions with Bernd Sturmfels helped very much to form the idea that seems to be the key to describing toric deformations: Those deformations are closely related to splittings of certain polyhedra into Minkowski sums. I am very greatful to all these people for their support.

Finally, I found the ultimative construction of the versal deformation for isolated, toric Gorenstein singularities during a one-year stay at M.I.T. during the academic year 1993/94. I would like to thank Richard Stanley for giving me the opportunity (and the DAAD for giving me the money) to spend a year at this very stimulating place.

This thesis consists mainly of revised versions of several manuscripts and papers. We have tried to bring the notations into a common standard, and, during the last year, we have added several new sections. Roughly we can give the following description:

- Sections (3.1)-(3.3) and (4.1)-(4.3) consist of the papers [Al 3] and [Al 5].
- Most parts of (3.4), (3.5), and (4.4) are new.
- Chapter 5 together with (2.2), (2.3) form the paper [Al 6].
- Chapter 6 is completely new.
- Chapter 7 (except (7.4.4), which is new) is contained in [Al 4].

Finally, I want to thank Herbert Kurke for constant support and encouragement.

Contents

Chapter 1

Introduction

1.1 Deformations

(1.1.1) In the very beginning, deformation theory appeared as the investigation of how complex structures may vary on a fixed compact, smooth manifold. In his famous paper "Theorie der abelschen Funktionen" (1857; cf. [Rm]) Riemann already mentioned the $3g-3$ moduli determining the complex structure of an algebraic curve ("Riemann surface") of genus $g \geq 2$.

Looking for an analogeous description in higher dimensions, Kodaira and Spencer started to develop the machinery of deformation theory (cf. [KoSp]). Because of the misery that (beginning in dimension two) descent moduli spaces do not always exist, they used a modified, weaker concept: The versal (or semi-universal) deformation $f : X \to S$ of a manifold $Y = X_0 = f^{-1}(0 \in S)$ "contains" all possible deformations; but even the minimal one no longer provides a one-to-one correspondence between fibers and complex structures.
Under the assumption that $H^2(X_0, \theta_{X_0}) = 0$ (i.e. no obstructions occur) they have shown that a miniversal deformation always exists. Later, allowing singular spaces for the base, Kuranishi (cf. [Ku]) completed their work and got rid of the H^2-assumption. The tangent space of the versal base space S equals $H^1(X_0, \theta_{X_0})$ (i.e. locally $S \subseteq H^1(X_0, \theta_{X_0})$), and the dual vector space $H^2(X_0, \theta_{X_0})^*$ provides the (non-linear) equations.

See [Kod] for a detailed description of these results, including the ups and downs during the process of developing this theory.

(1.1.2) In a similar manner, we may regard deformations of germs of analytic spaces (or equivalently, deformations of local, analytic $\mathbb{C}$-algebras). If $Y = (Y, 0)$ is such a germ (often called "singularity", since smooth germs are not the interesting ones), we define the following functor:

$$\mathrm{Def}_Y((T,0)) := \{g : Z \to T \text{ (flat), together with } g^{-1}(0) \xrightarrow{\sim} Y\} /_{\text{isomorphisms}}.$$

Definition: *A deformation $f : X \to S$ (i.e. $f \in Def_Y(S)$) is called versal, if*

(i) every other deformation $g : Z \to T$ of Y may be induced via base change from some analytic map $T \to S$, and

(ii) this map may be prescribed on closed, analytic subgerms $T' \subseteq T$.

Moreover, $f : X \to S$ is called mini-versal, if

(iii) not $T \to S$, but at least its differential map $T_{T,0} \to T_{S,0}$ is uniquely determined. (Equivalently: $T_{S,0} \to Def_Y(\mathbb{C}[\varepsilon]/_{\varepsilon^2})$ is an isomorphism.)

It is easy to see that the mini-versal deformation (if it exists) is uniquely determined. Y being a complete intersection, this is a family over a smooth base space obtained by certain perturbations of the defining equations. However, as soon as we leave this class of singularities, the structure of the versal family or even the base space will be more complicated. In (5.7.2) we will present a (three-dimensional) example of an almost rigid singularity admitting a fat point as base space of its versal deformation.
Since the whole deformation theory of a singularity is encoded in this construction, the mini-versal deformation (or just its base space) contains important information about the given germ Y. For instance, it may be regarded a source of numerical invariants.

(1.1.3) Good references for facts about deformation theory are Artin's Lecture notes [Art 2], the large introduction in Palamodov's paper [Pa], or Steven's thesis [St 2]. Nevertheless, we would like to mention some important points:

General facts. In [Sch 1] Schlessinger has investigated (in a more general setting) the restriction of functors such as Def_Y to the category of local Artinian algebras over $\mathbb{C}$. He got explicit conditions for the existence of a formal, formally versal object which are easy to check. In case of our deformation functor most of them are met automatically; it remains to have the vector space $\mathrm{Def}_Y(\mathbb{C}[\varepsilon]/_{\varepsilon^2})$ be finite-dimensional, which will be true at least for isolated singularities.
Moreover, Schlessinger's construction leads, on the one hand, to an explicit method of constructing the versal deformation up to some order via power series Ansatz and, on the other hand, to a very useful criterion of versality. The latter one has been explicitly described and used in [Arn] and [JS 4]; for details see (3.1).

As before in the deformation theory of compact analytic spaces, it was easier to prove the existence of an analytic, mini-versal deformations under the assumption that no obstructions occur. Tjurina did so in [Tju] for normal, isolated singularities (assuming $\mathrm{Ext}^2_Y(\Omega^1_Y, \mathcal{O}_Y) = 0$). Later, in 1972, Grauert treated the general case in [Gra].

Tangent space and obstructions. It is not difficult to show that the vector spaces T^1_Y and T^2_Y (in the form as they are used in (3.1)) equal $\mathrm{Def}_Y(\mathbb{C}[\varepsilon]/_{\varepsilon^2})$ or contain the obstructions for lifting deformations to a higher order, respectively. However, much more effort is necessary to obtain these vector spaces as the first two groups of a general cohomology theory for (commutative) algebras $R \to A$. The crucial step is to define the so-called cotangent complex $C^{\cdot}_{A/R}$ of A-modules. Then, one obtains the functors T^i and T_i via

$$T_i(A/R; M) := H_i(C^{\cdot}_{A/R} \otimes_A M) \;\text{ and }\; T^i(A/R; M) := H^i(\mathrm{Hom}_A(C^{\cdot}_{A/R}, M)) \,.$$

(M is an arbitrary A-module, and the relations to the above mentioned T^i_Y are $T^i_Y = T^i(\mathcal{O}_{Y,0}/\mathbb{C}; \mathcal{O}_{Y,0})$.) The advantages of this more general point of view are the following:

(1) In the explicit formulas of (3.1), T^1_Y and T^2_Y seem to depend on the special embedding of Y into a smooth space $\mathbb{C}^{w+1}$ and on some other things. Hence, it is a better way to show at once that the complex $C^{\cdot}_{A/R}$ is defined (up to homotopy) in a natural way.

(2) We get the notion of relative T^i's (instead just those for $\mathbb{C} \to \mathcal{O}_{Y,0}$).

(3) We obtain formulas for the behavior under base change and localization.

(4) There are two long exact sequences, either for changing the A-module M or the algebras (starting with ring homomorphisms $R \to A \to B$). Since $T_0(A/R; M) = \Omega^1_{A/R} \otimes_A M$ and $T^0(A/R; M) = \mathrm{Der}_R(A, M)$, the latter one extends the usual exact sequences for Kähler differentials or their duals (cf. [Ma]).

(5) There is a spectral sequence governing the relations between the T^i's and the T_i's. (Moreover, if the previous definition is extended to the level of sheaves, we also obtain a local-global spectral sequence.)

(6) The complex $\mathrm{Hom}_A(C^{\cdot}_{A/R}, A))$ has the additional structure of a commutative graded algebra which provides the same structure on $\oplus_i T^i(A/R; A)$ extending the Lie algebra structure on $\mathrm{Der}_R(A, A)$. In particular, we obtain the so-called cup product $T^1 \times T^1 \to T^2$ (cf. (4.1) for a description in terms of the T^i_Y's). Moreover, Massey products may be defined.

A frequently used reference is the paper [LiS] of Lichtenbaum-Schlessinger who defined the cotangent complex up to the second position (obtaining the right T^2's). For the whole complex see [An 2], [Ld], or [Qu] (with a categorial approach) or [Pa] which gives a rather down-to-earth description of the cotangent complex via the so-called Tjurina-resolvent. See §9 in [Qu] for a comparison of both definitions. A very readable summary of the properties one has to know (but without proofs) is the first section in [BeC].

Particular investigations. After the general framework is settled, it remains to ask for the shape of the versal deformation for given singularities or for certain classes of them - for instance for general theorems about the smoothness or about the number and dimension of irreducible components of the base space. What does it mean that there might be several components or that the base space might be non-reduced? The first example showing that the situation is not always as boring as in the complete intersection case was given by Pinkham (cf. [Pi 1]):

Example: Let $Y \subseteq \mathbb{C}^5$ be the cone over the rational normal curve of degree four. Y is $\mathbb{Q}$-Gorenstein, and it is given by the equations

$$\mathrm{rank}\begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \leq 1\,.$$

Destroying the symmetry by introducing three parameters $\underline{t} = (t_1, t_2, t_3)$ induces a flat family $Y_t \to \mathbb{C}^3$ defined by

$$\mathrm{rank}\begin{pmatrix} y_0 & y_1 + t_1 & y_2 + t_2 & y_3 + t_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \leq 1\,.$$

On the other hand, Y could be defined by the equations

$$\mathrm{rank}\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} \leq 1\,,$$

too. They provide a one-parameter deformation $Y_s \to \mathbb{C}$ via

$$\mathrm{rank}\begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 + s & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} \leq 1\,.$$

The versal deformation of Y equals the union of these two families. Its base space is the union of a hyperplane and a line in $\mathbb{C}^4$. (In particular, it is not possible to find any flat family over a smooth parameter space containing both deformations $Y_t \to \mathbb{C}^3$ and $Y_s \to \mathbb{C}$.)

We would like to give some examples of particular investigations. For a detailed survey of some of these developments we refer to the thesis [St 2].

- In [Sch 2] Schlessinger shows that isolated quotient singularities of dimension at least three are rigid.

- De Jong and van Straten investigated normal surface singularities via projecting them onto non-isolated hypersurfaces in $\mathbb{C}^3$. They compared the deformation theories of the former with a suitable adaption of that of the latter. As an application, they were able to determine the versal deformation of rational quadruple points (cf. [JS 1], [JS 2], [JS 3]).
- Using their own projection method, de Jong and van Straten have developed in [JS 5] the so-called picture method. It describes the versal deformation of two-dimensional sandwiched singularities (i.e. modifications of smooth germs) in terms of deformations of "pictures", i.e. of certain incidences between points and algebraic curves (e.g. lines).
- Which deformations of a singularity admit a simultaneous resolution? See [Wa 1], [Lf], or [Al 1] for investigations concerning this question. In case of rational surface singularities, Artin has shown in [Art 1] that those parameters form their own component of the versal base space.
- In [KoSh] Kollár and Shephard-Barron applied the theory of three-dimensional varieties to obtain for two-dimensional cyclic quotient singularities a one-to-one correspondence between so-called P-resolutions on the one hand, and components of the versal base space on the other hand. (This generalizes the concept of the Artin component mentioned above.) Using this approach and the results of [Ri], it was Arndt ([Arn]), Christophersen ([Ch 3]), and Stevens ([St 1]) who were able to give an explicit description of the components.
- Many investigations have been done for cones over projective varieties (e.g. projective curves). For computations of the dimension of T^1_Y see [Pi 1], [Mu], [Sch 3], and [Wa 5].

1.2 What is this paper about?

(1.2.1) The starting point was the following observation Christophersen made while investigating the versal deformation of two-dimensional cyclic quotient singularities (i.e. two-dimensional, affine toric varieties): *The total spaces over components of the versal base space are again toric.* This fact raises the following questions:

(i) Does this statement hold for higher-dimensional toric varieties as well?

(ii) Assume that we are given a polyhedral cone σ describing an affine toric variety Y. Is it possible to describe the total spaces over versal components just by combinatorial methods? (Given σ, what do the cones describing the total spaces look like?)

(iii) What does the whole versal deformation of an affine toric variety look like? This question not only touches questions about number and dimension of irreducible components; the base space might even be non-reduced. Moreover, the goal is not just to get equations (as you can get by using computer algebra systems), but to obtain a combinatorial description displaying patterns.

(iv) What does the deformation theory of hypersurfaces (or complete intersections) in affine toric varieties look like?

In this thesis, we deal with the first three questions; we definitely do not say anything about (iv). A key for all investigations is the notion of a Minkowski summand of some polyhedron (arising as affine cross cut of our cone σ). It occurs in the description of infinitesimal deformations (vector space T^1), of genuine one-parameter deformations, as well as in the construction of the versal deformation for toric Gorenstein singularities. Our results are the following:

- We describe the infinitesimal deformation theory of affine toric varieties by calculating T^1, T^2, and the cup product $T^1 \times T^1 \to T^2$ (see chapters 3 and 4).

- Those genuine deformations with toric total spaces (called "toric deformations") are investigated and described in chapter 7 (approaching question (ii)). Via the Kodaira-Spencer map, they are always contained in the "negative" part $\oplus_{R\in\sigma^\vee\cap M} T^1_Y(-R)$ of T^1_Y. In particular, we obtain that (i) is not true: As soon as there are non-negative infinitesimal deformations (for instance, if Y is the cone over the weighted projective space $\mathbb{P}(1,2,3)$), they cannot be toric.

- If Y is smooth in codimension two, then we can do even better: For each fixed $R\in\sigma^\vee\cap M$ we can describe the "versal deformation of degree $-R$" (see chapter 6). However, besides the non-negative degrees, another problem remains: It is not clear how to put these homogeneous, versal pieces together to obtain the whole versal deformation. This problem does not occur if T^1_Y is concentrated in one single degree (as is the case for isolated, three-dimensional Gorenstein singularities treated in chapter 5). In particular, for those toric varieties the above question (iii) is answered.

Now we give a more detailed guide about how this paper is organized:

(1.2.2) In chapter 2 we try to put the reader in the mood for polytopes, Minkowski summands, and toric varieties. In particular, sections (2.1) and (2.4) are used to remind the reader of basic notions and to fix notation.

On the other hand, in (2.2) (right before starting to talk about toric varieties), we present a construction of an affine scheme $\bar{\mathcal{M}}$ assigned to a lattice polytope Q. It reflects the several possibilities of splitting Q into Minkowski summands: Those decompositions involving only lattice summands (similar to a non-unique prime factorization) are encoded as irreducible components of $\bar{\mathcal{M}}_{\mathrm{red}}$ (cf. Theorem (2.2.5)); the "remaining" ones cause a non-reduced structure of $\bar{\mathcal{M}}$. Proofs are contained in section (2.3).

(1.2.3) In chapter 3 we begin to investigate deformation theory of toric varieties by doing the job which must always be done first: We "compute" the vector spaces T^1_Y and T^2_Y. Computing means presenting combinatorial formulas relating these spaces to the given datum, i.e. the cone σ representing the affine toric variety $Y=Y_\sigma$ we want to deform.

There are two different ways to present those formulas (connected by an exact sequence mentioned in (3.4.1)). The first version (Theorem (3.2.4)) is close to the language of equations and syzygies of Y_σ; it is very useful if we want keep control of the actual perturbations (cf. Theorem (3.2.6) and (3.2.7)). On the other hand, Theorems (3.4.1) and (3.4.3) are closer to the concept of Minkowski summands, which will be eventually the main tool to describe versal deformations. The latter formula is my favourite one, even if it looks a little bit ugly at first sight. However, if one excludes singularities which are not smooth in codimension two, then it reduces to Theorem (3.4.5).

Chapter 3 ends with applications of the T^i_Y-formulas obtained so far. We describe vanishing results for certain degrees of T^1_Y (cf. (3.5.2), (3.5.3)), and we treat the case of three-dimensional isolated singularities separately (cf. (3.5.4), (3.5.5)).

(1.2.4) Chapter 4 is the most technical one. Here we use our T^i_Y-formulas to describe the cup-product $T^1_Y\times T^1_Y\to T^2_Y$. Again, this can be done in the two different languages mentioned above (see Theorem (4.2.2) for the first version and Proposition (4.4.3) for the second one).

Knowing the cup product (in particular in its second version) gives the information about the versal base space up to the second order. Even if not needed for the proof of versality in chapters 5 and 7, it was the formula in Theorem (4.4.4) which gave us the first idea what the actual equations of the versal base space could look like. Hence, the calculation of (4.4.5) might serve in the same way to guess versal equations for the cases yet to be solved.

On the other hand, if Q is a lattice polytope with primitive edges, then we have defined in (2.2.2) the vector space $V(Q)/\underline{1}$. It can be interpreted in the following two different ways:

(i) $V(Q)/\underline{1}$ equals T^1 of the affine, toric Gorenstein singularity $Y_{\mathrm{Cone}(Q)}$ (cf. Corollary (5.1.2)), and, on the other hand,

(ii) $V(Q)/\underline{1}$ equals $\mathrm{Pic}Y_{\Sigma(Q)}/\mathcal{O}(1)$, if $Y_{\Sigma(Q)}$ denotes the projective toric variety induced by Q in the dual sense (see (2.4.3)).

Comparing with the results of Batyrev ([Bat]), the spirit of mirror symmetry seems to be involved here. Hence, it might be interesting whether the relation between (i) and (ii) extends to T^2 and higher Chow groups, respectively. Moreover, if it does somehow, we could use our knowledge of the cup product to compare it with the intersection pairing of divisors.

(1.2.5) In chapter 5 we investigate toric Gorenstein singularities Y which are smooth in codimension two. They may be described via cones over lattice polytopes Q (with primitive edges) embedded into height one.

We start in (5.1) with drawing from chapter 3 all possible information about T^1 and T^2 applied to this particular case. The main observation is that T^1_Y, if finite-dimensional, is concentrated in the single degree $-R^*$ (with $[R^* = 1]$ being the affine hyperplane containing Q). Since we have still problems combining deformations arising in different degrees (e.g. constructed as in chapters 6 or 7), it is exactly that fact which makes it possible to construct the whole versal deformation for Y.

Sections (5.2) and (5.3) describe that construction. Once we have the knowledge that deformations of Y are related in some way to Minkowski summands (cf. (1.2.7)), the following idea arises quite naturally: Use the "moduli space" of Minkowski summands (the cone $C(Q)$ defined in (2.2.2)) and its "universal family" (the tautological cone $\tilde{C}(Q)$ defined in (5.2.2)) to build up the first stage toward the versal deformation - just by applying the functor of transfering cones into affine toric varieties. The actual versal deformation is not toric at all (neither the base, nor the total space), but it emerges from our construction with just a little push. It turns out that the base space is exactly the $\bar{\mathcal{M}}(Q)$ as constructed in the very beginning (cf. section (2.2) and Theorem (5.3.1)).

Unfortunely, being a natural construction is not sufficient for being versal, so we have to present a proof for this fact. The proof is divided into computing the Kodaira-Spencer-map as well as the obstruction map (cf. sections (5.4) and (5.5), respectively).

We conclude this chapter by giving an explicit description of the components of the versal base space and its total spaces (cf. section (5.6)) and with several examples presented in section (5.7). (The example of the hexagon Q_6 defined in section (2.2) - yielding the cone over the Del Pezzo surface of degree 6 - accompanies the whole construction anyway.)

(1.2.6) Still assuming that Y is smooth in codimension two, in some sense the method of chapter 5 goes through also for non-Gorenstein varieties: Restricting ourselves to some fixed degree $R \geq 0$ (i.e. $R \in \sigma^\vee \cap M$), we describe in chapter 6 the adjustments to be made for obtaining the "versal deformation in degree $-R$". If (as it was in the Gorenstein case) T^1 is concentrated in one single so-called negative degree, then this equals the actual versal deformation.
On the other hand, even if T^1 consists of negative degrees only, we are not (yet) able to combine pieces sitting in different degrees together. See (6.6) for a conjecture concerning this question.

(1.2.7) Finally, in chapter 7 we present the whole problem from a different point of view. Historically it was my first approach toward deforming toric varieties: Hoping (from Christophersen's results in [Ch 3]) that total spaces over all or at least some components of the versal base space are toric again, we have created the corresponding notion of so-called toric deformations (cf. (7.1.1)) and have tried to classify them. The result is Theorem (7.2.5) which says roughly that toric deformations arise from dividing certain polytopes into Minkowski summands and putting them into parallel planes. This has been the starting point of the rather direct approach described in the earlier chapters 5 and 6.

However, chapter 7 has been included not only for nostalgic reasons - it is not just a part of the results obtained in the chapters 5 and 6: The approach of chapter 7 includes the more difficult case of singularities that might be singular in codimension two (such as two-dimensional cyclic quotient singularities or cones over singular surfaces). As we have already seen in Theorem (3.4.3), the description of T^1 of those singularities does not only involve the Minkowski summands (encoded in $V(Q)$), but also some additional parameters (contained in $W(Q)$). Now, the calculations of chapter 7 (including the description of the Kodaira-Spencer map in section (7.4), cf. (7.4.4)) show how they fit into the game.

On the other hand, Proposition (7.1.3) and Theorem (7.2.5) show that our several constructions involving Minkowski summands of affine cross cuts of σ have no chance to cover more than the case of negative deformations (i.e. $R \in \sigma^\vee \cap M$). If T^1 involves pieces from non-negative degrees also (e.g. Y equals the Gorenstein cone over the weighted projective space $\mathbb{P}(1,2,3)$), then we need completely new methods for constructing the versal deformation or just some genuine deformations (i.e. over reduced parameter spaces) in those degrees. Studying those deformations for non-isolated three-dimensional toric Gorenstein singularities would be important for a better understanding of three-dimensional flips (cf. [Re 2]).

Chapter 2

Convex cones, polytopes, and toric varieties

2.1 Basic notions

(2.1.1) We are going to introduce briefly the notions we need from convex geometry. It should be considered a good opportunity to fix notations, on the one hand, and to bring readers from algebraic geometry in the mood for cones and polytopes, on the other hand. Standard references for the details are [Grü] or the new book [Zi].

Convex cones: Throughout the paper we use the word *cone* for rational, convex, polyhedral cones. If N, M are two mutually dual, free Abelian groups of finite rank, then a cone $\sigma \subseteq N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ can be given either by its fundamental generators

$$\sigma = \langle a^1, \dots, a^M \rangle := \sum_{j=1}^{M} \mathbb{R}_{\geq 0} \cdot a^j \qquad (a^1, \dots, a^M \in N)$$

or by finitely many inequalities

$$\sigma = \{ a \in N_{\mathbb{R}} \mid \langle a, r^i \rangle \geq 0 \,;\, i = 1, \dots, K \} \qquad (r^1, \dots, r^K \in M)\,.$$

The elements $a^j \in N$ and $r^i \in M$ can be normalized by asking for primitive ones, i.e. not being proper multiples. (I hope the reader will not be confused by abuse of notation: M, N might stand for free Abelian groups as well as for some natural numbers. Moreover, we use the symbol $\langle \dots \rangle$ for both the pairing between the mutually dual lattices N, M and for indicating the generators of cones; "$<$" also denotes the face relation.)
The concept of duality interchanges both representations: The cone dual to σ is defined as

$$\sigma^{\vee} := \{ r \in M_{\mathbb{R}} \mid \langle a, r \rangle \geq 0 \;\text{ for all } a \in \sigma \}\,.$$

It has $r^1, \dots, r^K \in M$ as fundamental generators, or it can be given by the inequalities provided by $a^1, \dots, a^M \in N$.

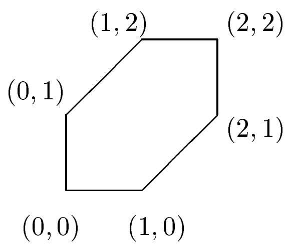
Polytopes and polyhedra: Let $(\boldsymbol{A}, \mathbb{L})$ be a finite-dimensional real vector (or affine) space with a lattice. (Rational) polyhedra in $(\boldsymbol{A}, \mathbb{L})$ are given as intersections of finitely many rational half spaces. If additionally compact, they will be called polytopes.

Definition: *For two polyhedra $Q', Q'' \subseteq \boldsymbol{A}$ we define their Minkowski sum as the polyhedron $Q' + Q'' := \{p' + p'' \mid p' \in Q', p'' \in Q''\}$. Obviously, this notion also makes sense for translation* *classes of polyhedra (which is important, if $\mathbb{A}$ is just an affine space).*

It is not difficult to see that the faces of $Q' + Q''$ equal the Minkowski sums of the corresponding faces (defined by the same hyperplane in $\mathbb{A}$) of Q' and Q''. In particular, up to dilatation, the set of edges of $Q' + Q''$ equals the union of the corresponding sets for Q' and Q'', respectively.

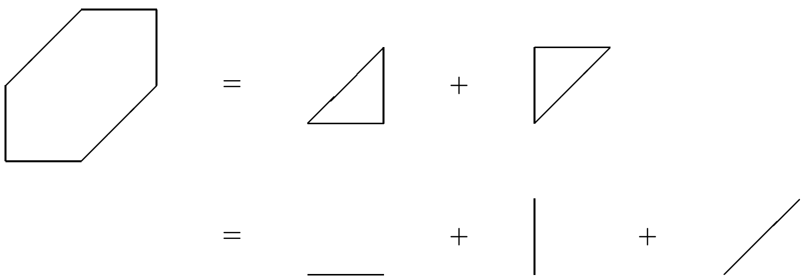
Example: Let us introduce the following example, which will be continued through the paper: For Q we take the plane hexagon

$$Q_6 := \operatorname{Conv}\{(0,0),(1,0),(2,1),(2,2),(1,2),(0,1)\} \subseteq \mathbb{R}^2.$$



hexagon Q_6

Our polytope Q_6 splits into



(2.1.2) Every polyhedron $Q \subseteq \mathbb{A}$ is decomposable into the Minkowski sum $Q = Q^c + Q^\infty$ of a (compact) polytope Q^c and the so-called cone of unbounded directions Q^∞. The latter one is uniquely determined by Q, the compact summand is not. However, we can take for Q^c the minimal one - given as the convex hull of the vertices of Q itself. (If Q was already compact, then $Q^c = Q$ and $Q^\infty = 0$.)

Cross cuts: Let us turn to the following special setting - it serves as an example at the moment, but it reflects a rather typical situation in the sequel.

Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone and fix some primitive element $R \in M$. Then $\mathbb{A} := [R = 1] := \{a \in N_{\mathbb{R}} \mid \langle a, R\rangle = 1\} \subseteq N_{\mathbb{R}}$ is an affine space with lattice $\mathbb{L} := [R = 1] \cap N$. We define the cross cut of σ in degree R as the polyhedron $Q := \sigma \cap [R = 1] \subseteq \mathbb{A}$. It has the cone of unbounded directions $Q^\infty = \sigma \cap [R = 0] \subseteq N_{\mathbb{R}}$.

On the other hand, we obtain the compact part Q^c of Q by describing its vertices. Obviously, they correspond exactly to those fundamental generators a^j of σ meeting $\langle a^j, R\rangle \geq 1$ - the actual vertices equal $\bar{a}^j = a^j / \langle a^j, R\rangle$.

Fundamental generators contained in $R^\perp$ can still be "seen" as edges in Q^∞, but those with $\langle \bullet, R\rangle < 0$ are "invisible" in Q. In particular, we can get the cone σ back from Q, if and only if $R \in \sigma^\vee$.

(2.1.3) One of the most frequently used notions will be that of *Minkowski summands* of a given polyhedron $Q \subseteq \boldsymbol{A}$. Of course, a Minkowski summand Q' of Q should be at least a summand in the usual sense (i.e. there has to be a Q'' such that $Q = Q' + Q''$). However, since $Q = Q' + Q^\infty$ for every $Q^c \subseteq Q' \subseteq Q$, this might not be enough; we would like to avoid additional face structure of Q' (not "coming" from Q). We take the following definition from [Sm]:

Definition: *A polyhedron Q' is called a Minkowski summand of Q, if there is a Q'' such that $Q = Q' + Q''$, and if $(Q')^\infty = Q^\infty$.*

That means, a Minkowski summand has always the same cone of unbounded directions and, up to dilatation, the same compact edges as the original polyhedron.

Remark: Sometimes in the literature, the notion "Minkowski summand" has a slightly different meaning. Speaking with our definition, it has been used to describe Minkowski summands not of Q, but of sufficiently high multiples of Q.
Those summands form a semigroup with cancellation property, i.e. it can be embedded into a group. Elements of that group can be represented as formal differences between two Minkowski summands - and we will call them *generalized Minkowski summands* (of multiples of Q).

(2.1.4) Finally, we would like to make some remarks about *fans*. They are a basic tool for describing non-affine toric varieties (gluing of cones translates directly into gluing affine pieces). Though we are dealing with the local case, i.e. with cones instead of fans, there is an interesting relation between fans and Minkowski summands we should mention. For further information and details we refer to arbitrary textbooks introducing toric varieties (as mentioned in section (2.4)).

Definition: *A (finite) fan Σ is a finite collection of cones in $N_{\mathbb{R}}$ such that*

(1) for every cone $\sigma \in \Sigma$ each of its faces belongs to Σ, and

(2) the intersection $\sigma \cap \tau$ of two cones $\sigma, \tau \in \Sigma$ is always a common face of both of them.

The set $|\Sigma| := \cup_{\sigma\in\Sigma}\sigma$ is called the support of Σ.

Let $Q \subseteq M_{\mathbb{R}}$ be a rational polyhedron. By $a \mapsto \min\langle a, Q\rangle$ we obtain a piecewise linear function defined on the cone $(Q^\infty)^\vee \subseteq N_{\mathbb{R}}$. Its maximal domains of linearity generate a fan $\Sigma(Q)$ supported on $(Q^\infty)^\vee$ (the *inner normal fan* of Q). If $\dim Q = \dim M_{\mathbb{R}}$, then each cone of $\Sigma(Q)$ admits an apex (in 0). Now, we have the following well known fact:

Proposition: *Let $Q, Q' \subseteq M_{\mathbb{R}}$ be rational polyhedra. Then, Q' is a Minkowski summand of some positive multiple of Q, if and only if $\Sigma(Q)$ is (a possibly trivial) subdivision of $\Sigma(Q')$ (i.e., if $|\Sigma(Q)| = |\Sigma(Q')|$, and every cone of $\Sigma(Q)$ is contained in some cone of $\Sigma(Q')$).*

Example: Looking at the hexagon Q_6 from (2.1.1), we see that $\Sigma(Q_6)$ is a plane fan built from six descent cones. The inner normal fans of the triangle summands of Q_6 can be obtained by deleting every other ray; the fans assigned to the three one-dimensional summands consist of two half spaces each.

2.2 The Minkowski scheme of a lattice polytope

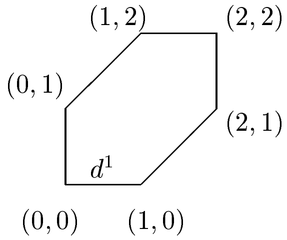
(2.2.1) Let $Q \subseteq \boldsymbol{A}$ be a lattice polytope, i.e. the vertices are contained in $\mathbb{L}$. We will assume that the edges do not contain any interior lattice points (cf. (2.3.6)), hence, after choosing orientations they are given by primitive vectors $d^1, \dots, d^N \in I\!L$.

Definition: *For every two-dimensional face ("2-face")* $\varepsilon < Q$ *we define its sign vector* $\underline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_N) \in \{0, \pm 1\}^N$ *by*

$$\varepsilon_i := \begin{cases} \pm 1 & \text{if } d^i \text{ is an edge of } \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

such that the oriented edges $\varepsilon_i \cdot d^i$ *fit to a cycle along the boundary of* ε*. This determines* $\underline{\varepsilon}$ *up to sign, and we choose one of both possibilities. In particular,* $\sum_i \varepsilon_i d^i = 0$*.*

Example: We consider our hexagon Q_6.



hexagon Q_6

Starting with $d^1 := \overline{(0,0)(1,0)}$, the anticlockwise oriented edges are denoted by $d^1, \dots, d^6$. As vectors they equal

$$\begin{array}{lll} d^1 = (1,0); & d^2 = (1,1); & d^3 = (0,1); \\ d^4 = (-1,0); & d^5 = (-1,-1); & d^6 = (0,-1). \end{array}$$

Q_6 is 2-dimensional, hence, it is its own unique 2-face $\varepsilon = Q$. For $\underline{Q}$ we take $\underline{Q} = (1, \dots, 1)$.

(2.2.2) We define the vector space $V(Q) \subseteq I\!R^N$ by

$$V := V(Q) := \{(t_1, \dots, t_N) \mid \sum_i t_i \, \varepsilon_i \, d^i = 0 \text{ for every 2-face } \varepsilon < Q\}.$$

Then, $C(Q) := V \cap I\!R^N_{\geq 0}$ is a rational, polyhedral cone in V, and its points correspond to the Minkowski summands of positive multiples of Q: Given a point $(t_1, \dots, t_N) \in C(Q)$, the corresponding polytope $Q_{\underline{t}}$ is built by using the edges $t_i \cdot d^i$ instead of the plain d^i contained in Q (cf. (5.2.1)). For a particular Minkowski summand Q' of a positive multiple of Q we will denote the corresponding point in the cone by $\varrho(Q') \in C(Q)$. (By the way, the vector space $V(Q)$ consists exactly of the generalized Minkowski summands mentioned in (2.1.3).)

Example: 1) $\varrho(t \cdot Q) = (t, \dots, t) \in C(Q) \subseteq V \subseteq I\!R^N$.

2) We look at the two different possibilities shown in (2.1.1) to decompose Q_6 into a Minkowski sum. Applying ϱ, these two equations become

$$\begin{aligned} (1,1,1,1,1,1) &= (1,0,1,0,1,0) + (0,1,0,1,0,1) \\ &= (1,0,0,1,0,0) + (0,0,1,0,0,1) + (0,1,0,0,1,0) \,. \end{aligned}$$

(2.2.3) For each 2-face $\varepsilon < Q$ and for each integer $k \geq 1$ we define the (vector valued) polynomial

$$g_{\varepsilon,k}(\underline{t}) := \sum_{i=1}^{N} t_i^k \, \varepsilon_i \, d^i \,.$$

Using coordinates of $\boldsymbol{A}$, the $g_{\varepsilon,k}(\underline{t})$ turn into regular polynomials; for each pair (ε, k) we will get two linearly independent ones. We obtain an ideal

$$\mathcal{J} := (g_{\varepsilon,k} \mid \varepsilon < Q \text{ is a 2-face, } k \geq 1) \subseteq \mathbb{C}[t_1, \ldots, t_N],$$

which defines an affine closed subscheme

$$\mathcal{M} := \operatorname{Spec} \mathbb{C}[\underline{t}] / \mathcal{J} \subseteq V_{\mathbb{C}} \subseteq \mathbb{C}^N.$$

Example: For our hexagon Q_6 mentioned in (2.2.1) we obtain

$$\mathcal{J} = \left(t_1^k + t_2^k - t_4^k - t_5^k,\ t_2^k + t_3^k - t_5^k - t_6^k \mid k \geq 1\right).$$

Of course, just finitely many equations are sufficient to generate the ideal $\mathcal{J}$ - but we can even give an effective criterion to see which equations can be dropped:

Proposition: *Let $\varepsilon < Q$ be a 2-face. Then, ε is contained in a two-dimensional subspace of $\boldsymbol{A}$, and this vector space comes with a natural lattice (the restriction of the big lattice $\mathbb{L}$).*
If ε is contained in two different strips defined by pairs of parallel lines of lattice-distance $\leq k_0$ each, then the equations $g_{\varepsilon,k}$ $(k > k_0)$ are contained in the ideal generated by $g_{\varepsilon,1}, \ldots, g_{\varepsilon,k_0}$.

Proof: cf. (2.3.3).

Corollary: *If Q is contained in n linearly independent strips (defined by pairs of parallel hyperplanes) of lattice-thickness $\leq k_0$, then all polynomials $g_{\varepsilon,k}$ with $k > k_0$ are superfluous.*

Example: Obviously, Q_6 is contained in at least three strips of thickness 2. Hence, $\mathcal{J}$ is generated in degree ≤ 2:

$$\mathcal{J} = \left(t_1 + t_2 - t_4 - t_5, \quad t_2 + t_3 - t_5 - t_6, \quad t_1^2 + t_2^2 - t_4^2 - t_5^2, \quad t_2^2 + t_3^2 - t_5^2 - t_6^2\right).$$

(2.2.4) Denote by ℓ the canonical projection

$$\ell : \mathbb{C}^N \twoheadrightarrow \mathbb{C}^N / \mathbb{C} \cdot (1, \ldots, 1) = \mathbb{C}^N / \mathbb{C} \cdot \varrho(Q).$$

On the level of regular functions this corresponds to the inclusion
$\mathbb{C}[t_i - t_j \mid 1 \leq i, j \leq N] \subseteq \mathbb{C}[t_1, \ldots, t_N]$.

Theorem: *(compare with Remark (5.2.5))*

(1) $\mathcal{J}$ is generated by polynomials from $\mathbb{C}[t_i - t_j]$, i.e. $\mathcal{M} = \ell^{-1}(\bar{\mathcal{M}})$ for some affine closed subscheme $\bar{\mathcal{M}} \subseteq V_{\mathbb{C}} / \mathbb{C} \cdot \varrho(Q) \subseteq \mathbb{C}^N / \mathbb{C} \cdot \varrho(Q)$.
($\bar{\mathcal{M}}$ is defined by the ideal $\mathcal{J} \cap \mathbb{C}[t_i - t_j]$.)

(2) $\mathcal{J} \subseteq \mathbb{C}[t_1, \ldots, t_N]$ is the smallest ideal that meets property (1) and, on the other hand, contains the "toric" equations

$$\prod_{i=1}^N t_i^{d_i^+} - \prod_{i=1}^N t_i^{d_i^-} \qquad \textit{with}$$

$\underline{d} \in \mathbb{Z}^N \cap span\{[\langle\varepsilon_1 d^1, c\rangle, \dots, \langle\varepsilon_N d^N, c\rangle] \mid \varepsilon < Q$ *2-face*, $c \in \boldsymbol{A}^*\}$.
(For an integer h we denote

$$h^+ := \begin{cases} h & \text{if } h \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad ; \quad h^- := \begin{cases} 0 & \text{if } h \geq 0 \\ -h & \text{otherwise} \end{cases} .)$$

Proof: cf. (2.3.4).

Example: Toric equations for Q_6 are for instance $t_1\, t_2 - t_4\, t_5$, $t_2\, t_3 - t_5\, t_6$, and $t_1\, t_6 - t_3\, t_4$.

(2.2.5) We want to describe the structure of the underlying reduced spaces of $\mathcal{M}$ or $\bar{\mathcal{M}}$.

First, we mention the following trivial observations concerning the cone $C(Q)$:

(i) Minkowski summands Q' of Q itself (instead of a positive multiple of Q) are characterized by the property $\varrho(Q) - \varrho(Q') \in \mathbb{R}^N_{\geq 0}$, i.e. all components of $\varrho(Q')$ have to be contained in the interval $[0, 1]$.

(ii) For a Minkowski summand Q' (of some positive multiple of Q) the property of being a lattice polytope is equivalent to the fact that $\varrho(Q') \in \mathbb{Z}^N$.

Now, let $Q = R_0 + \ldots + R_n$ be a decomposition of Q into a Minkowski sum of $n+1$ lattice polytopes. Then, the N-tuples $\varrho(R_0), \dots, \varrho(R_n)$ consist of numbers 0 and 1 only, and they sum up to $(1, \dots, 1)$. In particular, the $(n+1)$-plane $\mathbb{C} \cdot \varrho(R_0) + \ldots + \mathbb{C} \cdot \varrho(R_n) \subseteq \mathbb{C}^N$ (or its n-dimensional image via ℓ) is contained in $\mathcal{M}$ (in $\bar{\mathcal{M}}$, respectively).

Remark:

(1) That $(n+1)$-plane (or its image via ℓ) is given by the linear equations $t_i - t_j = 0$ (if d^i, d^j belong to a common summand R_k).

(2) Refinements of Minkowski decompositions (they form a partial ordered set) correspond to inclusions of the associated planes.

Theorem: *$\mathcal{M}_{\text{red}}$ and $\bar{\mathcal{M}}_{\text{red}}$ equal the union of those flats corresponding to maximal Minkowski decompositions of Q into lattice summands.*

Proof: cf. (2.3.5).

Example: $\mathcal{M}(Q_6)$ and $\bar{\mathcal{M}}(Q_6)$ are already reduced schemes (for non-reduced examples see section 5.7). Let us study them directly:

- The linear equations allow the following substitution:

$$\begin{array}{rcl} t &:=& t_1 \\ s_1 &:=& t_1 - t_3 \\ s_2 &:=& t_4 - t_2 \\ s_3 &:=& t_1 - t_4 \end{array} \qquad \begin{array}{rcl} t_1 &=& t \\ t_2 &=& t - s_2 - s_3 \\ t_3 &=& t - s_1 \\ t_4 &=& t - s_3 \\ t_5 &=& t - s_2 \\ t_6 &=& t - s_1 - s_3\,. \end{array}$$

- The two quadratic equations transform into $s_1\, s_3 = s_2\, s_3 = 0$.

In particular, $\bar{\mathcal{M}}$ is the union of a line and a 2-plane - corresponding to the Minkowski decompositions

$$\begin{array}{rcl} Q_6 & = & \text{Conv}\,\{(0,0),(1,0),(1,1)\} + \text{Conv}\,\{(0,0),(0,1),(1,1)\} \text{ and} \\ Q_6 & = & \text{Conv}\,\{(0,0),(1,0)\} + \text{Conv}\,\{(0,0),(0,1)\} + \text{Conv}\,\{(0,0),(1,1)\}. \end{array}$$

mentioned in (2.2.2).

(2.2.6) $\bar{\mathcal{M}}$ (or $\mathcal{M} = \ell^{-1}(\bar{\mathcal{M}})$) reflect the possibilities of constructing Minkowski decompositions of Q:

- The underlying reduced space encodes the decompositions of Q into lattice summands.
- Extremal decompositions into rational summands are hidden in the scheme structure of $\bar{\mathcal{M}}$. Its tangent space in 0 (the smallest affine space containing $\bar{\mathcal{M}}$) equals ${}^{V_{\mathbb{C}}}\!/_{\mathbb{C}\cdot\varrho(Q)}$ - it is the vector space arising from the cone $C(Q)$ of Minkowski summands by killing the summands homothetic to Q.

Therefore, we will call $\bar{\mathcal{M}}$ the (affine) *Minkowski scheme* of Q.

Remark: The ideals defining $\mathcal{M}$ and $\bar{\mathcal{M}}$ are homogeneous. Hence, there are projective versions of these schemes, too.

2.3 Proof of the previous statements

(2.3.1) Using vectors $c \in \mathbb{L}^*$ (or selected $c \in \boldsymbol{A}^*$) we can evaluate the edges $d^1, \dots, d^N$ to get integers

$$d_1 := \langle \varepsilon_1 d^1, c\rangle, \dots, d_N := \langle \varepsilon_N d^N, c\rangle$$

for every given 2-face $\varepsilon < Q$. Doing so, the statements of section (2.2) can be reduced to much simpler lemmas, which we will present here.

Then, all those lemmas are proved using the following recipe:

(i) Assume $d_i = \pm 1$ - then the lemmas reduce to well known facts concerning symmetric functions.

(ii) Move to the general case by specialization of variables.

(2.3.2) For the whole section (2.3) we use the following notations:

Let $d_1, \dots, d_N \in \mathbb{Z}$ such that $d_1, \dots, d_M \geq 0$, $d_{M+1}, \dots, d_N \leq 0$, and $\sum_{i=1}^N d_i = 0$.

$$\begin{array}{rcccl} g_k(\underline{t}) & := & g_{\underline{d},k}(\underline{t}) & := & \sum_{i=1}^N d_i\, t_i^k\,, \\ p(\underline{t}) & := & p_{\underline{d}}(\underline{t}) & := & t_1^{d_1}\cdot \ldots \cdot t_M^{d_M} - t_{M+1}^{d_{M+1}} \cdot \ldots \cdot t_N^{d_N}\,. \end{array}$$

Denote by σ_k and s_k the k-th elementary symmetric polynomial and the sum of the k-th powers of a given set of variables, respectively.

Remark: For $1 \leq i,j \leq M$ or $M+1 \leq i,j \leq N$, identifying the two variables t_i and t_j (i.e. switching from $\mathbb{C}[\underline{t}]$ to ${}^{\mathbb{C}[\underline{t}]}\!/_{t_i - t_j}$) yields the following situation:

- t_i, t_j are replaced by a common new variable $\tilde{t}$ (i.e. N is replaced by $N-1$),

- d_i, d_j are replaced by $\tilde{d} := d_i + d_j$, but
- $g_k(\underline{t}), p(\underline{t})$ keep their shapes in the new set up.

In particular, the general situation can always be obtained via factorization from the special case $d_1 = \ldots = d_M = 1$; $d_{M+1} = \ldots = d_N = -1$ (and $N = 2M$). Renaming $t_i = x_i$, $t_{M+i} = y_i$ ($i \leq M$) it looks like

$$\begin{aligned} g_k(\underline{x}, \underline{y}) &= \left(\sum_{i=1}^{M} x_i^k\right) - \left(\sum_{i=1}^{M} y_i^k\right) = s_k(\underline{x}) - s_k(\underline{y}), \\ p(\underline{x}, \underline{y}) &= (x_1 \cdot \ldots \cdot x_M) - (y_1 \cdot \ldots \cdot y_M) = \sigma_M(\underline{x}) - \sigma_M(\underline{y}). \end{aligned}$$

(2.3.3) Lemma: *If $k_0 := \sum_{i=1}^{M} d_i = -\sum_{i=M+1}^{N} d_i$, then the polynomials g_k ($k > k_0$) are $\mathbb{C}[\underline{t}]$-linear combinations of the $g_1, \ldots, g_{k_0}$.* (This implies Proposition (2.2.3).)

Proof: As previously discussed, we may regard the special case $d_i = \pm 1$. In particular, this implies $k_0 = M$.
Now, for an arbitrary k ($> M$), the expression $s_k(\underline{x})$ is a polynomial in either the $\sigma_1(\underline{x}), \ldots, \sigma_M(\underline{x})$ or the $s_1(\underline{x}), \ldots, s_M(\underline{x})$, say

$$s_k(\underline{x}) = P_k\left(s_1(\underline{x}), \ldots, s_M(\underline{x})\right).$$

Then,

$$g_k(\underline{x}, \underline{y}) = s_k(\underline{x}) - s_k(\underline{y}) = P_k\left(s_1(\underline{x}), \ldots, s_M(\underline{x})\right) - P_k\left(s_1(\underline{y}), \ldots, s_M(\underline{y})\right),$$

but for each monomial $s_1^{e_1}\, s_2^{e_2} \ldots s_M^{e_M}$ occurring in P_k, we have

$$\begin{aligned} & s_1(\underline{x})^{e_1} \cdot \ldots \cdot s_M(\underline{x})^{e_M} - s_1(\underline{y})^{e_1} \cdot \ldots \cdot s_M(\underline{y})^{e_M} = \\ &\quad = \textstyle\sum_{v=1}^{M} \sum_{i=1}^{e_v} [s_v(\underline{x}) - s_v(\underline{y})] \cdot s_1(\underline{x})^{e_1} \ldots s_{v-1}(\underline{x})^{e_{v-1}}\, s_v(\underline{x})^{i-1} \cdot \\ &\qquad\qquad \cdot s_v(\underline{y})^{e_v - i}\, s_{v+1}(\underline{y})^{e_{v+1}} \ldots s_M(\underline{y})^{e_M} \\ &\quad = \textstyle\sum_{v=1}^{M} g_v(\underline{x}, \underline{y}) \cdot \sum_{i=1}^{e_v} s_1(\underline{x})^{e_1} \ldots s_{v-1}(\underline{x})^{e_{v-1}}\, s_v(\underline{x})^{i-1} \cdot \\ &\qquad\qquad \cdot s_v(\underline{y})^{e_v - i}\, s_{v+1}(\underline{y})^{e_{v+1}} \ldots s_M(\underline{y})^{e_M}, \end{aligned}$$

which proves the lemma. □

(2.3.4) Lemma:

1. *The ideal $\mathcal{J} := (g_k \mid k \geq 1) \subseteq \mathbb{C}[t_1, \ldots, t_N]$ is generated by polynomials in $t_i - t_1$ ($i = 2, \ldots, N$) only.*
2. *$\mathcal{J}$ is the smallest ideal generated by polynomials in $t_i - t_1$, which additionally contains p.*

(This implies Theorem (2.2.4).)

Proof: (1) Replacing t_i by $t_i - t_1$ as arguments in g_k yields

$$\begin{aligned} g_k(t_1 - t_1, \ldots, t_N - t_1) &= \sum_{i=1}^{N} d_i\, (t_i - t_1)^k = \sum_{i=1}^{N} d_i \cdot \left(\sum_{v=0}^{k} (-1)^v\, t_1^v\, t_i^{k-v}\right) \\ &= \sum_{v=0}^{k} (-1)^v\, t_1^v \cdot \left(\sum_{i=1}^{N} d_i\, t_i^{k-v}\right) = \sum_{v=0}^{k} (-1)^v\, t_1^v\, g_{k-v}(\underline{t}). \end{aligned}$$

In particular, $(g_k(\underline{t}) \mid k \geq 1)$ and $(g_k(\underline{t} - t_1) \mid k \geq 1)$ are the same ideals in $\mathbb{C}[\underline{t}]$.

(2) The polynomial rings $\mathbb{C}[\underline{t}]$ and $\mathbb{C}[t_1, \underline{t} - t_1]$ are equal, i.e. each polynomial $q(\underline{t})$ can uniquely be written as

$$q(\underline{t}) = \sum_{v \geq 0} q_v(t_2 - t_1, \ldots, t_N - t_1) \cdot t_1^v.$$

Moreover, if $J \subseteq \mathbb{C}[\underline{t}]$ is an ideal generated by polynomials in $\underline{t} - t_1$ only, then for each $q(\underline{t}) \in J$ the components q_v are automatically contained in J, too.

Let us determine the components of the polynomial p - we will start with our special case again:

$$p(T + \underline{X}, T + \underline{Y}) = (T + X_1) \cdot \ldots \cdot (T + X_M) - (T + Y_1) \cdot \ldots \cdot (T + Y_M)$$

has $\sigma_k(\underline{X}) - \sigma_k(\underline{Y})$ as coefficient of T^{M-k} $(k = 1, \ldots, M)$. Now, there are a polynomial P_k and a non-vanishing rational number c_k (not depending on M) such that

$$\sigma_k(\underline{X}) = P_k(s_1(\underline{X}), \ldots, s_{k-1}(\underline{X})) + c_k \cdot s_k(\underline{X}).$$

As in the proof of the previous lemma we obtain

$$\begin{aligned}\sigma_k(\underline{X}) - \sigma_k(\underline{Y}) &= P_k(s_1(\underline{X}), \ldots, s_{k-1}(\underline{X})) - P_k(s_1(\underline{Y}), \ldots, s_{k-1}(\underline{Y})) + \\ &\qquad + c_k \cdot s_k(\underline{X}) - c_k \cdot s_k(\underline{Y}) \\ &= \sum_{v=1}^{k-1} q_v(\underline{X}, \underline{Y}) \cdot g_v(\underline{X}, \underline{Y}) + c_k \cdot g_k(\underline{X}, \underline{Y})\end{aligned}$$

for some coefficients q_v. Specialization - first by $T \mapsto x_1$, $X_i \mapsto x_i - x_1$, $Y_i \mapsto y_i - x_1$, then followed by the usual one - shows that the ideal generated by the components $p_v(\underline{t} - t_1)$ of p equals $\mathcal{J}$. □

(2.3.5) **Lemma:** *Let $\underline{\xi} = (\xi_1, \ldots, \xi_N) \in \mathbb{C}^N$ be a point such that $g_k(\underline{\xi}) = 0$ for each $k \geq 1$. Then, for every fixed $\xi \in \mathbb{C}$, we have $\sum_{\xi_i = \xi} d_i = 0$.* (This implies Theorem (2.2.5).)

Proof: The equations $\sum_{i=1}^N d_i \, \xi_i^k = 0$ present 0 as a linear combination of the vectors $(\xi_i, \xi_i^2, \xi_i^3, \ldots)$. On the other hand, it is the Vandermonde telling us that this linear combination has to be a trivial one, i.e. the sum of those coefficients d_i belonging to the same $(\xi, \xi^2, \xi^3, \ldots)$ vanishes. □

(2.3.6) The polytope Q was assumed to have primitive edges only. Actually, we never needed this fact neither in the previous lemmata nor in their proofs. It is only important to translate these results into the language of Minkowski summands used in section (2.2).

Dropping this condition, similar constructions are possible. However, by declaring some or just all lattice points contained in edges of Q to be additional, artificial vertices of Q, several possibilities arise with equal rights. The two extremal cases (add either no or all possible generalized vertices) seem to be the most interesting ones.

Remark:

(1) For a natural number $g \in \mathbb{N}$, the polytopes Q (with some fixed set of possibly artificial vertices) and $g \cdot Q$ (with the corresponding set of vertices) induce the same Minkowski scheme $\bar{\mathcal{M}}$.

(2) Let $Q_1 \subseteq Q_2$ be the same polytopes with different sets of generalized vertices. Then, $\bar{\mathcal{M}}_1$ is a closed subscheme of $\bar{\mathcal{M}}_2$. It is defined by identifying the variables associated to those generalized edges of Q_2 that are contained in the same generalized edge of Q_1.

Conjecture: Let Q be a lattice polytope such that each extremal Minkowski summand of Q is a lattice polytope, too. Then, using all generalized vertices of Q, the affine schemes $\mathcal{M}$ and $\bar{\mathcal{M}}$ are reduced.

In particular, if Q is an arbitrary lattice polytope (with primitive edges), then $\bar{\mathcal{M}}_Q$ could be embedded into some reduced $\bar{\mathcal{M}}_{g\cdot Q}$. The non-reduced structure of $\bar{\mathcal{M}}_Q$ would arise as a germ of components visible in $\bar{\mathcal{M}}_{g\cdot Q}$ only.

2.4 Toric Varieties

(2.4.1) Most books concerned with toric varieties define them as normal varieties Y containing some torus $(\mathbb{C}^*)^{\dim Y}$ as a dense, open subset such that its group structure may be extended to an action on the whole Y. Then, starting with the definitions $M :=$ character group, and $N :=$ set of 1-parameter subgroups of the torus, those varieties are completely described in terms of fans in $N_{\mathbb{R}}$. Standard references for an introduction into that subject are the survey article [Da] of Danilov, the first chapter of [Ke], or the books [Od] and [Fu 2].

Although very special, toric varieties are useful as examples or first test objects for general theories - many theorems in algebraic geometry have their combinatorial counterpart in the world of cones and polytopes. (For instance, in [Re 1] the main ideas of Mori theory can be seen very clearly.)
Since we are mainly interested in the deformation theory of (singular) germs of varieties (which is well understood for complete intersections), toric singularities become interesting for us by the following reason: The combinatorial description makes it possible to handle a huge amount (compared with the codimension) of equations in a rather efficient way.

We would like to give a brief sketch of the subject of toric varieties here. We start with the affine ones, then we will mention more global (and in particular projective) toric varieties. For that purpose, our point of view will be the following: Consider toric varieties just a special class of algebraic varieties defined by many equations, but encoded in some nice and easy combinatorics. For details and proofs we refer to the books mentioned above.

(2.4.2) *Affine toric varieties.*
Let M, N be two mutually dual, free Abelian groups of finite rank (as in (2.1.1)).

Definition: *If $\sigma \subseteq N_{\mathbb{R}}$ is a cone with apex, then we define by $Y_\sigma := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ its associated, affine toric variety. ($\mathbb{C}[\sigma^\vee \cap M]$ denotes the semigroup ring - obtained by regarding elements $r \in \sigma^\vee \cap M$ as exponents of some "abstract symbol" x.)*
Let $\sigma_1 \subseteq N^1_{\mathbb{R}}$, $\sigma_2 \subseteq N^2_{\mathbb{R}}$ be two cones. Then, a $\mathbb{Z}$-linear maps $f : N^1 \to N^2$ such that $f(\sigma_1) \subseteq \sigma_2$ induces an algebraic morphism $f : Y_1 \to Y_2$ in an obvious way. Those maps will be regarded the morphisms in the category of affine, toric varieties.

The semigroup $\sigma^\vee \cap M$ is generated by the finite set E of its irreducible elements. (E is often called the Hilbert basis of that semigroup.) Assigning to each element $r \in E$ a variable z_r, our affine toric variety Y_σ can be embedded into $\mathbb{C}^E$. It is defined by the binomial equations obtained by "raising" linear dependencies between the r's into the exponents of the z_r's. To give an example, the relation $r + 2s = 3t + u$ turns into $z_r\, z_s^2 = z_t^3\, z_u$.

Examples:

(1) The cone $\sigma := \mathbb{R}^k_{\geq 0} \subseteq \mathbb{R}^k$ (i.e. $N := \mathbb{Z}^k$) yields $\sigma^\vee \cap M = \mathbb{N}^k$, hence $Y_\sigma = \mathbb{C}^k$.

(2) Let $E \subseteq \sigma^\vee \cap M$ be the Hilbert basis for an arbitrary cone $\sigma \subseteq N_{I\!R}$. Then, assigning each $a \in N$ the E-tuple $(\langle a, r\rangle)_{r\in E} \in Z\!\!Z^E$, defines a $Z\!\!Z$-linear map $N \to Z\!\!Z^E$ putting σ into $I\!R^E_{\geq 0}$. At the level of toric varieties, this yields exactly the embedding $Y_\sigma \hookrightarrow I\!\!C^E$ described above.

(3) Let $n, q \in I\!N$ be relatively prime numbers. With $(\alpha, \beta) \mapsto (-q\,\alpha + \beta, n\,\alpha)$, we obtain a $Z\!\!Z$-linear map $f : Z\!\!Z^2 \to Z\!\!Z^2$ sending $I\!R^2_{\geq 0}$ onto $\sigma := \langle (1,0), (-q,n)\rangle \subseteq I\!R^2$. (By the way, any two-dimensional cone can be written that way.) At the toric level this means we have a morphism $\pi : I\!\!C^2 \to Y_\sigma$.
The dual cone equals $\sigma^\vee = \langle [0,1], [n,q]\rangle$, and we obtain for $f^*(\sigma^\vee \cap Z\!\!Z^2)$ the semigroup

$$f^*(\sigma^\vee \cap Z\!\!Z^2) = I\!N^2 \cap \{(r_1, r_2) \in Z\!\!Z^2 \mid r_1 + q\,r_2 \equiv 0 \bmod n\}\,.$$

Hence, the affine coordinate ring $I\!\!C[\sigma^\vee \cap Z\!\!Z^2]$ of Y_σ equals the subring of $I\!\!C[z_1, z_2]$ consisting of polynomials invariant under $\begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix}$ (with ξ being a primitive n^{th} root of unity). In particular, Y_σ is a cyclic quotient singularity, and π is the quotient map. For $n = 4$, $q = 1$ we obtain the example mentioned in (1.1.3).

We have seen that almost all two-dimensional cones yield singular toric varieties. This reflects the general situation - smooth, affine toric varieties are boring: If σ is a top-dimensional cone, then *Y_σ is smooth, if and only if σ is a simplex generated by a $Z\!\!Z$-basis of N* (i.e. the determinant of its fundamental generators has to be ± 1). If this is the case, then Y_σ is isomorphic to the affine space.

Remark: If $\sigma_1 \subseteq N^1_{I\!R}$, $\sigma_2 \subseteq N^2_{I\!R}$, then $\sigma_1 \times \sigma_2$ is a cone in $N^1_{I\!R} \oplus N^2_{I\!R}$. Moreover, we have $Y_{\sigma_1 \times \sigma_2} = Y_{\sigma_1} \times Y_{\sigma_2}$. In particular, if $\tau \subseteq N_{I\!R}$ is *not* top-dimensional, then $\tau = \tau \times \{0\} \subseteq \operatorname{span}_{I\!R}(\tau) \times$ complement. That means $Y_\tau = Y_{\tau \subseteq \operatorname{span} \tau} \times (I\!\!C^*)^{\operatorname{codim} \tau}$, and we see that the previous assumption about $\dim \sigma$ was not essential.

Finally, we would like to remark that special cones yield special toric varieties: If $Q \subseteq (\boldsymbol{A}, I\!L)$ is a lattice polytope, then we can define the cone $\sigma := I\!R_{\geq 0} \cdot (Q, 1)$ in $N_{I\!R} := \boldsymbol{A} \oplus I\!R$ (with $N := I\!L \oplus Z\!\!Z$). It is exactly the toric varieties defined by those cones which are Gorenstein (cf. (5.1.1)). We will treat those cones and varieties in chapter 5 - their deformation theory can be described in terms of Minkowski summands of Q.

(2.4.3) As already mentioned, morphisms between affine toric varieties arise from $Z\!\!Z$-linear maps $f : N^1 \to N^2$ such that $f(\sigma_1) \subseteq \sigma_2$. A very important special case is that of $f : N \to N$ being the identical map and σ_1 being a face of σ_2. If $r \in \sigma_2^\vee \cap M$ is actually cutting out that face (i.e. $\sigma_1 = \sigma_2 \cap r^\perp$), then $I\!\!C[\sigma_2^\vee \cap M]$ equals the localization of $I\!\!C[\sigma_1^\vee \cap M]$ by the element x^r. In particular, the induced map $Y_{\sigma_1} \to Y_{\sigma_2}$ is an open embedding (identifying the first variety with the open subset $[x^r \neq 0] \subseteq Y_{\sigma_1}$). Moreover, every open embedding in our category arises that way.

Definition: *If Σ is a fan in $N_{I\!R}$, then the toric variety Y_Σ is obtained by gluing together the affine pieces Y_σ ($\sigma \in \Sigma$) along common faces of Σ-cones.*
A map between toric varieties $Y_{\Sigma^1}, Y_{\Sigma^2}$ is given by a $Z\!\!Z$-linear $f : N^1 \to N^2$ such that for each $\sigma_1 \in \Sigma^1$ there is some $\sigma_2 \in \Sigma^2$ meeting $f(\sigma_1) \subseteq \sigma_2$.

Proposition: *The morphism $Y_{\Sigma^1} \to Y_{\Sigma^2}$ induced from $f : N^1 \to N^2$ is proper, if and only if $|\Sigma^1| = f^{-1}(|\Sigma^2|)$. In particular, a toric variety Y_Σ is complete, if and only if $|\Sigma| = N_{I\!R}$.*
Projectivity is equivalent to Σ being the inner normal fan of some polytope in $M_{I\!R}$ (cf. the upcoming (2.4.4)).

This criterion makes it easy to construct three-dimensional toric varieties that are complete, but not projective.

On the other hand, proper birational morphisms onto a toric variety Y_Σ correspond to polyhedral subdivisions of Σ. Hence, by Proposition (2.1.4), looking for projective contractions of a given projective toric variety (assigned to some polytope P) means looking for Minkowski summands (of P).

Examples:

(1) The k-dimensional fan spanned by the canonical basis vectors $e^1, \ldots, e^k$ of $\mathbb{Z}^k$ and $-e$ (with $e := e^1 + \ldots + e^k$) defines the projective space $\mathbb{P}^k$.

(2) The subdivision of the smooth cone $\sigma = \mathbb{R}^k_{\geq 0} = \langle e^1, \ldots, e^k \rangle$ into the union of $\sigma_i := \langle e^1, \ldots, \hat{e}^i, \ldots, e^k, e \rangle$ $(i = 1, \ldots, k)$ describes the blowing up of the origin in the affine k-space.

(3) If $\sigma = \langle a^1, \ldots, a^M \rangle$ is an arbitrary cone (with apex), then the normalized blowing up of Y_σ in the closed orbit is given by the subdivision of σ into the union of $\sigma_r := \{a \in \sigma \mid \langle a, r \rangle \leq \langle a, E \rangle\}$ (and r is runing through the Hilbert basis E of $\sigma^\vee \cap M$).

(4) Let σ be the three-dimensional cone over the unit square (contained in height one). Y_σ equals the cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$, i.e. the three-dimensional node. Now, the two possibilities of drawing diagonals into that given square correspond to the two small resolutions of Y_σ.

In general, every fan can be subdivided into a "smooth" fan. That means, every toric variety admits a toric desingularization.

Invertible sheaves on Y_Σ: Trivializing it on the torus, an invertible sheaf $\mathcal{L}$ is given by monomial generators $x^{r(\sigma)}$ on Y_σ for each cone $\sigma \in \Sigma$. In particular, $\Gamma(Y_\sigma, \mathcal{L}) = \oplus_{r \in r(\sigma) + (\sigma^\vee \cap M)} \mathbb{C} \cdot x^r$.

We define the polyhedron $\Delta(\mathcal{L}) := \cap_{\sigma \in \Sigma} \big(r(\sigma) + \sigma^\vee \big) \subseteq M_{\mathbb{R}}$. If Σ is complete, then $\Delta(\mathcal{L})$ is bounded (i.e. a polytope). By definition, the global sections of $\mathcal{L}$ form the vector space $\Gamma(Y_\Sigma, \mathcal{L}) = \oplus_{r \in M \cap \Delta(\mathcal{L})} \mathbb{C} \cdot x^r$ which will be finite-dimensional, if $\Delta(\mathcal{L})$ is a polytope. Moreover, $\mathcal{L}$ is generated by global sections, if and only if $r(\sigma) \in \Delta(\mathcal{L})$ for every cone $\sigma \in \Sigma$. If this is the case, then the $r(\sigma)$ form the vertices of that polyhedron, and Σ is a subdivision of the inner normal fan $\Sigma(\Delta(\mathcal{L}))$.

In particular, if $\Sigma = \Sigma(P)$ for some polytope P (i.e. if Y_Σ is projective), then globally generated invertible sheaves correspond to lattice polytopes that are Minkowski summands of some multiple of P. Hence, $\operatorname{Pic} Y_\Sigma = V(P)$, and $C(P)$ coincides with the ample cone (cf. (2.2.2) for the definition of V and C).

(2.4.4) Finally, we would like to make some additional remarks about projective toric varieties. Let $P \subseteq (\mathbb{A}, \mathbb{L})$ be a lattice polytope. Then, we can associate to P the polarized projective variety $(X_P, \mathcal{O}_X(P))$ in three different ways:

(1) Denote by $\Sigma := \Sigma(P)$ the inner normal fan (contained in $\mathbb{A}^*$) of the polytope P. We define $X_P := X_\Sigma$, and $\mathcal{O}_X(P)$ is just the ample invertible sheaf associated to P itself.
In particular, $h^0(X_P, \mathcal{O}_X(P)) = \#(P \cap \mathbb{L})$, and the corresponding linear system defines a map $X_P \to \mathbb{P}^{\#(P \cap \mathbb{L})-1}$.

(2) The homogeneous coordinates of $\mathbb{P}^{\#(P \cap \mathbb{L})-1}$ are in a one-to-one correspondence with the lattice points of P. We translate the *affine* relations between them (analogously to (2.4.2), by writing the relations multiplicatively) into homogeneous, monomial equations on $\mathbb{P}^{\#(P \cap \mathbb{L})-1}$. They define a projective subvariety $X'_P \subseteq \mathbb{P}^{\#(P \cap \mathbb{L})-1}$ which turns out to be the scheme theoretical image of X_P under the previously described map. In the other way around, we can obtain X_P from X'_P just by taking the normalization and $\mathcal{O}_X(P)$ by pulling back $\mathcal{O}(1)$.

(3) Embedding P via $P \subseteq \mathbb{A} \cong \mathbb{A} \times \{1\} \hookrightarrow \mathbb{A} \times \mathbb{R} =: M_{\mathbb{R}}$ in the next dimension (in a *dual way* as we did with Q at the end of (2.4.2)), we obtain rational, polyhedral cones $\sigma^\vee := \mathbb{R}_{\geq 0} \cdot P \subseteq M_{\mathbb{R}}$ and $\sigma := \sigma^{\vee\vee} \subseteq N_{\mathbb{R}}$. The latter one contains $a^* := (\underline{0}, 1)$ as an interior point; the equation $[a^* = 1]$ cuts out P from $\sigma^\vee$ again.
Moreover, a^* can be used to make the ring $\mathbb{C}[\sigma^\vee \cap M]$ $\mathbb{Z}$-graded ($\deg x^r := \langle a^*, r\rangle$), and we obtain $X_P = \operatorname{Proj} \mathbb{C}[\sigma^\vee \cap M]$. (In fact, it is not difficult to see that the homogeneous localizations by monomials yield those semigroup rings induced by the fan Σ.)

On the other hand, we can use the cone σ to define an affine toric variety $Y_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. Since $\mathbb{C}[\sigma^\vee \cap M] = \oplus_{i \geq 0} H^0(X_P, \mathcal{O}_X(P)^i)$ (as graded rings), Y equals the cone over $(X_P, \mathcal{O}_X(P))$. Let us elucidate this fact from the combinatorial view point:
The open subset $Y \setminus \{0\} \subseteq Y$ is given by the fan $\partial\sigma$ contained in $N_{\mathbb{R}}$, and the projection $\pi : Y \setminus \{0\} \to X_P$ corresponds to the $\mathbb{Z}$-linear map $\pi : N \longrightarrow\!\!\!\!\rightarrow N/_{\mathbb{Z} \cdot a^*} = \mathbb{A}^*$ (sending proper faces of σ onto Σ-cones).

Claim: π maps the facets of σ isomorphically (including the lattice structure) onto the top-dimensional cones of Σ. In particular, $Y \setminus \{0\}$ is *locally* isomorphic to $X_P \times \mathbb{C}^*$.

Proof: Considering just the real structure of the cones, there is no problem at all. On the other hand, let (for some facet $\tau < \sigma$) $a \in \tau \subseteq N_{\mathbb{R}}$ such that $\pi(a)$ belongs to the lattice ($\pi(a) \in \pi(\tau) \cap {}^N/\mathbb{Z}a^*$). The special shape of σ implies that τ is given as $\tau = \sigma \cap r^\perp$ for some vertex $r \in P$. In particular, $\langle a^*, r\rangle = 1$, and, if $\pi(a)$ is represented by some $b \in N$, then $b - \langle b, r\rangle a^* \in \pi^{-1}(\pi(\tau)) \cap r^\perp = \tau$. Hence, $a = b - \langle b, r\rangle a^*$ is an element of N. □

Chapter 3

T^1 and T^2 for affine toric varieties

3.1 The modules T^1 and T^2 in general

(3.1.1) For an affine scheme $Y = \text{Spec } A$, there are two important A-modules, T^1_Y and T^2_Y carrying information about its deformation theory: T^1_Y describes the infinitesimal deformations, and T^2_Y contains the obstructions for extending deformations of Y to larger base spaces (cf. [KPR], [LiS], or [Pa]; see also (1.1.3) in this thesis).
In case Y admits a versal deformation, T^1_Y is the tangent space of the versal base space S. Moreover, if $\mathcal{J}$ denotes the ideal defining $(S,0)$ as a closed subscheme of the germ $(T^1_Y, 0)$, the module $\left({}^{\mathcal{J}}\!/_{m_{T^1,0}\,\mathcal{J}}\right)^*$ can be canonically embedded into T^2_Y, i.e. $(T^2_Y)^*$-elements induce the equations defining S in T^1_Y.

The vector spaces T^i_Y come with a cup product $T^1_Y \times T^1_Y \to T^2_Y$. The associated quadratic form $T^1_Y \to T^2_Y$ describes the quadratic part of the elements of $\mathcal{J}$, i.e. it can be used to get a better approximation of the versal base space S as it is its tangent space (see section (4.1)).

(3.1.2) If Y is given as a closed subscheme of some $\mathbb{C}^{w+1}$, then the standard exact sequence of differentials can be completed on the right hand side yielding

$$0 \to \Theta_Y \to \Theta_{\mathbb{C}^{w+1}} \otimes \mathcal{O}_Y \to \mathcal{N}_{Y|\,\mathbb{C}^{w+1}} \to T^1_Y \to 0.$$

($\Theta_Y = \text{Der}_{\mathbb{C}}(A,A)$ denotes the tangent sheaf, i.e. the derivations from A to A;
$\Theta_{\mathbb{C}^{w+1}} = \bigoplus_{v=0}^{w} \mathbb{C}[\underline{z}] \cdot \frac{\partial}{\partial z_v} \cong \mathbb{C}[z_0, \dots, z_w]^{w+1}$; $\mathcal{N}_{Y|\,\mathbb{C}^{w+1}}$ denotes the normal sheaf of $Y \hookrightarrow \mathbb{C}^{w+1}$.)

Let $Y \subseteq \mathbb{C}^{w+1}$ be given by equations $f_1, \dots, f_m$, i.e. its ring of regular functions equals

$$A = {}^{P}\!/_{I} \quad \text{with } P = \mathbb{C}[z_0, \dots, z_w]\,;\; I = (f_1, \dots, f_m)\,.$$

Then, using $d : {}^{I}\!/_{I^2} \to A^{w+1}$ $(d(f_i) := (\frac{\partial f_i}{\partial z_0}, \dots \frac{\partial f_i}{\partial z_w}))$, the above sequence means that the vector space T^1_Y equals

$$T^1_Y = {}^{\text{Hom}_A({}^{I}\!/_{I^2}, A)}\Big/_{\text{Hom}_A(A^{w+1}, A)}\,.$$

Remark: (additional T^1-formulas)

(i) Let A be a reduced ring. Then, T^1_Y is also computable as $T^1_Y = \text{Ext}^1_A(\Omega^1_{A|\,\mathbb{C}}, A)$ ($\Omega^1_{A|\,\mathbb{C}} =$ sheaf of Kähler differentials, i.e. $\Theta_X = \text{Hom}(\Omega^1_{A|\,\mathbb{C}}, A)$; cf. [KPR], ch. 5). In fact, we will use this formula for our computation in section (3.3).

(ii) Let $Y = \operatorname{Spec} A$ be a normal surface singularity with the only singular point $0 \in Y$. Then, T^1_Y is the kernel of the map

$$H^1(Y\backslash 0, \Theta_{Y\backslash 0}) \to H^1(Y\backslash 0, \Theta_{\mathbb{C}^{w+1}|\, Y\backslash 0})$$

(cf. [Sch 2], §1). The embedding $T^1_Y \subseteq H^1(Y\backslash 0, \Theta_{Y\backslash 0})$ corresponds to the restriction map $\operatorname{Def}_Y \to \operatorname{Def}_{Y\backslash 0}$.

(3.1.3) Let $\mathcal{R} \subseteq P^m$ denote the P-module of relations between the equations $f_1, \ldots, f_m$. If $\{e^1, \ldots, e^m\}$ denotes the canonical basis of P^m, then $\mathcal{R}$ contains the so-called Koszul relations $\mathcal{R}_0 := \langle f_i\, e^j - f_j\, e^i \rangle$ as a submodule. Now, T^2_Y can be obtained as

$$T^2_Y = {\operatorname{Hom}_P({}^{\mathcal{R}}\!/_{\mathcal{R}_0}, A)} \Big/ {\operatorname{Hom}_P(P^m, A)} \quad .$$

Remark: Let A be a normal ring. Then, analogeously to T^1_Y, the vector space T^2_Y is computable as $T^2_Y = \operatorname{Ext}^2_A(\Omega^1_{A|\,\mathbb{C}}, A)$. (We may use the same reference [KPR]; though a slightly different notion of T^2_Y has been used there, both vector spaces coincide for normal singularities.)

(3.1.4) Finally, we construct the so-called obstruction map. It detects all possible infinitesimal extensions of a flat family to another one defined over some larger base space. We follow the explanation given in §4 of [JS 4].

Let $0 \to W \to R' \stackrel{\pi}{\to} R \to 0$ be a small extension of a local $\mathbb{C}$-algebra R, i.e. $W \cdot m_{R'} = 0$, and W is a finite-dimensional $\mathbb{C}$-vector space. Moreover, assume that we are given a flat R-algebra $A_R = R[\underline{z}]/(F_1, \ldots, F_m)$ deforming A; in particular, the equations $F_i(\underline{z})$ specialize to $f_i(\underline{z})$ via $R \twoheadrightarrow \mathbb{C}$. Then, we may construct an element $\lambda(A_R) \in T^2_Y \otimes_{\mathbb{C}} W$ in the following way:

(i) For any relation $s \in \mathcal{R} \subseteq \mathbb{C}[\underline{z}]^m$ we choose a lifting $S' \in R'[\underline{z}]^m$ such that $\sum_i S'_i(\underline{z}) \cdot F_i(\underline{z}) = 0$ in $R[\underline{z}]$.

(ii) Take any $F'_i(\underline{z}) \in R'[\underline{z}]$ lifting the equations F_i. Then, $\sum_i S'_i(\underline{z}) \cdot F'_i(\underline{z}) \in P \otimes_{\mathbb{C}} W \subseteq R'[\underline{z}]$, and we define $\lambda(A_R; s)$ as its image in $A \otimes_{\mathbb{C}} W$.

It is not hard to check that $\lambda(A_R) \in T^2_Y \otimes_{\mathbb{C}} W$ does not depend on the choices being made in the previous construction. Moreover, we obtain

Proposition:

(1) $\lambda(A_R) = 0$, if and only if the deformation A_R lifts to one over R'.

(2) The corresponding map $\lambda(A_R) : W^* \to T^2_Y$ ("obstruction map") is injective, if and only if A_R is not liftable to any deformation over a ring between R and R' (and different from R).

Applied to the following situation, we obtain a very useful criterion of versality:

Theorem: ([Arn]) *Let $R = \mathbb{C}\{\underline{t}\}/\mathcal{J}$ and $R' = \mathbb{C}\{\underline{t}\}/(\underline{t})\mathcal{J}$. Then, A_R is the mini-versal deformation of A, if and only if the Kodaira-Spencer-map is an isomorphism and the obstruction map is injective.*

3.2 The first T^i-formulas for toric varieties

(3.2.1) As far as I know, the deformation theory of affine toric varieties (e.g. calculation of its infinitesimal deformations) is hardly mentioned in the literature. There are just the papers [IsOd] and [LdSl]. The first one does not deal with toric singularities directly, but with varieties defined by monomial equations. The latter paper builds up a cohomology theory for semigroups; via a spectral sequence argument the main result is that this theory describes exactly the algebra cohomology (in particular T^1, T^2) of the corresponding toric variety. However, an explicit description of the vector spaces T^i is lacking except for the case of two-dimensional quotient singularities. We will describe T^1 and T^2 in such a way that it can be used for further investigations (such as describing Kodaira-Spencer or obstruction maps of certain deformations).

(3.2.2) We start with fixing (or recalling from (2.4.2)) the usual notations when dealing with affine toric varieties (cf. [Ke] or [Od]):

- Let M, N be mutually dual free Abelian groups, we denote by $M_{I\!R}, N_{I\!R}$ the associated real vector spaces obtained by base change with $I\!R$.

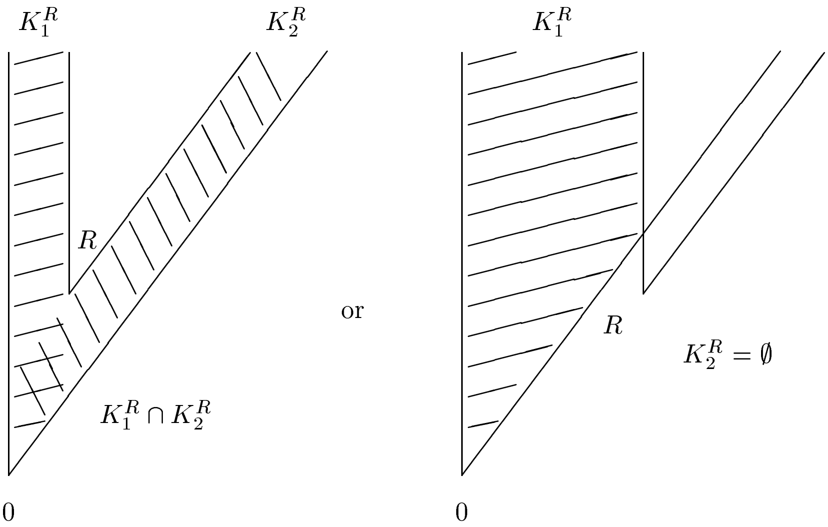
- Let $\sigma = \langle a^1, \dots, a^M \rangle \subseteq N_{I\!R}$ be a rational, polyhedral cone with apex in 0 - given by its fundamental generators.
  $\sigma^\vee := \{r \in M_{I\!R} \mid \langle \sigma, r \rangle \geq 0\} \subseteq M_{I\!R}$ is called the dual cone. It induces a *partial order* on the lattice M via $[\, a \geq b$ iff $a - b \in \sigma^\vee\,]$.

- $A := \mathbb{C}[\sigma^\vee \cap M]$ denotes the semigroup algebra. It is the ring of regular functions of the toric variety $Y = \operatorname{Spec} A$ associated to σ.
  The ring A itself as well as most of its important modules (such as T^1 and T^2) admit an M-(multi)grading. It is this grading which will make computations possible.

- Denote by $E \subset \sigma^\vee \cap M$ the minimal set of generators of this semigroup ("the Hilbert basis"). E equals the set of all indecomposable elements of $\sigma^\vee \cap M$. In particular, there is a surjection of semigroups $\pi : I\!N^E \longrightarrow\!\!\!\!\rightarrow \sigma^\vee \cap M$, and this fact translates into a closed embedding $Y \hookrightarrow \mathbb{C}^E$. To make the notations coherent with section (3.1), assume that $E = \{r^0, \dots, r^w\}$ consists of $w + 1$ elements (inducing coordinates $z_0, \dots, z_w$).

(3.2.3) To a fixed degree $R \in M$ we associate "thick facets" K_j^R of the dual cone $\sigma^\vee$ defined by

$$K_j^R := \{r \in \sigma^\vee \cap M \mid \langle a^j, r \rangle < \langle a^j, R \rangle\} \quad (j = 1, \dots, M).$$



Lemma: (properties of the sets K_j^R)

(1) $\langle a^j, R\rangle \leq 0$ *iff* $K_j^R = \emptyset$;
$\langle a^j, R\rangle = 1$ *iff* $K_j^R = [a^j$*-face of* $\sigma^\vee] \cap M$;
$\langle a^j, R\rangle \geq 2$ *iff* K_j^R *contains an interior point of* $\sigma^\vee$.

(2) K_j^R *is closed for descent, i.e.* $r \in K_j^R$ *and* $0 \leq s \leq r$ *imply* $s \in K_j^R$.

(3) $\cup_j K_j^R = (\sigma^\vee \cap M) \setminus (R + \sigma^\vee)$.

(4) Let $\langle a^1, \dots, a^k\rangle < \sigma$ *be a smooth face of* σ *(i.e.* $\{a^1, \dots, a^k\}$ *is a part of a* $\mathbb{Z}$*-basis of the lattice* N*). Then, for elements* $r_1, \dots, r_m \in \sigma^\vee \cap M$ *the conditions*

(i) $r_1, \dots, r_m \in K_1^R \cap \dots \cap K_k^R$ *and*
(ii) $\exists\, \ell \geq r_1, \dots, r_m : \ell \in K_1^R \cap \dots \cap K_k^R$

are equivalent.

Proof: The only non-trivial part is the last one. It is sufficient to prove it for $m = 2$; let $r_1, r_2 \in K_1^R \cap \dots \cap K_k^R$ be given.
Step 1: Since $\langle a^1, \dots, a^k\rangle < \sigma$ is a smooth face, there exist elements $s^1, \dots, s^k \in \sigma^\vee \cap M$ such that

$$\langle a^j, s^v\rangle = \delta_{jv} \qquad (1 \leq j, v \leq k).$$

Step 2: We may assume that $\langle a^j, r_1\rangle = \langle a^j, r_2\rangle$ *for* $j = 1, \dots, k$*:*
In fact, let $\langle a^j, r_1\rangle - \langle a^j, r_2\rangle = g_j \geq 0$. Then, r_2 could be corrected by $r_2 := r_2 + g_j \cdot s^j$.

Step 3: Let $s \in M$ be an interior point of the dual face $\langle a^1, \dots, a^k\rangle^* = (a^1)^\perp \cap \dots \cap (a^k)^\perp \cap \sigma^\vee$ of the cone $\sigma^\vee$, i.e.

$$\begin{array}{rcll} \langle a^j, s\rangle & = & 0 & \text{for } \; j = 1, \dots, k \;\text{ and} \\ \langle a^v, s\rangle & > & 0 & \text{for } \; v \notin \{1, \dots, k\}. \end{array}$$

Hence,

$$\begin{array}{rcll} \langle a^j, r_1 - r_2 + g \cdot s\rangle & = & 0 & \text{for } \; j = 1, \dots, k \;\text{ and} \\ \langle a^v, r_1 - r_2 + g \cdot s\rangle & \geq & 0 & \text{for } \; v \notin \{1, \dots, k\} \;\text{ and }\; g \gg 0, \end{array}$$

and we can define $\ell := r_1 + g\cdot s = r_2 + (r_1 - r_2 + g\cdot s)$. □

Mostly, we will need two special cases of the last part of the previous lemma:

Corollary: *Let* $r_1,\dots,r_m \in \sigma^\vee \cap M$.

(a) For $j = 1,\dots,M$ *we obtain that*

$$r_1,\dots,r_m \in K_j^R \quad \textit{iff} \quad \exists\, \ell \geq r_1,\dots,r_m : \ell \in K_j^R.$$

(b) Let Y_σ *be smooth in codimension 2; let* $\langle a^j, a^k\rangle < \sigma$ *be a two-dimensional face of the cone* σ. *Then, we obtain*

$$r_1,\dots,r_m \in K_j^R \cap K_k^R \quad \textit{iff} \quad \exists\, \ell \geq r_1,\dots,r_m : \ell \in K_j^R \cap K_k^R.$$

Intersecting those sets K_j^R with $E \subseteq \sigma^\vee \cap M$, we obtain the basic objects for describing the modules T_Y^i:

$$\begin{aligned} E_j^R &:= K_j^R \cap E = \{r \in E \mid \langle a^j, r\rangle < \langle a^j, R\rangle\}\,,\\ E_0^R &:= \bigcup_{j=1}^N E_j^R\,, \text{ and}\\ E_\tau^R &:= \bigcap_{a^j\in\tau} E_j^R \text{ for faces } \tau < \sigma\,. \end{aligned}$$

We obtain a complex $L(E^R)_\bullet$ of free Abelian groups via

$$L(E^R)_{-k} := \bigoplus_{\substack{\tau<\sigma\\ \dim\tau = k}} L(E_\tau^R)$$

with the usual differentials. ($L(\dots)$ denotes the free Abelian group of integral, linear dependencies.) The most interesting part ($k \leq 2$) can be written explicitly as

$$L(E^R)_\bullet : \quad \cdots \to \oplus_{\langle a^j,a^k\rangle<\sigma} L(E_j^R \cap E_k^R) \longrightarrow \oplus_j L(E_j^R) \longrightarrow L(E_0^R) \to 0\,.$$

(3.2.4) Theorem:

(1) $T_Y^1(-R) = H^0\left(L(E^R)_\bullet^* \otimes_{\mathbb{Z}} \mathbb{C}\right)$

(2) $T_Y^2(-R) \supseteq H^1\left(L(E^R)_\bullet^* \otimes_{\mathbb{Z}} \mathbb{C}\right)$

(3) Moreover, if Y *is smooth in codimension 2 (i.e. if the 2-faces* $\langle a^j, a^k\rangle < \sigma$ *are spanned by a part of a* $\mathbb{Z}$*-basis of the lattice* N*), then*

$$T_Y^2(-R) = H^1\left(L(E^R)_\bullet^* \otimes_{\mathbb{Z}} \mathbb{C}\right)\,.$$

(4) Module structure: If $x^s \in A$ *(i.e.* $s \in \sigma^\vee \cap M$*), then* $E_j^{R-s} \subseteq E_j^R$*, hence* $L(E^R)_\bullet^* \subseteq L(E^{R-s})_\bullet^*$. *The induced map in cohomology corresponds to the multiplication with* x^s *in the* A*-modules* T_Y^1 *and* T_Y^2*, respectively.*

The proof is contained in section (3.3).

Remark: The assumption made in (3) cannot be dropped: Taking for Y a two-dimensional cyclic quotient singularity given by some two-dimensional cone σ, then there are only two different sets E_1^R and E_2^R (for each $R \in M$). In particular, $H^1(L(E^R)^*_\bullet \otimes_{\mathbb{Z}} \mathbb{C}) = 0$.

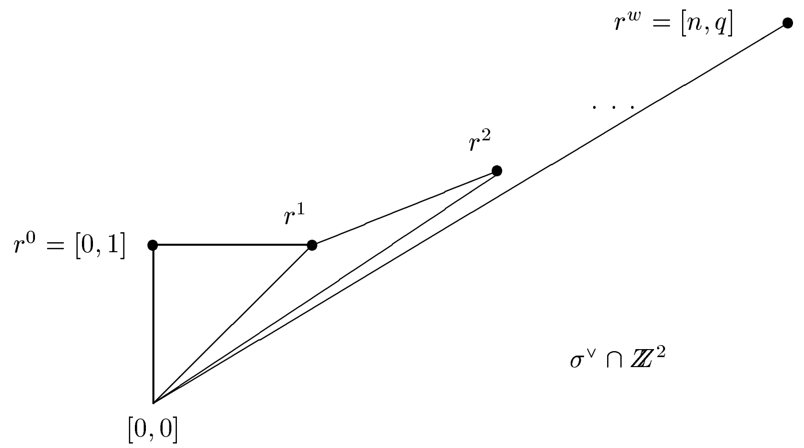
(3.2.5) Nevertheless, we can use two-dimensional cyclic quotient singularities (i.e. two-dimensional affine toric varieties, cf. example (3) in (2.4.2)) as an object to demonstrate how the T^1-formula of the previous theorem works.

If $Y(n,q)$ denotes the quotient of $\mathbb{C}^2$ by the $\mathbb{Z}/_{n\mathbb{Z}}$-action $\xi \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix}$, then $Y(n,q)$ is given by the cone $\sigma = \langle (1,0); (-q,n) \rangle \subseteq \mathbb{R}^2$. We develop $\frac{n}{n-q}$ into a (negative) continued fraction

$$\frac{n}{n-q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 \ddots - \cfrac{1}{a_{w-1}}}} \qquad (a_v \geq 2).$$

Then $E \subseteq \sigma^\vee \cap \mathbb{Z}^2$ is given as the set $E = \{r^0, \dots, r^w\} \subseteq \langle [0,1], [n,q] \rangle \cap \mathbb{Z}^2$ with

- $r^0 = [0,1]$, $r^1 = [1,1]$, $r^w = [n,q]$;
- $r^{v-1} + r^{v+1} = a_v \cdot r^v$ $(v = 1, \dots, w-1)$ (cf. [Ri] or [Od]).



Now, there are four different cases for the multidegree $R \in M \cong \mathbb{Z}^2$. (We assume $w \geq 2$, i.e. Y is not smooth; let $a^1 = (1,0)$ and $a^2 = (-q,n)$.):

(i) $R = r^1$ (or analogously $R = r^{w-1}$): We obtain $E_1^R = \{r^0\}$ and $E_2^R = \{r^2, \dots, r^w\}$, and the theorem yields $T_Y^1(-R) \cong \begin{cases} \mathbb{C} & \text{for} \quad w \geq 3 \\ 0 & \text{for} \quad w = 2. \end{cases}$

(ii) $R = r^v$ $(2 \leq v \leq w-2)$: We obtain $E_1^R = \{r^0, \dots, r^{v-1}\}$ and $E_2^R = \{r^{v+1}, \dots, r^w\}$, and the theorem yields $T_Y^1(-R) \cong \mathbb{C}^2$.

(iii) $R = p \cdot r^v$ $(1 \leq v \leq w-1,\ 2 \leq p < a_v$ for $w \geq 3$; or $v = 1 = w-1,\ 2 \leq p \leq a_1$ for $w = 2)$: We obtain $E_1^R = \{r^0, \dots, r^v\}$ and $E_2^R = \{r^v, \dots, r^w\}$, and the theorem yields $T_Y^1(-R) \cong \mathbb{C}$.

(iv) For the remaining $R \in M$, either $E_1^R \subseteq E_2^R$ or $E_2^R \subseteq E_1^R$ or $\#(E_1^R \cap E_2^R) \geq 2$. In all these cases the theorem yields $T_Y^1(-R) = 0$.

(3.2.6) We want to describe the isomorphisms connecting the general T^i-formulas of (3.1.2) and (3.1.3) with the toric ones given in (3.2.4).

$Y \hookrightarrow \mathbb{C}^{w+1}$ is given by the equations

$$f_{ab} := \underline{z}^a - \underline{z}^b \quad (a, b \in \mathbb{N}^{w+1} \text{ with } \pi(a) = \pi(b) \text{ in } \sigma^\vee \cap M),$$

and it is easier to deal with this infinite set of equations (which generates the ideal I as a $\mathbb{C}$-vector space) instead of selecting a finite number of them in some non-canonical way. In particular, for m of (3.1.2) and (3.1.3) we take

$$m := \{(a, b) \in \mathbb{N}^{w+1} \times \mathbb{N}^{w+1} \mid \pi(a) = \pi(b)\}.$$

The general T^i-formulas mentioned in (3.1.2) and (3.1.3) remain true.

Theorem: *For a fixed element $R \in M$ let $\varphi : L(E)_{\mathbb{C}} \to \mathbb{C}$ induce some element of*

$$\left(L(E_0^R) \Big/ \sum_j L(E_j^R) \right)^* \otimes_{\mathbb{Z}} \mathbb{C} \cong T_Y^1(-R) \quad \textit{(cf. Theorem (3.2.4)(1))}.$$

Then, the A-linear map

$$\begin{array}{ccc} I/I^2 & \longrightarrow & A \\ \underline{z}^a - \underline{z}^b & \mapsto & \varphi(a - b) \cdot x^{\pi(a) - R} \end{array}$$

provides the same element via the formula (3.1.2).

Again, the proof is contained in (3.3) (see the end of (3.3.5)).

Remark: A simple, but nevertheless important check shows that the map $(\underline{z}^a - \underline{z}^b) \mapsto \varphi(a-b) \cdot x^{\pi(a) - R}$ goes into A, indeed:
Assume $\pi(a) - R \notin \sigma^\vee$. Then, there exists an index j such that $\langle a^j, \pi(a) - R\rangle < 0$. Denoting by "supp q" (for a $q \in \mathbb{R}^E$) the set of those $r \in E$ providing a non-vanishing entry q_r, we obtain

$$\text{supp}\,(a - b) \subseteq \text{supp}\, a \cup \text{supp}\, b \subseteq E_j^R,$$

i.e. $\varphi(a - b) = 0$.

(3.2.7) The P-module $\mathcal{R} \subseteq P^m$ is generated by relations of two different types:

$$\begin{array}{lcll} r(a, b; c) & := & e^{a+c,\, b+c} - \underline{z}^c\, e^{a,b} & (a, b, c \in \mathbb{N}^{w+1};\ \pi(a) = \pi(b)) \quad \text{and} \\ s(a, b, c) & := & e^{b,c} - e^{a,c} + e^{a,b} & (a, b, c \in \mathbb{N}^{w+1};\ \pi(a) = \pi(b) = \pi(c)). \end{array}$$

($e^{\bullet,\bullet}$ denote the standard basis vectors of P^m.)

Theorem: *For a fixed element $R \in M$ let $\psi_j : L(E_j^R)_{\mathbb{C}} \to \mathbb{C}$ be linear maps such that $\psi_j = \psi_k$ on $L(E_j^R \cap E_k^R)_{\mathbb{C}} = L(E^R_{\langle a^j, a^k\rangle})_{\mathbb{C}}$ for each two-dimensional face $\langle a^j, a^k\rangle < \sigma$. In particular, they induce some element of*

$$\left(\frac{Ker\ (\oplus_j L(E_j^R) \longrightarrow L(E_0^R))}{im\ (\oplus_{\langle a^j, a^k\rangle < \sigma} L(E_j^R \cap E_k^R) \to \oplus_j L(E_j^R))} \right)^* \otimes_{\mathbb{Z}} \mathbb{C} \subseteq T_Y^2(-R) \quad \textit{(cf. (3.2.4)(2))}.$$

Then, the P-linear map

$$\begin{array}{rcll} {}^{\mathcal{R}}\!/_{\mathcal{R}_0} & \longrightarrow & A & \\ r(a,b;c) & \mapsto & \left\{\begin{array}{ll} \psi_j(a-b)\,x^{\pi(a+c)-R} & \textit{for } \pi(a)\in K_j^R;\ \pi(a+c)\geq R \\ 0 & \textit{for } \pi(a)\geq R \textit{ or } \pi(a+c)\in \bigcup_j K_j^R \end{array}\right. \\ s(a,b,c) & \mapsto & 0 & \end{array}$$

is correct defined, and via the formula (3.1.3) it induces the same element of T_Y^2.

For the proof we refer to (3.3). Nevertheless, we check the *correctness of the definition* of the P-linear map ${}^{\mathcal{R}}\!/_{\mathcal{R}_0} \to A$ instantly:

(i) If $\pi(a)$ is contained in two different sets K_j^R and K_k^R, then the two fundamental generators a^j and a^k can be connected by a sequence $a^{j_0},\dots,a^{j_p}$, such that

- $a^{j_0}=a^j$, $a^{j_p}=a^k$,
- $a^{j_{v-1}}$ and a^{j_v} are the edges of some 2-face of σ $(v=1,\dots,p)$, and
- $\pi(a)\in K_{j_v}^R$ for $v=0,\dots,p$.

Hence, supp $(a-b)\subseteq E_{j_{v-1}}^R\cap E_{j_v}^R$ $(v=1,\dots,p)$, and we obtain

$$\psi_j(a-b)=\psi_{j_1}(a-b)=\ldots=\psi_{j_{p-1}}(a-b)=\psi_k(a-b)\,.$$

(ii) There are three types of P-linear relations between the generators $r(\dots)$ and $s(\dots)$ of $\mathcal{R}$:

$$\begin{array}{rcl} 0 &=& \underline{z}^d\,r(a,b;c)-r(a,b;c+d)+r(a+c,b+c;d)\,, \\ 0 &=& r(b,c;d)-r(a,c;d)+r(a,b;d)-s(a+d,b+d,c+d)+\underline{z}^d\,s(a,b,c)\,, \\ 0 &=& s(b,c,d)-s(a,c,d)+s(a,b,d)-s(a,b,c)\,. \end{array}$$

Our map respects them all.

(iii) Finally, the typical element $(\underline{z}^a-\underline{z}^b)e^{cd}-(\underline{z}^c-\underline{z}^d)e^{ab}\in\mathcal{R}_0$ equals

$$-r(c,d;a)+r(c,d;b)+r(a,b;c)-r(a,b;d)-s(a+c,b+c,a+d)+s(b+c,a+d,b+d)\,.$$

It will be sent to 0, too.

3.3 Proof of the T^i-formulas

(3.3.1) We will use the sheaf $\Omega_Y^1=\Omega^1_{A|\mathbb{C}}$ of Kähler differentials (and the Ext-formulas mentioned in remarks (3.1.2) and (3.1.3)) for computing the modules T_Y^i.

The $\mathbb{C}$-linear map $\mathbb{C}[\sigma^\vee\cap M]\to M\otimes_{\mathbb{Z}}\mathbb{C}[M]$, $x^s\mapsto s\otimes x^s$ is a $\mathbb{C}$-derivation: It kills the constants, and for $s,t\in\sigma^\vee\cap M$ we have $x^s(t\otimes x^t)+x^t(s\otimes x^s)=(s+t)\otimes x^{s+t}$. Hence, by definition of the Kähler differentials, we obtain a $\mathbb{C}[S]$-linear map

$$\Omega_Y^1\longrightarrow M\otimes_{\mathbb{Z}}\mathbb{C}[M]\,,\quad d\,x^s\mapsto s\otimes x^s\,.$$

After killing torsion of Ω_Y^1, this gives an embedding: $M\otimes_{\mathbb{Z}}\mathbb{C}[M]$ can be identified with the module of Kähler differentials on the torus $(\mathbb{C}^*)^{\operatorname{rank} M}\cong\operatorname{Spec}\mathbb{C}[M]$, and the previous map corresponds to the restriction of differentials from Y onto the open subset $(\mathbb{C}^*)^{\operatorname{rank} M} \subseteq Y$. Since $\Omega^1_{(\mathbb{C}^*)^{\operatorname{rank} M}}$ is just a localization of Ω^1_Y, we obtain that the above map has exactly $\operatorname{tors}(\Omega^1_Y)$ as its kernel.

In particular, we prefer to use the module $\Omega^1_Y \big/ \operatorname{tors}(\Omega^1_Y) \subseteq M \otimes_{\mathbb{Z}} \mathbb{C}[M]$ instead of the original Ω^1_Y. Since $\operatorname{tors}(\Omega^1_Y)$ is concentrated at the singular locus of Y, the maps

$$\alpha_i : \operatorname{Ext}^i_A\left(\Omega^1_Y \big/ \operatorname{tors}(\Omega^1_Y), A\right) \hookrightarrow \operatorname{Ext}^i_A\left(\Omega^1_Y, A\right) \cong T^i_Y \quad (i = 1, 2)$$

are injective. Moreover, they are isomorphisms for

- $i = 1$, since Y is normal, and for
- $i = 2$, if Y is smooth in codimension 2.

(3.3.2) We will build a special M-graded, A-free resolution

$$\mathcal{E} \xrightarrow{d_E} \mathcal{D} \xrightarrow{d_D} \mathcal{C} \xrightarrow{d_C} \mathcal{B} \xrightarrow{d_B} \Omega^1_Y \big/ \operatorname{tors}(\Omega^1_Y) \to 0$$

to compute the groups Ext^1 and Ext^2. Denoting $\operatorname{supp} q := \{r \in E \mid q_r \neq 0\} \quad (q \in L(E))$,

$$L^2(E) := L(L(E))\,,\; L^3(E) := L(L^2(E)), \quad \text{and}$$

$$\operatorname{supp}^2\xi := \bigcup_{q \in supp\, \xi} \operatorname{supp} q \quad (\xi \in L^2(E)), \quad \operatorname{supp}^3\omega := \bigcup_{\xi \in supp\, \omega} \operatorname{supp}^2\xi \quad (\omega \in L^3(E))\,,$$

the resolution can be constructed as follows:

($\mathcal{B}$) Starting with

$$\mathcal{B} := \oplus_{r\in E}\, A \cdot B(r)\,, \quad d_B : B(r) \mapsto d\,x^r\,, \quad \text{and} \quad \deg B(r) := r\,,$$

our first task is to determine the kernel of $d_B : \mathcal{B} \to M \otimes_{\mathbb{Z}} \mathbb{C}[M]$ $(B(r) \mapsto r \otimes x^r)$. Defining $\deg(s \otimes x^r) := r$, the map d_B indeed respects the grading. Hence, it suffices to look for the M-homogeneous elements of the kernel. They are generated (as a $\mathbb{C}$-vector space) by

$$C(q;\ell) := \sum_{r\in E} q_r \cdot x^{\ell - r} B(r) \quad (q \in L(E) \subseteq \mathbb{Z}^E;\; \ell \geq \operatorname{supp} q)\,.$$

(In general, for a relation $q \in L(E)$, there is no smallest element $\ell \geq \operatorname{supp} q$. Hence, it is more naturally to regard *all* of those elements ℓ (as well as *all* relations $q \in L(E)$) instead of some finite, but non-canonical selection of generators. This leads us to consider the $\mathbb{C}$-linear basis of $\ker d_B$ instead of some A-linear generating system.)

($\mathcal{C}$) The previous formula suggests

$$\mathcal{C} := \oplus_{\substack{q\in L(E)\\ \ell \geq supp\, q}} A \cdot C(q;\ell)\,, \quad \deg C(q;\ell) := \ell$$

and says, how to define d_C. (Caution: Our policy of regarding *all* elements q and ℓ causes free A-modules of *infinite rank*.)
Now, there are two kinds of homogeneous generators of $\ker d_C$:

(i) $D(q;\ell,\eta) := C(q;\eta) - x^{\eta-\ell} \cdot C(q;\ell) \quad (q \in L(E)\,,\; \eta \geq \ell \geq \operatorname{supp} q)$ compare the different possibilities of choosing lattice points dominating $\operatorname{supp} q$, and

(ii) $D(\xi;\eta) := \sum_{q\in L(E)} \xi_q \cdot C(q;\eta) \quad (\xi \in L^2(E)\,,\ \eta \geq \operatorname{supp}^2\xi)$ is of the usual type.

($\mathcal{D}$) As usual, we define

$$\mathcal{D} := \left(\oplus_{\substack{q\in L(E)\\ \eta\geq\ell\geq supp\, q}} A\cdot D(q;\ell,\eta) \right) \oplus \left(\oplus_{\substack{\xi\in L^2(E)\\ \eta\geq supp^2\xi}} A\cdot D(\xi;\eta) \right), \quad \deg D(\ldots;\eta) := \eta\,,$$

and the the map d_D is induced by the formulas $(\mathcal{C})(i)$ and $(\mathcal{C})(ii)$. Its kernel is generated by

(i) $E(q;\ell,\eta,\mu) := D(q;\eta,\mu) - D(q;\ell,\mu) + x^{\mu-\eta}\cdot D(q;\ell,\eta)$
$(q \in L(E)\,,\ \mu \geq \eta \geq \ell \geq \operatorname{supp} q)$,

(ii) $E(\xi;\eta,\mu) := D(\xi;\mu) - x^{\mu-\eta}\cdot D(\xi;\eta) - \sum_{q\in L(E)} \xi_q \cdot D(q;\eta,\mu)$
$(\xi \in L^2(E)\,,\ \mu \geq \eta \geq \operatorname{supp}^2\xi)$,

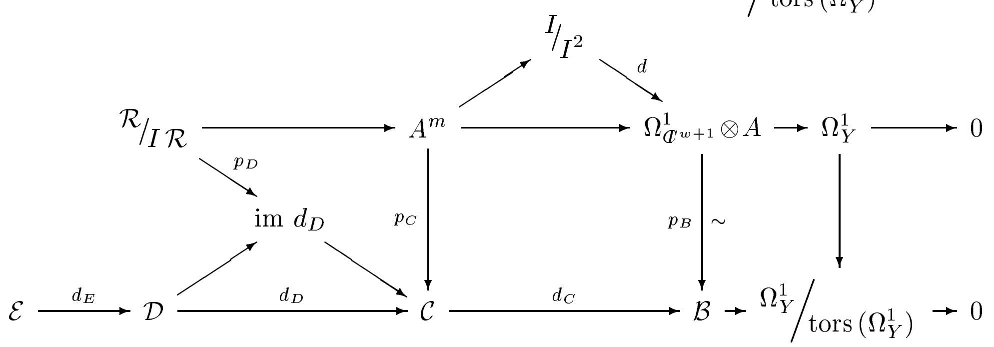
(iii) $E(\omega;\mu) := \sum_{\xi\in L^2(E)} \omega_\xi \cdot D(\xi;\mu) \quad (w \in L^3(E)\,,\ \mu \geq \operatorname{supp}^3 w)$.

($\mathcal{E}$) Finally, it is clear, how to define $\mathcal{E}$ and the map d_E. Moreover, the generators $E(\ldots;\mu)$ of $\mathcal{E}$ are defined to have degree $\mu \in M$.

(3.3.3) Combining the two exact sequences

$$\mathcal{R}/_{I\mathcal{R}} \longrightarrow A^m \longrightarrow I/_{I^2} \to 0 \quad\text{and}\quad I/_{I^2} \longrightarrow \Omega^1_{\mathbb{C}^{w+1}} \otimes A \longrightarrow \Omega^1_Y \to 0\,,$$

we get a complex (not exact at the place of A^m) involving Ω^1_Y. We will compare in the following commutative diagram this complex with the previous resolution of $\Omega^1_Y \big/ \operatorname{tors}(\Omega^1_Y)$:



Let us explain the three labeled vertical maps:

(B) $p_B : dz_r \mapsto B(r)$ is an isomorphism between two free A-modules of rank $w+1$.

(C) $p_C : e^{ab} \mapsto C(a-b;\pi(a))$. In particular, the image of this map is spanned by those $C(q,\ell)$ meeting $\ell \geq \bar{q} := \pi(q^+)$ (cf. (4.2.1)) which is stronger than just $\ell \geq \operatorname{supp} q$.

(D) Finally, p_D arises as pull back of p_C to $\mathcal{R}/_{I\mathcal{R}}$. It can be described by $r(a,b;c) \mapsto D(a-b;\pi(a),\pi(a+c))$ and $s(a,b,c) \mapsto D(\xi;\pi(a))$ (ξ denotes the relation $\xi = [(b-c)-(a-c)+(a-b)=0]$).

Remark: Starting with the typical $\mathcal{R}_0$-element mentioned in (3.2.7)(iii), the previous description of the map p_D yields 0 (even in $\mathcal{D}$).

(3.3.4) By (3.3.1) we get the A-modules T^i_Y by computing the cohomology of the complex dual to that of (3.3.2).

Denote by G one of the capital letters B, C, D, or E. Then, an element ψ of the dual free module $\mathcal{G}^* = (\bigoplus_G \mathbb{C}[\sigma^\vee \cap M] \cdot G)^*$ can be described by giving elements $g(x) \in \mathbb{C}[\sigma^\vee \cap M]$ to be the images of the generators G (g stands for b, c, d, or e, respectively).

For ψ to be homogeneous of degree $-R \in M$, $g(x)$ has to be a monomial of degree

$$\deg g(x) = -R + \deg G.$$

In particular, the corresponding complex coefficient $g \in \mathbb{C}$ (i.e. $g(x) = g \cdot x^{-R+\deg G}$) satisfies that

$$g \neq 0 \quad \text{implies} \quad -R + \deg G \geq 0 \quad (\text{i.e. } -R + \deg G \in \sigma^\vee).$$

(3.3.5) For computing $T^1_Y(-R)$, the interesting part of the dualized complex $(3.3.2)^*$ in degree $-R$ equals the complex of infinite-dimensional $\mathbb{C}$-vector spaces

$$\mathcal{B}^*_{-R} \xrightarrow{d^*_C} \mathcal{C}^*_{-R} \xrightarrow{d^*_D} \mathcal{D}^*_{-R}$$

with coordinates $\underline{b}$, $\underline{c}$, and $\underline{d}$, respectively:

$$\begin{array}{rcl} \mathcal{B}^*_{-R} & = & \{\underline{b(r)} \mid b(r) = 0 \text{ for } r - R \notin \sigma^\vee\} \subseteq \mathbb{C}^E \\ \mathcal{C}^*_{-R} & = & \{\underline{c(q;\ell)} \mid c(q;\ell) = 0 \text{ for } \ell - R \notin \sigma^\vee\} \subseteq \mathbb{C}^{\#\{C(q,\ell)\}} \\ \mathcal{D}^*_{-R} & = & \{[\underline{d(q;\ell,\eta)}, \underline{d(\xi;\eta)}] \mid \begin{array}{rcl} d(q;\ell,\eta) & = & 0 \text{ for } \eta - R \notin \sigma^\vee, \text{ and} \\ d(\xi;\eta) & = & 0 \text{ for } \eta - R \notin \sigma^\vee\}. \end{array} \end{array}$$

(The parameters run through the following sets: $r \in E$ for $\underline{b(r)}$; $q \in L(E)$, $\ell \geq \operatorname{supp} q$ for $\underline{c(q;\ell)}$; $q \in L(E)$, $\eta \geq \ell \geq \operatorname{supp} q$ for $\underline{d(q;\ell,\eta)}$; and $\xi \in L^2(E)$, $\eta \geq \operatorname{supp}^2 \xi$ for $\underline{d(\xi;\eta)}$.)

The differentials d^*_C and d^*_D can be described by

$$\begin{array}{rcl} c(q;\ell) & = & \sum_{r\in E} q_r \cdot b(r) \qquad \text{and} \\ d(q;\ell,\eta) & = & c(q;\eta) - c(q;\ell), \\ d(\xi;\eta) & = & \sum_{q\in L(E)} \xi_q \cdot c(q;\eta). \end{array}$$

Hence, we are able to compute $\ker d^*_D$ and $\operatorname{im} d^*_C$:

$$\begin{array}{rcl} \ker d^*_D & = & \{\underline{c(q,\ell)} \mid \begin{array}{l} \bullet\, c(q,\ell) = 0 \text{ if } \ell \not\geq R \\ \bullet\, c(q,\ell) \text{ does not depend on } \ell \\ \bullet\, \sum_{q\in L(E)} \xi_q \cdot c(q,\ell) = 0 \text{ for } \xi \in L^2(E)\} \end{array} \\ & = & \{\underline{c(q)} \mid \begin{array}{l} \bullet\, c(q) = 0 \text{ if } \exists \ell \geq \operatorname{supp} q : \ell \not\geq R \\ \bullet\, \sum_{q\in L(E)} \xi_q \cdot c(q) = 0 \text{ for } \xi \in L^2(E)\} \end{array} \\ & = & \{c \in L(E)^*_{\mathbb{C}} \mid c(q) = 0 \text{ if } \exists \ell \geq \operatorname{supp} q : \ell \not\geq R\} \end{array}$$

and

$$\operatorname{im} d^*_C = \{c \in L(E)^*_{\mathbb{C}} \mid \exists\, \underline{b(r)} : \begin{array}{l} \bullet\, b(r) = 0 \text{ if } r \not\geq R \\ \bullet\, c(q) = \sum_{r\in E} q_r \cdot b(r)\}. \end{array}$$

Involving the sets K^R_j introduced in (3.2.3) and using part (a) of the corollary contained in that section, we can reformulate the description of $\ker d^*_D$:

$$\begin{array}{rcl} \ker d^*_D & = & \{c \in L(E)^*_{\mathbb{C}} \mid c(q) = 0 \text{ if } \exists j, \exists \ell \geq \operatorname{supp} q : \ell \in K^R_j\} \\ & = & \{c \in L(E)^*_{\mathbb{C}} \mid c(q) = 0 \text{ if } \exists j : \operatorname{supp} q \subseteq K^R_j\} \\ & = & \ker \left[L(E)^*_{\mathbb{C}} \to \oplus_j L(E^R_j)^*_{\mathbb{C}}\right]. \end{array}$$

Moreover, we have

$$\begin{aligned}\operatorname{im} d_C^* &= \operatorname{im}\left[\mathbb{C}^{E\setminus E_0^R} \hookrightarrow \mathbb{C}^E \twoheadrightarrow L(E)^*_{\mathbb{C}}\right]\\ &= \ker\left[L(E)^*_{\mathbb{C}} \twoheadrightarrow L(E_0^R)^*_{\mathbb{C}}\right]\end{aligned}$$

(dualize the exact sequence $0 \to L(E_0^R) \to L(E) \to \mathbb{C}^{E\setminus E_0^R}$ to get the latter equality). Hence,

$$\begin{aligned}T^1(-R) = {}^{\ker d_D^*}\!/\operatorname{im} d_C^* &= \ker\left[L(E_0^R)^*_{\mathbb{C}} \to \oplus_j L(E_j^R)^*_{\mathbb{C}}\right]\\ &= \left(L(E_0^R)\Big/\sum_j L(E_j^R)\right)_* \otimes_{\mathbb{Z}} \mathbb{C}\,,\end{aligned}$$

and the first part of Theorem (3.2.4) is proved.

To compare this T^1-formula with that involving $\operatorname{Hom}({}^I/I^2, A)$ (i.e. to prove Theorem (3.2.6)), we regard the commutative diagram constructed in (3.3.3):
Given a linear map $\varphi : L(E)_{\mathbb{C}} \to \mathbb{C}$, we obtain (through the recent identifications) an element of $\mathcal{C}^*_{-R}$ - it maps $C(q;\ell)$ onto $\varphi(q)\cdot x^{\ell-R}$. Pulling via p_C^* this homomorphism back to A^m, it sends the basis vector e^{ab} to $\varphi(a-b)\cdot x^{\pi(a)-R}$.
On the other hand, it is the same e^{ab} that stands for the binomial equation $\underline{z}^a - \underline{z}^b \in I$. Hence, Theorem (3.2.6) is proved.

(3.3.6) Proving the formula for $T^2(-R)$ requires to consider the vector space

$$\mathcal{E}^*_{-R} = \{[\underline{e(q;\ell,\eta,\mu)}, \underline{e(\xi;\eta,\mu)}, \underline{e(\omega;\mu)}]\,|\,\text{each coordinate vanishes for } \mu - R \notin \sigma^\vee\}$$

(additionally to $\mathcal{C}^*_{-R}$ and $\mathcal{D}^*_{-R}$ mentioned in (3.3.5)). The differential d_E^* is given by

$$\begin{aligned}e(q;\ell,\eta,\mu) &= d(q;\eta,\mu) - d(q;\ell,\mu) + d(q;\ell,\eta),\\ e(\xi;\eta,\mu) &= d(\xi;\mu) - d(\xi;\eta) - \textstyle\sum_{q\in F}\xi_q\cdot d(q;\eta,\mu),\\ e(\omega;\mu) &= \textstyle\sum_{\xi\in G}\omega_\xi\cdot d(\xi;\mu),\end{aligned}$$

and we obtain

$$\begin{aligned}\ker d_E^* = \{[\underline{d(q;\ell,\eta)};\,\underline{d(\xi;\eta)}]\,|\; & d(q;\ell,\eta) = d(\xi;\eta) = 0 \text{ for } \eta - R \notin \sigma^\vee,\\ & d(q;\ell,\mu) = d(q;\ell,\eta) + d(q;\eta,\mu) \quad (\mu \ge \eta \ge \ell \ge \operatorname{supp} q),\\ & d(\xi;\mu) = d(\xi;\eta) + \textstyle\sum_q \xi_q\cdot d(q;\eta,\mu) \quad (\mu\ge\eta\ge\operatorname{supp}^2\xi),\\ & \textstyle\sum_{\xi\in L^2(E)}\omega_\xi\, d(\xi;\mu) = 0 \text{ for } \omega\in L^3(E) \text{ with } \mu \ge \operatorname{supp}^3\omega\,\},\\ \operatorname{im} d_D^* = \{[\underline{d(q;\ell,\eta)};\,\underline{d(\xi;\eta)}]\,|\;\exists c(q,\ell):\; & c(q,\ell) = 0 \text{ for } \ell - R\notin\sigma^\vee,\\ & d(q;\ell,\eta) = c(q;\eta) - c(q;\ell),\\ & d(\xi;\eta) = \textstyle\sum_{q\in L(E)}\xi_q\cdot c(q;\eta)\,\}.\end{aligned}$$

Again, these two vector spaces can be simplified through involving the sets K_j^R (or E_j^R). Let us define

$$\begin{aligned}W := \{[\underline{x_j(q)}_{(q\in L(E_j^R))}]\,|\; & x_j(q) = x_k(q) \text{ for } \bullet\,\langle a^j, a^k\rangle < \sigma \text{ is a 2-face and}\\ & \qquad\qquad\qquad\;\; \bullet\, q\in L(E_j^R\cap E_k^R)\,,\\ & \xi\in L^2(E_j^R) \text{ implies } \textstyle\sum_q \xi_q\cdot x_j(q) = 0\,\} \text{ and}\\ W' := \{[\underline{x(q)}_{(q\in\cup_j L(E_j^R))}]\,|\; & \xi\in L(\textstyle\bigcup_j L(E_j^R)) \text{ implies } \textstyle\sum_q\xi_q\cdot x(q) = 0\,\}.\end{aligned}$$

Lemma: *The linear map $W \to \ker d_E^*$ defined by*

$$\begin{aligned}d(q;\ell,\eta) &:= \begin{cases} x_j(q) & \textit{for} \quad \ell\in K_j^R,\; \eta \ge R\\ 0 & \textit{for} \quad \ell \ge R \;\; \textit{or} \;\; \eta\in\bigcup_j K_j^R\,;\end{cases}\\ d(\xi;\eta) &:= 0\end{aligned}$$

induces injective maps

$$W\big/_{W'} \hookrightarrow \ker d_E^*\big/_{\mathrm{im}\, d_D^*} \hookrightarrow T^2(-R)\,.$$

If Y is smooth in codimension 2, both of them will be isomorphisms. (We already have known that for the latter map, cf. (3.3.1).)

Proof: 1) The map $W \to \ker d_E^*$ is *correct defined*: On the one hand, an argument as used in (3.2.7)(i) shows that $\ell \in K_j^R \cap K_k^R$ would imply $x_j(q) = x_k(q)$. On the other hand, the image of $[x_j(q)]_{q\in L(E_j^R)}$ meets all conditions in the description of $\ker d_E^*$.

2) W' maps to $\mathrm{im}\, d_D^*$ (take $c(q,\ell) := x(q)$ for $\ell \geq R$ and $c(q,\ell) := 0$ otherwise).

3) The map between the two factor spaces is *injective*: Assume for $[x_j(q)]_{q\in L(E_j^R)}$ that there exist elements $c(q,\ell)$, such that

$$\begin{aligned} c(q;\ell) &= 0 \ \text{ for } \ell \in \bigcup_j K_j^R\,, \\ x_j(q) &= c(q;\eta) - c(q;\ell) \ \text{ for } \eta \geq \ell,\ \ell \in K_j^R,\ \eta \geq R\,, \\ 0 &= c(q;\eta) - c(q;\ell) \ \text{ for } \eta \geq \ell \text{ and } [\ell \geq R \text{ or } \eta \in \bigcup_j K_j^R]\,, \text{ and} \\ 0 &= \sum_q \xi_q \cdot c(q;\eta) \ \text{ for } \eta \geq \mathrm{supp}^2 \xi\,. \end{aligned}$$

In particular, $x_j(q)$ do not depend on j, and these elements have the property

$$\sum_q \xi_q \cdot x_\bullet(q) = 0 \ \text{ for } \xi \in L(\bigcup_j L(E_j^R)).$$

4) If Y is smooth in codimension 2, the map is *surjective* :
Given an element $[d(q;\ell,\eta),\, d(\xi;\eta)] \in \ker d_E^*$, there exist complex numbers $c(q;\eta)$ such that:

(i) $d(\xi;\eta) = \sum_q \xi_q \cdot c(q;\eta)$,

(ii) $c(q;\eta) = 0$ for $\eta \notin R + \sigma^\vee$ (i.e. $\eta \in \bigcup_j K_j^R$).

(Do this separately for each η and distinguish between the cases $\eta \in R + \sigma^\vee$ and $\eta \notin R + \sigma^\vee$.)
In particular, $[c(q;\eta) - c(q;\ell),\, d(\xi;\eta)] \in \mathrm{im}\, d_D^*$. Hence, up to now, we have seen that we may assume $d(\xi;\eta) = 0$.

Let us choose some sufficiently high degree $\ell^* \geq E$. Then,

$$x_j(q) := d(q;\ell,\eta) - d(q;\ell^*,\eta)$$

(with $\ell \in K_j^R$, $\ell \geq \mathrm{supp}\, q$ (cf. Corollary (3.2.3)(a)), and $\eta \geq \ell, \ell^*, R$) defines some preimage:

(i) It is independent from the choice of η: Using a different η' generates the difference $d(q;\eta,\eta') - d(q;\eta,\eta')$.

(ii) It is independent from $\ell \in K_j^R$: Choosing another $\ell' \in K_j^R$ with $\ell' \geq \ell$ would add the summand $d(q;\ell,\ell')$ which is 0; for the general case use Corollary (3.2.3)(a).

(iii) If $\langle a^j, a^k\rangle < \sigma$ is a 2-face with $\mathrm{supp}\, q \subseteq L(E_j^R) \cap L(E_k^R)$, then by Corollary (3.2.3)(b) we may choose an $\ell \in K_j^R \cap K_k^R$ achieving $x_j(q) = x_k(q)$.

(iv) For $\xi \in L^2(E_j^R)$ we have

$$\sum_q \xi_q \cdot d(q;\ell,\eta) = \sum_q \xi_q \cdot d(q;\ell^*,\eta) = 0\,,$$

and this gives the corresponding relation for the $x_j(q)'$s.

(v) Finally, if we apply to $\underline{[x_j(q)]} \in W$ the linear map $W \to \ker d_E^*$, the result differs from $[d(q;\ell,\eta),0] \in \ker d_E^*$ by the $\operatorname{im} d_D^*$-element built from

$$c(q;\ell) := \begin{cases} d(q;\ell,\eta) - d(q;\ell^*,\eta) & \text{if } \ell \geq R \\ 0 & \text{otherwise}\,. \end{cases}$$

□

(3.3.7) Now, it is easy to complete the proofs for Theorem (3.2.4) (part 2 and 3) and Theorem (3.2.7):

First, for a tuple $\underline{[x_j(q)]}_{q\in L(E_j^R)}$, the condition

$$\xi \in L^2(E_j^R) \text{ implies } \sum_q \xi_q \cdot x_j(q) = 0$$

is equivalent to the fact that the coordinates $x_j(q)$ are induced by elements $x_j \in L(E_j^R)^*_{\mathbb{C}}$.
The other condition for elements of W just says that for 2-faces $\langle a^j, a^k\rangle < \sigma$ there is $x_j = x_k$ on $L(E_j^R \cap E_k^R)_{\mathbb{C}} = L(E_j^R)_{\mathbb{C}} \cap L(E_k^R)_{\mathbb{C}}$. In particular, we obtain

$$W = \ker\Big(\oplus_j L(E_j^R)^*_{\mathbb{C}} \to \oplus_{\langle a^j,a^k\rangle<\sigma} L(E_j^R \cap E_k^R)^*_{\mathbb{C}}\Big)\,.$$

In the same way we get

$$W' = \left(\sum_j L(E_j^R)_{\mathbb{C}}\right)^*\,,$$

and our T^2-formula is proven.

Moreover, if $\psi_j : L(E_j^R)_{\mathbb{C}} \to \mathbb{C}$ are linear maps defining an element of W, then they induce the following A-linear map on $\mathcal{D}$ (even on $\operatorname{im} d_D$):

$$\begin{array}{rcl} D(q;\ell,\eta) & \mapsto & \begin{cases} \psi_j(q)\cdot x^{\eta-R} & \text{for} \quad \ell \in K_j^R,\ \eta \geq R \\ 0 & \text{for} \quad \ell \geq R \ \text{ or } \ \eta \in \bigcup_j K_j^R \end{cases} \\ D(\xi;\eta) & \mapsto & 0\,. \end{array}$$

Now, looking at the diagram of (3.3.3), this translates exactly into the claim of Theorem (3.2.7).

3.4 An alternative to the complex $L(E^R)_\bullet$

(3.4.1) Let $R \in M$ be fixed for the whole section. The complex $L(E^R)_\bullet$ introduced in (3.2.3) fits naturally into the exact sequence

$$0 \to L(E^R)_\bullet \longrightarrow (\mathbb{Z}^{E^R})_\bullet \longrightarrow \operatorname{span}(E^R)_\bullet \to 0$$

of complexes built in the same way as $L(E^R)_\bullet$, i.e.

$$(\mathbb{Z}^{E^R})_{-k} := \oplus_{\substack{\tau<\sigma \\ dim\,\tau=k}} \mathbb{Z}^{E^R_\tau} \qquad \text{and} \qquad \operatorname{span}(E^R)_{-k} := \oplus_{\substack{\tau<\sigma \\ dim\,\tau=k}} \operatorname{span}(E^R_\tau)\,.$$

Lemma: *The complex $(\mathbb{Z}^{E^R})_\bullet$ is exact.*

Proof: The complex $(\mathbb{Z}^{E^R})_\bullet$ can be decomposed into a direct sum

$$(\mathbb{Z}^{E^R})_\bullet = \bigoplus_{r\in E} (\mathbb{Z}^{E^R})(r)_\bullet$$

showing the contribution of each $r \in E$. The complexes occurring as summands are defined as

$$\begin{aligned}(\mathbb{Z}^{E^R})(r)_{-k} \quad &:= \quad \oplus_{\substack{\tau<\sigma \\ dim\,\tau=k}} \left\{ \begin{array}{ll} \mathbb{Z} = \mathbb{Z}^{\{r\}} & \text{for } r \in E^R_\tau \\ 0 & \text{otherwise} \end{array} \right\} \\ &= \quad \mathbb{Z}^{\#\{\tau \,|\, dim\,\tau=k;\, r\in E^R_\tau\}}\,.\end{aligned}$$

Denote by H^+ the halfspace

$$H^+ := \{a \in N_{\mathbb{R}} \mid \langle a, r\rangle < \langle a, R\rangle\} \subseteq N_{\mathbb{R}}.$$

Then, for $\tau \neq 0$, the fact that $r \in E^R_\tau$ is equivalent to $\tau \setminus \{0\} \subseteq H^+$. On the other hand, $r \in E^R_0$ corresponds to the condition $\sigma \cap H^+ \neq \emptyset$.
In particular, $(\mathbb{Z}^{E^R})(r)_\bullet$, shifted by one place, equals the complex for computing the reduced homology of the topological space $\cup\{\tau \mid \tau \setminus \{0\} \subseteq H^+\} \subseteq \sigma$ cut by some affine hyperplane. Since this space is contractable, the complex is exact. □

As a corollary, we obtain

Theorem: *The complexes $L(E^R)^*_\bullet$ and span$(E^R)^*_\bullet[1]$ are quasiisomorphic. In particular, under the usual assumptions (cf. Theorem (3.2.4)), we obtain*

$$T^i_Y(-R) = H^i\left(\operatorname{span}(E^R)^*_\bullet \otimes_{\mathbb{Z}} \mathbb{C}\right)\,.$$

For later use, we mention explicitly how to change from the $L(\ldots)$- to the span-language at the T^1-level: Starting with $q \in L(E^R_0)$ (i.e. $\sum_v q_v r^v = 0$), we split it into $q = \sum_j q^j$ $(q^j \in \mathbb{Z}^{E^R_j})$ and take $v^j := -\sum_v q^j_v r^v$ for the elements of span E^R_j. Obviously, they sum up to $0 \in M$.
(The strange sign in the definition of v^j is, of course, not necessary - it just makes some upcoming formulas nicer.)

Example: In the case of two-dimensional cyclic quotient singularities (cf. (3.2.5)), the formula for $T^1_Y(-R)$ comes down to

$$T^1_Y(-R) = \left((\operatorname{span}_{\mathbb{C}} E^R_1) \cap (\operatorname{span}_{\mathbb{C}} E^R_2) \Big/ \operatorname{span}_{\mathbb{C}}(E^R_1 \cap E^R_2) \right)^*\,.$$

The several possibilities for R (as mentioned in (3.2.5)) translate into

(i) $\operatorname{span}_{\mathbb{C}} E^R_1 = (a^1)^\perp$, $\operatorname{span}_{\mathbb{C}} E^R_2 = \mathbb{C}^2$ (or $(a^2)^\perp$, if $w = 2$), and $\operatorname{span}_{\mathbb{C}} E^R_{12} = 0$.

(ii) $\operatorname{span}_{\mathbb{C}} E^R_1 = \operatorname{span}_{\mathbb{C}} E^R_2 = \mathbb{C}^2$, and $\operatorname{span}_{\mathbb{C}} E^R_{12} = 0$.

(iii) $\mathrm{span}_{\mathbb{C}} E_1^R = \mathrm{span}_{\mathbb{C}} E_2^R = \mathbb{C}^2$, and $\mathrm{span}_{\mathbb{C}} E_{12}^R = \mathbb{C} \cdot R$.

(iv) $\mathrm{span}_{\mathbb{C}} E_{12}^R = \mathrm{span}_{\mathbb{C}} E_1^R$ (or $\mathrm{span}_{\mathbb{C}} E_{12}^R = \mathrm{span}_{\mathbb{C}} E_2^R$).

Remark: The reason that makes the recent formula easier to handle than the old one is the simpler description of $\mathrm{span}_{\mathbb{C}} E_j^R$ (compared with $L(E_j^R)_{\mathbb{C}}$):

$$\mathrm{span}_{\mathbb{C}} E_j^R = \begin{cases} 0 & \text{if } \langle a^j, R\rangle \le 0 \\ (a^j)^\perp & \text{if } \langle a^j, R\rangle = 1 \\ M_{\mathbb{C}} & \text{if } \langle a^j, R\rangle \ge 2\,. \end{cases}$$

Moreover, if Y is smooth in codimension two, then $\mathrm{span}_{\mathbb{C}} E_{jk}^R = (\mathrm{span}_{\mathbb{C}} E_j^R) \cap (\mathrm{span}_{\mathbb{C}} E_k^R)$ holds for 2-faces $\langle a^j, a^k\rangle < \sigma$.

(3.4.2) We are going to use the previous theorem to develop an alternative description of $T_Y^1(-R)$ which relates infinitesimal deformations of Y (of degree $-R$) to Minkowski summands of the polyhedron $Q := \sigma \cap [R = 1]$. (It neither needs to be compact, nor its vertices to be lattice points.) If Y is smooth in codimension two, then this relation will be quite straight.

Remark: For $\langle a^j, R\rangle \ge 1$, we define $\bar{a}^j := {}^{a^j}\!/\langle a^j, R\rangle$. Those points form exactly the set of vertices of Q. Moreover, $\langle a^j, R\rangle = 1$, if and only if $\bar{a}^j$ is a lattice point (i.e. $\bar{a}^j \in N$).

Nevertheless, even in this more general situation, we are able to keep using the definition of $C(Q) \subseteq V(Q)$ (cf. (2.2.2)) - we just have to deal with the compact faces only:
Denoting by $d^1, \dots, d^N \in R^\perp \subseteq N_{\mathbb{R}}$ the compact edges of Q, we can assign to each compact 2-face $\varepsilon < Q$ its sign vector $\underline{\varepsilon} \in \{0, \pm 1\}^N$ in the usual way (cf. (2.2.1)). Then,

$$V(Q) := \{(t_1, \dots, t_N) \mid \sum_i t_i\, \varepsilon_i\, d^i = 0 \;\text{ for every compact 2-face } \varepsilon < Q\}.$$

Again, the points of the cone $C(Q) := V(Q) \cap \mathbb{R}^N_{\ge 0}$ parametrize exactly the Minkowski summands (in the sense of Definition (2.1.3)) of positive multiples of Q. Moreover, we need the additional vector space

$$W(Q) := \mathbb{R}^{\#\{\text{vertices not belonging to N}\}}$$

with coordinates s_j for each vertex $\bar{a}^j \in Q \setminus N$ (i.e. $\langle a^j, R\rangle \ge 2$).

(3.4.3) **Theorem:** *For each compact edge $d^{jk} = \overline{\bar{a}^j \bar{a}^k}$ there is a subspace $U_{jk} \subseteq V(Q)^* \oplus W(Q)^*$ belonging to one of the following three (mutually overlapping) types:*

(0) $U_{jk} = 0$,

(1) U_{jk} is generated by $s_j - s_k$ (drop those elements, i.e. set $U_{jk} = 0$, if not both coordinates exist), or

(2) U_{jk} is generated by $t_{jk} - s_j$, $t_{jk} - s_k$ (again, drop generators that do not make sense).

(The actual type of U_{jk} depends on the edge $\overline{\bar{a}^j \bar{a}^k}$ only and will be discussed in the sequel. Anyway, if Y is smooth in codimension two, then type (2) applies for each edge.)

Using that notation, we have

$$T_Y^1(-R) = \ker \left[V_{\mathbb{C}}(Q) \oplus W_{\mathbb{C}}(Q) \longrightarrow \big(\sum_{d^{jk}} U_{jk}\big)^* \right] \Big/ \;\mathbb{C} \cdot (1, \dots, 1; 1, \dots, 1)\,.$$

In some sense, the vector space $V(Q)$ (encoding the Minkowski summands) can be considered the main tool to describe infinitesimal deformations. The elements of $W(Q)$ can - depending on the type of the U_{jk}'s - either be additional parameters, or they imply dependencies between elements of $V(Q)$ (excluding Minkowski summands not fitting in a certain type).

Now, we start describing how the type of U_{jk} depends on the shape of the particular edge $\overline{\bar{a}^j\bar{a}^k}$ (only being important in case Y has two-codimensional singularities). The easiest way to do so seems to be the following one:
Replace Q by the line segment $\overline{\bar{a}^j\bar{a}^k}$ and take for U_{jk} exactly that type yielding the right $T^1(-R)$-dimension for the two-dimensional cyclic quotient singularity induced by the cone $< a^j, a^k >$ (cf. (i)-(iv) in (3.2.5) or (3.4.1)).
If one prefers a more explicit description, then denote by g^{jk} the affine line (in $N_{\mathbb{R}}$) containing the edge $\overline{\bar{a}^j\bar{a}^k}$ of Q and distinguish between the following two cases:

(a) *The line g^{jk} contains lattice points (equivalently: R is primitive, when regarded as an element of $M/(a^j,a^k)^\perp$):*

  (a.1) If the edge $\overline{\bar{a}^j\bar{a}^k}$ contains an *interior* lattice point, then we assign to it the space U_{jk} of type (0).

  (a.2) Otherwise, of type (2).

(b) *The line g^{jk} contains no lattice points at all:*

  Denote by $p \in \mathbb{N} \setminus \{0\}$ the smallest number such that lattice points are contained in $p \cdot g^{jk}$ (i.e. $R \equiv p \cdot R^0 \pmod{(a^j,a^k)^\perp}$ for some $R^0 \in M$ being primitive in $M/(a^j,a^k)^\perp$). In particular, $p \geq 2$ and $p | \langle a^j, R\rangle; \langle a^k, R\rangle$. Hence, both coordinates s_j and s_k are well defined. Now, in any case, we assign to our edge at least the equation $s_j = s_k$ - that means, type (0) will be excluded for U_{jk}. Actually, we assign to $\overline{\bar{a}^j\bar{a}^k}$ the subspace U_{jk} of

  (b.1) type (1), if the closed line segment $p \cdot \overline{\bar{a}^j\bar{a}^k}$ contains at least p *interior* lattice points (or at least $p-1$ ones, if both bounds $p\,\bar{a}^j$ and $p\,\bar{a}^k$ belong to N themselves),

  (b.2) type (2), otherwise. (This case implies that d^{jk} has (lattice-) length smaller than one.)

Remark:

- If $R \in M$ is not primitive, then neither it is in the factors $M/(a^j,a^k)^\perp$. In particular, we are always in case (b), and all coordinates s_j have to be equal on $T^1_Y(-R)$. Setting them all zero (we can do this mod $\mathbb{C} \cdot (\underline{1},\underline{1})$), we can consider $T^1_Y(-R)$ a subspace of $V_{\mathbb{C}}(Q)$.
  The condition for a (complexified) Minkowski summand belonging to $T^1_Y(-R)$ is the following: Whenever an edge d^{jk} of Q is so "short" fitting into case (b.2), then it has to be erased in the Minkowski summand.

- If Y is smooth in codimension two, then for each edge d^{jk} the elements a^j, a^k form a part of a $\mathbb{Z}$-basis for the lattice N. In particular, the line segment $p \cdot \overline{\bar{a}^j\bar{a}^k}$ does not contain any *interior* lattice point. Hence, it fits into case (a.2) or (b.2) - always inducing U_{jk} of type (2).

(3.4.4) **Proof** (of the previous theorem):
Step 1: From Theorem (3.4.1) we know that $T^1_Y(-R)$ equals the complexification of the cohomology of the complex

$$N_{\mathbb{R}} \to \oplus_j \left(\operatorname{span}_{\mathbb{R}} E_j^R\right)^* \to \oplus_{\langle a^j,a^k\rangle<\sigma} \left(\operatorname{span}_{\mathbb{R}} E_{jk}^R\right)^* .$$

According to the remark of (3.4.1), we can represent an element of $\oplus_j \left(\operatorname{span}_{\mathbb{R}} E_j^R\right)^*$ by a family of

- $b^j \in N_{I\!R}$, if $\langle a^j, R\rangle \geq 2$ and
- $b^j \in {}^{N_{I\!R}}/I\!R \cdot a^j$, if $\langle a^j, R\rangle = 1$.

Dividing by the image of $N_{I\!R}$ means to shift this family by common vectors $b \in N_{I\!R}$. More interesting is the condition for families $\{b^j\}$ mapping onto 0: For each compact edge $\overline{\bar{a}^j, \bar{a}^k} < Q$ the functions b^j and b^k have to be equal on $\operatorname{span}_{I\!R} E^R_{jk}$. Since

$$(a^j, a^k)^\perp \subseteq \operatorname{span}_{I\!R} E^R_{jk} \subseteq (\operatorname{span}_{I\!R} E^R_j) \cap (\operatorname{span}_{I\!R} E^R_k),$$

we obtain as a necessary condition $b^j - b^k \in I\!R a^j + I\!R a^k$ immediately. However, in case the cone $\langle a^j, a^k\rangle$ is singular, the actual behavior of $\operatorname{span}_{I\!R} E^R_{jk}$ requires a closer look.

Step 2: We introduce new "coordinates":

- $\bar{b}^j := b^j - \langle b^j, R\rangle \bar{a}^j \in R^\perp$ (being well defined even in the case $\langle a^j, R\rangle = 1$);
- $s_j := -\langle b^j, R\rangle$ for j meeting $\langle a^j, R\rangle \geq 2$ (inducing an element of $W(Q)$).

The shift of the b^j's by an element $b \in N_{I\!R}$ (i.e. $(b^j)' = b^j + b$) appears in the new coordinates as follows:

$$\begin{array}{rcl} (\bar{b}^j)' &=& (b^j)' - \langle (b^j)', R\rangle \bar{a}^j \;=\; b^j + b - \langle b^j, R\rangle \bar{a}^j - \langle b, R\rangle \bar{a}^j \\ &=& \bar{b}^j + b - \langle b, R\rangle \bar{a}^j\,, \\ s'_j &=& -\langle (b^j)', R\rangle \;=\; s_j - \langle b, R\rangle\,. \end{array}$$

In particular, an element $b \in R^\perp$ does not change the s_j's but shifts the $\bar{b}^j$'s inside the hyperplane $R^\perp$. Hence, the set of the $\bar{b}^j$'s should be considered modulo translation (inside $R^\perp$) only.

On the other hand, the condition $b^j - b^k \in I\!R a^j + I\!R a^k$ changes into $\bar{b}^j - \bar{b}^k \in I\!R \bar{a}^j + I\!R \bar{a}^k$ or even $\bar{b}^j - \bar{b}^k \in I\!R(\bar{a}^j - \bar{a}^k)$ (look at the values of R). Hence, the $\bar{b}^j$'s form the vertices of an (at least generalized) Minkowski summand of Q. Modulo translation, this summand is completely described by the dilatation factors defined by

$$\bar{b}^j - \bar{b}^k = t_{jk} \cdot (\bar{a}^j - \bar{a}^k)\,.$$

Now, the remaining part of the action of $b \in N_{I\!R}$ comes down to an action of $\langle b, R\rangle$ only:

$$\begin{array}{rcll} t'_{jk} &=& t_{jk} - \langle b, R\rangle & \text{and} \\ s'_j &=& s_j - \langle b, R\rangle & \text{(as we already have known).} \end{array}$$

Up to now, we have obtained that $T^1_Y(-R) \subseteq V_{I\!\!C}(Q) \oplus W_{I\!\!C}(Q)/(\underline{1}, \underline{1})$.

Step 3: Actually, the elements b^j and b^k have to be equal on $\operatorname{span}_{I\!R} E^R_{jk}$ which may be a larger space than just $(a^j, a^k)^\perp$. To measure the difference we consider the factor $\operatorname{span}_{I\!R} E^R_{jk} \big/ (a^j, a^k)^\perp$ contained in the two-dimensional vector space ${}^{M_{I\!R}}\big/(a^j, a^k)^\perp = \operatorname{span}_{I\!R}(a^j, a^k)^*$. Since this factor coincides with the set $\operatorname{span}_{I\!R} E^{\bar{R}}_{jk}$ assigned to the two-dimensional cone $\langle a^j, a^k\rangle \subseteq \operatorname{span}_{I\!R}(a^j, a^k)$ ($\bar{R}$ denotes the image of R in $\operatorname{span}_{I\!R}(a^j, a^k)^*$), we may assume that $\sigma = \langle a^1, a^2\rangle$ (i.e. $j = 1$, $k = 2$) represents a two-dimensional cyclic quotient singularity. In particular, we have just to discuss the four cases (i)-(iv) from the example contained in (3.2.5) and (3.4.1):

Lemma: Let $\sigma = \langle a^1, a^2\rangle$ be a two-dimensional cone as in (3.2.5).

(1) Let $R \in \operatorname{int} \sigma^\vee \cap M$ be primitive. Then, R is indecomposable (i.e. an element of $\{r^1, \ldots, r^{w-1}\}$), if and only if the line segment $\overline{\bar{a}^1 \bar{a}^2} = \sigma \cap [R = 1]$ contains an *interior* lattice point.

(2) Assume $R = r^v$ ($v \in \{1, \dots, w-1\}$). Then, the number a_v (defined by $r^{v-1} + r^{v+1} = a_v \cdot r^v$) is one bigger than the number of *interior* lattice points of $\overline{\bar{a}^1 \bar{a}^2}$.

Proof: Both statements are contained in the explanations of [Od], §1.6. Nevertheless, we present the short proof here directly:

(1) If $R = r^v$, then $\{r^v, r^{v+1}\}$ is a $\mathbb{Z}$-basis of M, and the equations $\langle a^*, r^v\rangle = \langle a^*, r^{v+1}\rangle = 1$ define an interior lattice point a^* of σ.

For the reverse implication, assume that $a^* \in \operatorname{int}\sigma \cap N \cap [R = 1]$. Then, a decomposition $R = R_1 + R_2$ inside $\sigma^\vee \cap M$ would (w.l.o.g.) imply that $\langle a^*, R_1\rangle = 0$ - contradicting the fact that a^* belongs to the interior of σ.

(2) An interior lattice point a^* of $\overline{\bar{a}^1 \bar{a}^2}$ is characterized by the conditions

$$\langle a^*, r^v\rangle = 1\,;\ \langle a^*, r^{v+1}\rangle \in \mathbb{Z}\,;\ \langle a^*, r^{v+1}\rangle \geq 1\,;\ \langle a^*, r^{v-1}\rangle \geq 1\,.$$

Since $r^{v-1} + r^{v+1} = a_v \cdot r^v$, those points are parametrized by integers g ($= \langle a^*, r^{v+1}\rangle$) meeting $1 \leq g \leq a_v - 1$. □

Now, let us start with case (a), i.e. we assume that R is a primitive element of M.

By the first part of the previous lemma we know that (a.1) is a synonymous for $R \in \{r^1, \dots, r^{w-1}\}$. Those degrees were discussed in (i) and (ii) of (3.2.5) and (3.4.1) - we have obtained $\operatorname{span}_{\mathbb{R}} E_{12}^R = 0$. The lack of any additional equations for the b^j's means that U_{12} has to be zero, i.e. of type (0).

The remaining primitive degrees yield $T^1_Y(-R) = 0$ or, equivalently,

$$\operatorname{span}_{\mathbb{R}} E_{12}^R = (\operatorname{span}_{\mathbb{R}} E_1^R) \cap (\operatorname{span}_{\mathbb{R}} E_2^R)\,.$$

Hence, at least after a suitable lifting of b^j to $N_{\mathbb{R}}$, we can use the equation $b^1 = b^2$ in $N_{\mathbb{R}}$:

$$\bar{b}^1 = b^1 - \langle b^1, R\rangle \bar{a}^1 = b^2 - \langle b^1, R\rangle \bar{a}^1 = \bar{b}^2 - \langle b^1, R\rangle \bar{a}^1 + \langle b^2, R\rangle \bar{a}^2$$

yields

$$t_{12} \cdot (\bar{a}^1 - \bar{a}^2) = \bar{b}^1 - \bar{b}^2 = -\langle b^1, R\rangle \bar{a}^1 + \langle b^2, R\rangle \bar{a}^2\,.$$

In particular, as far as defined, the variables s_j ($j = 1, 2$) equal t_{12}. This is encoded in the vector space U_{12} of type (2).

For case (b), we assume that $R = p \cdot R^0$ with some primitive $R^0 \in M$. Using the first part of the previous lemma again, we see that condition (b.1) implies $R^0 \in \{r^1, \dots, r^{w-1}\}$. Moreover, it is the second part of that lemma telling us that (b.1) is actually equivalent to (iii) from (3.2.5) and (3.4.1).

Hence, assuming (b.1), we obtain $\operatorname{span}_{\mathbb{R}} E_{12}^R = \mathbb{R} \cdot R$ as the space where b^1 and b^2 have to be equal on. That means $s_1 = s_2$, and we need U_{12} of type (1).

On the other hand, the class (b.2) implies $T^1_Y(-R) = 0$ again. Similarly to the case (a.1), we have to choose type (2) for U_{12}.

Theorem (3.4.3) (including the characterization of the vector spaces U_{jk}) is proven. □

(3.4.5) Finally, we assume that Y is smooth in codimension two. Then, U_{jk} is always of the (maximal) type (2), and the additional coordinates s_j of $W(Q)$ are completely determined by those of $V(Q)$ (but still enforcing some dependencies between them). In particular, as a corollary from (3.4.3), we obtain

Theorem: *If Y is smooth in codimension two, then $T^1_Y(-R) \subseteq {}^{V_{\mathbb{C}}(Q)}/_{\mathbb{C}} \cdot (\underline{1})$ is given by the equations*

$$t_{jk} = t_{kl} \quad \textit{(if } d^{jk},\ d^{kl} \textit{ are compact edges with a common non-lattice vertex } \bar{a}^k\textit{).}$$

That means, infinitesimal deformations of Y are in one-to-one correspondence to those equivalence classes of Minkowski summands of Q keeping the stars of non-lattice vertices in Q unchanged up to homothety.

(3.4.6) Still assuming Y to be smooth in codimension two, we try to illuminate the behavior of $T^2_Y(-R)$. Let us start with the following commutative diagram built from the first terms of the span-complex defined in (3.4.1):

$$\begin{array}{ccccc} & & & & \oplus_{\varepsilon<\sigma}\big(\cap_{a^j\in\varepsilon}\operatorname{span}E^R_j\big)^* \\ & & & \nearrow & \downarrow \\ \oplus_{a^j\in\sigma}\big(\operatorname{span}E^R_j\big)^* & \longrightarrow & \oplus_{\langle a^j,a^k\rangle<\sigma}\big(\operatorname{span}E^R_{jk}\big)^* & & \\ & & & \searrow & \downarrow \\ & & & & \oplus_{\varepsilon<\sigma}\big(\operatorname{span}E^R_\varepsilon\big)^* \end{array}$$

($\varepsilon<\sigma$ stands for three-dimensional faces of σ.) The vertical map is surjective; it would be an isomorphism, if Y was smooth in codimension three. According to Theorem (3.4.1), $T^2_Y(-R)$ equals the cohomology of the (complexified) bottom row. In particular, there is a map

$$\Psi : T^2_Y(-R) \to \bigoplus_{\varepsilon<\sigma}\Big(\cap_{a^j\in\varepsilon}\operatorname{span}E^R_j\Big/\operatorname{span}E^R_\varepsilon\Big)^* \subseteq \bigoplus_{\varepsilon<\sigma}\big(\cap_{a^j\in\varepsilon}\operatorname{span}_{\mathbb{C}}E^R_j\big)^*.$$

"evaluating" this vector space.

Proposition: *If $R\in M$ is always different from 1 on the fundamental generators a^j of σ (e.g. equals a proper multiple of some other degree), or if $\dim\sigma=3$, then Ψ is injective.*

Proof: All we have to show is exactness of the top row in the previous diagram. Since this is obvious in the three-dimensional case, let us assume $\langle a^j,R\rangle\neq 1$ for $j=1,\dots,M$. Then, the Abelian groups $\operatorname{span}E^R_j$ involved in that complex are either M or 0, depending on $\langle a^j,R\rangle>0$ or not. Similarly to the proof of Lemma (3.4.1) this means that our complex measures the reduced homology ($\otimes_{\mathbb{Z}}M$) of the topological space $\cup\{\tau\mid\tau\setminus\{0\}\subseteq[R>0]\}\subseteq\sigma$ cut by some affine hyperplane. □

Corollary: *If R does not yield 1 on the a^j's, then Ψ provides an embedding*

$$\Psi : T^2_Y(-R)\hookrightarrow N_{\mathbb{C}}^{\{\varepsilon<\sigma\,\mid\,R>0\text{ on }\varepsilon\}}.$$

Moreover, its image in the ε-th summand is killed by $\operatorname{span}_{\mathbb{C}}E^R_\varepsilon$.

3.5 First applications

(3.5.1) For an $R\in M$ we denote by $\tau(R)$ the smallest face of σ containing all generators a^j meeting $\langle a^j,R\rangle\geq 1$. (In particular, $\langle a^k,R\rangle\leq 0$ for $a^k\in\sigma\setminus\tau$.)
If $Y(R)$ stands for the affine variety assigned to $\tau(R)$ (considered as a top-dimensional cone in $\operatorname{span}_{\mathbb{R}}\tau(R)=\tau(R)-\tau(R)$), then $Y(R)\times(\mathbb{C}^*)^{\dim\sigma-\dim\tau(R)}$ is an open subset of Y.

Proposition: *Let $\bar{R}$ be the image of R in $(\mathrm{span}_{I\!R}\tau(R))^* = M/_{\tau(R)^\perp}$. Then, the infinitesimal deformations of Y in degree $-R$ are caused by those of $Y(R)$, i.e. $T^1_Y(-R) = T^1_{Y(R)}(-\bar{R})$.*

Proof: The polyhedra $Q := \sigma \cap [R = 1] \subseteq N_{I\!R}$ and $\bar{Q} := \tau \cap [\bar{R} = 1] \subseteq \mathrm{span}_{I\!R}\tau(R) \subseteq N_{I\!R}$ are built from the same vertices ($\bar{a}^j$ with $\langle a^j, R\rangle \geq 1$) but different cones of unbounded directions ($\sigma \cap R^\perp$ for Q and $\tau \cap \bar{R}^\perp = \tau \cap R^\perp$ for $\bar{Q}$).
In particular, the compact faces of Q and $\bar{Q}$ are the same, and we obtain

$$V(Q) = V(\bar{Q})\,,\; W(Q) = W(\bar{Q})\,,\; \text{ and } U_{jk}(Q) = U_{jk}(\bar{Q})\,. \qquad \Box$$

Corollary: *If T^1_Y is finite-dimensional, then for all proper faces $\tau < \sigma$ the toric variety Y_τ associated to the cone τ is rigid.*

Proof: Let $\bar{R} \in M/_{\tau^\perp}$ (represented by some $R \in M$) such that $T^1_{Y_\tau}(-\bar{R}) \neq 0$. If $r \in \sigma^\vee \cap M$ defines the face τ (i.e. $\tau = \sigma \cap r^\perp$), then $R - k\cdot r$ still represents $\bar{R}$, and, moreover, $\tau(R - k\cdot r) \subseteq \tau$ for $k \gg 0$. In particular, $T^1_Y(-[R - k\cdot r]) = T^1_{Y_\tau}(-\bar{R}) \neq 0$ for those (infinitely many) k. $\Box$

Since none of the two-dimensional cyclic quotient singularities are rigid, we finally obtain:
At least three-dimensional affine toric varieties have an infinite-dimensional T^1 unless they are smooth in codimension two.

(3.5.2) Assume that Y is smooth in codimension two.

Proposition: *$T^1_Y(-R) \neq 0$ implies that $\{a^j \mid \langle a^j, R\rangle \leq 1\}$ is not contained in any proper face of σ.*

Proof: Assume that $\{a^j \mid \langle a^j, R\rangle \leq 1\} \subseteq \tau$. We choose some vertex $\bar{a} \in Q$ not coming from τ (in particular, not being contained in the lattice N); from (3.4.5) we know that the same dilatation factor, say t, has to be assigned to all edges containing $\bar{a}$.
Now, we consider the remaining compact edges of Q. Those which are not contained in τ can be connected via a path along edges through non-τ-vertices with $\bar{a}$. In particular, they obtain the same dilatation factor t.
On the other hand, if we are given a compact edge d contained in τ, then there is at least one two-dimensional compact face ε not contained in τ but containing this given edge d. Then, we already know that the dilatation factors of its edges (except that of d) equal t. However, using the equation for ε in the definition of $V(Q)$ (cf. (3.4.2)), this has to hold for d, too. $\Box$

Corollary: *There are just finitely many $R \geq 0$ such that $T^1_Y(-R) \neq 0$.*

Proof: There are just finitely many different choices for selecting elements a^j to yield $\langle \bullet, R\rangle =$ 0 or 1. Fix on of those choices and denote

$$H^0 := \{a^j \mid \langle a^j, R\rangle = 0\}\,; \quad H^1 := \{a^j \mid \langle a^j, R\rangle = 1\}\,.$$

We already know that $H^0 \cup H^1$ may not be contained in any proper face of σ. We distinguish two cases:

(a) $\mathrm{span}_{I\!R}(H^0 \cup H^1) = N_{I\!R}$: Then, R (if there is any) is uniquely determined by H^0 and H^1.

(b) $\mathrm{span}_{I\!R}(H^0 \cup H^1) =: H \subseteq N_{I\!R}$: Then, H contains interior points of σ - let b be one of them. Since the value of R on b is determined by H^0 and H^1, we can show that the value of R on the fundamental generators a^j is bounded: On the one hand, we have $\langle a^j, R\rangle \geq 0$. On the other hand, to each a^j we can associate the line through a^j and b. Moreover, we can choose some point $(a^j)' \in \sigma$ sitting at this line - but at the other side of b as a^j does. Then, the condition $\langle (a^j)', R\rangle \geq 0$ gives an upper bound for $\langle a^j, R\rangle$. □

Remark: Degrees $R \in M$ such that $R \geq 0$ are, at least in some sense, easier to handle. In section (4.4) it is exactly that class making a nicer description of the cup product possible. Moreover, only for $R \in \sigma^\vee \cap M$ the cone σ can be reconstructed from the polyhedron Q (via $\sigma = \overline{\mathbb{R}\cdot Q}$). The latter is the main reason for the construction of the versal deformation working well in those degrees (cf. chapter 6).
On the other hand, in dimension three, the modules $T^2_Y(-k\,R)$ ($k \geq 1$) vanish unless R is contained in $\operatorname{int}\sigma^\vee$ (cf. the upcoming section (3.5.4)). Hence, deformations of those degrees are unobstructed.

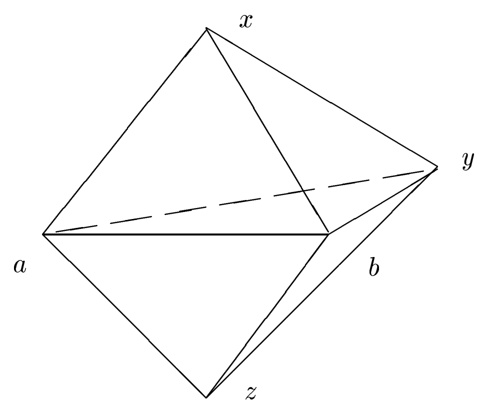
(3.5.3) Proposition: *We still assume that Y is smooth in codimension two. Let $R \in \operatorname{int}\sigma^\vee \cap M$. If every 2-face of $Q := \sigma \cap [R = 1]$ contains at most three vertices belonging to the lattice, then $T^1_Y(-R) = 0$.*

Proof: Since R is an interior point of $\sigma^\vee$, the polyhedron Q is compact.
Now, for every 2-face $\varepsilon < Q$, the assumption of containing at most three lattice points as vertices means that there are in fact only three different dilatation factors left in ε. On the other hand, triangles having mutually equal angles are homothetical. It is that simple fact (encoded in the ε-equation of $V(Q)$) telling that even those three dilatation factors have to be equal. □

Corollary: *Let Y be smooth in codimension two. If the three-dimensional faces of σ are simplicial cones (i.e. built from three fundamental generators), then $T^1_Y(-R) = 0$ for each $R \in \operatorname{int}\sigma^\vee \cap M$.*

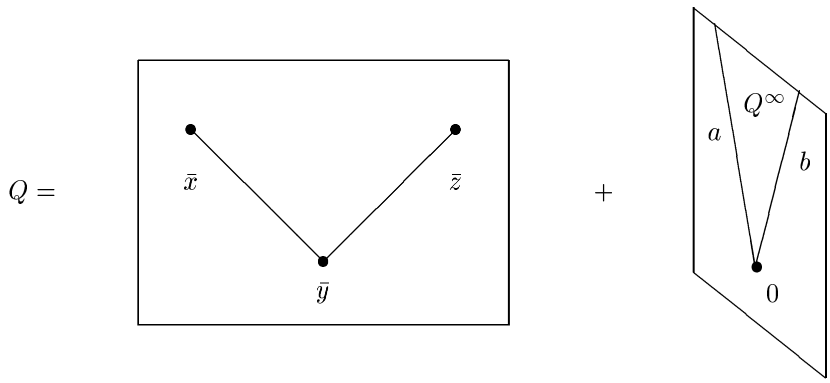
However, it is not always true that those cones (even if smooth in codimension three) provide rigid singularities:

Example: Let $\sigma = \langle a, b, x, y, z\rangle$ be the (four-dimensional) cone over some double tetrahedron.



$$\begin{aligned} a &= (1,0,0,0)\\ b &= (0,1,0,0)\\ y &= (0,0,1,0)\\ x &= (1,1,1,-1)\\ z &= (1,1,1,1) \end{aligned}$$

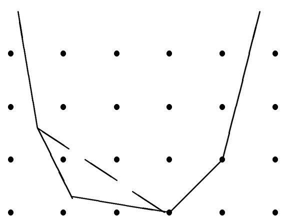
(The two partial cones generated by a, b, y, x and a, b, y, z, respectively, are smooth. Hence, so are the facets of σ.)
Let $R = [0, 0, 1, 0]$. Then, the compact part of Q just consists of two edges (containing the lattice-vertices $\bar{x}, \bar{y}, \bar{z}$).



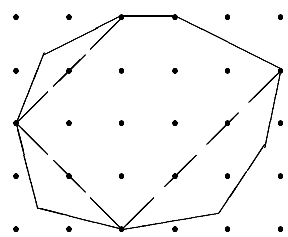
In particular, $\dim T^1_Y(-R) = 1$.
(Moreover, for me, it seems to be that all examples for non-rigid, four-dimensional, isolated toric singularities are more or less of that type.)

(3.5.4) *Three-dimensional toric varieties with isolated singularities:*
Let $\sigma = \langle a^1, \dots, a^M \rangle$ be a three-dimensional cone with smooth facets, i.e. every pair $\{a^j, a^{j+1}\}$ ($j = 1, \dots, M$; $M+1 := 1$) can be completed to some $\mathbb{Z}$-basis of the lattice N. As usual, fixing some degree $R \in M$ provides the polygon $Q := \sigma \cap [R = 1]$ with its cone of infinite directions Q^∞. Now, we construct a second polygon $\bar{Q}$ by cutting all compact corners containing non-lattice vertices:

$$\bar{Q} := \operatorname{conv}\{\bar{a}^j \in Q \mid \langle a^j, R \rangle = 1, \text{ or } \bar{a}^j \text{ is contained in some non-compact edge}\} + Q^\infty \subseteq Q\,.$$



Two examples
for $\bar{Q} \subseteq Q$



Proposition: *Infinitesimal deformations of Y in degree $-R$ correspond to (complexified) Minkowski summands of $\bar{Q}$ (modulo homothety). Actually,*

$$\dim T^1_Y(-R) = \begin{cases} \#\{\bar{a}^j \mid \bar{a}^j \in N, \ i.e.\ \langle a^j, R\rangle = 1\} - 3 & \text{if } R > 0 \\ \#\{\bar{a}^j \mid \bar{a}^j \in N, \ \text{not contained in a non-compact edge}\} & \text{if } R \not> 0\,. \end{cases}$$

For the obstructions in degree $-R$, we obtain

$$T^2_Y(-R)^* = \left. \cap_{j=1}^M (\operatorname{span}_{\mathbb{C}} E^R_j) \middle/ \operatorname{span}_{\mathbb{C}}(\cap_{j=1}^M E^R_j) \right.\,,$$

which is a subquotient of $M_{\mathbb{C}}$. In particular, $T^2_Y(-R)$ is at most three-dimensional and vanishes unless $R > 0$.

Proof: The first part is a direct consequence of Theorem (3.4.5). For the T^2-formula we could either look at (3.4.6), or we take Theorem (3.4.1) and use the obvious fact

$$\ker\left[\oplus_j \operatorname{span} E^R_{j\,j+1} \to \oplus_j \operatorname{span} E^R_j\right] = \cap_{j=1}^M \operatorname{span} E^R_j\,. \qquad \square$$

Remark:

(1) Assume that those edges of $\bar{Q}$ containing interior lattice points admit lattice points as vertices, too. Then, the cones $\sigma = \overline{I\!R_{\geq 0} \cdot Q}$ and $\bar{\sigma} := \overline{I\!R_{\geq 0} \cdot \bar{Q}}$ define affine toric varieties with the same infinitesimal deformations in degree $-R$.

(2) It is good to know obstructions in proper multiples of some fixed degree. For $R \in M$ and $k \geq 2$ we obtain

$$T^2_Y(-kR)^* = \begin{cases} M_{\mathbb{C}} / \operatorname{span}_{\mathbb{C}}(\cap_{j=1}^M E_j^{kR}) & \text{if } R > 0 \\ 0 & \text{if } R \not> 0\,. \end{cases}$$

Moreover, we know that $R \in \operatorname{span}_{\mathbb{C}}(\cap_{j=1}^M E_j^{kR})$ i.e. $T^2_Y(-kR)$ can be at most two-dimensional.

Example: Let $(\alpha_1, -\alpha_2, \alpha_3, -\alpha_4) \in \mathbb{Z}^4$ be a primitive lattice point with $\alpha_j > 0$ $(j = 1,2,3,4)$. Then, $\mathbb{Z}^4 / \mathbb{Z} \cdot (\alpha_1, -\alpha_2, \alpha_3, -\alpha_4) \cong \mathbb{Z}^3$, i.e. we obtain an exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^4 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow 0 \\ & 1 & \mapsto & (\alpha_1, -\alpha_2, \alpha_3, -\alpha_4) & & & \\ & & & e^j & \longmapsto & a^j & . \end{array}$$

Now, the cone $\sigma := \langle a^1, a^2, a^3, a^4 \rangle \subseteq I\!R^3$ admits the following properties:

- $\alpha_1 a^1 + \alpha_3 a^3 = \alpha_2 a^2 + \alpha_4 a^4$
- There exists an $R \in M$ such that $\langle a^j, R \rangle = g_j$, if and only if $\alpha_1 g_1 + \alpha_3 g_3 = \alpha_2 g_2 + \alpha_4 g_4$. (The fundamental generators $a^j \in \mathbb{Z}^3$ are in the most general position that is possible when (1) holds.) We denote those R by $R \sim [g_1, g_2, g_3, g_4]$.
- Y has an isolated singularity (i.e. is smooth in codimension two), if and only if $\gcd(\alpha_j, \alpha_{j+1}) = 1$ $(j = 1, \ldots, 4,\ \alpha_5 := \alpha_1)$.

For the behavior of T^1_Y, there are three mutually excluding cases:

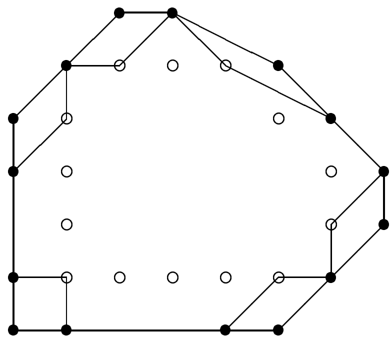
(i) $\alpha_1 \in \alpha_2 + \alpha_4 + I\!N\alpha_2 + I\!N\alpha_4 + I\!N\alpha_3$ or $\alpha_3 \in \alpha_2 + \alpha_4 + I\!N\alpha_2 + I\!N\alpha_4 + I\!N\alpha_1$ (implying $\alpha_1 \geq \alpha_2 + \alpha_4$ or $\alpha_3 \geq \alpha_2 + \alpha_4$, respectively):
Then, for every $R \in M$ (and there is at least one) shaped as $R \sim [1, \geq 1, \leq 0, \geq 1]$ or $R \sim [\leq 0, \geq 1, 1, \geq 1]$, we have $\dim T^1_Y(-R) = 1$.

(ii) Analogously, there is the case $\alpha_2 \in \alpha_1 + \alpha_3 + I\!N\alpha_1 + I\!N\alpha_3 + I\!N\alpha_4$ or $\alpha_4 \in \alpha_1 + \alpha_3 + I\!N\alpha_1 + I\!N\alpha_3 + I\!N\alpha_2$.

(iii) Finally, we assume $\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4$. This condition is equivalent to Y being Gorenstein (cf. (5.1.1)) and implies that the whole space T^1_Y will be one-dimensional (concentrated in degree $R \sim [1,1,1,1]$).

Y will be rigid, if the numbers $\alpha_1, \ldots, \alpha_4$ do not satisfy any of the previous three conditions.

(3.5.5) *Cones over smooth, projective, toric surfaces:*
Let $P \subseteq I\!R^2$ be a lattice polygon defining some projective, toric surface X_P (cf. (2.4.4)). That surface will be smooth, if and only if the vectors along adjacent edges form $\mathbb{Z}$-bases of $\mathbb{Z}^2$. (Equivalently: The lattice parallelograms at the corners of P have to have volume one each.)



Polygon P with parallelograms of volume one drawn at some corners

We use the notation of (2.4.4): $\sigma \subset N_{\mathbb{R}} := \mathbb{R}^3$ is the dual of the cone over P (embedded in an affine plane of height one in $\mathbb{R}^3$); $Y = Y_\sigma$ is the cone (in the sense of algebraic geometry) over X_P. Since Y is a cone, its affine coordinate ring as well as modules such as T^1_Y admit a natural $\mathbb{Z}$-grading. It is induced from our more frequently used M-(multi)grading via $\langle a^*, \bullet \rangle$.

Proposition: *Except P is (as a lattice polytope) isomorphic to the unit square (yielding $Y = [xy - zw = 0] \subseteq \mathbb{C}^3$), we have*

$$(T^1_Y)_- := \oplus_{k\geq 1} T^1_Y(-k) = T^1_Y(-1) = \oplus_{R\in P} T^1_Y(-R)\,.$$

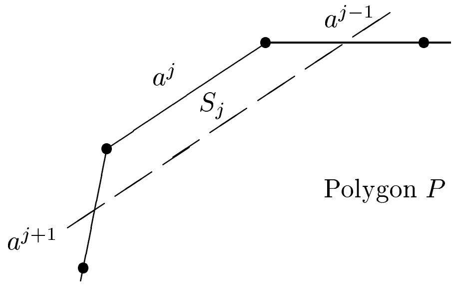
Proof: *Step 1: In the direct sum $(T^1_Y)_- = \oplus_{\langle a^*, R\rangle \geq 1} T^1_Y(-R)$ there occur only elements $R \in \sigma^\vee \cap M$:*
Assume that $T^1_Y(-R) \neq 0$ and $\langle a^*, R\rangle \geq 1$. The first condition implies that there is an $a^j \in \sigma$ meeting

$$\langle a^j, R\rangle = 1\,, \quad \langle a^{j-1}, R\rangle,\ \langle a^{j+1}, R\rangle \geq 1\,;$$

in particular, R is non-negative on the cone $\sigma' := \langle a^{j-1}, a^j, a^{j+1}, a^* - a^j\rangle \subseteq N_{\mathbb{R}}$. Hence, it remains to show that $\sigma \subseteq \sigma'$:

Let $a \in \sigma \cap N$ but $a \notin \sigma'$. Since a cannot be contained in the σ-edges emerging from a^j, the sum $a + a^j$ has to be even an interior point of σ, i.e. $a - (a^* - a^j) = (a + a^j) - a^* \in \sigma \cap N$. Continuing this process, there are just two different possibilities:
Case 1: $a - k(a^* - a^j) \in \sigma \cap N$ for every $k \in \mathbb{N}$ - but this would imply that $a^j - a^* = -(a^* - a^j) \in \sigma$ (and a^j would not longer be a fundamental generator of σ).
Case 2: There is a $k \in \mathbb{N}$ such that $a - k(a^* - a^j) \in \sigma'$. Hence, $a = [a - k(a^* - a^j)] + k(a^* - a^j) \in \sigma'$.

Step 2: $\langle a^*, R\rangle \geq 2$ (and $R \in \sigma^\vee \cap M$) implies $T^1_Y(-R) = 0$:
Assuming that $\langle a^*, R\rangle \geq 2$, the point $\bar{R} := R/\langle a^*, R\rangle$ is contained in the polytope P, and each equation $\langle a^j, R\rangle = 1$ implies that $\langle a^j, \bar{R}\rangle = 1/\langle a^*, R\rangle \leq 1/2$.
The strips $S_j := \{\bar{R}\,|\ \langle a^*, \bar{R}\rangle = 1;\ 0 \leq \langle a^j, \bar{R}\rangle \leq 1/2\}$ along the edges $(a^j)^\perp$ of P intersect the adjacent edges in the first half on the way to the next lattice point.

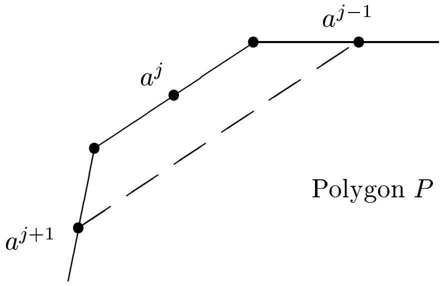


Polygon P

In particular, neither there could be $\langle a^j, R\rangle = 1$ (i.e. $\bar{R} \in S_j$) and $\bar{R} \in (a^k)^\perp$ at the same time for non-adjacent a^j, a^k, nor (up to the exception made in the proposition) $\bar{R}$ could be contained in four different strips $S_\bullet$. Hence, $T^1_Y(-R) = 0$. □

Which of the cones over smooth, projective, toric surfaces admit non-trivial, negative deformations? We fix some lattice point $R \in P$ and ask for conditions for $T^1_Y(-R) \neq 0$:

(i) Assume that $R \in \partial P$:
Then, there has to be some a^j such that $\langle a^j, R\rangle = 1$ and $\langle a^{j-1}, R\rangle$, $\langle a^{j+1}, R\rangle \geq 1$. Hence, R is as well a boundary point of P as an interior point of the (in the picture dashed) line connecting the two lattice points on ∂P next to the edge $(a^j)^\perp$.



In particular, this dashed line has to be an edge of P by itself, i.e. P is contained in a strip of (lattice-) thickness one.
P is a quadrangle with (after choosing suitable coordinates) vertices $[0,0]$, $[a,0]$, $[b,1]$, $[0,1]$ ($a, b \in \mathbb{Z}$; $1 \leq a \leq b$). It does not contain any interior lattice points, but each lattice point R (except the four vertices) on ∂P provides a one-dimensional $T^1_Y(-R)$.
Y is a scroll given by the cone $\sigma = \langle (1,0,0), (0,1,0), (-1, b-a, a), (0,-1,1)\rangle$ (i.e. $\alpha_1 = \alpha_3 = 1$, $\alpha_2 = b$, $\alpha_4 = a$ in the language of Example (3.5.4)) or by the equations

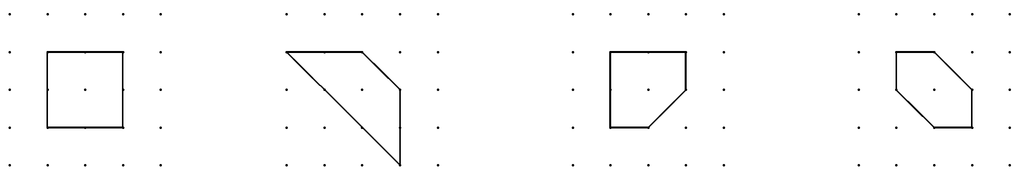
$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & \dots & y_{a-1} & z_0 & \dots & z_{b-1} \\ y_1 & y_2 & \dots & y_a & z_1 & \dots & z_b \end{pmatrix} \leq 1\,.$$

Except for $a = b = 1$ (yielding $[xy - zw = 0] \subseteq \mathbb{C}^3$ again), we obtain $\dim(T^1_Y)_- = a + b - 2$. Moreover, it is possible to write down the negative part of the versal deformation (with parameters s_v, t_μ) directly:

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 + s_1 & \dots & y_{a-1} + s_{a-1} & z_0 & z_1 + t_1 & \dots & z_{b-1} + t_{b-1} \\ y_1 & y_2 & \dots & y_a & z_1 & z_2 & \dots & z_b \end{pmatrix} \leq 1\,.$$

(ii) Assume that R is an interior lattice point of P:
In a similar way as in (i), we obtain first that P has to be contained in strip of (lattice-) thickness two. Actually, for P and $R \in \operatorname{int} P$ there are the following possibilities:

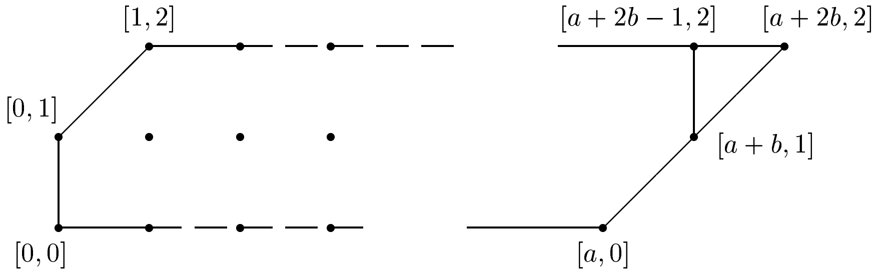
(ii.1) We have four special polygons containing exactly one interior lattice point R^* each. In either case, Y is Gorenstein and $T^1_Y = (T^1_Y)_-$ is concentrated in degree $-R^*$ (of dimension 1, 1, 2, and 3, respectively).



Polygon P'_4 Polygon P_4 Polygon P_5 Polygon P_6

The versal base space of toric Gorenstein singularities will be constructed in chapter 5. P_4' corresponds to $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2))$, and the polygons P_4, P_5, and P_6 induce Del Pezzo surfaces of degree 8, 7, and 6, respectively; for a detailed discussion of the latter three examples we refer to (5.7.1) (also showing the duals Q_i of P_i).

(ii.2) Moreover, there are also two series of polygons: pentagons with vertices $[1,2]$, $[0,1]$, $[0,0]$, $[a,0]$, and $[a+2b,2]$ and hexagons with the two vertices $[a+b,1]$ and $[a+2b-1,2]$ instead of $[a+2b,2]$ (with $a,b \in \mathbb{Z};\ a,b \geq 1$).



Except for $a = b = 1$, the vector space $(T^1_Y)_-$ is one-dimensional in the pentagon case (concentrated in degree $-[1,1]$) and two-dimensional in the hexagon case (concentrated in the degrees $-[1,1]$ and $-[a+b-1,1]$). For $a = b = 1$, we obtain our polygons P_5 and P_6 from (ii.1) again.

Chapter 4

The cup product

4.1 The cup product in general

(4.1.1) Using the notation of section (3.1), it is well known (cf. [Ld], (5.1.5)) that the cup product $T^1 \times T^1 \to T^2$ can be defined in the following way:

(i) Starting with an $\varphi \in \mathrm{Hom}_A({}^{I}\!/_{I^2}, A)$, we lift the images of the f_i obtaining elements $\tilde{\varphi}(f_i) \in P$.

(ii) Given a relation $r \in \mathcal{R}$, the linear combination $\sum_i r_i\, \tilde{\varphi}(f_i)$ vanishes in A, i.e. it is contained in the ideal $I \subseteq P$. Denote by $\lambda(\varphi) \in P^m$ any set of coefficients such that

$$\sum_i r_i\, \tilde{\varphi}(f_i) + \sum_i \lambda_i(\varphi)\, f_i = 0 \quad \text{in } P.$$

(Of course, λ depends on r also.)

(iii) If $\varphi, \psi \in \mathrm{Hom}_A({}^{I}\!/_{I^2}, A)$ represent two elements of T^1_Y, then we define for each relation $r \in \mathcal{R}$

$$(\varphi \cup \psi)(r) := \sum_i \lambda_i(\varphi)\, \psi(f_i) + \sum_i \varphi(f_i)\, \lambda_i(\psi) \ \in A\,.$$

Remark: The definition of the cup product does not depend on the choices we made:

(a) Choosing a $\lambda'(\varphi)$ instead of $\lambda(\varphi)$ yields $\lambda'(\varphi) - \lambda(\varphi) \in \mathcal{R}$, i.e. in A we obtain the same result.

(b) Let $\tilde{\varphi}'(f_i)$ be different liftings to P. Then, the difference $\tilde{\varphi}'(f_i) - \tilde{\varphi}(f_i)$ is contained in I, i.e. it can be written as some linear combination

$$\tilde{\varphi}'(f_i) - \tilde{\varphi}(f_i) = \sum_j t_{ij}\, f_j\,.$$

Hence,

$$\sum_i r_i\, \tilde{\varphi}'(f_i) = \sum_i r_i\, \tilde{\varphi}(f_i) + \sum_{i,j} t_{ij}\, r_i\, f_j\,,$$

and we can define $\lambda'_j(\varphi) := \lambda_j(\varphi) - \sum_i t_{ij}\, r_i$ (corresponding to $\tilde{\varphi}'$ instead of $\tilde{\varphi}$). Then, we obtain for the cup product

$$(\varphi \cup \psi)'(r) - (\varphi \cup \psi)(r) = -\sum_i r_i \cdot \left(\sum_j t_{ij}\, \psi(f_j) \right)\,,$$

but this expression comes from some map $P^m \to A$.

(4.1.2) The associated to the cup product quadratic form $T^1 \to T^2$ can also be obtained in the following way: Considering the exact sequence

$$0 \to \mathbb{C} \stackrel{\cdot \varepsilon^2}{\longrightarrow} \mathbb{C}[\varepsilon]/\varepsilon^3 \longrightarrow \mathbb{C}[\varepsilon]/\varepsilon^2 \to 0\,,$$

we may ask for lifting infinitesimal first-order deformations ξ (elements of T^1) to the second order. Hence, using the construction of (3.1.4) ($W := \mathbb{C}$), each $\xi \in T^1$ is assigned the obstruction $\lambda(\xi) \in T^2$ governing this construction. It is not difficult to see that $\lambda(\xi) = \xi \cup \xi$.

As a corollary, we obtain that the associated to the cup product quadratic form $\lambda : T^1 \to T^2$ describes the equations of the versal base space up to the second order.

4.2 The cup product for toric varieties

(4.2.1) Turning to the particular case of toric varieties (use the notation of (3.2)), the cup product $T^1_Y \times T^1_Y \to T^2_Y$ respects the M-grading. It splits into pieces

$$T^1_Y(-R) \times T^1_Y(-S) \longrightarrow T^2_Y(-R-S) \quad (R, S \in M)\,.$$

To describe these maps in our combinatorial language, we choose some set-theoretical section $\Phi : M \to \mathbb{Z}^{w+1}$ of the $\mathbb{Z}$-linear map

$$\begin{array}{rcl} \pi : \mathbb{Z}^{w+1} & \longrightarrow & M \\ a & \mapsto & \sum\limits_v a_v\, r^v \end{array}$$

with the additional property $\Phi(\sigma^\vee \cap M) \subseteq \mathbb{N}^{w+1}$.

Let $q \in L(E) \subseteq \mathbb{Z}^{w+1}$ be an integral relation between the generators of the semigroup $\sigma^\vee \cap M$. We introduce the following notations:

- $q^+, q^- \in \mathbb{N}^{w+1}$ denote the positive and the negative part of q, respectively. (With other words: $q = q^+ - q^-$ and $\sum_v q_v^-\, q_v^+ = 0$.)
- $\bar{q} := \pi(q^+) = \sum_v q_v^+\, r^v = \sum_v q_v^-\, r^v = \pi(q^-) \in M$.
- If $\varphi, \psi : L(E) \to \mathbb{Z}$ are linear maps and $R, S \in M$, then we define

  $$t_{\varphi,\psi,R,S}(q) := \varphi(q) \cdot \psi\left(\Phi(\bar{q} - R) + \Phi(R) - q^-\right) + \psi(q) \cdot \varphi\left(\Phi(\bar{q} - S) + \Phi(S) - q^+\right)\;,$$

  or, more generally, for $(\alpha, \beta) \in m$

  $$\begin{aligned} t_{\varphi,\psi,R,S}(\alpha,\beta) := \;& \varphi(\alpha - \beta) \cdot \psi\left(\Phi(\pi(\alpha) - R) + \Phi(R) - \beta\right) + \\ & + \psi(\alpha - \beta) \cdot \varphi\left(\Phi(\pi(\alpha) - S) + \Phi(S) - \alpha\right)\;. \end{aligned}$$

  (Both expressions are related via $t(q) = t(q^+, q^-)$.)

Assume that φ and ψ vanish on $\sum_j L(E_j^R)$ and $\sum_j L(E_j^S)$, respectively. In particular, they represent elements from $T^1_Y(-R)$ and $T^1_Y(-S)$ via Theorem (3.2.4).

Lemma: *Let $\alpha, \beta, \gamma \in \mathbb{N}^E$ with $\pi(\alpha) = \pi(\beta) = \pi(\gamma)$.*

(1) $t(\alpha,\beta) = -t(\beta,\alpha)$*; in particular* $t(-q) = -t(q)$ *for* $q \in L(E)$*.*

(2) $t(\alpha,\beta) = t(\alpha-\beta)$ *as long as* $\pi(\alpha) \in \bigcup_j K_j^{R+S}$*. (Do not mix up that property with* $t(q) = t(q^+, q^-)$ *being valid without further assumptions.)*

(3) $t(\beta,\gamma) - t(\alpha,\gamma) + t(\alpha,\beta) = 0$*.*

Proof: (1) $t(\alpha,\beta) + t(\beta,\alpha) = \varphi(\alpha-\beta)\,\psi(\alpha-\beta) + \psi(\alpha-\beta)\,\varphi(\beta-\alpha) = 0$.

(2) It is enough to show that $t(\alpha+r,\beta+r) = t(\alpha,\beta)$ for $r \in I\!N^E$, $\pi(\alpha+r) \in \bigcup_j K_j^{R+S}$. The difference of these two terms equals

$$\begin{aligned} t(\alpha+r,\beta+r) - t(\alpha,\beta) \;=\; & \varphi(\alpha-\beta)\cdot\psi[\Phi(\pi(\alpha+r)-R) - \Phi(\pi(\alpha)-R) - r]+ \\ & +\psi(\alpha-\beta)\cdot\varphi[\Phi(\pi(\alpha+r)-S) - \Phi(\pi(\alpha)-S) - r]\,, \end{aligned}$$

and both summands vanish: For instance, on the one hand, $\pi(\alpha) \in \bigcup_j K_j^R$ would cause $\varphi(\alpha-\beta) = 0$, and, on the other hand, $\pi(\alpha) - R \geq 0$ and $\pi(r + \Phi(\pi(\alpha)-R)) \in \bigcup_j K_j^S$ imply $\psi[\Phi(\pi(\alpha+r)-R) - \Phi(\pi(\alpha)-R) - r] = 0$.

(3) By extending φ and ψ to linear maps $\mathbb{C}^E \to \mathbb{C}$, we obtain

$$\begin{aligned} t(\alpha,\beta) = \; & [\varphi(\alpha-\beta)\,\psi\,(\Phi(\pi(\alpha)-R) + \Phi(R)) + \psi(\alpha-\beta)\,\varphi\,(\Phi(\pi(\alpha)-S) + \Phi(S))]+ \\ & +[\varphi(\beta)\,\psi(\beta) - \varphi(\alpha)\,\psi(\alpha)]. \end{aligned}$$

Now, since $\pi(\alpha) = \pi(\beta) = \pi(\gamma)$, both types of summands add up to 0 separately in the expression $t(\beta,\gamma) - t(\alpha,\gamma) + t(\alpha,\beta)$. □

Remark: The previous lemma does not imply that $t(q)$ is $\mathbb{Z}$-linear in q. The assumption for $\pi(\alpha)$ made in (2) is really essential.

(4.2.2) **Theorem:** *Assume that* Y *is smooth in codimension 2.*
Let $R, S \in M$*, and let* $\varphi, \psi : L(E)_{\mathbb{C}} \to \mathbb{C}$ *be linear maps vanishing on* $\sum_j L(E_j^R)_{\mathbb{C}}$ *and* $\sum_j L(E_j^S)_{\mathbb{C}}$*, respectively. In particular, they define elements* $\varphi \in T_Y^1(-R)$*,* $\psi \in T_Y^1(-S)$ *(which involves a slight abuse of notations).*

(1) Each integral relation $q \in L(E_j^{R+S})$ *admits a decomposition* $q = \sum_l q^l$ *such that* $q^l \in L(E_j^{R+S})$ *and, moreover,* $\langle a^j, \bar{q}^l\rangle < \langle a^j, R+S\rangle$ *(i.e.* $\bar{q}^l \in K_j^{R+S}$*).*

(2) $\sum_l q^l = 0$ *(with* $\bar{q}^l \in K_j^{R+S}$*) implies* $\sum_l t(q^l) = 0$*. In particular, the decomposition of (1) induces a well defined element* $t^j_{\varphi,\psi,R,S} \in L(E_j^{R+S})^*$ *via* $t^j(q) := \sum_l t(q^l)$*.*

(3) For adjacent a^j, a^k *the relations* $q \in L(E_j^{R+S} \cap E_k^{R+S})$ *admit decompositions* $q = \sum_l q^l$ *that work for both* j *and* k*. In particular,* $t^j(q) = t^k(q)$*.*

(4) According to Theorem (3.2.4), the family $(t^j_{\varphi,\psi,R,S}) \in \oplus_j L(E_j^{R+S})^*$ *defines an element of* $T_Y^2(-R-S)$*. It equals the cup product* $\varphi \cup \psi$*.*

The proof of the theorem is contained in (4.3).

Remark: Replacing all the terms $\Phi(\bullet)$ in the t's of the previous formula for $\varphi \cup \psi$ by arbitrary liftings from M to $\mathbb{Z}^{w+1}$, the result in $T_Y^2(-R-S)$ will be unchanged as long as we obey the following two rules:

(i) Use always (for all q, q^l, and j) the *same liftings* of R and S to $\mathbb{Z}^{w+1}$.

(ii) Elements of $\sigma^\vee \cap M$ always have to be lifted to $\mathbb{N}^{w+1}$.

Proof: Replacing $\Phi(R)$ by $\Phi(R)+d$ ($d \in L(E)$) at each occurrence changes all maps $(\varphi \cup \psi)_j = t^j_{\varphi,\psi,R,S}$ by the summand $\psi(d)\cdot\varphi(\bullet)$. However, this additional linear map comes from $L(E)^*$, hence it is trivial on $\operatorname{Ker}\big(\oplus_j L(E_j^{R+S}) \to L(E_0^{R+S}) \subseteq L(E)\big)$.

Let us look at the terms $\Phi(\bar{q}-R)$ in $t(q)$ now: Unless $\bar{q} \geq R$, the factor $\varphi(q)$ vanishes (cf. Remark (3.2.6)). On the other hand, the expression $t(q)$ is never used for those relations q satisfying $\bar{q} \geq R+S$ (cf. conditions for the q^l's). Hence, we may assume that

$$(\bar{q}-R) \geq 0 \;\text{ and, moreover, } (\bar{q}-R) \in \bigcup_j K_j^S\,.$$

Now, each two liftings of $(\bar{q}-R)$ to $\mathbb{N}^{w+1}$ differ by an element of $\operatorname{Ker}\psi$ only (apply the method of Remark (3.2.6) again), in particular, they cause the same result for $t(q)$. □

4.3 Proof of the cup product formula

(4.3.1) Let us fix an element $j \in \{1,\dots,M\}$. Since $\sigma^\vee \cap M$ contains elements r with $\langle a^j, r\rangle = 1$, some of them must be contained in the generating set E, too. We choose one of these elements and call it $r(j)$.
Now, to each $r \in E$ we associate some relation $p^j(r) \in L(E)$ by

$$p^j(r) := e^r - \langle a^j, r\rangle \cdot e^{r(j)} + [\text{suitable element of } \mathbb{Z}^{E\cap(a^j)^\perp}]\,.$$

The two essential properties of these special relations are

(i) $\langle a^j, \overline{p^j(r)}\rangle = \langle a^j, r\rangle$ (see (4.2.1) for the definition of $\bar{q}$ for a relation q), and

(ii) if $q \in L(E)$ is any relation, then q and $\sum_{r\in E} q_r \cdot p^j(r)$ differ by some element of $L(E\cap(a^j)^\perp)$ only.

In particular, this proves (1). For (3), let a^j, a^k be two adjacent edges of σ. We adapt the construction of the elementary relations $p^j(r)$; instead of the $r(j)$'s, we will use elements $r(j,k) \in E$ characterized by the property

$$\langle a^j, r(j,k)\rangle = 1\,,\; \langle a^k, r(j,k)\rangle = 0\,.$$

(Those elements exist, since Y is assumed to be smooth in codimension 2.) Now, we define

$$p^{jk}(r) := e^r - \langle a^j, r\rangle \cdot e^{r(j,k)} - \langle a^k, r\rangle \cdot e^{r(k,j)} + [\text{suitable element of } \mathbb{Z}^{E\cap(a^j)^\perp\cap(a^k)^\perp}]\,.$$

These special $p(r)$'s meet the usual properties (i) and (ii) - but for the two different indices j and k at the same time. In particular, if $q \in L(E)$ is any relation, then q and $\sum_{r\in E} q_r \cdot p^{jk}(r)$ differ by some element of $L(E\cap(a^j)^\perp\cap(a^k)^\perp)$ only.

(4.3.2) *Claim:* Let $q^l \in L(E)$ be relations such that $\sum_l \langle a^j, \bar{q}^l\rangle < \langle a^j, R+S\rangle$. Then, $\sum_l t(q^l) = t(\sum_l q^l)$.

(*Proof:* We can restrict ourselves to the case of two summands, q^1 and q^2. Then, by Lemma (4.2.1),

$$\begin{aligned} t(q^1)+t(q^2) &= t\left((q^1)^+,(q^1)^-\right)+t\left((q^2)^+,(q^2)^-\right) \\ &= t\left((q^1)^+ + (q^2)^+,(q^1)^- + (q^2)^+\right)+t\left((q^2)^+ + (q^1)^-,(q^2)^- + (q^1)^-\right) \\ &= t\left((q^1)^+ + (q^2)^+,(q^2)^- + (q^1)^-\right) \\ &= t(q^1+q^2)\,. \qquad \Box) \end{aligned}$$

In particular, if $\sum_l q^l = 0$ (with $\bar{q}^l \in K_j^{R+S}$), then this applies for the special decompositions

$$q^l = \sum_r q^l_r \cdot p^j(r) + q^{0,l} \quad (q^{0,l} \in L(E\cap (a^j)^\perp))$$

of the summands q^l themselves. We obtain

$$\sum_{q^l_r>0} q^l_r \cdot t\left(p^j(r)\right) + t(q^{0,l}) = t\left(\sum_{q^l_r>0} q^l_r\, p^j(r) + q^{0,l}\right) =: t(q^{1,l})$$

and

$$\sum_{q^l_r<0} q^l_r \cdot t\left(p^j(r)\right) = t\left(\sum_{q^l_r<0} q^l_r\, p^j(r)\right) =: t(q^{2,l})\,.$$

Up to elements of $E\cap(a^j)^\perp$, the relations $q^{1,l}$ and $q^{2,l}$ are connected by the common

$$(q^{1,l})^- = -q^{1,l}_{r(j)} \cdot e^{r(j)} = \langle a^j, \bar{q}^l\rangle \cdot e^{r(j)} = q^{2,l}_{r(j)} \cdot e^{r(j)} = (q^{2,l})^+\,.$$

Hence, Lemma (4.2.1) yields

$$\sum_r q^l_r \cdot t\left(p^j(r)\right) + t(q^{0,l}) = t(q^{1,l}) + t(q^{2,l}) = t\left(q^{1,l}+q^{2,l}\right) = t(q^l)\,,$$

and we conclude

$$\begin{aligned} \sum_l t(q^l) &= \sum_l \left(\sum_r q^l_r \cdot t\left(p^j(r)\right) + t(q^{0,l})\right) \\ &= \sum_r \left(\sum_l q^l_r\right) t\left(p^j(r)\right) + t\left(\sum_l q^{0,l}\right) \qquad \text{(cf. previous claim)} \\ &= 0 + t\left(\sum_l q^l - \sum_{l,r} q^l_r\, p^j(r)\right) \\ &= 0\,. \end{aligned}$$

(4.3.3) It remains to relate the family $(t^j_{\varphi,\psi,R,S})$ to the cup product $\varphi\cup\psi$, i.e. to show (4). Fix an $R\in M$, and let $\varphi\in L(E)^*_{\mathbb{C}}$ induce some element (also denoted by φ) of $T^1_Y(-R)$. Using the notations of (4.1.1) we can take

$$\tilde{\varphi}(f_{\alpha\beta}) := \varphi(\alpha-\beta)\cdot \underline{z}^{\Phi(\pi(\alpha)-R)}$$

for the auxiliary P-elements needed to compute the $\lambda(\varphi)$'s (cf. Theorem (3.2.6)).

Now, we have to distinguish between the two several types of relations generating the P-module $\mathcal{R}\subseteq P^m$ (cf. (3.2.7)):

(r) Regarding the relation $r(a,b;c)$ we obtain

$$\begin{aligned}\sum_{(\alpha,\beta)\in m} r(a,b;c)_{\alpha\beta}\cdot\tilde{\varphi}(f_{\alpha\beta}) &= \tilde{\varphi}(f_{a+c,b+c}) - \underline{z}^c\,\tilde{\varphi}(f_{ab}) \\ &= \varphi(a-b)\cdot\left(\underline{z}^{\Phi(\pi(a+c)-R)} - \underline{z}^{c+\Phi(\pi(a)-R)}\right) \\ &= \varphi(a-b)\cdot f_{\Phi(\pi(a+c)-R),\,c+\Phi(\pi(a)-R)}\,.\end{aligned}$$

In particular,

$$\lambda_{\alpha\beta}^{r(a,b;c)}(\varphi) = \begin{cases} \varphi(a-b) & \text{for } [\alpha,\beta] = [c+\Phi(\pi(a)-R),\,\Phi(\pi(a+c)-R)] \\ 0 & \text{otherwise}\,.\end{cases}$$

(s) The corresponding result for the relation $s(a,b,c)$ is much nicer:

$$\begin{aligned}\sum_{(\alpha,\beta)\in m} s(a,b,c)_{\alpha\beta}\cdot\tilde{\varphi}(f_{\alpha\beta}) &= \tilde{\varphi}(f_{bc}) - \tilde{\varphi}(f_{ac}) + \tilde{\varphi}(f_{ab}) \\ &= [\varphi(b-c)-\varphi(a-c)+\varphi(a-b)]\cdot\underline{z}^{\Phi(\pi(a)-R)} \\ &= 0\,.\end{aligned}$$

In particular, $\lambda^{s(a,b,c)}(\varphi) = 0$.

Now, let R, S, φ, and ψ as in the assumption of Theorem (4.2.2). Using formula (4.1.1)(iii), our previous computations yield $(\varphi\cup\psi)(s(a,b,c)) = 0$ and

$$\begin{aligned}(\varphi\cup\psi)(r(a,b;c)) &= \textstyle\sum_{\alpha,\beta}\lambda_{\alpha\beta}^{r(a,b;c)}(\varphi)\cdot\psi(f_{\alpha\beta}) + \sum_{\alpha,\beta}\varphi(f_{\alpha\beta})\cdot\lambda_{\alpha\beta}^{r(a,b;c)}(\psi) \\ &= \varphi(a-b)\cdot\psi\,(c+\Phi(\pi(a)-R)-\Phi(\pi(a+c)-R))\cdot x^{\pi(c+\Phi(\pi(a)-R))-S} + \\ &\quad +\psi(a-b)\cdot\varphi\,(c+\Phi(\pi(a)-S)-\Phi(\pi(a+c)-S))\cdot x^{\pi(c+\Phi(\pi(a)-S))-R} \\ &= [\varphi(a-b)\cdot\psi\,(c+\Phi(\pi(a)-R)-\Phi(\pi(a+c)-R)) + \\ &\quad +\psi(a-b)\cdot\varphi\,(c+\Phi(\pi(a)-S)-\Phi(\pi(a+c)-S))]\cdot x^{\pi(a+c)-R-S}\,.\end{aligned}$$

Unless $\pi(a+c)\geq R+S$, both summand in the brackets will vanish. (Use the same argument as done in the proof of the second part of Lemma (4.2.1).)

(4.3.4) To apply Theorem (3.2.7) we would like to remove the argument c from the big coefficient of $x^{\pi(a+c)-R-S}$. This will be done by adding a suitable coboundary T.

Define the P-linear map $T\in\mathrm{Hom}(P^m, A)$ by

$$T : e^{\alpha\beta}\mapsto\begin{cases} t(\alpha,\beta)\,x^{\pi(\alpha)-R-S} & \text{for } \pi(\alpha)\geq R+S \\ 0 & \text{otherwise}\,.\end{cases}$$

Pulling back T to $\mathcal{R}\subseteq P^m$ yields (in case of $\pi(a+c)\geq R+S$)

$$\begin{aligned}T(r(a,b;c)) &= \begin{cases} [t(a+c,b+c)-t(a,b)]\cdot x^{\pi(a+c)-R-S} & \text{for } \pi(a)\geq R+S \\ t(a+c,b+c)\cdot x^{\pi(a+c)-R-S} & \text{otherwise}\end{cases} \\ &= \begin{cases} -(\varphi\cup\psi)(r(a,b;c)) & \text{for } \pi(a)\geq R+S \\ t(a,b)\,x^{\pi(a+c)-R-S} - (\varphi\cup\psi)(r(a,b;c)) & \text{otherwise}\end{cases}\end{aligned}$$

and $T(s(a,b,c)) = 0$ (by (2) of the previous lemma).

On the other hand, T yields a trivial element of $T^2_Y(-R-S)$, i.e. inside this group we may replace $\varphi\cup\psi$ by $(\varphi\cup\psi)+T$ to obtain

$$\begin{aligned}(\varphi\cup\psi)(r(a,b;c)) &= \begin{cases} t(a,b)\cdot x^{\pi(a+c)-R-S} & \text{for } \pi(a)\in\bigcup_j K_j^{R+S};\ \pi(a+c)\geq R+S \\ 0 & \text{otherwise}, \end{cases} \\ (\varphi\cup\psi)(s(a,b,c)) &= 0\,. \end{aligned}$$

Having Theorem (3.2.7) in mind, this formula for $\varphi\cup\psi$ is exactly what we were looking for: Given an $r(a,b;c)$ with $\pi(a)\in K_j^{R+S}$, let us compute $t^j_{\varphi,\psi,R,S}(q:=a-b)$. We do not need to split $q=a-b$ into a sum $q=\sum_l q^l$ - the element q itself already satisfies the condition

$$\langle a^j,\bar{q}\rangle \leq \langle a^j,\pi(a)\rangle < \langle a^j, R+S\rangle.$$

In particular, $t^j_{\varphi,\psi,R,S}(a-b)=t_{\varphi,\psi,R,S}(a-b)=t_{\varphi,\psi,R,S}(a,b)$, and we are done.

4.4 Alternative cup product formula

(4.4.1) We are going to describe the cup product using the description of T^1_Y and T^2_Y given in section (3.4). Unfortunately, this has been possible only for degrees $R,S\geq 0$.

Lemma:

(1) If $S\geq 0$ (i.e. $S\in\sigma^\vee\cap M$), then $t(q)=0$ unless $\bar{q}\geq R$.

(2) If $R,S\geq 0$ and $\langle a^j, R+S\rangle\geq 1$, then t vanishes on $L(E\cap(a^j)^\perp)$.

Proof: (1) As used many times, the property $\bar{q}\in\bigcup_j E_j^R$ implies $\varphi(q)=0$. Now, we can distinguish between two cases:
Case 1: $\bar{q}\in\bigcup_j E_j^S$. We obtain $\psi(q)=0$, in particular, both summands of $t(q)$ vanish.
Case 2: $\bar{q}\geq S$. Then, $\bar{q}-S$, $S\in\sigma^\vee\cap M$, and Φ lifts these elements to $\mathbb{N}^{w+1}$. Now, the condition $\bar{q}\in\bigcup_j E_j^R$ implies that $\varphi\left(\Phi(\bar{q}-S)+\Phi(S)-q^+\right)=0$.
(2) is a direct consequence of the first part: Since $\langle a^j,R+S\rangle\geq 1$, elements $q\in L(E\cap(a^j)^\perp)$ cannot meet both $\bar{q}\geq R$ and $\bar{q}\geq S$. □

We will assume $R,S\geq 0$ (and Y being smooth in codimension two) for the whole section.

Remark: In the special case of $R\geq S\geq 0$ we can choose liftings $\Phi(R)\geq\Phi(S)\geq 0$ in $\mathbb{N}^{w+1}$. In case of $\bar{q}\geq R$ we may assume that $q^+\geq\Phi(R)$ is true in $\mathbb{N}^{w+1}$ (the general q's are differences of those ones). Then, t can be computed as the product $t(q)=\varphi(q)\,\psi(q)$.

Proof: We can choose $\Phi(\bar{q}-R):=q^+-\Phi(R)$ and $\Phi(\bar{q}-S):=q^+-\Phi(S)$. Then, the claim follows straight forward. □

(4.4.2) Assume that we start with elements of $T^1_Y(-R)$ and $T^1_Y(-S)$ that are given (via Theorem (3.4.1)) by linear maps $\varphi^j:\operatorname{span}E_j^R\to\mathbb{Z}$ and $\psi^j:\operatorname{span}E_j^S\to\mathbb{Z}$ $(j=1,\ldots,M)$, respectively. (In particular, $\varphi^j=\varphi^k$ and $\psi^j=\psi^k$ for two-dimensional faces $\langle a^j,a^k\rangle<\sigma$.)

We define $\varphi:\mathbb{Z}^E\to\mathbb{Z}$ as zero on $\mathbb{Z}^{E\setminus E_0^R}$ and as $\varphi(e^r):=\varphi^j(r)$ for $r\in E_j^R$. Following the remark right after Theorem (3.4.1), the restriction of $-\varphi$ to $L(E_0^R)\subseteq\mathbb{Z}^E$ equals (via Theorem (3.2.4)) the same element of $T^1_Y(-R)$ as we have started with. (Of course, the same arguments apply to

ψ^j and ψ.)
By abuse of notation we will sometimes write r for $e^r \in \mathbb{Z}^E$ (e.g. $\varphi(r)$ for $\varphi(e^r)$). However, keep always in mind that the arguments of φ^j are elements of $\operatorname{span} E_j^R \subseteq M$, but the arguments of φ come from $\mathbb{Z}^E$.

We need the notions $r(j)$, $r(j,k)$, $p^j(r)$, and $p^{jk}(r)$ introduced in (4.3.1). Moreover, we define

$$\ell(j,k) := e^{r(j,k)} - e^{r(j)} + [\text{suitable element of } \mathbb{Z}^{E \cap (a^j)^\perp}] \subseteq L(E)\,.$$

Remark:

(i) $p^{jk}(r) = p^{kj}(r)$ and

(ii) $p^j(r) - p^{jk}(r) \equiv \langle a^j, r\rangle \cdot \ell(j,k) \pmod{L(E \cap (a^j)^\perp)}$.

Then, we can describe the cup product $\varphi \cup \psi \in T_Y^2(-R-S)$ of the two given infinitesimal deformations as a certain map $(\varphi \cup \psi) : \oplus_{\langle a^j, a^k\rangle < \sigma} \operatorname{span} E_{jk}^{R+S} \to \mathbb{Z}$:

Proposition: *Let $\langle a^j, a^k\rangle < \sigma$ be a two-dimensional face. Then, the map $(\varphi \cup \psi)_{jk}$ can be given as $(\varphi \cup \psi)_{jk} = t[\ell(k,j)] \cdot a^k - t[\ell(j,k)] \cdot a^j \in N$. (The data of φ and ψ have been involved in the definition of t.)*
Moreover, that particular $(\varphi \cup \psi)_{jk}$ does not depend from the special choice of elements $r(j,k)$ or $L(E \cap (a^j)^\perp)$-summands in the definition of $\ell(j,k)$. On the other hand, changing $r(j)$ means changing $(\varphi \cup \psi)_{jk}$ by a coboundary (coming from $\oplus_j(\operatorname{span} E_j^{R+S})^*$).

Proof: Of course, we will use Theorem (4.2.2). Assume that we are given elements $q^{jk} \in \operatorname{span} E_{jk}^{R+S}$ (for two-faces $\langle a^j, a^k\rangle < \sigma$) representing a class q in $H^2\left(\operatorname{span}(E^R)_\bullet\right)$. Then, its image in $\oplus_j L(E_j^{R+S})$ can be described as follows (cf. (3.4.1)): Represent q^{jk} as $q^{jk} = \sum_{r \in E} q_r^{jk} \cdot r$ with $q_\bullet^{jk} \in \mathbb{Z}^{E_{jk}^{R+S}}$ and take

$$q_\bullet^j := \sum_{a^k \text{adjacent to } a^j} \operatorname{sgn}\langle j,k\rangle \cdot q_\bullet^{jk} \quad \in \ L(E_j^{R+S}) \ \subseteq \ \mathbb{Z}^E\,.$$

(The sign $\operatorname{sgn}\langle j,k\rangle$ measures the difference between two orientations of $\langle a^j, a^k\rangle$: On the one hand, each face of σ is oriented somehow from the very beginning, and, on the other hand, considering a^j as the "first" vertex of $\langle a^j, a^k\rangle$ yields a second one.)

According to Theorem (4.2.2) we obtain

$$(\varphi \cup \psi)(q) \ = \ \sum_{j=1}^M t^j(q_\bullet^j) \ = \ \sum_{j=1}^M \left[\sum_{r \in E} q_r^j \cdot t[p^j(r)] + t[L(E \cap (a^j)^\perp)\text{-element}]\right].$$

Since $E_j^{R+S} = \emptyset$ (i.e. the j-th summand vanishes) unless $\langle a^j, R+S\rangle \geq 1$, we may apply Lemma (4.4.1)(2) killing the second t-summands. Hence,

$$\begin{aligned}(\varphi \cup \psi)(q) \ &= \ \sum_{j=1}^M \sum_{r \in E} \sum_{a^k \text{adj. to } a^j} \operatorname{sgn}\langle j,k\rangle \cdot q_r^{jk} \cdot t[p^j(r)] \\ &= \ \sum_{\langle a^j, a^k\rangle < \sigma} \ \sum_{r \in E} q_r^{jk} \cdot \left(t[p^k(r)] - t[p^j(r)]\right).\end{aligned}$$

On the other hand, if $q_r^{jk} \neq 0$ (i.e. $\langle a^j, r\rangle < \langle a^j, R+S\rangle$), then by Theorem (4.2.2)(2) and part (ii) of the previous remark we know $t[p^j(r)] = t[p^{jk}(r)] + \langle a^j, r\rangle \cdot t[\ell(j,k)]$. Doing the same thing with $t[p^k(r)]$ and using $p^{jk} = p^{kj}$, we obtain

$$t[p^k(r)] - t[p^j(r)] = \langle a^k, r\rangle \cdot t[\ell(k,j)] - \langle a^j, r\rangle \cdot t[\ell(j,k)]\,.$$

Finally, we investigate the influence of the choices being made in the definition of $\ell(j,k)$:
1) $\ell(j,k)$ has been defined up to relations from $L(E\cap(a^j)^\perp)$ only. However, if $\langle a^j, R+S\rangle \geq 2$, then those summands have no influence to $t[\ell(j,k)]$ (cf. Theorem (4.2.2)(2) and Lemma (4.4.1)(2)). On the other hand, if $\langle a^j, R+S\rangle \leq 1$, then $E_{jk}^{R+S} \subseteq (a^j)^\perp$, i.e. the coefficient of a^j in $(\varphi\cup\psi)_{jk}$ has no meaning, anyway.
2) Let $r'(j,k) \in E$ be another element meeting $\langle a^j, r'(j,k)\rangle = 1$ and $\langle a^k, r'(j,k)\rangle = 0$. Then, there is a relation

$$d(j,k) := e^{r'(j,k)} - e^{r(j,k)} + [\text{suitable element of } \mathbb{Z}^{E\cap(a^j,a^k)^\perp}] \subseteq L(E\cap(a^k)^\perp)$$

providing $\ell'(j,k) = \ell(j,k) + d(j,k)$. In particular, at least in the case $\langle a^j, R+S\rangle \geq 2$, we have $t[\ell'(j,k)] = t[\ell(j,k)] + t[d(j,k)]$ and $t[d(j,k)] = 0$.
3) Similarly, changing $r(j)$ into $r'(j)$ yields a relation $d(j) = e^{r'(j)} - e^{r(j)} + \ldots \in L(E)$, and we obtain $t[\ell'(j,k)] = t[\ell(j,k)] - t[d(j)]$. □

(4.4.3) We are going to replace the rather strange elements $t[\ell(j,k)]$. In case of $\langle a^j, R\rangle = \langle a^k, R\rangle = \langle a^j, S\rangle = \langle a^k, S\rangle = 1$, we define (writing $R, S \in \mathbb{N}^E$ for $\Phi(R)$ and $\Phi(S)$, respectively)

$$\begin{aligned}\varphi^{jk} &:= \varphi(S) - \varphi^j[r(k,j)] - \varphi^k[r(j,k)] + \varphi^{j \text{ or } k}[r(j,k)+r(k,j)-S]\\ &= \varphi[S-r(k,j)] - \varphi^k[S-r(k,j)]\\ &= \varphi[S-r(j,k)] - \varphi^j[S-r(j,k)]\end{aligned}$$

and, similarly,

$$\psi^{jk} := \psi(R) - \psi^j[r(k,j)] - \psi^k[r(j,k)] + \psi^{j \text{ or } k}[r(j,k)+r(k,j)-R]\,.$$

(The latter two descriptions of φ^{jk} show that this number (and analogously ψ^{jk}) does not depend on the particular choices of $r(j,k)$ and $r(k,j)$.)

Proposition: *The elements $(\varphi\cup\psi)_{jk} \in (\text{span}\, E_{jk}^{R+S})^*$ vanish unless either*

(1) $\langle a^j, R\rangle = \langle a^k, R\rangle = \langle a^j, S\rangle = \langle a^k, S\rangle = 1$ (yielding $(\varphi\cup\psi)_{jk} = \varphi^{jk}\cdot\psi^{jk}\cdot(a^j - a^k)$) or

(2) $R = S \in E$ and

(i) $\langle a^j, R\rangle = 1$, $\langle a^k, R\rangle \geq 2$ (yielding $(\varphi\cup\psi)_{jk} = \varphi^k(R)\cdot\psi^k(R)\cdot a^j$) or

(ii) $\langle a^j, R\rangle \geq 2$, $\langle a^k, R\rangle = 1$ (yielding $(\varphi\cup\psi)_{jk} = -\varphi^j(R)\cdot\psi^j(R)\cdot a^k$).

Remark: If $R = S \notin E$ (in particular, R has to be decomposable in $\sigma^\vee\cap M$) and $\langle a^j, R\rangle = 1$, $\langle a^k, R\rangle \geq 2$, then the previous proposition implies that $(\varphi\cup\psi)_{jk} = 0$.
On the other hand, all calculations remain valid after enlarging E (actually, we have never used minimality of E). In particular, we may turn R into an "artificial" element of E yielding a possibly different value for $(\varphi\cup\psi)_{jk}$. Looking like a contradiction, this fact can be explained as follows: First, one has to be careful anyway, since with E also the functions φ and ψ will change. On the other hand, there is a coboundary "connecting" the results of those two different approaches.

Proof (of the proposition): *Step 1:* $t[\ell(j,k)] = 0$ *unless* $\langle a^j, R\rangle = \langle a^j, S\rangle = 1$*:*
Using Lemma (4.4.1)(1), we know that $t[\ell(j,k)] \neq 0$ implies $\overline{\ell(j,k)} \geq R, S$. In particular, $\langle a^j, R\rangle, \langle a^j, S\rangle \leq 1$, and we have already mentioned that in case of $\langle a^j, R+S\rangle \leq 1$ (i.e. $E_{jk}^{R+S} \subseteq (a^j)^\perp$) the coefficient $t[\ell(j,k)]$ of a^j does not matter.

Step 2: Assume $\langle a^j, R\rangle = \langle a^j, S\rangle = 1$*. Then (using a special choice for* $r(j)$*), we obtain*

$$t[\ell(j,k)] = \big(\varphi[S - r(j,k)] - \varphi^j[S - r(j,k)]\big) \cdot \big(\psi[r(j,k) - R] - \psi^j[r(j,k) - R]\big)\,.$$

(In particular, we are done in the case of $\langle a^j, R\rangle = \langle a^k, R\rangle = \langle a^j, S\rangle = \langle a^k, S\rangle = 1$*.)*
We may choose $r(j) := R$, even on the $\mathbb{N}^E$-level ($r(j) := S$ would yield the same result). Then, we apply the general definition of t in (4.2.1) to the relation $\ell(j,k) = e^{r(j,k)} - e^R + s^+(j,k) - s^-(j,k)$ (with $s^+, s^- \in \mathbb{N}^{E\cap(a^j)^\perp}$). The first summand in t vanishes, since the argument of ψ equals $\Phi([R + s^-] - R) + \Phi(R) - e^R - s^- = 0$ ($\Phi(R)$ and e^R are just synonyms, and for $\Phi(s^-)$ we may choose s^- itself by Remark (4.2.2)). For the second t-summand, we consider

$$\begin{aligned}\psi\big(r(j,k) - R + s^+(j,k) - s^-(j,k)\big) &= \psi\big(r(j,k) - R\big) + \psi^j\big(s^+(j,k) - s^-(j,k)\big)\\ &= \psi\big(r(j,k) - R\big) - \psi^j\big(r(j,k) - R\big)\end{aligned}$$

and

$$\begin{aligned}\varphi\big(\Phi[\bar\ell - S] + S - [r(j,k) + s^+(j,k)]\big) &= \varphi\big(S - r(j,k)\big) + \varphi^j\big(\Phi[\bar\ell - S] - s^+(j,k)\big)\\ &= \varphi\big(S - r(j,k)\big) - \varphi^j\big(S - r(j,k)\big)\,.\end{aligned}$$

Step 3: $t[\ell(j,k)] = 0$ *unless* $\langle a^k, R\rangle = \langle a^k, S\rangle$*:*
Assume $\langle a^k, R\rangle > \langle a^k, S\rangle$. We will use that $S - r(j,k) - \langle a^k, S\rangle \cdot r(k,j) \in (a^j, a^k)^\perp$ and call this element $s(j,k)$. Hence,

$$\begin{aligned}\varphi\big(S - r(j,k)\big) &= \varphi^k\big(S - r(j,k)\big) \quad (\text{since } \langle a^k, R\rangle > \langle a^k, S\rangle)\\ &= \varphi^k\big(\langle a^k, S\rangle \cdot r(k,j) + s(j,k)\big)\\ &= \varphi^j\big(\langle a^k, S\rangle \cdot r(k,j) + s(j,k)\big) \quad (\text{if } \langle a^k, R\rangle \geq 2,\ \text{then } \varphi^j = \varphi^k \text{ on } (a^j)^\perp \subseteq E_{jk}^R;\\ &\qquad\qquad \text{if } \langle a^k, R\rangle = 1,\ \text{then } \langle a^k, S\rangle = 0)\\ &= \varphi^j\big(S - r(j,k)\big)\,.\end{aligned}$$

Step 4: Assume that $\langle a^j, R\rangle = \langle a^j, S\rangle = 1$ *and* $\langle a^k, R\rangle = \langle a^k, S\rangle \geq 2$*.*
We obtain $\varphi^k\big(S - r(j,k)\big) = \varphi^j\big(S - r(j,k)\big)$ as in the previous step and continue

$$\varphi\big(S - r(j,k)\big) - \varphi^j\big(S - r(j,k)\big) = \varphi\big(S - r(j,k)\big) - \varphi^k\big(S - r(j,k)\big) = \varphi(S) - \varphi^k(S)\,.$$

Hence,

$$t[\ell(j,k)] = -\big(\varphi(S) - \varphi^k(S)\big) \cdot \big(\psi(R) - \psi^k(R)\big)\,.$$

Now, we distinguish between the following cases:

(i) *Assume* $R \neq S$. In particular, there must be some $l \in \sigma$ such that w.l.o.g. $\langle a^l, R\rangle > \langle a^l, S\rangle$ (we may even assume a^l being adjacent to a^k). Hence, $\varphi(S) = \varphi^l(S) = \varphi^k(S)$ (the latter equality follows from $S \in (\operatorname{span} E_k^R) \cap (\operatorname{span} E_l^R)$) implying $t[\ell(j,k)] = 0$.

(ii) *Assume* $R = S$ *is not contained in* E. Hence, $R = R_1 + R_2$ splits even on the $\mathbb{N}^E$-level. In particular, $\langle a^k, R_1\rangle \leq \langle a^k, R\rangle$ and moreover, if they are equal, there is an adjacent a^l such that $\langle a^l, R_1\rangle < \langle a^l, R\rangle$. Similarly to (i), this means that $\varphi(R_1) = \varphi^k(R_1)$, and we obtain $\varphi(R) = \varphi(R_1) + \varphi(R_2) = \varphi^k(R_1) + \varphi^k(R_2) = \varphi^k(R)$.

(iii) *Assume* $R = S$ is *contained in* E. Obviously, this implies $\varphi(R) = \psi(R) = 0$. □

(4.4.4) *Assume* $R = S$ (≥ 0).
We are computing the cup product $T^1_Y(-R) \times T^1_Y(-R) \to T^2_Y(-2R)$. According to the special shape of the degree $2R$ (a proper multiple of an element of M), we may use the embedding $\Psi : T^2_Y(-2R) \hookrightarrow N_{\mathbb{C}}^{\{\varepsilon<\sigma \,|\, R>0 \text{ on } \varepsilon\}}$ (cf. (3.4.6)) to display our result. For a three-dimensional face $\varepsilon = \langle a^1, \ldots, a^m \rangle < \sigma$ (with $R > 0$ on ε) we have $(\varphi \cup \psi)_\varepsilon := \Psi_\varepsilon(\varphi \cup \psi) = \sum_{j=1}^m (\varphi \cup \psi)_{j\,j+1}$ (with $m + 1 := 1$).

On the other hand, we would like to deal with elements of $T^1_Y(-R)$ as Minkowski summands of $Q := \sigma \cap [R = 1]$. Recalling the recipe of (3.4.4) and (3.4.5), the collection of maps $\varphi^j : \operatorname{span} E^R_j \to \mathbb{Z}$ corresponds to a Minkowski summand $\underline{t} \in V_{\mathbb{C}}(Q)/\underline{1}$ in the following way:

- t_{jk} are the dilatation factors for the edges, i.e. if the Minkowski summand is given by its vertices $\bar{b}^j \in R^\perp$, then $\bar{b}^j - \bar{b}^k = t_{jk} \cdot (\bar{a}^j - \bar{a}^k)$ (with $\bar{a}^j = a^j / \langle a^j, R \rangle$).
- If $\langle a^j, R \rangle \geq 2$, then t_{jk} does not depend on k and will be denoted by s_j, too. (Set $s_j := 0$, if $\langle a^j, R \rangle = 1$.)
- With $b^j = \bar{b}^j - s_j\, \bar{a}^j$ (hence $s_j = -\langle b^j, R \rangle$), we have $\varphi^j = \langle b^j, \bullet \rangle$.

(Take $t'_{jk}, s'_j, b'^j, \bar{b}'^j$ for the data corresponding to ψ.)

Theorem: *Let* $\underline{t}, \underline{t}' \in T^1_Y(-R) \subseteq V_{\mathbb{C}}(Q)/\underline{1}$. *Then, if* $\varepsilon = \operatorname{conv}\{\bar{a}^1, \ldots, \bar{a}^m\} < Q$ *is a compact, two-dimensional face, the* ε*-component of the cup product is given by*

$$(\underline{t} \cup \underline{t}')_\varepsilon := \Psi_\varepsilon(\underline{t} \cup \underline{t}') = \sum_{j=1}^m t_{j\,j+1}\, t'_{j\,j+1} \cdot (\bar{a}^j - \bar{a}^{j+1}) \in R^\perp \subseteq N_{\mathbb{C}}\,.$$

In particular, focusing on the degree-$(-R)$-part of the versal deformation of Y_σ, the quadratic equations look exactly (up to the new exponents 2) like the linear ones (defining $V_{\mathbb{C}}(Q)$ as a subspace of $\mathbb{C}^N$, cf. (3.4.2)).

Proof: We have just to translate the result obtained in the previous proposition into the right language. Let us assume $R \in E$.
First, if $\langle a^j, R \rangle = \langle a^k, R \rangle = 1$, then

$$\begin{aligned}
\varphi^{jk} &= \varphi(R) - \varphi^j[r(k,j)] - \varphi^k[r(j,k)] + \varphi^{j \text{ or } k}[r(j,k) + r(k,j) - R] \\
&= 0 - \langle b^j, r(k,j) \rangle - \langle b^k, r(j,k) \rangle + \langle b^j, r(j,k) + r(k,j) - R \rangle \\
&= \langle b^j - b^k,\, r(j,k) \rangle - \langle b^j, R \rangle \\
&= t_{jk} \cdot \langle a^j - a^k,\, r(j,k) \rangle - 0 \;=\; t_{jk}\,.
\end{aligned}$$

Hence, $(\varphi \cup \psi)_{j\,k} = t_{jk}\, t'_{jk} \cdot (a^j - a^k)$.

On the other hand, if for instance $\langle a^j, R \rangle = 1$, $\langle a^k, R \rangle \geq 2$, then

$$(\varphi \cup \psi)_{j\,k} = \langle b^k, R \rangle \cdot \langle b'^k, R \rangle \cdot a^j = s_k \cdot s'_k \cdot a^j$$

(and $(\varphi \cup \psi)_{j\,k} = -s_j\, s'_j \cdot a^k$ if $\langle a^j, R \rangle \geq 2$, $\langle a^k, R \rangle = 1$).

Since edges with $\langle a^j, R \rangle, \langle a^k, R \rangle \geq 2$ yield $(\varphi \cup \psi)_{j\,k} = 0$, we are done by taking the sum over all edges of $\varepsilon < Q$. □

(4.4.5) *Assume* $R \neq S$ (and put both into E).
In that case, Ψ does not need to be injective anymore. Nevertheless, it pays to analyze what Ψ is

doing with the cup product of elements from $T^1_Y(-R)$ and $T^1_Y(-S)$. We might find at least *some* of the quadratic equations for the versal base space with mixed degrees.

Definition: $a^j \in \sigma$ will be called a 1-vertex, if $\langle a^j, R\rangle = \langle a^j, S\rangle = 1$. Moreover, a 1-edge is a two-dimensional face $\langle a^j, a^k\rangle < \sigma$ connecting two 1-vertices a^j and a^k. (It has been only the 1-edges contributing a non-trivial $(\varphi \cup \psi)_{jk}$.)

Lemma: *Let $\underline{t} \in T^1_Y(-R) \subseteq V_{\mathbb{C}}(Q)/\underline{1}$. Then, for each 1-vertex $a^j \in \sigma$, there is a value $t_j \in \mathbb{C}$ such that*

(1) If $a^j, a^k \in \sigma$ are 1-vertices that can be connected within $[R = S] \subseteq \sigma$ by some chain of edges (i.e. 2-faces of σ), then $t_j = t_k$. In particular, this holds for 1-edges $\langle a^j, a^k\rangle < \sigma$.

(2) If $\langle a^j, a^k\rangle < \sigma$ is a 1-edge, then $\varphi^{jk} = t_{jk} - t_j = t_{jk} - t_k$.

(3) Moreover, in case of $R \not\leq S$, we have $t_j = t_{jl}$ for all vertices $a^l \in [R \geq S] \subseteq \sigma$ adjacent to a^j, but not forming a 1-edge with a^j.

Proof: Let us first assume that $R \not\leq S$. In particular, there is at least one "neighbor" a^l of a^j being contained in $[R > S] \subseteq \sigma$, and we define $t_j := t_{jl}$.

To prove (3), we take another $a^{l'}$ (even $\langle a^{l'}, R\rangle = \langle a^{l'}, S\rangle$ will be allowed, if $\langle a^{l'}, R\rangle \geq 2$). Using an induction argument, we may assume that a^j, a^l, and $a^{l'}$ are contained in a common three-dimensional face $\varepsilon \subseteq [R \geq S] \subseteq \sigma$. Then, we have $R > 0$ on ε, and R has value 1 in at most three different vertices including a^j. (Indeed, in $[R \geq S]$ the fact $\langle a^\bullet, R\rangle = 1$ implies $\langle a^\bullet, S\rangle = 0$ or $a^\bullet \in [R = S]$.) Hence, the linear equations defining $V(Q)$ (cf. (3.4.2)) imply that all dilatation factors for edges in ε are equal.

To prove (1) in the case $R \not\leq S$ (i.e. we may use (3)), it is sufficient to deal with 1-edges $\langle a^j, a^k\rangle < \sigma$. Again, we choose some three-dimensional face $\varepsilon < \sigma$ containing $\langle a^j, a^k\rangle$ (and the remaining vertices from $[R > S]$) and look for vertices with $R = 1$: If there are at most three of them, then we are done as in the previous discussion (implying not only $t_l = t_k$, but also $t_j = t_{jk}$). On the other hand, if $\langle a^j, R\rangle = \langle a^k, R\rangle = \langle a^l, R\rangle = \langle a^{l'}, R\rangle = 1$, then $\langle a^l, S\rangle = \langle a^{l'}, S\rangle = 0$, and it is the $V(Q)$-equation for ε evaluated in S bringing the result.

Now, we consider the case $R \leq S$. Defining $t_j := \varphi^j(S)$, we would like to prove (1): We may assume that on the edges (in $[R = S] \subseteq \sigma$) between a^j and a^k there are no further 1-vertices. In particular, all dilatation factors are equal, and we call them t_{jk}. Then,

$$t_j - t_k = \varphi^j(S) - \varphi^k(S) = \langle b^j - b^k, S\rangle = t_{jk} \cdot \langle a^j - a^k, S\rangle = 0\,.$$

Finally, one obtains (2) for both cases ($R \not\leq S$ and $R \leq S$) by straight computation (like in the proof of Theorem (4.4.4)). □

Theorem: *Let $\underline{t} \in T^1_Y(-R)$ and $\underline{t}' \in T^1_Y(-S)$. Then, if $\varepsilon = \langle a^1, \dots, a^m\rangle < \sigma$ is a three-dimensional face, the Ψ_ε-image of the cup product is given by*

(1) $\Psi_\varepsilon(\underline{t} \cup \underline{t}') = \sum_{j=1}^m t_{j\,j+1}\, t'_{j\,j+1} \cdot (\bar{a}^j - \bar{a}^{j+1}) \in (R,S)^\perp \subseteq N_{\mathbb{C}}$, if ε is contained in $[R = S]$. (In particular, this result fits into (4.4.4).)

(2) $\Psi_\varepsilon(\underline{t} \cup \underline{t}') = 0$, otherwise.

Proof: In the first case ($\varepsilon \subseteq [R = S]$) we use Proposition (4.4.3) and the second part of the previous lemma to obtain

$$\Psi_\varepsilon(\underline{t} \cup \underline{t}') = \sum_{\text{1-edges of } \varepsilon} (t_{jk} - t_j) \cdot (t'_{jk} - t'_j) \cdot (\bar{a}^j - \bar{a}^k)\,.$$

Moreover, part (1) of the previous lemma tells that t_j does not depend on j - let us call it t_ε (and do the same thing for the t'-factor). On the other hand, since $R \neq S$, we may assume (w.l.o.g.) that $R \not\leq S$. In particular, t_ε equals all dilatation factors of non-1-edges of ε (by (3) of the same lemma). Hence,

$$\begin{aligned}\Psi_\varepsilon(\underline{t} \cup \underline{t}') &= \sum_{j=1}^{m} (t_{j\,j+1} - t_\varepsilon) \cdot (t'_{j\,j+1} - t'_\varepsilon) \cdot (\bar{a}^j - \bar{a}^{j+1}) \\ &= \sum_{j=1}^{m} t_{j\,j+1}\, t'_{j\,j+1}\, (\bar{a}^j - \bar{a}^{j+1}) \,-\, t_\varepsilon \cdot \sum_{j=1}^{m} t'_{j\,j+1}\, (\bar{a}^j - \bar{a}^{j+1}) - \\ &\qquad - t'_\varepsilon \cdot \sum_{j=1}^{m} t_{j\,j+1}\, (\bar{a}^j - \bar{a}^{j+1}) \,+\, t_\varepsilon\, t'_\varepsilon \cdot \sum_{j=1}^{m} (\bar{a}^j - \bar{a}^{j+1}) \\ &= \sum_{j=1}^{m} t_{j\,j+1}\, t'_{j\,j+1}\, (\bar{a}^j - \bar{a}^{j+1}) \,-\, 0 \,-\, 0 \,+\, 0\,.\end{aligned}$$

Finally, let us assume $R \geq S$, $R \neq S$ on ε. Then, there is at most one 1-edge $\langle a^j, a^k\rangle$ in ε, and we know $\Psi_\varepsilon(\underline{t} \cup \underline{t}') = (t_{jk} - t_j) \cdot (t'_{jk} - t'_j) \cdot (\bar{a}^j - \bar{a}^k)$. On the other hand, similarly to the proof of the previous lemma, there are at most three different dilatation factors for edges in ε: t_{jk} for $\langle a^j, a^k\rangle$, some t for a possible edge being contained in $[S = 0]$, and t_j for the remaining ones. In particular, $t_{jk} = t = t_j$. □

Corollary: *If* $\dim \sigma = 3$*, then we have a complete description of the cup product* $T^1_Y(-R) \times T^1_Y(-S) \to T^2_Y(-R-S)$ *(*$R, S \geq 0$*):*

$$\underline{t} \cup \underline{t}' = \begin{cases} \sum_{j=1}^{m} t_{j\,j+1}\, t'_{j\,j+1}\, (\bar{a}^j - \bar{a}^{j+1}) \in R^\perp \subseteq N_{\mathbb{C}} & (\textit{if } R = S) \\ 0 \quad (\textit{if } R \neq S)\,. \end{cases}$$

Proof: On the one hand, σ is its only 3-face. On the other hand, we know that $\Psi = \Psi_\sigma$ is always injective in the three-dimensional case. □

Chapter 5

Toric Gorenstein singularities

5.1 $\mathbb{Q}$-Gorenstein cones

(5.1.1) The aim of this chapter is to apply the previous results to the special case of toric ($\mathbb{Q}$-) Gorenstein singularities. On the one hand, this notion is the next one if we are looking for a wider class than that of complete intersections (which yields no interesting deformation theory). On the other hand, the property "$\mathbb{Q}$-Gorenstein" admits a very clear description in the language of toric varieties and convex cones. (As far as I know, this was first mentioned by Ishida, [Ish], Theorem 7.7.)

In general, the dualizing sheaf ω on a Cohen-Macaulay variety can be obtained as

(i) $\omega_Z := \Omega_Z^{\dim Z}$ (sheaf of the highest differential forms), if Z is smooth, and

(ii) $\pi_*\omega_Y := \underline{\mathrm{Hom}}_{\mathcal{O}_Z}(\pi_*\mathcal{O}_Y,\, \omega_Z)$ for a flat, finite map $\pi : Y \to Z$.

Definition: A variety Y is called ($\mathbb{Q}$-) Gorenstein if (the reflexive hull of some tensor power of) ω_Y is an invertible sheaf on Y.

Since toric varieties are normal, the dualizing sheaf can be obtained as the push forward of the canonical sheaf on its smooth part. Hence, in our special situation, ω_Y equals the T(orus)-invariant, complete fractional ideal that is given by the order function mapping each fundamental generator onto $1 \in \mathbb{Z}$ (cf. Theorem I/9 in [Ke]). In particular, we obtain the following

Fact: *Let $Y = \mathrm{Spec}\, \mathbb{C}[\overset{\vee}{\sigma} \cap M]$ be an affine toric variety given by a cone $\sigma = \langle a^1, \dots, a^M\rangle$. Then, Y is $\mathbb{Q}$-Gorenstein, if and only if there is a primitive element $R^* \in M$ and a natural number $g \in \mathbb{N}$ such that*

$$\langle a^j,\, R^*\rangle = g \quad \text{for each } j = 1, \dots, M.$$

Y is Gorenstein, if and only if additionally $g = 1$.

In particular, toric $\mathbb{Q}$-Gorenstein singularities are obtained by putting a lattice polytope $Q \subseteq \mathbb{A}$ into the affine hyperplane $\mathbb{A} \times \{g\} \subseteq \mathbb{A} \times \mathbb{R} =: N_{\mathbb{R}}$ and defining $\sigma := \mathrm{Cone}(Q)$. Then, the canonical degree R^* equals $[\underline{0}, 1]$.

(5.1.2) The crucial point making $\mathbb{Q}$-Gorenstein cones easier to handle is the following: Affine conditions for fundamental generators a^j (such as "$\langle a^j, R\rangle = c$") can be translated into homogeneous ones (as "$\langle a^j, g\,R - c\,R^*\rangle = 0$").

Lemma: *Let σ be a $\mathbb{Q}$-Gorenstein cone which is smooth in codimension two. For each $R \in M$, the existence of some a^j meeting $\langle a^j, R\rangle \geq 2$ implies $T^1_Y(-R) = 0$.*

Proof: We consider the set $\tau := \{a^j \mid \langle a^j, R\rangle \leq 1\}$ (which does not need to represent any face of σ). It is built from exactly those fundamental generators of σ that are contained in the half space $[\langle \bullet, g\,R - R^*\rangle \leq 0]$.
Now, we can proceed along the lines of the proof of Proposition (3.5.2). □

Let us fix some R such that $T^1_Y(-R) \neq 0$. Using the notation of (3.5.1), our previous lemma implies that R yields exactly 1 on each fundamental generator of $\tau(R)$ (and, by definition of $\tau(R)$, non-positive values outside). There are just two different cases:

- $\tau(R) = \sigma$: That fact is equivalent to $R = R^*$, $g = 1$ (i.e. Y is Gorenstein and $R = R^*$ is the particular degree cutting out the lattice polytope Q). Moreover, $T^1_Y(-R^*) = V_{\mathbb{C}}(Q)/\mathbb{C}\cdot \underline{1}$.
- $\tau(R) \neq \sigma$: Then, $T^1_Y(-R) = V_{\mathbb{C}}(\tau(R) \cap [R = 1])/\mathbb{C}\cdot \underline{1}$ (the classes of Minkowski summands of a proper face of Q). Since (3.5.1), the whole T^1_Y is infinite-dimensional.

Corollary: *Let σ be an at least three-dimensional $\mathbb{Q}$-Gorenstein cone. Assume that T^1_Y is finite-dimensional (for instance, if Y has an isolated singularity). Then,*

(1) Y is rigid unless it is Gorenstein.

(2) If Y is Gorenstein ($g = 1$), then $T^1_Y = T^1_Y(-R^*) = V_{\mathbb{C}}(Q)/\mathbb{C}\cdot \underline{1}$.

(5.1.3) Let $Q \subseteq \boldsymbol{A}$ be a lattice polytope; via affine embedding into height one, it provides a Gorenstein cone $\sigma = \mathbb{R}_{\geq 0}\cdot Q \subseteq \boldsymbol{A} \times \mathbb{R} = N_{\mathbb{R}}$.

Proposition:

(1) Y is smooth in codimension two, if and only if the edges of Q do not contain interior lattice points (i.e. they are represented by primitive elements $d^i \in N \cap R^\perp$).

(2) Assume, moreover, $\dim Y \geq 4$ (i.e. $\dim Q \geq 3$). If $\dim T^1_Y < \infty$, then every 2-face of Q has to be a triangle (with three primitive edges).

(3) If every 2-face of Q is a triangle with three primitive edges (for instance, if Y is smooth in codimension three), then Y will be rigid.

Proof: The first part is straight forward. The second part follows from (3.5.1) - each polygon with at least four edges admits non-trivial Minkowski summands.
For the last part, we start with the homogeneous part $T^1_Y(-R^*)$ - it vanishes since (3.5.3). If $R \in M$ is an arbitrary degree, we use (3.5.1) telling that $T^1_Y(-R) = T^1_{Y(R)}(-\bar{R})$. Moreover, since (3.5.2) and (5.1.2), $Y(R)$ is a Gorenstein singularity of smaller dimension, and $\bar{R}$ is the corresponding canonical degree (called R^* for Y). Hence, we have $T^1_{Y(R)}(-\bar{R}) = 0$ again. □

In particular, if we were interested in *isolated* Gorenstein singularities, then the three-dimensional case would be the only interesting one. However, this case should be considered an important example for the upcoming investigations.

Remark: Just to be fashionable, we want to mention the following fact: As explained in (2.4.4), we may assign to Q also the projective variety $X_{\Sigma(Q)}$. Then, Corollary (5.1.2) says that $T^1_Y = \operatorname{Pic}(X_{\Sigma(Q)})/\mathcal{O}_X(1)$, i.e. some kind of mirror symmetry seems to be involved.

5.2 The tautological cone over $C(Q)$

(5.2.1) Let $Q \subseteq \boldsymbol{A}$ be a lattice polytope with primitive edges. We are going to investigate the deformation theory of $Y = Y_\sigma$ in detail.
In (2.2.2) we have introduced the cone $C(Q)$ of Minkowski summands of $I\!R_{\geq 0} \cdot Q$. For an element $(t_1, \dots, t_N) \in C(Q)$ the corresponding summand $Q_{\underline{t}}$ was built by the edges $t_i \cdot d^i$ $(i = 1, \dots, N)$. However, defining $Q_{\underline{t}}$ as a particular polytope inside its translation class requires a closer look:

Assume that $0 \in \boldsymbol{A}$ coincides with some vertex of the lattice polytope Q. Then, each vertex a of Q can be reached from there by some walk along the edges of Q - we obtain

$$a = \sum_{i=1}^{N} \lambda_i \, d^i \;\; \text{for some } \underline{\lambda} = (\lambda_1, \dots, \lambda_N)\,, \;\lambda_i \in Z\!\!\!Z.$$

Now, given an element $\underline{t} \in C(Q)$, we can define the corresponding vertex $a_{\underline{t}}$ (and finally the polytope $Q_{\underline{t}}$ as the convex hull of all of them) by

$$a_{\underline{t}} := \sum_{i=1}^{N} t_i \, \lambda_i \, d^i.$$

(The linear equations defining $V(Q) = \operatorname{span} C(Q)$ ensure that this definition does not depend on the particular path from 0 to a through the 1-skeleton of Q.)

In the whole chapter 5 we use the notation of section (2.2).

(5.2.2) **Definition:** *The tautological cone $\tilde{C}(Q) \subseteq \boldsymbol{A} \times V =: \tilde{N}_{I\!R} \subseteq \boldsymbol{A} \times I\!R^N$ is defined as*

$$\tilde{C}(Q) := \{(a, \underline{t}) \,|\, \underline{t} \in C(Q);\; a \in Q_{\underline{t}}\}.$$

Remark: $\tilde{C}(Q)$ is (as $C(Q)$) a rational, polyhedral cone. It is generated by the pairs $(a^j_{\underline{t}^l}, \underline{t}^l)$ with

- a^j is a vertex of Q and
- $\underline{t}^l$ is a fundamental generator of $C(Q)$.

(This follows from the simple rule $(a_{\underline{t}+\underline{t}'}, \underline{t} + \underline{t}') = (a_{\underline{t}}, \underline{t}) + (a_{\underline{t}'}, \underline{t}')$ for a vertex $a \in Q$ and $\underline{t}, \underline{t}' \in C(Q)$.)

With $\sigma = \operatorname{Cone}(Q) \subseteq \boldsymbol{A} \times I\!R$ (putting Q into the hyperplane $(t = 1)$), we obtain a fiber product diagram of rational polyhedral cones:

$$\begin{array}{ccc} [\sigma \subseteq \boldsymbol{A} \times I\!R] & \stackrel{i}{\hookrightarrow} & [\tilde{C}(Q) \subseteq \boldsymbol{A} \times V] \\ \downarrow \mathrm{pr}_{I\!R} & & \downarrow \mathrm{pr}_V \\ I\!R_{\geq 0} & \stackrel{\cdot \underline{\varrho}(Q)}{\hookrightarrow} & [C(Q) \subseteq V] \end{array}$$

(The vertical maps are projections onto the $I\!R$- and the V-component, respectively. The inclusion i is given by $(t \cdot a;\, t) \mapsto (t \cdot a;\, t, \dots, t)$.)

(5.2.3) The three cones $\sigma = \operatorname{Cone}(Q) \subseteq \boldsymbol{A} \times I\!R$, $\tilde{C}(Q) \subseteq \boldsymbol{A} \times V$, and $C(Q) \subseteq V$ define affine toric varieties called Y, X, and S, respectively. Using the abbreviations

$$M := I\!L^* \times Z\!\!\!Z, \quad M_{I\!R} := \boldsymbol{A}^* \times I\!R, \quad \tilde{M} := I\!L^* \times V^*_{Z\!\!\!Z}, \quad \tilde{M}_{I\!R} := \boldsymbol{A}^* \times V^*\,,$$

the corresponding rings of regular functions are $A(Y) = \mathbb{C}[\sigma^\vee \cap M]$, $A(X) = \mathbb{C}[\tilde{C}(Q)^\vee \cap \tilde{M}]$, and $A(S) = \mathbb{C}[C(Q)^\vee \cap V_{\mathbb{Z}}^*]$. These varieties come with the following maps:

(i) The diagram of (5.2.2) induces a fiber product diagram

$$\begin{array}{ccc} Y & \stackrel{i}{\hookrightarrow} & X \\ \downarrow & & \downarrow\pi \\ \mathbb{C} & \hookrightarrow & S\,. \end{array}$$

Both horizontal maps are closed embeddings. (These claims will be checked in (5.2.5) and (5.2.8)(1).)

(ii) $C(Q) = V \cap \mathbb{R}^N_{\geq 0}$ is contained in $\mathbb{R}^N_{\geq 0}$, and this inclusion provides a morphism $p : S \to \mathbb{C}^N$ inducing functions $t_1, \ldots, t_N$ on S. The composition $\mathbb{C} \hookrightarrow S \stackrel{p}{\to} \mathbb{C}^N$ sends t to $(t, \ldots, t)$.

We will use the map $\pi : X \to S$ to construct the versal deformation of Y.

(5.2.4) To study the toric varieties Y, X, and S it is important to understand the dual cones of $\sigma, \tilde{C}(Q)$, and $C(Q)$, respectively. Let us start with the dual cone of σ:

To each non-trivial $c \in \mathbb{L}^*$ we associate a vertex $a(c)$ of Q and an integer $\eta_0(c)$ meeting the properties

$$\begin{array}{rcl} \langle Q, -c\rangle & \leq & \eta_0(c) \qquad \text{and} \\ \langle a(c), -c\rangle & = & \eta_0(c). \end{array}$$

For $c = 0$ we define $a(0) := 0 \in \mathbb{L}$ and $\eta_0(0) := 0 \in \mathbb{Z}$.

Remark:

(1) With respect to Q, $c \neq 0$ is the inner normal vector of the affine supporting hyperplane $[\langle \bullet, -c\rangle = \eta_0(c)]$ through $a(c)$. In particular, $\eta_0(c)$ is uniquely determined, while $a(c)$ is not.

(2) Since $0 \in Q$, the integers $\eta_0(c)$ are non-negative.

By the definition of η_0, we have

$$\partial\sigma^\vee \cap M = \{[c, \eta_0(c)] \mid c \in \mathbb{L}^*\}\,.$$

Moreover, if $c^1, \ldots, c^w \in \mathbb{L}^* \setminus 0$ are those elements producing irreducible pairs $[c, \eta_0(c)]$ (i.e. not allowing any non-trivial lattice decomposition $[c, \eta_0(c)] = [c', \eta_0(c')] + [c'', \eta_0(c'')]$), then the elements

$$[\underline{0}, 1],\ [c^1, \eta_0(c^1)], \ldots, [c^w, \eta_0(c^w)]$$

form the minimal generator set E for $\sigma^\vee \cap M$ as a semigroup. Among them are all pairs $[c, \eta_0(c)]$ describing facets (i.e. top dimensional faces) of Q.

E has always been used to obtain a closed embedding $Y \hookrightarrow \mathbb{C}^{w+1}$. To emphasize the special role of the element $[\underline{0}, 1] \in E$, we will denote the corresponding coordinate of $\mathbb{C}^{w+1}$ by t rather than the usual z_0 (keeping $z_1, \ldots, z_w$ unchanged).

Example: We continue our example Q_6 from section (2.2). Here, the facets of Q_6 equal its edges $d^1, \ldots, d^6$, and they are sufficient for producing all irreducible pairs $[c^1, \eta_0(c^1)], \ldots, [c^6, \eta_0(c^6)]$. We have

$$\begin{array}{rclrclrcl} c^1 & = & [0,1], & c^2 & = & [-1,1], & c^3 & = & [-1,0], \\ c^4 & = & [0,-1], & c^5 & = & [1,-1], & c^6 & = & [1,0]\,. \end{array}$$

The corresponding vertices are (for instance)

$$a(c^6) = a(c^1) = (0,0), \quad a(c^2) = a(c^3) = (2,1), \quad a(c^4) = a(c^5) = (1,2),$$

and we obtain

$$\eta_0(c^1) = 0,\ \eta_0(c^2) = 1,\ \eta_0(c^3) = 2,\ \eta_0(c^4) = 2,\ \eta_0(c^5) = 1,\ \eta_0(c^6) = 0\,.$$

(5.2.5) Thinking of $C(Q)$ as a cone in $\mathbb{R}^N$ instead of V allows dualizing the equation $C(Q) = \mathbb{R}^N_{\geq 0} \cap V$ to get $C(Q)^\vee = \mathbb{R}^N_{\geq 0} + V^\perp$. Hence, for $C(Q)$ as a cone in V we obtain

$$C(Q)^\vee = {\mathbb{R}^N_{\geq 0} + V^\perp}\big/{V^\perp} = \text{im}\,[\mathbb{R}^N_{\geq 0} \longrightarrow V^*].$$

The surjection $\mathbb{R}^N_{\geq 0} \twoheadrightarrow C(Q)^\vee$ induces a map $\mathbb{N}^N \longrightarrow C(Q)^\vee \cap V^*_{\mathbb{Z}}$, which does not need to be surjective at all. This leads to the following definition:

Definition: *On $V^*_{\mathbb{Z}}$ we introduce a partial ordering "$\geq$" by*

$$\underline{\eta} \geq \underline{\eta}' \quad \Longleftrightarrow \quad \underline{\eta} - \underline{\eta}' \in \text{im}\,[\mathbb{N}^N \to V^*_{\mathbb{Z}}] \subseteq C(Q)^\vee \cap V^*_{\mathbb{Z}}.$$

On the geometric level, the non-saturated semigroup im $[\mathbb{N}^N \to V^*_{\mathbb{Z}}] \subseteq C(Q)^\vee \cap V^*_{\mathbb{Z}}$ corresponds to the scheme theoretical image $\bar{S}$ of $p : S \to \mathbb{C}^N$, and $S \to \bar{S}$ is its normalization (cf. (5.3.2)). The equations of $\bar{S} \subseteq \mathbb{C}^N$ are collected in the kernel of

$$\mathbb{C}[t_1, \ldots, t_N] = \mathbb{C}[\mathbb{N}^N] \stackrel{\varphi}{\longrightarrow} \mathbb{C}[C(Q)^\vee \cap V^*_{\mathbb{Z}}] \subseteq \mathbb{C}[V^*_{\mathbb{Z}}],$$

and it is easy to see that

$$\begin{array}{rcl} \ker \varphi & = & \left(\prod_{i=1}^N t_i^{d_i^+} - \prod_{i=1}^N t_i^{d_i^-} \;\middle|\; \underline{d} \in \mathbb{Z}^N \cap V^\perp \right) \qquad \text{with} \\ V^\perp & = & \text{span}\left\{ [\langle \varepsilon_1 d^1, c\rangle, \ldots, \langle \varepsilon_N d^N, c\rangle] \,\middle|\, \varepsilon < Q \text{ is a 2-face, } c \in \boldsymbol{A}^* \right\}. \end{array}$$

Remark: Using our new notations, we can reformulate Theorem (2.2.4) now: $\mathcal{M} \subseteq \mathbb{C}^N$ is the largest closed subscheme that is contained in $\bar{S}$ and additionally comes from ${\mathbb{C}^N}\big/{\mathbb{C} \cdot \varrho(Q)}$ via ℓ.

On the other hand, dualizing the embedding $\mathbb{R}_{\geq 0} \hookrightarrow C(Q)$ yields

$$\begin{array}{ccc} C(Q)^\vee \cap V^*_{\mathbb{Z}} & \twoheadrightarrow & \mathbb{N} \\ \underline{\eta} & \mapsto & \sum_i \eta_i \end{array}$$

at the level of semigroups. This map is surjective, even after restricting to the subset im $[\mathbb{N}^N \to V^*_{\mathbb{Z}}]$: All coordinate functions t_i map onto $1 \in \mathbb{N}$.

Geometrically this means that both maps $\mathbb{C} \to S$ and $\mathbb{C} \to \bar{S}$ are closed embeddings, and the corresponding ideals are $\left(x^{\underline{\eta}} - x^{\underline{\eta}'} \;\middle|\; \underline{\eta}, \underline{\eta}' \in C(Q)^\vee \cap V^*_{\mathbb{Z}} \text{ with } \sum_i \eta_i = \sum_i \eta'_i\right)$ and $(t_i - t_j \,|\, 1 \leq i, j \leq N)$, respectively. In particular, we got a first contribution to proof the claims made in (5.2.3)(i).

(5.2.6) In the next two sections we take a closer look at the dualized cone $\tilde{C}(Q)^\vee$.

Definition: *For $c \in \mathbb{L}^*$ choose some path from $0 \in Q$ to $a(c) \in Q$ through the 1-skeleton of Q and let $\underline{\lambda}^c = (\lambda_1^c, \ldots, \lambda_N^c) \in \mathbb{Z}^N$ be the vector counting how often (and in which direction) we went through each particular edge (cf. (5.2.1)). Then,*

$$\underline{\eta}(c) := \left[-\lambda_1^c \langle d^1, c\rangle, \ldots, -\lambda_N^c \langle d^N, c\rangle\right] \in \mathbb{Z}^N$$

defines an element $\underline{\eta}(c) \in V_{\mathbb{Z}}^*$ not depending on the special choice of the path $\underline{\lambda}^c$.

(Let $\tilde{\underline{\lambda}}^c$ be a different path from 0 to $a(c)$; it will differ from $\underline{\lambda}^c$ by some linear combination $\sum_{\varepsilon<Q} g_\varepsilon \, \underline{\varepsilon}$ ($g_\varepsilon \in \mathbb{Z}$ for 2-faces $\varepsilon < Q$) only. In particular,

$$\tilde{\lambda}_i^c \langle d^i, c\rangle - \lambda_i^c \langle d^i, c\rangle = \sum_{\varepsilon<Q} g_\varepsilon \langle \varepsilon_i \, d^i, \, c\rangle,$$

and we obtain $\underline{\eta}(c)_{\tilde{\lambda}} - \underline{\eta}(c)_\lambda \in V^\perp$.)

Lemma:

(i) $\underline{\eta}(0) = 0 \in V_{\mathbb{Z}}^*$.

(ii) For all $c \in \mathbb{L}^*$ we have $\underline{\eta}(c) \geq 0$ (in the sense of Definition (5.2.5)).

(iii) $\underline{\eta}$ is convex: $\sum_v g_v \, \underline{\eta}(c^v) \geq \underline{\eta}(\sum_v g_v \, c^v)$ for natural numbers $g_v \in \mathbb{N}$.

(iv) $\sum_{i=1}^N \eta_i(c) = \eta_0(c)$ for arbitrary $c \in \mathbb{L}^*$.

Proof: (ii) $a(c)$ is a vertex of Q providing minimal value of the linear function $\langle \bullet, c\rangle$. In particular, we can choose a path $\underline{\lambda}^c$ from $0 \in Q$ to $a(c)$ such that this function decreases in each step, i.e. $\lambda_i^c \langle d^i, c\rangle \leq 0$ ($i = 1, \ldots, N$).

(iii) We define the following paths through the 1-skeleton of Q:

- $\underline{\lambda} :=$ path from $0 \in Q$ to $a(\sum_v g_v \, c^v) \in Q$,
- $\underline{\mu}^v :=$ path from $a(\sum_v g_v \, c^v) \in Q$ to $a(c^v) \in Q$ such that $\mu_i^v \langle d^i, c^v\rangle \leq 0$ for each $i = 1, \ldots, N$.

Then, $\underline{\lambda}^v := \underline{\lambda} + \underline{\mu}^v$ is a path from $0 \in Q$ to $a(c^v)$, and for $i = 1, \ldots, N$ we obtain

$$\begin{aligned}\sum_v g_v \, \eta_i(c^v) - \eta_i\left(\sum_v g_v \, c^v\right) &= -\sum_v g_v \, (\lambda_i + \mu_i^v) \, \langle d^i, c^v\rangle + \lambda_i \left\langle d^i, \sum_v g_v \, c^v \right\rangle \\ &= -\sum_v g_v \, \mu_i^v \, \langle d^i, c^v\rangle \geq 0 \, .\end{aligned}$$

(iv) By definition of $\underline{\lambda}^c$ we have $\sum_{i=1}^N \lambda_i^c \, d^i = a(c)$. In particular,

$$\sum_{i=1}^N \eta_i(c) = -\sum_{i=1}^N \langle \lambda_i^c \, d^i, \, c\rangle = -\langle a(c), \, c\rangle = \eta_0(c).$$

□

Example: In our hexagon Q_6 we choose the following paths from $(0,0)$ to the vertices $a(c^1), \ldots, a(c^6)$, respectively:

$$\underline{\lambda}^6 = \underline{\lambda}^1 := \underline{0}, \quad \underline{\lambda}^2 = \underline{\lambda}^3 := [1,1,0,0,0,0], \quad \underline{\lambda}^4 = \underline{\lambda}^5 := [1,1,1,1,0,0] \, .$$

They provide

$$\begin{array}{llllll} \underline{\eta}(c^1) = [0,0,0,0,0,0]\,, & \underline{\eta}(c^2) = [1,0,0,0,0,0]\,, & \underline{\eta}(c^3) = [1,1,0,0,0,0]\,, \\ \underline{\eta}(c^4) = [0,1,1,0,0,0]\,, & \underline{\eta}(c^5) = [-1,0,1,1,0,0]\,, & \underline{\eta}(c^6) = [0,0,0,0,0,0]\,. \end{array}$$

Since $[1,0,-1,-1,0,1] = [\langle d^1, [1,-1]\rangle, \dots, \langle d^6, [1,-1]\rangle] \in V^\perp$, the vector $\underline{\eta}(c^5)$ can be transformed into $[0,0,0,0,0,1]$.

Remark: The definitions of $a(c), \eta_0(c)$, and $\underline{\eta}(c)$ also make sense for general $c \in \boldsymbol{A}^*$. Then, $\eta_0(c) \in I\!R$ and $\underline{\eta}(c) \in V^*$ do not need to be contained in the lattices anymore. The previous lemma will keep valid (even for $g_v \in I\!R_{\geq 0}$ in (iii)), if the relation "≥ 0" is replaced by the weaker version "$\in C(Q)^\vee$".

(5.2.7) Proposition:

(1) $\tilde{C}(Q)^\vee = \{ [c,\underline{\eta}] \in \boldsymbol{A}^* \times V^* \mid \underline{\eta} - \underline{\eta}(c) \in C(Q)^\vee \}$

(2) In particular, $[c, \underline{\eta}(c)] \in \tilde{C}(Q)^\vee$*, and moreover, it is the only preimage of* $[c, \eta_0(c)] \in \sigma^\vee$ *via the surjection* $i^\vee : \tilde{C}(Q)^\vee \longrightarrow \sigma^\vee$*.*

(3) $[c^1, \underline{\eta}(c^1)], \dots, [c^w, \underline{\eta}(c^w)]$ *and* $C(Q)^\vee \cap V^*_{Z\!\!Z}$ *(embedded as* $[0, C(Q)^\vee]$*) generate the semigroup* $\tilde{C}(Q)^\vee \cap \tilde{M}$*. (For recalling the definition of the* $c^1, \dots, c^w$*, cf. (5.2.4).)*

Proof: (1) Let $[c, \underline{\eta}] \in \boldsymbol{A}^* \times V^*$ be given; if some representative of $\underline{\eta}$ in $I\!R^N$ is needed, then it will be denoted by the same name. We have the following equivalences:

$$\begin{aligned} [c,\underline{\eta}] \in \tilde{C}(Q)^\vee \quad &\Longleftrightarrow \quad \langle (Q_{\underline{t}}, \underline{t}), [c, \underline{\eta}] \rangle \geq 0 \quad \text{for each } \underline{t} \in C(Q) \\ &\Longleftrightarrow \quad \langle Q_{\underline{t}}, c\rangle + \langle \underline{t}, \underline{\eta} \rangle \geq 0 \quad \text{for each } \underline{t} \in C(Q) \\ &\Longleftrightarrow \quad \langle a(c)_{\underline{t}}, c\rangle + \langle \underline{t}, \underline{\eta} \rangle \geq 0 \quad \text{for each } \underline{t} \in C(Q). \end{aligned}$$

Using some path $\underline{\lambda}^c$ we obtain:

$$\begin{aligned} [c,\underline{\eta}] \in \tilde{C}(Q)^\vee \quad &\Longleftrightarrow \quad \sum_{i=1}^N t_i\, \lambda_i^c \langle d^i, c\rangle + \langle \underline{t}, \underline{\eta} \rangle \geq 0 \quad \text{for each } \underline{t} \in C(Q) \\ &\Longleftrightarrow \quad \sum_{i=1}^N t_i \cdot \left(\lambda_i^c \langle d^i, c\rangle + \eta_i \right) \geq 0 \quad \text{for each } \underline{t} \in C(Q) \\ &\Longleftrightarrow \quad \left[\lambda_1^c \langle d^1, c\rangle + \eta_1, \dots, \lambda_N^c \langle d^N, c\rangle + \eta_N \right] \in C(Q)^\vee. \end{aligned}$$

(2) By part (1) we know that for a $[c, \underline{\eta}] \in \tilde{C}(Q)^\vee$ it is possible to choose $I\!R^N$-representatives for $\underline{\eta}, \underline{\eta}(c)$ such that $\eta_i \geq \eta_i(c)$ for $i = 1, \dots, N$.
On the other hand, the two equalities $\sum_i \eta_i(c) = \eta_0(c)$ (cf. (iv) of the previous lemma) and $\sum_i \eta_i = \eta_0(c)$ (corresponding to the fact $[c, \underline{\eta}] \mapsto [c, \eta_0(c)]$) imply $\underline{\eta} = \underline{\eta}(c)$ then.

(3) Let $[c, \underline{\eta}] \in \tilde{C}(Q)^\vee$. Then, $[c, \eta_0(c)]$ is representable as a non-negative linear combination $[c, \eta_0(c)] = \sum_{v=1}^w p_v\, [c^v, \eta_0(c^v)]$ ($p_v \in I\!N$ if $c \in I\!L^*$). Since both elements $[c, \underline{\eta}(c)]$ and $\sum_v p_v [c^v, \underline{\eta}(c^v)]$ are preimages of $[c, \eta_0(c)]$ via $i^\vee$, they must be equal by (2), and we obtain

$$\begin{aligned} [c, \underline{\eta}] &= [c, \underline{\eta}(c)] + [0, \underline{\eta} - \underline{\eta}(c)] \\ &= \textstyle\sum_v p_v\, [c^v, \underline{\eta}(c^v)] + [0, \underline{\eta} - \underline{\eta}(c)]\,. \end{aligned}$$

□

(5.2.8) Finally, we will take a short look at the geometrical situation reached at this point.

(1) The linear map
$$\begin{array}{ccc} \tilde{C}(Q)^\vee \cap \tilde{M} & \longrightarrow\!\!\!\!\!\to & \sigma^\vee \cap M \\ [c,\, \underline{\eta}] & \mapsto & [c,\, \sum_i \eta_i] \end{array}$$
is surjective ($[c, \underline{\eta}(c)] \mapsto [c, \eta_0(c)]$; $[0, e_i] \mapsto [0, 1]$). Since
$$x^{[c,\underline{\eta}]} - x^{[c,\underline{\eta}']} = x^{[c,\underline{\eta}(c)]} \cdot (x^{[0,\underline{\eta}-\underline{\eta}(c)]} - x^{[0,\underline{\eta}'-\underline{\eta}(c)]}),$$
the kernel of the corresponding homomorphism between the semigroup algebras equals the ideal
$$\left(x^{[0,\underline{\eta}]} - x^{[0,\underline{\eta}']} \,\middle|\, \sum_i \eta_i = \sum_i \eta_i' \right).$$
In particular, the map $Y \hookrightarrow X$ is a closed embedding. Moreover, comparing with the similar statement concerning $C(Q)^\vee$ and $\mathbb{N}$ at the end of (5.2.5), we obtain that the diagram of (5.2.3)(i) is a fiber product diagram, indeed.

(2) The elements $[c^1, \underline{\eta}(c^1)], \ldots, [c^w, \underline{\eta}(c^w)] \in \tilde{C}(Q)$ induce some regular functions $Z_1, \ldots, Z_w$ on X. They define a closed embedding $X \hookrightarrow \mathbb{C}^w \times S$ lifting the embedding $Y \hookrightarrow \mathbb{C}^{w+1}$ of (5.2.4). Moreover, for $v = 1, \ldots, w$, Z_v is the only monomial function lifting z_v from Y to X.

We have obtained the following commutative diagram:
$$\begin{array}{ccccccc} Y & \hookrightarrow & \mathbb{C}^w \times \mathbb{C} & = & \mathbb{C}^w \times \mathbb{C} & & \\ \downarrow & \otimes & \downarrow & & \downarrow \Delta & & \\ X & \hookrightarrow & \mathbb{C}^w \times S & \xrightarrow{p} & \mathbb{C}^w \times \mathbb{C}^N & & \\ & & \downarrow & & \downarrow & & \\ & & S & \xrightarrow{p} & \mathbb{C}^N & \xrightarrow{\ell} & \mathbb{C}^{N-1}. \end{array}$$

5.3 A flat family over $\bar{\mathcal{M}}$

(5.3.1) **Theorem:** *Denote by $\bar{X}$ and $\bar{S}$ the scheme theoretical images of X and S in $\mathbb{C}^w \times \mathbb{C}^N$ and $\mathbb{C}^N$, respectively. Then,*

(1) $X \to \bar{X}$ and $S \to \bar{S}$ are the normalization maps.

(2) $\pi : X \to S$ induces a map $\bar{\pi} : \bar{X} \to \bar{S}$, and π can be recovered from $\bar{\pi}$ via base change $S \to \bar{S}$.

(3) Restricting to $\mathcal{M} \subseteq \bar{S}$ (cf. section (2.2)) and composing with ℓ turns $\bar{\pi}$ into a family
$$\bar{X} \times_{\bar{S}} \mathcal{M} \xrightarrow{\bar{\pi}} \mathcal{M} \xrightarrow{\ell} \bar{\mathcal{M}}.$$
It is flat in $0 \in \bar{\mathcal{M}} \subseteq \mathbb{C}^{N-1}$, and the special fiber equals Y.

The proof of this theorem will fill section (5.3).

(5.3.2) The ring of regular functions $A(\bar{S})$ is given as the image of the map $\mathbb{C}[t_1, \ldots, t_N] \to A(S)$. Since $\mathbb{Z}^N \longrightarrow\!\!\!\!\!\to V^*_{\mathbb{Z}}$ is surjective, the rings $A(\bar{S}) \subseteq A(S) \subseteq \mathbb{C}[V^*_{\mathbb{Z}}]$ have the same field of fractions.
On the other hand, while t-monomials with negative exponents might be involved in $A(S)$, the surjectivity of $\mathbb{R}^N_{\geq 0} \longrightarrow\!\!\!\!\!\to C(Q)^\vee$ tells us that sufficiently high powers of those monomials always

come from $A(\bar{S})$. In particular, $A(S)$ is normal over $A(\bar{S})$.

$A(\bar{X})$ is given as the image $A(\bar{X}) = \mathrm{im}\,(\mathbb{C}[Z_1,\dots,Z_w,t_1,\dots,t_N] \to A(X))$. Since $A(X)$ is generated by $Z_1,\dots,Z_w$ over its subring $A(S)$ (cf. Proposition (5.2.7)(3)), the same arguments as for S and $\bar{S}$ apply. Hence, Part (1) of the previous theorem is proved.

(5.3.3) Recalling that $z_1,\dots,z_w$, $t \in A(Y)$ stand for the monomials with exponents $[c^1, \eta_0(c^1)]$, ..., $[c^w, \eta_0(c^w)]$, $[0,1] \in \sigma^\vee \cap M$, respectively, the equations of $Y \subseteq \mathbb{C}^{w+1}$ (cf. (3.2.6)) are shaped as

$$\begin{aligned} f_{(a,b,\alpha,\beta)}(\underline{z},t) \quad &:= \quad t^\alpha \prod_{v=1}^{w} z_v^{a_v} - t^\beta \prod_{v=1}^{w} z_v^{b_v} \\ \text{with} \quad & a,b \in \mathbb{N}^w : \; \textstyle\sum_v a_v\, c^v = \sum_v b_v\, c^v \quad \text{and} \\ & \alpha,\beta \in \mathbb{N} : \; \textstyle\sum_v a_v\, \eta_0(c^v) + \alpha = \sum_v b_v\, \eta_0(c^v) + \beta\,. \end{aligned}$$

Defining $c := \sum_v a_v\, c^v = \sum_v b_v\, c^v$, we can lift them to the following elements of $A(\bar{S})[Z_1,\dots,Z_w]$ (described by using the map $\mathbb{C}[Z_1,\dots,Z_w,\, t_1,\dots,t_N] \longrightarrow A(\bar{S})[Z_1,\dots,Z_w]$):

$$F_{(a,b,\alpha,\beta)}(\underline{Z},\underline{t}) := f_{(a,b,\alpha,\beta)}(\underline{Z},t_1) - \underline{Z}^{[c,\underline{\eta}(c)]} \cdot \left(\underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v)} - \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v)}\right) \cdot \underline{t}^{-\underline{\eta}(c)}\,.$$

Remark:

(1) The symbol $\underline{Z}^{[c,\underline{\eta}(c)]}$ means $\prod_{v=1}^w Z_v^{p_v}$ with natural numbers $p_v \in \mathbb{N}$ such that $[c,\underline{\eta}(c)] = \sum_v p_v\, [c^v, \underline{\eta}(c^v)]$, or equivalently $[c,\eta_0(c)] = \sum_v p_v\, [c^v, \eta_0(c^v)]$. This condition does not determine the coefficients p_v uniquely - choose one of the possibilities. (Choosing other coefficients q_v with the same property yields

$$Z_1^{p_1} \cdot \ldots \cdot Z_w^{p_w} - Z_1^{q_1} \cdot \ldots \cdot Z_w^{q_w} = F_{(p,q,0,0)}(\underline{Z},\underline{t}) = f_{(p,q,0,0)}(\underline{Z},t)\,,$$

anyway.)

(2) By part (iii) of Lemma (5.2.6), we have $\sum_v a_v \underline{\eta}(c^v)$, $\sum_v b_v \underline{\eta}(c^v) \geq \underline{\eta}(c)$. In particular, representatives of the $\underline{\eta}$'s can be chosen such that all t-exponents occurring in monomials of F are non-negative, i.e. F defines an element of $A(\bar{S})[Z_1,\dots,Z_w]$, indeed.

Lemma: *The polynomials $F_{(a,b,\alpha,\beta)}$ generate $Ker(A(\bar{S})[\underline{Z}] \to A(X))$, i.e. they can be used as equations for $\bar{X} \subseteq \mathbb{C}^w \times \bar{S}$.*

Proof: Mapping F into $A(X) = \oplus_{[c,\underline{\eta}]}\, \mathbb{C}x^{[c,\underline{\eta}]}$ ($[c,\underline{\eta}]$ runs through all elements of $\tilde{C}(Q)^\vee \cap (\mathbb{L}^* \times V_{\mathbb{Z}}^*)$; $Z_v \mapsto x^{[c^v,\underline{\eta}(c^v)]}$, $t_i \mapsto x^{[0,e_i]}$) yields

$$\begin{aligned} F_{(a,b,\alpha,\beta)} \quad &= \quad \left(t_1^\alpha \textstyle\prod_v Z_v^{a_v} - \underline{Z}^{[c,\underline{\eta}(c)]}\, \underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v) - \underline{\eta}(c)}\right) - \\ &\qquad - \left(t_1^\beta \textstyle\prod_v Z_v^{b_v} - \underline{Z}^{[c,\underline{\eta}(c)]}\, \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v) - \underline{\eta}(c)}\right) \\ &\mapsto \quad \left(x^{\alpha[0,e_1] + \sum_v a_v [c^v,\underline{\eta}(c^v)]} - x^{[c,\underline{\eta}(c)] + \alpha[0,e_1] + \sum_v a_v[0,\underline{\eta}(c^v)] - [0,\underline{\eta}(c)]}\right) - \\ &\qquad - \left(x^{\beta[0,e_1] + \sum_v b_v [c^v,\underline{\eta}(c^v)]} - x^{[c,\underline{\eta}(c)] + \beta[0,e_1] + \sum_v b_v[0,\underline{\eta}(c^v)] - [0,\underline{\eta}(c)]}\right) \\ &= \quad 0 \,-\, 0 = 0\,. \end{aligned}$$

On the other hand, $\operatorname{Ker}(A(\bar{S})[\underline{Z}] \to A(X))$ is obviously generated by the binomials

$$\underline{t}^{\underline{\eta}} Z_1^{a_1} \cdot \ldots \cdot Z_w^{a_w} - \underline{t}^{\underline{\mu}} Z_1^{b_1} \cdot \ldots \cdot Z_w^{b_w} \quad \text{such that}$$
$$\textstyle\sum_v a_v[c^v, \underline{\eta}(c^v)] + [0, \underline{\eta}] = \sum_v b_v[c^v, \underline{\eta}(c^v)] + [0, \underline{\mu}],$$
$$\text{i.e.} \quad \bullet \; c := \textstyle\sum_v a_v c^v = \sum_v b_v c^v$$
$$\bullet \; \textstyle\sum_v a_v \underline{\eta}(c^v) + \underline{\eta} = \sum_v b_v \underline{\eta}(c^v) + \underline{\mu}.$$

However,

$$\begin{aligned} \underline{t}^{\underline{\eta}} \underline{Z}^a - \underline{t}^{\underline{\mu}} \underline{Z}^b &= \underline{t}^{\underline{\eta}} \cdot \left(\textstyle\prod_v Z_v^{a_v} - \underline{Z}^{[c,\underline{\eta}(c)]} \, \underline{t}^{\sum_v a_v \underline{\eta}(c^v) - \underline{\eta}(c)} \right) - \\ &\quad - \underline{t}^{\underline{\mu}} \cdot \left(\textstyle\prod_v Z_v^{b_v} - \underline{Z}^{[c,\underline{\eta}(c)]} \, \underline{t}^{\sum_v b_v \underline{\eta}(c^v) - \underline{\eta}(c)} \right) \\ &= \underline{t}^{\underline{\eta}} \cdot F_{(a,p,0,\alpha)} - \underline{t}^{\underline{\mu}} \cdot F_{(b,p,0,\beta)} \end{aligned}$$

with $p \in \mathbb{N}^w$ such that $\sum_v p_v[c^v, \underline{\eta}(c^v)] = [c, \underline{\eta}(c)]$, $\alpha = \sum_v a_v \eta_0(c^v) - \eta_0(c)$, and $\beta = \sum_v b_v \eta_0(c^v) - \eta_0(c)$. □

(5.3.4) Using exponents $\underline{\eta}, \underline{\mu} \in \mathbb{Z}^N$ (instead of $\mathbb{N}^N$), the binomials $\underline{t}^{\underline{\eta}} \underline{Z}^a - \underline{t}^{\underline{\mu}} \underline{Z}^b$ generate the kernel of the map

$$A(S)[\underline{Z}] = A(\bar{S})[\underline{Z}] \otimes_{A(\bar{S})} A(S) \longrightarrow A(\bar{X}) \otimes_{A(\bar{S})} A(S) \longrightarrow A(X).$$

Since $\underline{Z}^a \otimes \underline{t}^{\underline{\eta}} - \underline{Z}^b \otimes \underline{t}^{\underline{\mu}} = \underline{Z}^{[c,\underline{\eta}(c)]} \otimes \left(\underline{t}^{\sum_v a_v \underline{\eta}(c^v) - \underline{\eta}(c) + \underline{\eta}} - \underline{t}^{\sum_v b_v \underline{\eta}(c^v) - \underline{\eta}(c) + \underline{\mu}} \right) = 0$ in $A(\bar{X}) \otimes_{A(\bar{S})} A(S)$, this implies that the surjection $A(\bar{X}) \otimes_{A(\bar{S})} A(S) \longrightarrow A(X)$ is injective, too. In particular, part (2) of our theorem is proved.

(5.3.5) We are going to use the following well known criterion of flatness:

Theorem: *Let $\tilde{\pi} : \tilde{X} \hookrightarrow \mathbb{C}^{w+1} \times \bar{\mathcal{M}} \longrightarrow \bar{\mathcal{M}}$ be a map with special fiber $Y = \tilde{\pi}^{-1}(0)$; in particular, $Y \subseteq \mathbb{C}^{w+1}$ is defined by the restrictions to $0 \in \bar{\mathcal{M}}$ of the equations defining $\tilde{X} \subseteq \mathbb{C}^{w+1} \times \bar{\mathcal{M}}$. Then, $\tilde{\pi}$ is flat, if and only if each linear relation between the (restricted) equations for Y lifts to some linear relation between the original equations for $\tilde{X}$.*

Proof: According to [Ma], (20.C), Theorem 49, flatness of $\tilde{\pi}$ in $0 \in \bar{\mathcal{M}}$ is equivalent to the vanishing of $\operatorname{Tor}_1^{\mathcal{O}_{\bar{\mathcal{M}},0}}((\tilde{\pi}_* \mathcal{O}_{\tilde{X}})_0, \mathbb{C})$ ($\mathbb{C}$ becomes an $\mathcal{O}_{\bar{\mathcal{M}},0}$-module via evaluating in $0 \in \bar{\mathcal{M}}$).
Using the embedding $\tilde{X} \hookrightarrow \mathbb{C}^{w+1} \times \bar{\mathcal{M}}$ (together with the defining equations and linear relations between them) we obtain an $\mathcal{O}_{\bar{\mathcal{M}},0}[Z_0, \ldots, Z_w]$-free (hence $\mathcal{O}_{\bar{\mathcal{M}},0}$-free) resolution of $(\tilde{\pi}_* \mathcal{O}_{\tilde{X}})_0$ up to the second term. Now, the condition that relations between Y-equations lift to those between $\tilde{X}$-equations is equivalent to the fact that our (partial) resolution remains exact under $\otimes_{\mathcal{O}_{\bar{\mathcal{M}},0}} \mathbb{C}$. □

For our special situation take $\tilde{X} := \bar{X} \times_{\bar{S}} \mathcal{M}$ (and $\bar{\mathcal{M}} := \bar{\mathcal{M}}$, $Y := Y$); in (5.3.3) we have seen how the equations defining $Y \hookrightarrow \mathbb{C}^w \times \mathbb{C}$ can be lifted to those defining $\bar{X} \hookrightarrow \mathbb{C}^w \times \bar{S}$, hence $\bar{X} \times_{\bar{S}} \mathcal{M} \hookrightarrow \mathbb{C}^w \times \mathcal{M} \xrightarrow{\sim} \mathbb{C}^w \times \mathbb{C} \times \bar{\mathcal{M}}$.
In particular, to show (3) of Theorem (5.3.1), we have just to take the linear relations between the $f_{(a,b,\alpha,\beta)}$'s and lift them to relations between the $F_{(a,b,\alpha,\beta)}$'s.

According to the special shape of our generator set E, there are three types of relations between the $f_{(a,b,\alpha,\beta)}$'s (in (3.2.7), type (i) was called $s(\ldots)$; the types (ii) and (iii) will fit into the $r(\ldots)$-relations):

(i) $f_{(a,r,\alpha,\gamma)} + f_{(r,b,\gamma,\beta)} = f_{(a,b,\alpha,\beta)}$
with • $\sum_v a_v c^v = \sum_v r_v c^v = \sum b_v c^v$ and
• $\sum_v a_v \eta_0(c^v) + \alpha = \sum_v r_v \eta_0(c^v) + \gamma = \sum_v b_v \eta_0(c^v) + \beta$.
For this relation, the same equation between the F's is true.

(ii) $t \cdot f_{(a,b,\alpha,\beta)} = f_{(a,b,\alpha+1,\beta+1)}$ lifts to $t_1 \cdot F_{(a,b,\alpha,\beta)} = F_{(a,b,\alpha+1,\beta+1)}$.

(iii) $\underline{z}^r \cdot f_{(a,b,\alpha,\beta)} = f_{(a+r,b+r,\alpha,\beta)}$.
With $c := \sum_v a_v c^v = \sum_v b_v c^v$, $\tilde{c} := c + \sum_v r_v c^v$ we obtain

$$\begin{aligned}
&\underline{Z}^r \cdot F_{(a,b,\alpha,\beta)} - F_{(a+r,b+r,\alpha,\beta)} = \\
&= \underline{Z}^{[\tilde{c},\underline{\eta}(\tilde{c})]} \cdot \left(\underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v) + \sum_v r_v \underline{\eta}(c^v)} - \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v) + \sum_v r_v \underline{\eta}(c^v)}\right) \cdot \underline{t}^{-\underline{\eta}(\tilde{c})} - \\
&\qquad - \underline{Z}^{[c,\underline{\eta}(c)]}\, \underline{Z}^r \cdot \left(\underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v)} - \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v)}\right) \cdot \underline{t}^{-\underline{\eta}(c)} \\
&= \left(\underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v) - \underline{\eta}(c)} - \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v) - \underline{\eta}(c)}\right) \cdot \\
&\qquad \left(\underline{t}^{\underline{\eta}(c) + \sum_v r_v \underline{\eta}(c^v) - \underline{\eta}(\tilde{c})}\, \underline{Z}^{[\tilde{c},\underline{\eta}(\tilde{c})]} - \underline{Z}^{[c,\underline{\eta}(c)]}\, \underline{Z}^r\right).
\end{aligned}$$

Now, the inequalities

$$\sum_v a_v \underline{\eta}(c^v),\ \sum_v b_v \underline{\eta}(c^v) \geq \underline{\eta}(c) \quad\text{and}\quad \underline{\eta}(c) + \sum_v r_v \underline{\eta}(c^v) - \underline{\eta}(\tilde{c}) \geq 0$$

imply that

– the first factor is contained in the ideal defining $0 \in \bar{\mathcal{M}}$, and
– the second factor is an equation of $\bar{X} \subseteq \mathbb{C}^w \times \bar{S}$ (called $F_{(q,p+r,\xi,0)}$ in (5.5.3)).

In particular, we have found a lift for the third relation, too.

The proof of Theorem (5.3.1) is complete.

(5.3.6) **Example:** The singularity Y_6 induced by the hexagon Q_6 equals the cone over the Del Pezzo surface of degree 6 (obtained by blowing up three points of $(\mathbb{P}^2, \mathcal{O}(3))$). As a closed subset of $\mathbb{C}^7$, it is given by the following 9 equations:

$$\begin{aligned}
&f_{(e_1,e_6+e_2,1,0)} = z_1\,t - z_6\,z_2, && f_{(e_2,e_1+e_3,1,0)} = z_2\,t - z_1\,z_3, && f_{(e_3,e_2+e_4,1,0)} = z_3\,t - z_2\,z_4,\\
&f_{(e_4,e_3+e_5,1,0)} = z_4\,t - z_3\,z_5, && f_{(e_5,e_4+e_6,1,0)} = z_5\,t - z_4\,z_6, && f_{(e_6,e_5+e_1,1,0)} = z_6\,t - z_5\,z_1,\\
&f_{(\underline{0},e_1+e_4,2,0)} = t^2 - z_1\,z_4, && f_{(\underline{0},e_2+e_5,2,0)} = t^2 - z_2\,z_5, && f_{(\underline{0},e_3+e_6,2,0)} = t^2 - z_3\,z_6\,.
\end{aligned}$$

Then, the construction described in (5.3.3) yields the liftings

$$\begin{aligned}
F_{(e_1,e_6+e_2,1,0)} &= (Z_1\,t_1 - Z_6\,Z_2) - Z_1(t_1 - t_1) &&= Z_1\,t_1 - Z_6\,Z_2,\\
F_{(e_2,e_1+e_3,1,0)} &= (Z_2\,t_1 - Z_1\,Z_3) - Z_2(t_1^2 - t_1\,t_2)\,t_1^{-1} &&= Z_2\,t_2 - Z_1\,Z_3,\\
F_{(e_3,e_2+e_4,1,0)} &= (Z_3\,t_1 - Z_2\,Z_4) - Z_3(t_1^2 t_2 - t_1\,t_2\,t_3)\,t_1^{-1} t_2^{-1} &&= Z_3\,t_3 - Z_2\,Z_4,\\
F_{(e_4,e_3+e_5,1,0)} &= (Z_4\,t_1 - Z_3\,Z_5) - Z_4(t_1\,t_2\,t_3 - t_2\,t_3\,t_4)\,t_2^{-1} t_3^{-1} &&= Z_4\,t_4 - Z_3\,Z_5,\\
F_{(e_5,e_4+e_6,1,0)} &= (Z_5\,t_1 - Z_4\,Z_6) - Z_5(t_1\,t_6 - t_2\,t_3)\,t_6^{-1} &&= Z_5\,t_5 - Z_4\,Z_6,\\
F_{(e_6,e_5+e_1,1,0)} &= (Z_6\,t_1 - Z_5\,Z_1) - Z_6(t_1 - t_6) &&= Z_6\,t_6 - Z_5\,Z_1,\\
F_{(\underline{0},e_1+e_4,2,0)} &= (t_1^2 - Z_1\,Z_4) - (t_1^2 - t_2\,t_3) = t_2\,t_3 - Z_1\,Z_4 &&= t_5\,t_6 - Z_1\,Z_4,\\
F_{(\underline{0},e_2+e_5,2,0)} &= (t_1^2 - Z_2\,Z_5) - (t_1^2 - t_3\,t_4) &&= t_3\,t_4 - Z_2\,Z_5,\\
F_{(\underline{0},e_3+e_6,2,0)} &= (t_1^2 - Z_3\,Z_6) - (t_1^2 - t_1\,t_2) &&= t_1\,t_2 - Z_3\,Z_6\,.
\end{aligned}$$

Together with the four equations mentioned at the end of (2.2.3), they describe a family contained in $\mathbb{C}^6 \times \mathbb{C}^6 \xrightarrow{\text{pr}_2} \mathbb{C}^6/\mathbb{C} \cdot (1,\ldots,1)$.

5.4 The Kodaira-Spencer map

(5.4.1) To each vertex $a^j \in Q$ (or equally named fundamental generator $a^j := (a^j, 1) \in \sigma$) and each element $R \in M$ we have associated the subset

$$E_j^R = E_{a^j}^R = \{r \in E \mid \langle a^j, r\rangle < \langle a^j, R\rangle\}\,.$$

For the special degree $R^* = [\underline{0}, 1] \in M$ (corresponding to the affine hyperplane containing Q) the associated subsets of $E = \{[c^1, \eta_0(c^1)], \ldots, [c^w, \eta_0(c^w)], [\underline{0}, 1]\}$ equal

$$E_j^{R^*} = E \cap (a^j)^\perp = \{[c^v, \eta_0(c^v)] \mid \langle a^j, -c^v\rangle = \eta_0(c^v)\}\,.$$

In (5.2.6), for each $c \in \mathbb{L}^*$, we have defined the linear form $\underline{\eta}(c) \in V_{\mathbb{Z}}^*$. (Restricted to the cone $C(Q)$, it maps $\underline{t}$ to $\operatorname{Max}\langle Q_{\underline{t}}, -c\rangle = \langle a(c)_{\underline{t}}, -c\rangle$.) This induces the following bilinear map:

$$\begin{array}{rccccc} \Phi: & {}^{V_{\mathbb{Z}}}\!/_{(1,\ldots,1)} & \times & L(E \cap \partial\sigma^\vee) & \longrightarrow & \mathbb{Z} \\ & \underline{t} & , & q & \mapsto & \sum_{v,i} t_i\, q_v\, \eta_i(c^v)\,. \end{array}$$

(Indeed, for $\underline{t} := \underline{1}$ we obtain $\sum_{v,i} q_v\, \eta_i(c^v) = \sum_v q_v\, \eta_0(c^v) = 0$ since $q \in L(E \cap \partial\sigma^\vee)$.) Moreover, if q comes from one of the submodules $L(E_j^{R^*}) \subseteq L(E \cap \partial\sigma^\vee)$, we obtain

$$\begin{aligned} \Phi(\underline{t}, q) &= \sum_v q_v \cdot \operatorname{Max}\langle Q_{\underline{t}}, -c^v\rangle = \sum_v q_v \cdot \langle a^j_{\underline{t}}, -c^v\rangle \\ &= \langle a^j_{\underline{t}}, -\sum_v q_v c^v\rangle = 0\,. \end{aligned}$$

(5.4.2) **Theorem:** *The Kodaira-Spencer map of the family $\bar{X} \times_{\bar{S}} \mathcal{M} \to \bar{\mathcal{M}}$ of section (5.3) equals the map*

$$T_0\bar{\mathcal{M}} = {}^{V_{\mathbb{C}}}\!/_{(1,\ldots,1)} \longrightarrow \left(L_{\mathbb{C}}(E \cap \partial\sigma^\vee) \Big/ \sum_j L_{\mathbb{C}}(E_j^{R^*}) \right)^* = T_Y^1(-R^*)$$

induced by the previous pairing. Moreover, composing this map with the isomorphism $T_Y^1(-R^*) \xrightarrow{\sim} V_{\mathbb{C}}/\underline{1}$ provided in (3.4.3) (see also (5.1.2)) yields the identity.

Proof: Using the same symbol $\mathcal{J}$ for the ideal $\mathcal{J} \subseteq \mathbb{C}[t_1, \ldots, t_N]$ as well as for the intersection $\mathcal{J} \cap \mathbb{C}[t_i - t_j \mid 1 \le i, j \le N]$ (cf. (2.2.4)), our family corresponds to the flat $\mathbb{C}[t_i - t_j]/_{\mathcal{J}}$-module ${}^{\mathbb{C}[\underline{Z}, \underline{t}]}\!/_{(\mathcal{J}, F_\bullet(\underline{Z}, \underline{t}))}$.

Now, we fix a non-trivial tangent vector $\underline{t}^0 \in V_{\mathbb{C}}$. Via $t_i \mapsto t + t_i^0\, \varepsilon$ it induces the infinitesimal family given by the flat $\mathbb{C}[\varepsilon]/_{\varepsilon^2}$-module

$$A_{\underline{t}^0} := {}^{\mathbb{C}[\underline{z}, t, \varepsilon]}\Big/_{(\varepsilon^2, F_\bullet(\underline{z}, t + \underline{t}^0\, \varepsilon))}\,.$$

To obtain the associated $A(Y)$-linear map $I/_{I^2} \to A(Y)$ ($I := (f_\bullet(\underline{z}, t))$ denotes the ideal of Y in $\mathbb{C}^{w+1}$), we have to compute the images of $f_\bullet(\underline{z}, t)$ in $\varepsilon\, A(Y) \subseteq A_{\underline{t}^0}$ and divide them by ε:

Using the notations of (5.3.3), in $A_{\underline{t}^0}$ it holds

$$\begin{aligned} 0 &= F_{(a,b,\alpha,\beta)}(\underline{z}, t + \underline{t}^0\, \varepsilon) \\ &= f_{(a,b,\alpha,\beta)}(\underline{z}, t + t_1^0\, \varepsilon) - \\ &\quad -\underline{z}^{[c, \underline{\eta}(c)]} \cdot \left((t + \underline{t}^0\, \varepsilon)^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v) - \underline{\eta}(c)} - (t + \underline{t}^0\, \varepsilon)^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v) - \underline{\eta}(c)} \right)\,. \end{aligned}$$

The relation $\varepsilon^2 = 0$ yields

$$f_{(a,b,\alpha,\beta)}(\underline{z}, t + t_1^0\varepsilon) = f_{(a,b,\alpha,\beta)}(\underline{z}, t) + \varepsilon \cdot (\alpha\, t^{\alpha-1}\, t_1^0\, \underline{z}^a - \beta\, t^{\beta-1}\, t_1^0\, \underline{z}^b)\,,$$

and similarly we can expand the other terms. Eventually, we obtain

$$\begin{aligned} f_{(a,b,\alpha,\beta)}(\underline{z}, t) &= -\varepsilon\, t_1^0\,(\alpha\, t^{\alpha-1}\,\underline{z}^a - \beta\, t^{\beta-1}\,\underline{z}^b) \,+\, \varepsilon\,\underline{z}^{[c,\underline{\eta}(c)]}\, t^{\alpha+\sum_v a_v \eta_0(c^v)-\eta_0(c)-1}\cdot \\ &\qquad \cdot \left[t_1^0\,(\alpha-\beta) + \textstyle\sum_i t_i^0\,\left(\sum_v (a_v - b_v)\eta_i(c^v)\right)\right] \\ &= \varepsilon \cdot x^{\sum_v a_v [c^v, \eta_0(c^v)] + [0,\alpha-1]} \cdot \left(\sum_i t_i^0 \left(\sum_v (a_v - b_v)\eta_i(c^v)\right)\right)\,. \end{aligned}$$

(In $\varepsilon\, A(Y)$ we were able to replace the variables t and z_v by $x^{[\underline{0},1]}$ and $x^{[c^v,\eta_0(c^v)]}$, respectively.)

On the other hand, we use Theorem (3.2.6): Fixing $R^* \in M$, it is the element of $L_{\mathbb{C}}(E\cap\partial\sigma^\vee)^*$ given by $q \mapsto \sum_{i,v} t_i^0\, q_v\, \eta_i(c^v)$ that corresponds to the infinitesimal deformation of $T^1_Y(-R^*)$ defined by the map

$$\begin{aligned} {}^{I}\big/_{I^2} &\longrightarrow A(Y) \\ t^\alpha\,\underline{z}^a - t^\beta\,\underline{z}^b &\mapsto \left(\sum_{i,v} t_i^0\,(a_v - b_v)\eta_i(c^v)\right)\cdot x^{\sum_v a_v[c^v,\eta_0(c^v)]+[0,\alpha-1]}\,. \end{aligned}$$

Finally, we have to combine Φ with the isomorphism $\left(L(E_0^{R^*})/\sum_j L(E_j^{R^*})\right)^* \xrightarrow{\sim} V(Q)/\underline{1}$ provided in section (3.4). In (3.4.1), we have related relations $q \in L(E_0^{R^*})$ to elements $(v^j) \in \ker\left(\oplus_j \operatorname{span} E_j^{R^*} \to M\right)$ via $q = \sum_j q^j$ $(q^j \in \mathbb{Z}^{E_j})$ and $v^j = -\sum_v q_v^j\, r^v$. Hence, similarly to the calculations preceding the theorem, we obtain

$$\Phi(\underline{t}, q) = \sum_j \Phi(\underline{t}, q^j) = -\sum_j \langle a^j_{\underline{t}}, \sum_v q_v^j\, c^v\rangle = \sum_j \langle (a^j_{\underline{t}}, 0),\, v^j\rangle\,.$$

Using the language of (3.4.4), this means that the Kodaira-Spencer map assigns to $\underline{t} \in V(Q)/\underline{1}$ the elements $b^j = (a^j_{\underline{t}}, 0) \in N$ $(j = 1,\dots,M)$ - just forming the vertices of the Minkowski summand $Q_{\underline{t}}$ in $(R^*)^\perp$. □

(5.4.3) Corollary: *If Y is a three-dimensional, isolated Gorenstein singularity, then the Kodaira-Spencer map $T_0\bar{\mathcal{M}} \to T^1_Y$ is an isomorphism.*
(Of course, this is right in higher dimensions, too. However, in (5.1.3) we have seen that those singularities are rigid anyway.)

Proof: From section (5.1) we know that $T^1_Y(-R^*) = T^1_Y$. □

5.5 The obstruction map

(5.5.1) In this section we build up the obstruction map as it was described in (3.1.4). As before,

$$\mathcal{J} = (g_{\varepsilon,k}(\underline{t} - t_1)\,|\,\varepsilon < Q,\, k \geq 1) = (g_{\underline{d},k}(\underline{t} - t_1)\,|\,\underline{d} \in V^\perp \cap \mathbb{Z}^N,\, k \geq 1) \subseteq \mathbb{C}[t_i - t_j]$$

denotes the homogeneous ideal of the base space $\bar{\mathcal{M}}$. Let

$$\tilde{\mathcal{J}} := (t_i - t_j)_{i,j}\cdot\mathcal{J} + \mathcal{J}_1\cdot\mathbb{C}[t_i - t_j] \subseteq \mathbb{C}[t_i - t_j\,|\,1 \leq i,j \leq N]\,.$$

Then, $W := \mathcal{J}/\tilde{\mathcal{J}}$ is a finite-dimensional, $\mathbb{Z}$-graded vector space ($W = \oplus_{k\geq 2} W_k$, and W_k is generated by the polynomials $g_{\underline{d},k}(\underline{t} - t_1)$). It comes as the kernel in the exact sequence

$$0 \to W \longrightarrow \mathbb{C}[t_i - t_j]\big/\tilde{\mathcal{J}} \longrightarrow \mathbb{C}[t_i - t_j]\big/\mathcal{J} \to 0\,.$$

Identifying t with t_1 and $\underline{z}$ with $\underline{Z}$, the tensor product with $\mathbb{C}[\underline{z}, t]$ (over $\mathbb{C}$) yields the important exact sequence

$$0 \to W \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, t] \longrightarrow \mathbb{C}[\underline{Z}, \underline{t}]\big/\tilde{\mathcal{J}} \cdot \mathbb{C}[\underline{Z}, \underline{t}] \longrightarrow \mathbb{C}[\underline{Z}, \underline{t}]\big/\mathcal{J} \cdot \mathbb{C}[\underline{Z}, \underline{t}] \to 0\,.$$

Now, let s be any relation with coefficients in $\mathbb{C}[\underline{z}, t]$ between the equations $f_{(a,b,\alpha,\beta)}$, i.e.

$$\sum s_{(a,b,\alpha,\beta)}\, f_{(a,b,\alpha,\beta)} = 0 \quad \text{in } \mathbb{C}[\underline{z}, t]\,.$$

By flatness of our family (cf. (5.3.5)), the components of s can be lifted to $\mathbb{C}[\underline{Z}, \underline{t}]$ obtaining an $\tilde{s}$ such that

$$\lambda(s) := \sum \tilde{s}_{(a,b,\alpha,\beta)}\, F_{(a,b,\alpha,\beta)} \mapsto 0 \quad \text{in } \mathbb{C}[\underline{Z}, \underline{t}]\big/\mathcal{J} \cdot \mathbb{C}[\underline{Z}, \underline{t}]\;.$$

In particular, each relation $s \in \mathcal{R}$ induces some element $\lambda(s) \in W \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, t]$, which is well defined after the additional projection to $W \otimes_{\mathbb{C}} A(Y)$. This procedure describes a certain element $\lambda \in T^2_Y \otimes_{\mathbb{C}} W = \mathrm{Hom}(W^*, T^2_Y)$ called the obstruction map.

Theorem: *The obstruction map $\lambda : W^* \to T^2_Y$ is injective.*

Corollary: *If $\dim T^1_Y < \infty$, our family equals the versal deformation of Y. In general, we could say that it is "versal in degree $-R^*$".*

Proof: In (5.4.2) we have proved that the Kodaira-Spencer map is an isomorphism (at least onto the homogeneous piece $T^1_Y(-R^*)$). By Theorem (3.1.4), this fact combined with injectivity of the obstruction map implies versality. □

The remaining part of section (5.5) contains the proof of the previous theorem.

(5.5.2) We have to improve the notations of the sections (5.2) and (5.3). Since $\bar{\mathcal{M}} \subseteq \bar{S} \subseteq \mathbb{C}^N$, we were able to use the toric equations (cf. (2.2.4)) during computations modulo $\mathcal{J}$. In particular, the exponents $\underline{\eta} \in \mathbb{Z}^N$ of $\underline{t}$ needed be known modulo $V^\perp$ only; it was enough to define $\underline{\eta}(c)$ as elements of $V^*_{\mathbb{Z}}$.
However, to compute the obstruction map, we have to deal with the smaller ideal $\tilde{\mathcal{J}} \subseteq \mathcal{J}$. Let us start with refining the definitions of (5.2.6):

(i) For each vertex $a \in Q$ we choose the following paths through the 1-skeleton of Q:

- $\underline{\lambda}(a) :=$ path from $0 \in Q$ to $a \in Q$.
- $\underline{\mu}^v(a) :=$ path from $a \in Q$ to $a(c^v) \in Q$ such that $\mu^v_i(a)\langle d^i, c^v\rangle \leq 0$ for each $i = 1, \ldots, N$.
- $\underline{\lambda}^v(a) := \underline{\lambda}(a) + \underline{\mu}^v(a)$ is then a path from $0 \in Q$ to $a(c^v)$ which depends on a.

(ii) For each $c \in \mathbb{L}^*$ we use the vertex $a(c)$ to define

$$\underline{\eta}^c(c) := \left[-\lambda_1(a(c))\langle d^1, c\rangle, \ldots, -\lambda_N(a(c))\langle d^N, c\rangle\right] \in \mathbb{Z}^N$$

and

$$\underline{\eta}^c(c^v) := \left[-\lambda^v_1(a(c))\langle d^1, c^v\rangle, \ldots, -\lambda^v_N(a(c))\langle d^N, c^v\rangle\right] \in \mathbb{Z}^N\,.$$

(iii) For each $c \in \mathbb{L}^*$ we fix a representation $c = \sum_v p_v^c\, c^v$ ($p_v^c \in \mathbb{N}$) such that $\eta_0(c) = \sum_v p_v^c\, \eta_0(c^v)$. (That means, c is represented only by those generators c^v that define faces of Q containing the face defined by c itself.)

Remark: Let $a \in \mathbb{N}^w$. Denoting $c := \sum_v a_v c^v$ we obtain $\sum_v a_v\, \underline{\eta}^c(c^v) - \underline{\eta}^c(c) \in \mathbb{N}^N$ by arguments as in Lemma (5.2.6).
Moreover, for the special representation $c = \sum_v p_v^c c^v$, the equation $\sum_v p_v^c\, \underline{\eta}^c(c^v) = \underline{\eta}^c(c)$ is true.

Now, we improve the definition of the polynomials $F_\bullet(\underline{Z}, \underline{t})$ given in (5.3.3). Let $a, b \in \mathbb{N}^w, \alpha, \beta \in \mathbb{N}$ such that $([a,\alpha],[b,\beta]) \in m \subseteq \mathbb{N}^{w+1} \times \mathbb{N}^{w+1}$, i.e.

$$c := \sum_v a_v\, c^v = \sum_v b_v\, c^v \quad \text{and} \quad \sum_v a_v\, \eta_0(c^v) + \alpha = \sum_v b_v\, \eta_0(c^v) + \beta\,.$$

Then,

$$F_{(a,b,\alpha,\beta)}(\underline{Z}, \underline{t}) := f_{(a,b,\alpha,\beta)}(\underline{Z}, t_1) - \underline{Z}^{\underline{p}^c} \cdot \left(\underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}^c(c^v) - \underline{\eta}^c(c)} - \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}^c(c^v) - \underline{\eta}^c(c)} \right)\,.$$

(5.5.3) We have to discuss the same three types of relations as we did in (5.3.5). Since there is only one single element $c \in \mathbb{L}$ involved in the relations (i) and (ii), computing modulo $\tilde{\mathcal{J}}$ instead of $\mathcal{J}$ makes no difference in these cases - we always obtain $\lambda(s) = 0$.

Let us regard the relation $s := \left[\underline{z}^r \cdot f_{(a,b,\alpha,\beta)} - f_{(a+r,b+r,\alpha,\beta)} = 0\right]$ $(r \in \mathbb{N}^w)$. We will use the following notations:

- $c := \sum_v a_v\, c^v = \sum_v b_v\, c^v$; $\underline{p} := \underline{p}^c$; $\underline{\eta} := \underline{\eta}^c$;
- $\tilde{c} := \sum_v (a_v + r_v)\, c^v = \sum_v (b_v + r_v)\, c^v = \sum_v (p_v + r_v)\, c^v$; $\underline{q} := \underline{p}^{\tilde{c}}$; $\underline{\tilde{\eta}} := \underline{\eta}^{\tilde{c}}$;
- $\xi := \sum_i \left((\sum_v (p_v + r_v) \tilde{\eta}_i(c^v)) - \tilde{\eta}_i(\tilde{c}) \right) = \sum_v (p_v + r_v) \eta_0(c^v) - \eta_0(\tilde{c})\,.$

Using the same lifting of s to $\tilde{s}$ as in (5.3.5) yields

$$\begin{aligned}
\lambda(s) &= \underline{Z}^r \cdot F_{(a,b,\alpha,\beta)} - F_{(a+r,b+r,\alpha,\beta)} - \\
&\qquad - \left(\underline{t}^{\alpha e_1 + \sum_v a_v \underline{\eta}(c^v) - \underline{\eta}(c)} - \underline{t}^{\beta e_1 + \sum_v b_v \underline{\eta}(c^v) - \underline{\eta}(c)} \right) \cdot F_{(q, p+r, \xi, 0)} \\
&= -\underline{Z}^{p+r} \cdot \left(\underline{t}^{\alpha e_1 + \sum_v (a_v - p_v) \underline{\eta}(c^v)} - \underline{t}^{\beta e_1 + \sum_v (b_v - p_v) \underline{\eta}(c^v)} \right) + \\
&\quad + \underline{Z}^q \cdot \left(\underline{t}^{\alpha e_1 + \sum_v (a_v + r_v - q_v) \underline{\tilde{\eta}}(c^v)} - \underline{t}^{\beta e_1 + \sum_v (b_v + r_v - q_v) \underline{\tilde{\eta}}(c^v)} \right) - \\
&\quad - \left(\underline{t}^{\alpha e_1 + \sum_v (a_v - p_v) \underline{\eta}(c^v)} - \underline{t}^{\beta e_1 + \sum_v (b_v - p_v) \underline{\eta}(c^v)} \right) \cdot \left(\underline{Z}^q\, \underline{t}^{\sum_v (p_v + r_v - q_v) \underline{\tilde{\eta}}(c^v)} - \underline{Z}^{p+r} \right) \\
&= \underline{Z}^q \cdot \left(\underline{t}^{\alpha e_1 + \sum_v (a_v + r_v - q_v) \underline{\tilde{\eta}}(c^v)} - \underline{t}^{\alpha e_1 + \sum_v (p_v + r_v - q_v) \underline{\tilde{\eta}}(c^v) + \sum_v (a_v - p_v) \underline{\eta}(c^v)} \right) - \\
&\quad - \underline{Z}^q \cdot \left(\underline{t}^{\beta e_1 + \sum_v (b_v + r_v - q_v) \underline{\tilde{\eta}}(c^v)} - \underline{t}^{\beta e_1 + \sum_v (p_v + r_v - q_v) \underline{\tilde{\eta}}(c^v) + \sum_v (b_v - p_v) \underline{\eta}(c^v)} \right)\,.
\end{aligned}$$

As in (5.3.5)(iii), we can see that $\lambda(s)$ vanishes modulo $\mathcal{J}$ (or even in $A(\bar{S})$) - just identify $\underline{\eta}$ and $\underline{\tilde{\eta}}$.

(5.5.4) In (5.5.1) we already mentioned the isomorphism

$$W \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, t] \xrightarrow{\sim} \mathcal{J} \cdot \mathbb{C}[\underline{Z}, \underline{t}] \Big/ \tilde{\mathcal{J}} \cdot \mathbb{C}[\underline{Z}, \underline{t}]$$

obtained by identifying t with t_1 and $\underline{z}$ with $\underline{Z}$. Now, with $\lambda(s)$, we have obtained an element of the right hand side, which has to be interpreted as an element of $W \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, t]$.

Lemma: *Let $A, B \in \mathbb{N}^N$ such that $\underline{d} := A - B \in V^\perp$ (i.e. $\underline{t}^A - \underline{t}^B \in \mathcal{J} \cdot \mathbb{C}[\underline{Z}, \underline{t}]$). Then, via the previously mentioned isomorphism, $\underline{t}^A - \underline{t}^B$ corresponds to the element*

$$\sum_{k \geq 1} c_k \cdot g_{\underline{d},k}(\underline{t} - t_1) \cdot t^{k_0 - k} \in W \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, t]$$

($k_0 := \sum_i A_i$; c_k are the constants occurred in (2.3.4)). In particular, the coefficients from W_k vanish for $k > k_0$.

Proof: First, we remark that it is allowed to assume that $A = \underline{d}^+$, $B = \underline{d}^-$, i.e. $\underline{t}^A - \underline{t}^B = p_{\underline{d}}(\underline{t})$ (cf. (2.3.2)). (Otherwise we could write this binomial as

$$\underline{t}^A - \underline{t}^B = \underline{t}^C \cdot \left(\underline{t}^{\underline{d}^+} - \underline{t}^{\underline{d}^-} \right) \quad (C \in \mathbb{N}^N),$$

and since

$$\underline{t}^C = (t_1 + [\underline{t} - t_1])^C \equiv t_1^{\sum_i C_i} \pmod{(t_i - t_j)},$$

we would obtain

$$\underline{t}^A - \underline{t}^B \equiv t_1^{\sum_i C_i} \cdot \left(\underline{t}^{\underline{d}^+} - \underline{t}^{\underline{d}^-} \right) \pmod{\tilde{\mathcal{J}}}.)$$

In (2.3.4) we have seen that

$$p_{\underline{d}}(\underline{t}) = \sum_{k=1}^{k_0} t_1^{k_0 - k} \cdot \left(\sum_{v=1}^{k-1} q_{v,k}(\underline{t} - t_1) \cdot g_{\underline{d},v}(\underline{t} - t_1) + c_k \cdot g_{\underline{d},k}(\underline{t} - t_1) \right)$$

(with $k_0 := \sum_i d_i^+$). Since $q_{v,k}(\underline{t} - t_1) \in (t_i - t_j) \cdot \mathbb{C}[t_i - t_j]$, this implies

$$p_{\underline{d}}(\underline{t}) \equiv \sum_{k=1}^{k_0} t_1^{k_0 - k} \cdot c_k \cdot g_{\underline{d},k}(\underline{t} - t_1) \pmod{\tilde{\mathcal{J}}}.$$

On the other hand, for $k > k_0$, Lemma (2.3.3) tells us that $g_{\underline{d},k}(\underline{t} - t_1)$ is a $\mathbb{C}[t_i - t_j]$-linear combination of the elements $g_{\underline{d},1}(\underline{t} - t_1), \dots, g_{\underline{d},k_0}(\underline{t} - t_1)$. Then, the degree k part of the corresponding equation shows $g_{\underline{d},k}(\underline{t} - t_1) \in \tilde{\mathcal{J}}$. □

Corollary: *Transferred to $W \otimes_{\mathbb{C}} \mathbb{C}[\underline{z}, t]$, the element $\lambda(s)$ equals*

$$\sum_{k \geq 1} c_k \cdot g_{\underline{d},k}(\underline{t} - t_1) \cdot \underline{z}^q \cdot t^{k_0 - k} \quad \textit{with} \quad \begin{array}{rcl} \underline{d} & := & \sum_v (a_v - b_v) \cdot \left(\underline{\tilde{\eta}}(c^v) - \underline{\eta}(c^v) \right), \\ k_0 & := & \alpha + \sum_v (a_v + r_v)\, \eta_0(c^v) - \eta_0(\tilde{c}). \end{array}$$

The coefficients vanish for $k > k_0$.

Proof: We apply the previous lemma to both the a- and the b-summand of the $\lambda(s)$-formula of (5.5.3). For the first one we obtain

$$\begin{array}{rcl} \underline{d}^{(a)} & = & [\alpha e_1 + \sum_v (a_v + r_v - q_v)\, \underline{\tilde{\eta}}(c^v)] - \\ & & \quad - [\alpha e_1 + \sum_v (p_v + r_v - q_v)\, \underline{\tilde{\eta}}(c^v) + \sum_v (a_v - p_v)\, \underline{\eta}(c^v)] \\ & = & \sum_v (a_v - p_v) \cdot \left(\underline{\tilde{\eta}}(c^v) - \underline{\eta}(c^v) \right) \qquad \text{and} \\ k_0 & = & \sum_i \left(\alpha e_1 + \sum_v (a_v + r_v - q_v)\, \underline{\tilde{\eta}}(c^v) \right)_i \\ & = & \alpha + \sum_v (a_v + r_v - q_v)\, \eta_0(c^v) \;=\; \alpha + \sum_v (a_v + r_v)\, \eta_0(c^v) - \eta_0(\tilde{c}). \end{array}$$

k_0 has the same value for both the a- and b-summand, and

$$\begin{array}{rcl}\underline{d}=\underline{d}^{(a)}-\underline{d}^{(b)} & = & \sum_v (a_v-p_v)\cdot\big(\underline{\tilde{\eta}}(c^v)-\underline{\eta}(c^v)\big)-\sum_v (b_v-p_v)\cdot\big(\underline{\tilde{\eta}}(c^v)-\underline{\eta}(c^v)\big)\\ & = & \sum_v (a_v-b_v)\cdot\big(\underline{\tilde{\eta}}(c^v)-\underline{\eta}(c^v)\big)\,.\end{array}$$

□

(5.5.5) Now, we try to approach the obstruction map λ from the opposite direction. Using the description of T^2_Y given in (3.2.4) we construct an element of $T^2_Y\otimes_{\mathbb{C}} W$, that afterwards will turn out to equal λ.

For $\rho\in\mathbb{Z}^N$ induced from some path along the edges of Q, we will denote

$$\underline{d}(\rho,c):=[\langle\rho_1\,d^1,\,c\rangle,\dots,\langle\rho_N\,d^N,\,c\rangle]\in\mathbb{Z}^N$$

the vector showing the behavior of $c\in\mathbb{L}^*$ passing each particular edge. If, moreover, ρ comes from a closed path, then $\underline{d}(\rho,c)$ will be contained in $V^\perp$.
On the other hand, for each $k\geq 1$, we can use the $\underline{d}$'s from $V^\perp$ to get elements $g_{\underline{d},k}(\underline{t}-t_1)\in W_k$ generating this vector space. Composing both procedures we obtain, for each closed path $\rho\in\mathbb{Z}^N$, a map

$$\begin{array}{rcccl} g^{(k)}(\rho,\bullet): & \mathbb{A}^* & \longrightarrow & V^\perp & \longrightarrow W_k\\ & c & & \mapsto & g_{\underline{d}(\rho,c),k}(\underline{t}-t_1)\,.\end{array}$$

Remark:

(1) Taking the sum over all 2-faces we get a surjective map

$$\sum_{\varepsilon<Q} g^{(k)}(\underline{\varepsilon},\bullet):\oplus_{\varepsilon<Q}\mathbb{A}^*\otimes_{\mathbb{R}}\mathbb{C}\longrightarrow\!\!\!\!\!\rightarrow W_k\,.$$

(2) Let $c\in\mathbb{L}^*$ (having integer coordinates is very important here). If $\rho^1,\rho^2\in\mathbb{Z}^N$ are two paths each connecting vertices $a,b\in Q$ such that

- $|\langle a,c\rangle-\langle b,c\rangle|\leq k-1$ and
- c is monotone along both paths (i.e. $\langle\rho^1_i\,d^i,c\rangle;\ \langle\rho^2_i\,d^i,c\rangle\geq 0$ for $i=1,\dots,N$),

then $\rho^1-\rho^2\in\mathbb{Z}^N$ will be a closed path yielding $g^{(k)}(\rho^1-\rho^2,\,c)=0$ in W_k.

Proof: The reason for (1) is the fact that the elements $\underline{d}(\varepsilon,c)$ ($\varepsilon<Q$ 2-face; $c\in\mathbb{L}^*$) generate $V^\perp$ as a vector space.
For the proof of (2), we consider $\underline{d}:=\underline{d}(\rho^1-\rho^2,\,c)$. Since $d_i=\langle\rho^1_i\,d^i,\,c\rangle-\langle\rho^2_i\,d^i,\,c\rangle$ is the difference of two non-negative integers, we obtain $d_i^+\leq\langle\rho^1_i\,d^i,\,c\rangle$. Hence,

$$\sum_i d_i^+\leq\sum_i\langle\rho^1_i\,d^i,\,c\rangle=\langle b,c\rangle-\langle a,c\rangle\leq k-1\,,$$

and as in (5.5.4) we obtain $g_{\underline{d},k}(\underline{t}-t_1)\in\tilde{\mathcal{J}}$ by Lemma (2.3.3). □

Considering the sets E_j for the degrees $R:=k\,R^*,\ k\geq 2$, we obtain

$$E_j^{kR^*}=\{[c^v,\eta_0(c^v)]\,|\,\langle a^j,c^v\rangle+\eta_0(c^v)\leq k-1\}\cup\{R^*\}\subseteq\sigma^\vee\cap M\,.$$

Then, we can define the following linear maps :

$$\begin{array}{rrcl}\psi_j^{(k)}: & L(E_j^{kR^*}) & \longrightarrow & W_k\\ & q & \mapsto & \sum_v q_v\cdot g^{(k)}\big(\underline{\lambda}(a^j)+\underline{\mu}^v(a^j)-\underline{\lambda}(a(c^v)),\,c^v\big)\,.\end{array}$$

(The q-coordinate corresponding to $R^* \in E_j^{kR^*}$ is not used in the definition of $\psi_j^{(k)}$.)

Lemma: *Let $\langle a^j, a^l\rangle < Q$ be an edge of the polyhedron Q. Then, on $L(E_j^{kR^*} \cap E_l^{kR^*}) = L(E_j^{kR^*}) \cap L(E_l^{kR^*})$, the maps $\psi_j^{(k)}$ and $\psi_l^{(k)}$ coincide.*
In particular (cf. Theorem (3.2.4)), the $\psi_j^{(k)}$'s induce a linear map $\psi^{(k)} : T_Y^2(-kR^*)^* \to W_k$.

Proof: Let $q \in L(E_j^{kR^*} \cap E_l^{kR^*})$. Moreover, we denote by $\rho^{jl} \in \mathbb{Z}^N$ the path consisting of the single edge running from a^j to a^l. Then,

$$\begin{aligned}\psi_j^{(k)}(q) - \psi_l^{(k)}(q) &= \textstyle\sum_v q_v \cdot g^{(k)}\left(\underline{\lambda}(a^j) + \underline{\mu}^v(a^j) - \underline{\lambda}(a^l) + \underline{\mu}^v(a^l),\, c^v\right)\\ &= g^{(k)}\left(\underline{\lambda}(a^j) - \underline{\lambda}(a^l) + \rho^{jl},\, \textstyle\sum_v q_v\, c^v\right) + \\ &\qquad + \textstyle\sum_v q_v \cdot g^{(k)}\left(\underline{\mu}^v(a^j) - \underline{\mu}^v(a^l) - \rho^{jl},\, c^v\right),\end{aligned}$$

and both summands vanish for several reasons. The first one is killed simply by the equality $\sum_v q_v\, c^v = 0$. For the second one we can use (2) of the previous remark: If $q_v \neq 0$, then the assumption about q implies the inequalities

$$0 \leq \langle a^j, c^v\rangle - \langle a(c^v), c^v\rangle\,;\; \langle a^l, c^v\rangle - \langle a(c^v), c^v\rangle \leq k-1\,.$$

Hence, assuming w.l.o.g. $\langle a^j, c^v\rangle \geq \langle a^l, c^v\rangle$, we can take $\rho^1 := -\underline{\mu}^v(a^l) - \rho^{jl}$ and $\rho^2 := -\underline{\mu}^v(a^j)$ to see that $g^{(k)}\left(\underline{\mu}^v(a^j) - \underline{\mu}^v(a^l) - \rho^{jl},\, c^v\right) = 0$. □

(5.5.6) Proposition: $\sum_{k\geq 1} c_k\, \psi^{(k)}$ *equals λ^*, the adjoint of the obstruction map.*

Proof: Using Theorem (3.2.7), we can find an element of $\mathrm{Hom}({}^{\mathcal{R}}\!/_{\mathcal{R}_0}, W_k \otimes A(Y))$ representing $\psi^{(k)} \in T_Y^2 \otimes W_k$ - it sends relations of type (i) (cf. (5.3.5)) to 0 and deals with relations of type (ii) and (iii) in the following way:

$$[\underline{z}^r\, t^\gamma \cdot f_{(a,b,\alpha,\beta)} - f_{(a+r,b+r,\alpha+\gamma,\beta+\gamma)} = 0] \mapsto \psi_j^{(k)}(a-b)\cdot x^{\sum_v (a_v+r_v)[c^v,\eta_0(c^v)]+(\alpha+\gamma-k)R^*}\,,$$

if

$$\langle (Q,1),\, \sum_v (a_v + r_v)\,[c^v, \eta_0(c^v)] + (\alpha+\gamma-k)R^*\rangle \geq 0\,,$$

and j is such that

$$\langle (a^j,1),\, \sum_v a_v\,[c^v,\eta_0(c^v)] + (\alpha-k)R^*\rangle < 0\,;$$

otherwise the relation is sent onto 0 (in particular, if there is not any j meeting the desired condition).

On Q, the linear forms $c := \sum_v a_v\, c^v$ and $\tilde{c} = \sum_v (a_v + r_v)c^v$ admit their minimal values at the vertices $a(c)$ and $a(\tilde{c})$, respectively. Hence, we can transform the previous formula into

$$\begin{aligned}&[\underline{z}^r\, t^\gamma \cdot f_{(a,b,\alpha,\beta)} - f_{(a+r,b+r,\alpha+\gamma,\beta+\gamma)} = 0] \mapsto \psi_{a(c)}^{(k)}(a-b)\cdot x^{\sum_v (a_v+r_v)[c^v,\eta_0(c^v)]+(\alpha+\gamma-k)R^*}\\ &\text{if}\quad \textstyle\sum_v (a_v+r_v)\eta_0(c^v) - \eta_0(\tilde{c}) + (\alpha+\gamma-k) = \\ &\qquad\quad = \langle (a(\tilde{c}),1),\, \textstyle\sum_v (a_v+r_v)\,[c^v,\eta_0(c^v)] + (\alpha+\gamma-k)R^*\rangle \geq 0\,,\\ &\qquad \textstyle\sum_v a_v\,\eta_0(c^v) - \eta_0(c) + (\alpha-k) = \\ &\qquad\quad = \langle (a(c),1),\, \textstyle\sum_v a_v\,[c^v,\eta_0(c^v)] + (\alpha-k)R^*\rangle < 0\end{aligned}$$

(and mapping onto 0 otherwise).

Adding the coboundary $h \in \operatorname{Hom}(\mathbb{C}[\underline{z},t]^m,\, W_k \otimes A(Y))$

$$h_{(a,\alpha),(b,\beta)} := \begin{cases} \psi^{(k)}_{a(c)}(a-b)\cdot x^{\sum_v a_v[c^v,\eta_0(c^v)]+(\alpha-k)R^*} & \text{for } \sum_v a_v\,\eta_0(c^v)-\eta_0(c)+\alpha\ge k\,,\\ 0 & \text{otherwise}\end{cases}$$

does not change the class in $T^2_Y(-kR^*)\otimes W_k$ (still representing $\psi^{(k)}$), but improves the representative from $\operatorname{Hom}(^{\mathcal{R}}\!/_{\mathcal{R}_0},\, W_k\otimes A(Y))$. It still maps type-(i)-relations to 0, and moreover

$$\begin{aligned}&[\underline{z}^r\,t^\gamma\cdot f_{(a,b,\alpha,\beta)} - f_{(a+r,b+r,\alpha+\gamma,\beta+\gamma)} = 0]\mapsto\\ &\mapsto \begin{cases}\left(\psi^{(k)}_{a(c)}(a-b)-\psi^{(k)}_{a(\tilde c)}(a-b)\right)\cdot x^{\sum_v (a_v+r_v)[c^v,\eta_0(c^v)]+(\alpha+\gamma-k)R^*} & \text{for } k_0+\gamma\ge k\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

(with $k_0=\alpha+\sum_v(a_v+r_v)\,\eta_0(c^v)-\eta_0(\tilde c)$).

By definition of $\psi^{(k)}_j$ and $g^{(k)}$ we obtain

$$\begin{aligned}&\psi^{(k)}_{a(c)}(a-b)-\psi^{(k)}_{a(\tilde c)}(a-b) =\\ &\quad= \textstyle\sum_v (a_v-b_v)\cdot g^{(k)}\left(\underline{\lambda}(a(c))+\underline{\mu}^v(a(c))-\underline{\lambda}(a(\tilde c))-\underline{\mu}^v(a(\tilde c)),\; c^v\right)\\ &\quad= \textstyle\sum_v (a_v-b_v)\cdot g^{(k)}\left(\underline{\lambda}^v(a(c))-\underline{\lambda}^v(a(\tilde c)),\; c^v\right)\\ &\quad= g_{\underline{d},\,k}(\underline{t}-t_1)\quad \text{with }\ \underline{d}\ =\ \textstyle\sum_v (a_v-b_v)\cdot \underline{d}\left(\underline{\lambda}^v(a(c))-\underline{\lambda}^v(a(\tilde c)),\, c^v\right)\\ &\qquad\qquad\qquad\qquad\qquad\ \ =\ \textstyle\sum_v (a_v-b_v)\cdot\left(\underline{\tilde\eta}(c^v)-\underline{\eta}(c^v)\right)\,,\end{aligned}$$

and this completes our proof. Indeed,

- for relations of type (ii) (i.e. $r=0;\ \gamma=1$) we know $c=\tilde c$, hence, those relations map onto 0;
- for relations of type (iii) (i.e. $\gamma=0$) we can compare the previous formula with the result obtained in Corollary (5.5.4): The coefficients coincide, and the monomial $\underline{z}^q\,t^{k_0-k}\in\mathbb{C}[\underline{z},t]$ maps onto $x^{\sum_v (a_v+r_v)[c^v,\eta_0(c^v)]+(\alpha+\gamma-k)R^*}\in A(Y)$.

□

(5.5.7) It remains to show that the summands $\psi^{(k)}$ of λ^* are indeed surjective maps from $T^2_Y(-kR^*)^*$ to W_k. We will do so by composing them with auxiliary surjective maps $p^k:\oplus_{\varepsilon<Q}\boldsymbol{A}^*\otimes_{\mathbb{R}}\mathbb{C}\longrightarrow\!\!\!\!\!\rightarrow T^2_Y(-kR^*)^*$ yielding $\psi^{(k)}\circ p^k=\sum_{\varepsilon<Q} g^{(k)}(\underline{\varepsilon},\bullet)$. Then, the result follows from the first part of Remark (5.5.5).

The degree kR^* ($k\ge 2$) is always different from 1 on the fundamental generators of σ. In particular, it meets the assumption made in Corollary (3.4.6) yielding an injection $\Psi: T^2_Y(-kR^*)\hookrightarrow\oplus_{\varepsilon<Q}N_{\mathbb{C}}$. Moreover, since $R^*=[\underline{0},1]\in E^{kR^*}_j$, the image of Ψ is killed by R^*, and we can use

$$\Psi^*:\oplus_{\varepsilon<Q}M_{\mathbb{C}}\longrightarrow\!\!\!\!\!\rightarrow\oplus_{\varepsilon<Q}M_{\mathbb{C}}/_{\mathbb{C}\,\cdot\,R^*}=\oplus_{\varepsilon<Q}\boldsymbol{A}^*_{\mathbb{C}}\longrightarrow\!\!\!\!\!\rightarrow T^2_Y(-kR^*)^*$$

for the auxiliary map p^k. Taking a closer look at that construction, we can give an explicit description of p^k; eventually we will be able to compute $\psi^{(k)}\circ p^k$:

Let us fix some 2-face $\varepsilon<Q$. Assume that $d^1,\dots,d^m$ are its counterclockwise oriented edges, i.e. the sign vector $\underline{\varepsilon}$ looks like $\varepsilon_i=1$ for $i=1,\dots,m$ and $\varepsilon_j=0$ otherwise. Moreover, we denote the vertices of $\varepsilon<Q$ by $a^1,\dots,a^m$ such that d^i runs from a^i to a^{i+1} ($m+1:=1$).

Starting with a $[c,\eta_0]\in M$ (and, as just mentioned, only the $c\in\mathbb{L}^*$ is essential) we have to proceed as follows:

(i) For $i = 1, \ldots, m$ we represent $[c, \eta_0]$ as a linear combination of elements of $E_i^{kR^*} \cap E_{i+1}^{kR^*}$. (This corresponds to the lifting from $\mathrm{span}(E^R)_\bullet$ to $(\mathbb{Z}^{E^R})_\bullet$.)

$$[c, \eta_0] = \sum_v q_{iv}\, [c^v, \eta_0(c^v)] + q_i\, [\underline{0}, 1]\,,$$

and $q_{iv} \neq 0$ implies $[c^v, \eta_0(c^v)] \in E_i^{kR*} \cap E_{i+1}^{kR^*}$, i.e.

$$\langle a^i, c^v \rangle + \eta_0(c^v) \leq k-1\,; \quad \langle a^{i+1}, c^v \rangle + \eta_0(c^v) \leq k-1\,.$$

(ii) We map the result to $\oplus_{i=1}^m \mathbb{Z}^{E_i^{kR^*}}$ by taking successive differences (corresponding to the application of the differential in the complex $(\mathbb{C}^{E^R})_\bullet$). The result is automatically contained in $\ker\big(\oplus_i L(E_i^{kR^*}) \to L(E)\big)$, and its i-th summand is the linear relation

$$\sum_v (q_{i,v} - q_{i-1,v}) \cdot [c^v, \eta_0(c^v)] + (q_i - q_{i-1}) \cdot [\underline{0}, 1] = 0\,.$$

(iii) Finally, we apply $\psi^{(k)}$ to obtain

$$\begin{aligned}
\psi^{(k)}(p^k(c)) &= \textstyle\sum_{i=1}^m \sum_v (q_{i,v} - q_{i-1,v}) \cdot g^{(k)}\left(\underline{\lambda}(a^i) - \underline{\lambda}(a(c^v)) + \underline{\mu}^v(a^i),\ c^v\right)\\
&= \textstyle\sum_{i,v} g^{(k)}\left(\underline{\lambda}(a^i) - \underline{\lambda}(a(c^v)) + \underline{\mu}^v(a^i),\ q_{i,v}\, c^v\right) -\\
&\qquad - \textstyle\sum_{i,v} g^{(k)}\left(\underline{\lambda}(a^{i+1}) - \underline{\lambda}(a(c^v)) + \underline{\mu}^v(a^{i+1}),\ q_{i,v}\, c^v\right)\\
&= \textstyle\sum_{i,v} g^{(k)}\left(\underline{\lambda}(a^i) - \underline{\lambda}(a^{i+1}) + \underline{\mu}^v(a^i) - \underline{\mu}^v(a^{i+1}),\ q_{i,v}\, c^v\right)\ .
\end{aligned}$$

Similar to the proof of Lemma (5.5.5) we introduce the path ρ^i consisting of the single edge d^i only. Then, if $q_{iv} \neq 0$ and w.l.o.g. $\langle a^i, c^v \rangle \geq \langle a^{i+1}, c^v \rangle$, the pair of paths $\underline{\mu}^v(a^i)$ and $\underline{\mu}^v(a^{i+1}) + \rho^i$ meets the assumption of Remark (5.5.5)(2) (cf. (i)). Hence, we can proceed as follows:

$$\begin{aligned}
\psi^{(k)}(p^k(c)) &= \textstyle\sum_{i,v} g^{(k)}\left(\underline{\lambda}(a^i) - \underline{\lambda}(a^{i+1}) + \rho^i,\ q_{iv}\, c^v\right) +\\
&\qquad + \textstyle\sum_{i,v} g^{(k)}\left(\underline{\mu}^v(a^i) - \underline{\mu}^v(a^{i+1}) - \rho^i,\ q_{iv}\, c^v\right)\\
&= \textstyle\sum_{i=1}^m g^{(k)}\left(\underline{\lambda}(a^i) - \underline{\lambda}(a^{i+1}) + \rho^i,\ \sum_v q_{iv}\, c^v\right)\\
&= \textstyle\sum_{i=1}^m g^{(k)}\left(\underline{\lambda}(a^i) - \underline{\lambda}(a^{i+1}) + \rho^i,\ c\right)\\
&= g^{(k)}\left(\textstyle\sum_{i=1}^m \rho^i,\ c\right)\\
&= g^{(k)}(\underline{\varepsilon},\ c)\,.
\end{aligned}$$

Hence, Theorem (5.5.1) is proven.

5.6 The components of the reduced versal family

(5.6.1) The components of the reduced base space $\bar{\mathcal{M}}_{red}$ correspond to maximal decompositions of Q into a Minkowski sum $Q = R_0 + \ldots + R_n$ with lattice polytopes $R_k \subseteq \boldsymbol{A}$ as summands. Intersections of components are obtained by the finest Minkowski decompositions of Q, that are coarser than all the involved maximal ones.

Theorem: *Fix such a Minkowski decomposition. Then, the corresponding component (or intersection of components) $\bar{\mathcal{M}}_0$ is isomorphic to $\mathbb{C}^{n+1}/\mathbb{C} \cdot (1, \ldots, 1)$, and the restriction $X_0 \to \bar{\mathcal{M}}_0$ of the versal family can be described as follows:*

(i) *Define the cone*

$$\tilde\sigma := Cone\left(\bigcup_{k=0}^{n}(R_k \times \{e^k\})\right) \subseteq \mathbf{A}\times I\!R^{n+1}\,,$$

it contains $\sigma = Cone(Q\times\{1\}) \subseteq N_{I\!R} = \mathbf{A}\times I\!R$ *via the diagonal embedding* $\mathbf{A}\times I\!R \hookrightarrow \mathbf{A}\times I\!R^{n+1}$ $((a,1)\mapsto(a;1,\dots,1))$. *The inclusion* $\sigma\subseteq\tilde\sigma$ *induces a closed embedding of the affine toric varieties defined by these cones - this gives* $Y\hookrightarrow X_0$.

(ii) *The projection* $\mathbf{A}\times I\!R^{n+1} \twoheadrightarrow I\!R^{n+1}$ *provides* $n+1$ *regular functions on* X_0, *i.e. we obtain a map* $X_0\to \mathbb{C}^{n+1}$. *Composing this morphism with* $\ell: \mathbb{C}^{n+1} \twoheadrightarrow \mathbb{C}^{n+1}/\mathbb{C}\cdot(1,\dots,1)$ *yields the flat map* $X_0\to\bar{\mathcal{M}}_0$.

We will use (5.6.2) and (5.6.3) to prove the theorem.

(5.6.2) We already know (cf. (2.2.5)) that both the space

$$\bar{\mathcal{M}}_0 = \mathbb{C}^{n+1}/\mathbb{C}\cdot(1,\dots,1) \subseteq \mathbb{C}^N/\mathbb{C}\cdot(1,\dots,1)$$

and its pullback $\mathcal{M}_0\subseteq\mathbb{C}^N$ are given by the equations $t_i - t_j = 0$ (if d^i, d^j belong to some common summand R_k of Q).

There is a chain of inclusions $\mathcal{M}_0\subseteq\mathcal{M}\subseteq\bar S\subseteq\mathbb{C}^N$, and the map $\mathcal{M}_0\hookrightarrow\bar S$ factorizes through an embedding $\mathcal{M}_0\hookrightarrow S$. It is given by the surjection of $\mathbb{C}$-algebras

$$\begin{array}{ccc}\mathbb{C}[C(Q)^\vee\cap V^*_{\mathbb{Z}}] & \twoheadrightarrow & \mathbb{C}[T_0,\dots,T_n]\\ t_i & \mapsto & T_k \text{ with } d^i\in R_k\end{array}$$

coming from the linear map

$$\begin{array}{ccccc} I\!R^{n+1} & \hookrightarrow & V & \hookrightarrow & I\!R^N\,.\\ e^k & \mapsto & \sum_{d^i\in R_k} e^i\,. & & \end{array}$$

The matrix of $I\!R^{n+1}\hookrightarrow I\!R^N$ equals the incidence matrix between edges and Minkowski summands of Q, and the space $\mathcal{M}_0 = \mathbb{C}^{n+1}$ corresponds to the cone $I\!R^{n+1}_{\ge 0} = I\!R^{n+1}\cap C(Q)$.

(5.6.3) The family $X_0\to\bar{\mathcal{M}}_0$ arises from $\bar X\times_{\bar S}\mathcal{M}\to\bar{\mathcal{M}}$ via base change $\bar{\mathcal{M}}_0\hookrightarrow\bar{\mathcal{M}}$. We obtain

$$\begin{array}{rclclcl} X_0 & = & (\bar X\times_{\bar S}\mathcal{M})\times_{\bar{\mathcal{M}}}\bar{\mathcal{M}}_0 & = & \bar X\times_{\bar S}(\mathcal{M}\times_{\bar{\mathcal{M}}}\bar{\mathcal{M}}_0) & = & \bar X\times_{\bar S}\mathcal{M}_0\\ & = & \bar X\times_{\bar S}(S\times_S\mathcal{M}_0) & = & (\bar X\times_{\bar S} S)\times_S\mathcal{M}_0 & = & X\times_S\mathcal{M}_0\,.\end{array}$$

Hence, with $\tilde\sigma$ as defined is the theorem, it remains to show that

$$\begin{array}{ccc}\mathbb{C}[\tilde C(Q)^\vee\cap(I\!L^*\times V^*_{\mathbb{Z}})] & \stackrel{\psi}{\longrightarrow} & \mathbb{C}[\tilde\sigma^\vee\cap(I\!L^*\times\mathbb{Z}^{n+1})]\\ \cup\!| & & \cup\!|\\ \mathbb{C}[C(Q)^\vee\cap V^*_{\mathbb{Z}}] & \twoheadrightarrow & \mathbb{C}[I\!N^{n+1}] = \mathbb{C}[T_0,\dots,T_n]\end{array}$$

is a tensor product diagram:

(i) $\tilde\sigma$ is the preimage of $I\!R^{n+1}_{\ge0}\subseteq C(Q)$ via the projection $\tilde C(Q)\twoheadrightarrow C(Q)$. In particular, $\tilde\sigma\subseteq\tilde C(Q)$ causes a surjective map $\psi_{I\!R}:\tilde C(Q)^\vee\twoheadrightarrow\tilde\sigma^\vee$.

(ii) To show surjectivity at the level of lattices (i.e. surjectivity of ψ) we start with an element $[c,\underline{\eta}] \in \tilde{C}(Q)^\vee$ and suppose its image $\psi_{I\!R}([c,\underline{\eta}])$ to be contained in $\tilde{\sigma}^\vee \cap (I\!L^* \times Z\!\!Z^{n+1})$.
In particular, $c \in I\!L^*$, and we obtain $[c,\underline{\eta}(c)] \in \tilde{C}(Q)^\vee \cap (I\!L^* \times V^*_{Z\!\!Z})$ implying that

$$[0,\underline{\eta}-\underline{\eta}(c)] = [c,\underline{\eta}] - [c,\underline{\eta}(c)] \in [0, C(Q)^\vee] \subseteq \tilde{C}(Q)^\vee$$

maps to an element of $I\!N^{n+1} \subseteq \tilde{\sigma}^\vee \cap (I\!L^* \times Z\!\!Z^{n+1})$.
On the other hand, surjectivity of $C(Q)^\vee \cap V^*_{Z\!\!Z} \longrightarrow\!\!\!\!\to I\!N^{n+1}$ causes that this element can be reached by some $[0,\underline{\mu}] \in [0,\, C(Q)^\vee \cap V^*_{Z\!\!Z}]$, too.
Hence, $[c,\underline{\eta}(c)+\underline{\mu}] \in \tilde{C}(Q)^\vee \cap (I\!L^* \times V^*_{Z\!\!Z})$ is a lattice-preimage of $\psi_{I\!R}([c,\underline{\eta}])$.

(iii) The same methods applies for showing that $\ker\psi$ is generated by the same elements as $\ker(\mathbb{C}[C(Q)^\vee \cap V^*_{Z\!\!Z}] \to \mathbb{C}[I\!N^{n+1}])$:
If $[c^1,\underline{\eta}^1]$, $[c^2,\underline{\eta}^2] \in \tilde{C}(Q)^\vee \cap (I\!L^* \times V^*_{Z\!\!Z})$ have the same image in $\mathbb{C}[\tilde{\sigma}^\vee \cap (I\!L^* \times Z\!\!Z^{n+1})]$ (i.e. $x^{[c^1,\underline{\eta}^1]} - x^{[c^2,\underline{\eta}^2]} \in \ker\psi$), then $c^1 = c^2$, and the elements

$$\underline{\mu}^1 := \underline{\eta}^1 - \underline{\eta}(c^1),\; \underline{\mu}^2 := \underline{\eta}^2 - \underline{\eta}(c^2) \in C(Q)^\vee \cap V^*_{Z\!\!Z}$$

have the same image in $\mathbb{C}[I\!N^{n+1}]$.
In particular, $x^{\underline{\mu}^1} - x^{\underline{\mu}^2} \in \ker(\mathbb{C}[C(Q)^\vee \cap V^*_{Z\!\!Z}] \to \mathbb{C}[I\!N^{n+1}])$, and

$$x^{[c^1,\underline{\eta}^1]} - x^{[c^2,\underline{\eta}^2]} = x^{[c^1,\underline{\eta}(c^1)]} \cdot \left(x^{\underline{\mu}^1} - x^{\underline{\mu}^2}\right)\,.$$

(5.6.4) **Example:** At the end of (2.2.5) we presented two decompositions of Q_6 into a Minkowski sum of lattice summands. Let us describe now the restrictions of the versal family to the associated components of $\bar{\mathcal{M}}$:

(i) Putting the two triangles R_0, R_1 into two parallel planes contained in $I\!R^3$ yields an octahedron as the convex hull of the whole configuration. Then, $\tilde{\sigma}$ is the (4-dimensional) cone over this octahedron

$$\tilde{\sigma} = \langle (0,0;1,0),\, (1,0;1,0),\, (1,1;1,0),\, (0,0;0,1),\, (0,1;0,1),\, (1,1;0,1)\rangle\,.$$

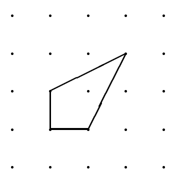
(ii) Looking at the second decomposition, we have to put three line segments R_0, R_1, R_2 on three parallel 2-planes in general position inside the affine space $I\!R^4$. Taking the convex hull of this configuration yields a 4-dimensional polytope that is dual to (triangle)×(triangle).
Again, $\tilde{\sigma}$ is the (5-dimensional) cone over this polytope

$$\tilde{\sigma} = \langle (0,0;1,0,0),\, (1,0;1,0,0),\, (0,0;0,1,0),\, (0,1;0,1,0),\, (0,0;0,0,1),\, (1,1;0,0,1)\rangle\,.$$

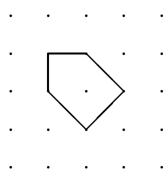
The total spaces over the components arise as the toric varieties defined by $\tilde{\sigma}$. In our example, they equal the cones over $I\!P^1 \times I\!P^1 \times I\!P^1$ and $I\!P^2 \times I\!P^2$, respectively.

5.7 Examples

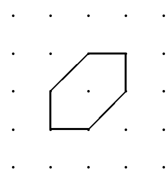
(5.7.1) Three examples of toric Gorenstein singularities arise as cones over the Del Pezzo surfaces obtained by blowing up $(I\!P^2, \mathcal{O}(3))$ in one, two, or three points, respectively. They correspond to the following polygons:



Polygon Q_4



Polygon Q_5



Polygon Q_6

Let us discuss these three examples:

(iv) The edges equal

$$d^1 = (1,0),\ d^2 = (1,2),\ d^3 = (-2,-1),\ d^4 = (0,-1)\,,$$

and they imply the following equations of the versal base space as closed subscheme of $\mathbb{C}^4/_{\mathbb{C}\cdot(1,1,1,1)}$:

$$t_1 + t_2 = 2t_3, \quad t_3 + t_4 = 2t_2, \quad t_1^2 + t_2^2 = 2t_3^2, \quad t_3^2 + t_4^2 = 2t_2^2\,.$$

Using the two linear equations, only two coordinates $t := t_1$, $\varepsilon := t_1 - t_3$ survive. (We get the t_i's back by $t_1 = t$, $t_2 = t - 2\varepsilon$, $t_3 = t - \varepsilon$, $t_4 = t - 3\varepsilon$.) Then, the two quadratic equations transform into $2\varepsilon^2 = 0$, i.e. the versal base space is a fat point.
On the other hand, Q_4 does not allow any splitting into a Minkowski sum involving lattice summands only. This reflects the triviality of the underlying reduced space. (Cf. (5.7.2).)

(v) The polygon Q_5 allows the decomposition into the sum of a triangle and a line segment. In particular, the reduced base space of the versal deformation of Y_5 has to be a line.
We compute the true base space: $d^1 = (1,1)$, $d^2 = (-1,1)$, $d^3 = (-1,0)$, $d^4 = (0,-1)$, $d^5 = (1,-1)$ yield the equations

$$t_1 - t_3 = t_2 - t_5 = t_4 - t_1 \quad \text{and} \quad t_1^2 - t_3^2 = t_2^2 - t_5^2 = t_4^2 - t_1^2\,.$$

With $t := t_1$, $s_1 := t_1 - t_3$, $s_2 := t_1 - t_2$ and $t_1 = t$, $t_2 = t - s_2$, $t_3 = t - s_1$, $t_4 = t + s_1$, $t_5 = t - s_1 - s_2$, they turn into

$$s_1^2 = 2s_1s_2 = 0\,.$$

(vi) This example has been spread in the paper.

(5.7.2) We will use the polygon $Q_4 := \operatorname{Conv}\{(0,0), (1,0), (2,2), (0,1)\}$ of (5.7.1)(iv) for a more detailed demonstration how the theory works. In particular, we will describe the versal family of Y_4 over $\operatorname{Spec} \mathbb{C}[\varepsilon]/_{\varepsilon^2}$:

(1) The (t,ε)-coordinates of V correspond to the linear map

$$\begin{pmatrix} 1 & 0 \\ 1 & -2 \\ 1 & -1 \\ 1 & -3 \end{pmatrix} : \mathbb{R}^2 \xrightarrow{\sim} V \hookrightarrow \mathbb{R}^4\,.$$

We obtain

$$\begin{aligned} C(Q_4) &= \{(a,b) \in \mathbb{R}^2 \mid a \ge 0,\ a - 2b \ge 0,\ a - b \ge 0,\ a - 3b \ge 0\} \\ &= \{(a,b) \in \mathbb{R}^2 \mid a \ge 0,\ a - 3b \ge 0\} \\ &= \langle [1,0],\, [1,-3] \rangle^\vee = \langle (0,-1),\, (3,1) \rangle \subseteq \mathbb{R}^2\,, \end{aligned}$$

and the map $\mathbb{N}^4 \to C(Q_4)^\vee \cap V^*_{\mathbb{Z}}$ sends e_1, e_2, e_3, e_4 to $[1,0]$, $[1,-2]$, $[1,-1]$, $[1,-3]$, respectively. In particular, this map is surjective, i.e. $S_4 = \bar{S}_4$ and $X_4 = \bar{X}_4$.

(2) To compute the tautological cone $\tilde{C}(Q_4)$, we need the Minkowski summands associated to the two fundamental generators of $C(Q_4)$:

$$(Q_4)_{(0,-1)} = \text{Conv}\{(0,0),\,(2,4),\,(0,3)\},\; (Q_4)_{(3,1)} = \text{Conv}\{(0,0),\,(3,0),\,(4,2)\}\,.$$

Hence,

$$\tilde{C}(Q_4) = \langle(0,0;0,-1);\,(2,4;0,-1);\,(0,3;0,-1);\,(0,0;3,1);\,(3,0;3,1);\,(4,2;3,1)\rangle\,.$$

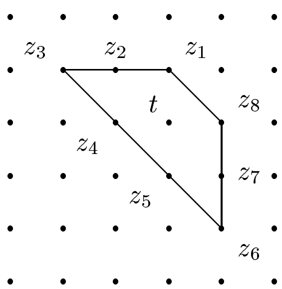
(3) Now, we have all information to obtain the versal family of $Y_4 = \text{Spec}\,\mathbb{C}[\text{Cone}(Q_4)^\vee \cap \mathbb{Z}^3]$:

- Restrict the family $\text{Spec}\,\mathbb{C}[\tilde{C}(Q_4)^\vee \cap \mathbb{Z}^4] \to \text{Spec}\,\mathbb{C}[C(Q_4)^\vee \cap \mathbb{Z}^2] \subseteq \mathbb{C}^4$ to the subspace $\mathbb{C}^2 \simeq V_{\mathbb{C}} \subseteq \mathbb{C}^4$, i.e. use the (t,ε)-coordinates instead of (t_1,t_2,t_3,t_4).
- Compose the result with the projection $\mathbb{C}^2 \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C}^1$ $((t,\varepsilon)\mapsto\varepsilon)$. That means, we do no longer regard t as a coordinate of the base space.
- Finally, we restrict our family to the closed subscheme defined by the equation $\varepsilon^2 = 0$.

(4) To obtain equations, we could either take a closer look to the family constructed so far, or we can proceed more directly as described in (5.2.6) and (5.3.3):

- Computing the minimal generator set of the semigroup $\text{Cone}\,(Q_4)^\vee \cap \mathbb{Z}^3$, we get the elements $[c^v;\eta_0(c^v)]$:

$$\begin{array}{lll} [c^1;\eta_0^1] = [0,1;0], & [c^2;\eta_0^2] = [-1,1;1], & [c^3;\eta_0^3] = [-2,1;2], \\ [c^4;\eta_0^4] = [-1,0;2], & [c^5;\eta_0^5] = [0,-1;2], & [c^6;\eta_0^6] = [1,-2;2], \\ [c^7;\eta_0^7] = [1,-1;1], & [c^8;\eta_0^8] = [1,0;0]. & \end{array}$$



Polygon $Q_4^\vee$

Together with $[0,0;1]$, they induce coordinates $z_1,\ldots,z_8,t$ on Y_4, i.e. we have obtained an embedding $Y_4 \hookrightarrow \mathbb{C}^9$.
(The sum of the three components of the vectors are always 1. In the figure we have drawn the first two coordinates.)

- $Y_4 \subseteq \mathbb{C}^9$ is defined by the following 20 equations:

$$\begin{array}{llll} t^2 - z_4z_8, & t^2 - z_1z_5, & t^2 - z_2z_7, & z_1t - z_2z_8, \\ z_2t - z_3z_8, & z_2t - z_1z_4, & z_3t - z_2z_4, & z_4t - z_3z_7, \\ z_4t - z_2z_5, & z_5t - z_4z_7, & z_5t - z_2z_6, & z_6t - z_5z_7, \\ z_7t - z_5z_8, & z_7t - z_1z_6, & z_8t - z_1z_7, & z_1z_3 - z_2^2, \\ z_3z_5 - z_4^2, & z_4z_6 - z_5^2, & z_6z_8 - z_7^2, & z_3z_6 - z_4z_5. \end{array}$$

- Choosing paths from $(0,0) \in Q_4$ to the other vertices, we obtain the list

$$\underline{\eta}^1 = [0,0,0,0],\; \underline{\eta}^2 = [1,0,0,0],\; \underline{\eta}^3 = [2,0,0,0],$$

$$\underline{\eta}^4 = [1,1,0,0] = [0,0,2,0],\; \underline{\eta}^5 = [0,2,0,0] = [0,0,1,1],$$

$$\underline{\eta}^6 = [0,0,0,2],\; \underline{\eta}^7 = [0,0,0,1],\; \underline{\eta}^8 = [0,0,0,0].$$

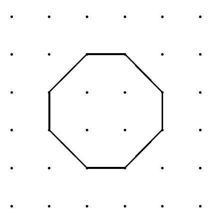
– Now, we can lift our 20 equations to the ring $\mathbb{C}[Z_1, \ldots, Z_8, t_1, \ldots, t_4]$:

$$\begin{array}{llll} t_1t_2 - Z_4Z_8, & t_2^2 - Z_1Z_5, & t_1t_4 - Z_2Z_7, & Z_1t_1 - Z_2Z_8, \\ Z_2t_1 - Z_3Z_8, & Z_2t_2 - Z_1Z_4, & Z_3t_2 - Z_2Z_4, & Z_4t_3 - Z_3Z_7, \\ Z_4t_2 - Z_2Z_5, & Z_5t_3 - Z_4Z_7, & Z_5t_2 - Z_2Z_6, & Z_6t_3 - Z_5Z_7, \\ Z_7t_3 - Z_5Z_8, & Z_7t_4 - Z_1Z_6, & Z_8t_4 - Z_1Z_7, & Z_1Z_3 - Z_2^2, \\ Z_3Z_5 - Z_4^2, & Z_4Z_6 - Z_5^2, & Z_6Z_8 - Z_7^2, & Z_3Z_6 - Z_4Z_5. \end{array}$$

– Finally, we restrict the family to the versal base space by switching to the (t, ε)-coordinates and obeying the equation $\varepsilon^2 = 0$. Moreover, t is no longer a coordinate of the base space:

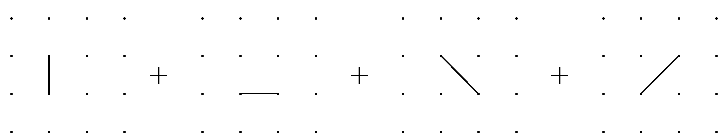
$$\begin{array}{lll} t(t-2\varepsilon) - z_4z_8, & t(t-4\varepsilon) - z_1z_5, & t(t-3\varepsilon) - z_2z_7, \\ z_1t - z_2z_8, & z_2t - z_3z_8, & z_2(t-2\varepsilon) - z_1z_4, \\ z_3(t-2\varepsilon) - z_2z_4, & z_4(t-\varepsilon) - z_3z_7, & z_4(t-2\varepsilon) - z_2z_5, \\ z_5(t-\varepsilon) - z_4z_7, & z_5(t-2\varepsilon) - z_2z_6, & z_6(t-\varepsilon) - z_5z_7, \\ z_7(t-\varepsilon) - z_5z_8, & z_7(t-3\varepsilon) - z_1z_6, & z_8(t-3\varepsilon) - z_1z_7, \\ z_1z_3 - z_2^2, & z_3z_5 - z_4^2, & z_4z_6 - z_5^2, \\ z_6z_8 - z_7^2, & z_3z_6 - z_4z_5. & \end{array}$$

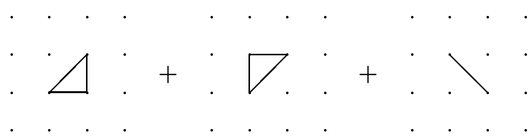
(5.7.3) At last we want to present an example involving more than only quadratic equations for the versal base space. Let Q_8 be the "regular" lattice 8-gon, it is contained in two strips of lattice thickness 3.

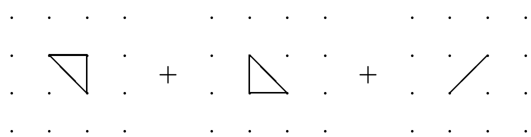


Polygon Q_8

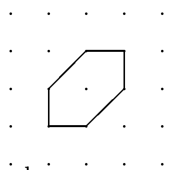
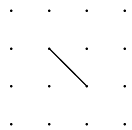
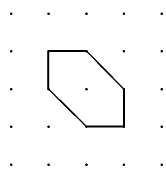
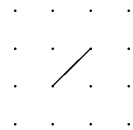
Q_8 admits three maximal Minkowski decompositions into a sum of lattice summands:

(i) Q_8 =


(ii) Q_8 =


(iii) Q_8 =


The decompositions (i), (ii) and (i), (iii) are refinements of the coarser decompositions

$Q_8 =$  $+$  and $Q_8 =$  $+$  ,

respectively.

These facts translate directly into the geometry of the reduced base space of the versal deformation of Q_8:

- It is embedded into some affine space $\mathbb{C}^5$ and equals the union of a 3-plane with two 2-planes (through $0 \in \mathbb{C}^5$).
- The two 2-planes each have a common line with the 3-dimensional component. However, they intersect each other in $0 \in \mathbb{C}^5$ only.

On the other hand, we can write down the equations of the true versal base space (as a closed subscheme of $\mathbb{C}^8/_{\mathbb{C}\cdot(1,\dots,1)}$):

$$t_1^k + t_2^k + t_8^k = t_4^k + t_5^k + t_6^k, \quad t_2^k + t_3^k + t_4^k = t_6^k + t_7^k + t_8^k \quad (k = 1, 2, 3)\,.$$

Chapter 6

Deformations of negative degree

6.1 Negative degrees

(6.1.1) The construction of chapter 5 works in a more general case: Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone providing an affine toric variety that is smooth in codimension two. Then, for a fixed $R \in \sigma^\vee \cap M$, we will obtain the "versal deformation of degree $-R$", i.e. that part of the entire versal deformation which remains after setting zero all T^1_Y-variables of degree different from $-R$. Those degrees comming from $-(\sigma^\vee \cap M)$ we will call (strictly) *negative degrees.*
In case that T^1_Y is concentrated in one single negative degree $-R$ (as is for three-dimensional, isolated toric Gorenstein singularities with $R = R^*$), the versal deformation of degree $-R$ is just versal.

(6.1.2) To obtain the whole versal deformation of Y, two further steps would be necessary:

(1) Non-negative degrees have to be considered, too. As already discussed in (3.5.2) and (3.5.3), the negative degrees are special in some sense - for instance, the cone σ can be reconstructed from the cross cut $Q := \sigma \cap [R = 1]$.

(2) If there are two different $R, S \in M$ (or even $R, S \in \sigma^\vee \cap M$), then it is not clear how to put the homogeneous pieces together getting a family containing both.
(It works in many examples, and a possible idea might roughly speaking be the following: After proceeding in degree $-R$ analogously to the Gorenstein case, lift $S \in \sigma^\vee \cap M$ to an element of $\tilde{C}(Q)^\vee \cap (\mathbb{L}^* \times V^*_{\mathbb{Z}})$ and try to deform the singularity given by the cone $\tilde{C}(Q)$ in that lifted degree. However, one of the problems arising that way is: We cannot stay inside the class of singularities which are smooth in codimension two.)

(6.1.3) The generalization from the Gorenstein case is rather straight. Nevertheless, we have preferred to present that special case first: Some of the notations have become simpler, making clearer the idea of the construction.

To avoid boring repetitions in the upcoming sections, we will proceed as follows: We will just mention how notions or statements have to be changed to fit into the general case. Whenever a calculation or a proof remain essentially the same, we will omit it (or just emphasize the adjustments to be made). We will even use the same numbering as in the previous chapter.

6.2 The tautological cone

(6.2.1) Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone such that the two-dimensional faces $\langle a^j, a^k \rangle < \sigma$ are smooth

(i.e. $\{a^j, a^k\}$ could be extended to a $\mathbb{Z}$-basis of N). Let $R \in \sigma^\vee \cap M$ be a primitive element which will be fixed through the whole chapter. (According to Remark (3.4.3), non-primitive R's would yield $T^1_Y(-R) = 0$ anyway.)
The affine hyperplane $\mathbf{A} := [\langle \bullet, R\rangle = 1] \subseteq N_{\mathbb{R}}$ containing the lattice points $\mathbb{L} := \mathbf{A} \cap N$ has (lattice-) distance one from the origin. We define $Q := \sigma \cap [R = 1]$ - it is a polyhedron in $\mathbf{A}$ with $Q^\infty := \sigma \cap R^\perp$ as its cone of unbounded directions. We fix one of the $\mathbb{L}$-vertices of Q to be the origin $0 \in \mathbf{A}$ - if there are not any vertices contained in the lattice at all, $T^1_Y(-R)$ will vanish again.

According to the general description of $T^1_Y(-R)$ given in Theorem (3.4.5), we have to replace the vector space $V(Q)$ (which for general polyhedra Q was defined in (3.4.2)) by its subspace

$$V'(Q) := \{(t_1, \dots, t_N) \mid \begin{array}{l} \bullet\ \sum_i t_i\, \varepsilon_i\, d^i = 0 \ \text{ for every compact 2-face } \varepsilon < Q \\ \bullet\ t_i = t_j \text{ if } d^i, d^j \text{ are compact edges containing a common non-} \\ \qquad \text{lattice vertex of } Q\,\}\,. \end{array}$$

Of course, we also will use $C'(Q) := V'(Q) \cap \mathbb{R}^N_{\geq 0}$ instead of $C(Q) \subseteq V(Q)$. For $\underline{t} \in C'(Q)$, the corresponding Minkowski summand $Q_{\underline{t}}$ is defined as

$$Q_{\underline{t}} = \operatorname{conv}\{\bar{a}_{\underline{t}} \mid \bar{a} \in Q \text{ is a vertex}\} \;+\; Q^\infty\,.$$

Nevertheless, just to simplify notations, we are going to use the usual symbols $V(Q)$ and $C(Q)$ for the new objects $V'(Q)$ and $C'(Q)$, respectively.

(6.2.2) Since Q does not need to be compact (if $R \in \partial\sigma^\vee$), we have to take the closure of $\mathbb{R}_{\geq 0} \cdot Q \subseteq N_{\mathbb{R}}$ to get the cone σ back. Similarly, we define

$$\tilde{C}(Q) := \overline{\{(a, \underline{t}) \mid \underline{t} \in C(Q);\ a \in Q_{\underline{t}}\}} \;=\; \{(a, \underline{t}) \mid \underline{t} \in C(Q);\ a \in Q_{\underline{t}}\} \;+\; (Q^\infty, 0)\,.$$

It is generated by the usual pairs $(\bar{a}^j_{\underline{t}^l}, \underline{t}^l)$ (cf. (5.2.2)) plus the additional elements $(a^k, 0)$ with $a^k \in Q^\infty = \sigma \cap R^\perp$.

(6.2.3) The diagrams of cones (given in (5.2.2)) and the corresponding affine toric varieties (cf. (5.2.3)) remain unchanged. However, the latter one might fail to be a fiber product diagram, as it will be explained in (6.2.8).

(6.2.4) To describe the dual cone $\sigma^\vee$ we will still use the vertices $a(c)$ and scalars $\eta_0(c)$ as we did in (5.2.4). However,

- just to emphasize that in the non-Gorenstein case they might be different from the corresponding fundamental generators $a(c) \in \sigma$, we prefer to call those vertices $\bar{a}(c) \in Q$ from now on (i.e. $\bar{a}(c) = a(c)/\langle a(c), R\rangle$), and
- they are only defined for $c \in (Q^\infty)^\vee \subseteq \mathbf{A}^*$, and - even for $c \in \mathbb{L}^* \cap (Q^\infty)^\vee$ - it will be no longer true that the $\eta_0(c)$'s have to be integers; we just know $\eta_0(c) \in \mathbb{R}_{\geq 0}$.

Denote by $\eta_0^*(c)$ the smallest integer greater or equal than $\eta_0(c)$. Then,

$$\sigma^\vee = \{[c, \eta_0(c)] \mid c \in (Q^\infty)^\vee\} \;+\; \mathbb{R}_{\geq 0} \cdot [\underline{0}, 1]$$

and

$$\sigma^\vee \cap M = \{[c, \eta_0^*(c)] \mid c \in \mathbb{L}^* \cap (Q^\infty)^\vee\} \;+\; \mathbb{N} \cdot [\underline{0}, 1]\,.$$

($[\underline{0}, 1]$ equals the element $R \in M$ fixed in the beginning.) In particular, analogously to (5.2.4), we can choose the generating set $E \subseteq \sigma^\vee \cap M$ as some

$$E = \{[\underline{0}, 1], [c^1, \eta_0^*(c^1)], \dots, [c^w, \eta_0^*(c^w)]\}.$$

(It just might happen that $[\underline{0}, 1]$ splits inside $\sigma^\vee \cap M$. Then, the generating set E will not be a minimal one.)

(6.2.5) According to the changes of $V = V(Q)$ in (6.2.1), we have to adapt the descriptions of $V^\perp$ (yielding the equations of $\bar{S} \subseteq \mathbb{C}^N$) and V^*:

$$\begin{aligned} V^\perp = \operatorname{span} \{ & \bullet [\langle \varepsilon_1 d^1, c\rangle, \dots, \langle \varepsilon_N d^N, c\rangle] \mid \varepsilon < Q \text{ is a compact 2-face, } c \in \mathbb{A}^*; \\ & \bullet [0, \dots, 1_i, \dots, -1_j, \dots, 0] \text{ if } d^i, d^j \text{ contain a common non-lattice vertex of } Q \}. \end{aligned}$$

To deal with the dual space V^*, the following point of view might be useful: In the previous chapter we have described its elements by using the surjection $\mathbb{R}^N \twoheadrightarrow V^*$. In particular, an element $\underline{\eta} \in V^*$ was given by coordinates η_i corresponding to the edges d^i of Q.
Now, caused by the additional equations $t_i - t_j$, we may collect the edges into several "connected components": Two edges are said to be directly connected, if they contain a common non-lattice vertex of Q. For each component built that way, not the single coordinates but just their sum is important to know.

(6.2.6) For $c \in (Q^\infty)^\vee \subseteq \mathbb{A}^*$ we take the usual definition of $\underline{\eta}(c) \in C(Q)^\vee \subseteq V^*$ via paths $\underline{\lambda}^c \in \mathbb{Z}^N$. Except for statements concerning the lattice (in the sense of Remark (5.2.6)), Lemma (5.2.6) remains true in the general case.
On the other hand, if $c \in \mathbb{L}^* \cap (Q^\infty)^\vee$, it is the equation $\sum_i \eta_i(c) = \eta_0(c)$ showing that we cannot expect $\underline{\eta}(c)$ belonging to $V^*_{\mathbb{Z}}$.

Definition:

(1) Let $\bar{a} \in Q$ be a vertex *not* contained in the lattice $\mathbb{L}$. Then, we denote by $e[\bar{a}] \in V^*_{\mathbb{Z}}$ the element represented by $[0, \dots, 1_i, \dots, 0] \in \mathbb{Z}^N$ for some compact edge d^i containing $\bar{a}$. (In the language of (6.2.5), $e[\bar{a}]$ yields 1 at the "connected component" containing $\bar{a}$, and 0 at the remaining components.)

(2) $\underline{\eta}^*(c) := \underline{\eta}(c) + (\eta_0^*(c) - \eta_0(c)) \cdot e[\bar{a}(c)] \in V^*$. (If $\bar{a}(c) \in \mathbb{L}$, then $\eta_0^*(c) - \eta_0(c) = 0$ implies that we do not need $e[\bar{a}(c)]$ in that case.)

Lemma: *Let $c \in \mathbb{L}^* \cap (Q^\infty)^\vee$. Then,*

(i) $\underline{\eta}^*(c) \in V^*_{\mathbb{Z}}$. *This element equals* $\underline{\eta}(c)$ *if and only if* $\langle \bar{a}(c), c\rangle \in \mathbb{Z}$ *(in particular, if* $\bar{a}(c) \in \mathbb{L}$*).*

(ii) $\underline{\eta}^*(c) \geq 0$ *in the sense that* $\underline{\eta}^*(c) \in im[\mathbb{N}^N \to V^*_{\mathbb{Z}}] \subseteq C(Q)^\vee \cap V^*_{\mathbb{Z}}$.

(iii) For $c^v \in \mathbb{L}^* \cap (Q^\infty)^\vee$ *and* $g_v \in \mathbb{N}$ *we have* $\sum_v g_v \, \underline{\eta}^*(c^v) \geq \underline{\eta}^*(\sum_v g_v \, c^v)$.

(iv) $\sum_{i=1}^N \eta_i^*(c) = \eta_0^*(c)$.

Proof: The last part is a direct consequence of both the corresponding statement for $\underline{\eta}(c)$ and the definition of $\underline{\eta}^*(c)$.
Now, for $c \in \mathbb{L}^* \cap (Q^\infty)^\vee$, we will show that $\underline{\eta}^*(c) \in V^*$ can be represented by a vector of $\mathbb{R}^N$ having just non-negative integers as coordinates. (However, according to (6.2.5), we may consider elements $\underline{\eta} \in \mathbb{R}^N$ in the factor $\mathbb{R}^N / \operatorname{span}\{t_i - t_j \mid d^i, d^j \text{ connected}\}$. In particular, we can speak about coordinates of $\underline{\eta}$ assigned to certain "components" of the edge set, e.g. assigned to certain

non-lattice vertices of Q.) This will imply (i) and (ii) of the lemma.

Proceeding as in the proof of Lemma (5.2.6), we choose some path along the edges of Q passing $\bar{a}^0 = 0, \ldots, \bar{a}^p = \bar{a}(c)$ and decreasing the value of c at each step. It provides some vector $\underline{\lambda}^c \in \mathbb{Z}^N$ yielding $\underline{\eta}(c)$ via $\eta_i(c) := -\lambda_i^c \langle d^i, c\rangle$.
Denote by $\bar{a}^{j_0}, \ldots, \bar{a}^{j_q}$ ($\{j_0, \ldots, j_q\} \subseteq \{0, \ldots, p\}$) the $\mathbb{L}$-vertices on the path. Then, for $v = 1, \ldots, q$, the edges between $\bar{a}^{j_{v-1}}$ and $\bar{a}^{j_v}$ (say $d^{i_1}, \ldots, d^{i_k}$) belong to the same "component". In particular, not the single $\eta^*_{i_1}(c), \ldots, \eta^*_{i_k}(c)$ but their sums have to be considered:

$$\sum_{\mu=1}^{k} \eta^*_{i_\mu}(c) = \sum_{\mu=1}^{k} \eta_{i_\mu}(c) = \langle -\sum_{\mu=1}^{k} \lambda^c_{i_\mu} d^{i_\mu}, c\rangle = \langle \bar{a}^{j_{v-1}} - \bar{a}^{j_v}, c\rangle \in \mathbb{N}\,.$$

If $\bar{a}(c)$ belongs to the lattice $\mathbb{L}$, then we are done. Otherwise, there might be at most one non-integer coordinate (assigned to $\bar{a}(c) \notin \mathbb{L}$) in $\underline{\eta}^*(c)$. However, this cannot be the case, since the sum taken over all coordinates of $\underline{\eta}^*(c)$ yields the integer $\eta^*_0(c)$.

To prove (iii) we start as in (5.2.6). Then, inside the vector space $\mathbb{R}^N/\mathrm{span}\{t_i - t_j \mid d^i, d^j \text{ connected}\}$, we obtain the (componentwise) inequality

$$\sum_v g_v\, \underline{\eta}^*(c^v) \;\geq\; \sum_v g_v\, \underline{\eta}(c^v) \;\geq\; \underline{\eta}\,(\sum_v g_v\, c^v)\,.$$

On the other hand, $\underline{\eta}\,(\sum_v g_v\, c^v)$ and $\underline{\eta}^*\,(\sum_v g_v\, c^v)$ might differ in at most one coordinate (assigned to $\bar{a}(\sum_v g_v\, c^v)$). If so, then by definition of $\underline{\eta}^*$ the latter one equals the smallest integer not smaller than the first one. Hence, we are done, since the left hand side of our inequality involves integers only. □

(6.2.7) Proposition:

(1) $\tilde{C}(Q)^\vee = \big\{ [c, \underline{\eta}] \in (Q^\infty)^\vee \times V^* \subseteq \mathbb{A}^* \times V^* \,\big|\, \underline{\eta} - \underline{\eta}(c) \in C(Q)^\vee \big\}$

(2) In particular, $[c, \underline{\eta}(c)] \in \tilde{C}(Q)^\vee$*; it is the only preimage of* $[c, \eta_0(c)] \in \sigma^\vee$ *via the surjection* $i^\vee : \tilde{C}(Q)^\vee \longrightarrow \sigma^\vee$.
Moreover, for $c \in \mathbb{L}^* \cap (Q^\infty)^\vee$*, it holds* $[c, \underline{\eta}^*(c)] \in \tilde{C}(Q)^\vee \cap \tilde{M}$*. These elements are liftings of* $[c, \eta^*_0(c)] \in \sigma^\vee \cap M$ *- but, in general, they are not the only ones (even to* $\tilde{C}(Q)^\vee \cap \tilde{M}$*).*

(3) $[c^1, \underline{\eta}^*(c^1)], \ldots, [c^w, \underline{\eta}^*(c^w)]$ *and* $C(Q)^\vee \cap V^*_{\mathbb{Z}}$ *(embedded as* $[0, C(Q)^\vee]$*) generate the semigroup* $\Gamma := \{[c, \underline{\eta}] \in (\mathbb{L}^* \cap (Q^\infty)^\vee) \times V^*_{\mathbb{Z}} \mid \underline{\eta} - \underline{\eta}^*(c) \in C(Q)^\vee\} \subseteq \tilde{C}(Q)^\vee \cap \tilde{M}$.
($\tilde{C}(Q)^\vee \cap \tilde{M}$ *is the saturation of that subsemigroup.)*

Proof: We just say some words to the third part. First, the condition "$\underline{\eta} - \underline{\eta}^*(c) \in C(Q)^\vee$" indeed describes a semigroup; this is a consequence of (iii) of Lemma (6.2.6).
On the other hand, let $[c, \underline{\eta}^*(c)]$ be given. Using some representation $[c, \eta^*_0(c)] = \sum_{v=1}^{w} p_v\, [c^v, \eta^*_0(c^v)]$ ($p_v \in \mathbb{N}$), we obtain by the same lemma

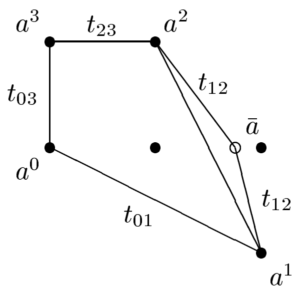
$$\sum_v p_v\, \underline{\eta}^*(c^v) - \underline{\eta}^*(c) = \sum_v p_v\, \underline{\eta}^*(c^v) - \underline{\eta}^*(\sum_v p_v\, c^v) \in C(Q)^\vee \quad \text{(or even} \geq 0).$$

Since, at the same time, the sum taken over all coordinates of that difference vanishes, the whole difference has to be zero. □

Example: We present an example showing that there might be liftings of $[c, \eta_0^*(c)]$ different from $[c, \underline{\eta}^*(c)]$ (i.e. Γ is a proper subsemigroup of $\tilde{C}(Q)^\vee \cap \tilde{M}$). Let $\sigma \subseteq I\!R^3$ be generated from

$$a^0 := (0,0;1)\,,\; a^1 := (2,-1;1)\,,\; a^2 := (1,1;1)\,,\; a^3 := (0,1;1)\,,\quad \text{and } a := (7,0;4)\,.$$

We consider the degree $R = [0,0;1]$. Then, Q is a pentagon with four vertices $a^0, \ldots, a^3 \in I\!L$ and the non-lattice vertex $\bar{a} = (7/4, 0)$ (if $a^0 \in Q$ coincides with the origin).



The equations for $V(Q)$ are

$$t_{01} \cdot (2,-1) + t_{12} \cdot [(-1/4, 1) + (-3/4, 1)] + t_{23} \cdot (-1,0) + t_{03} \cdot (-1,0) = (0,0)\,,$$

i.e. $t_{23} = 2\,t_{01} - t_{12}$ and $t_{03} = 2\,t_{12} - t_{01}$. Hence, $V(Q) = I\!R^2$ (with coordinates t_{01} and t_{12}), and the cone $C(Q) \subseteq V(Q)$ is given by the inequalities $t_{12} \le 2\,t_{01} \le 4\,t_{12}$.

Now, let $c := [-2,-1]$. The vertex with minimal c-value is $\bar{a}(c) = \bar{a} = (7/4, 0)$, hence $\eta_0(c) = 7/2$ and $\eta_0^*(c) = 4$. Choosing the path from a^0 to $\bar{a}$ via a^1, we obtain $\underline{\eta}(c) = 3\,t_{01} + 1/2\,t_{12}$ and $\underline{\eta}^*(c) = 3\,t_{01} + t_{12}$.
On the other hand, $\underline{\eta}' := 4\,t_{01}$ is another lifting of $\eta_0^*(c)$ (belonging to $\tilde{C}(Q)^\vee$ since $\underline{\eta}' - \underline{\eta}(c) = t_{01} - 1/2\,t_{12} \in \overline{C(Q)^\vee}$).

(6.2.8) The statements of (5.2.8) remain true, if the semigroup $\tilde{C}(Q)^\vee \cap \tilde{M}$ will be replaced by its (non-saturated) subgroup Γ mentioned in the last part of Proposition (6.2.7) (and X by $X' := \operatorname{Spec} \Gamma$, respectively).
Then, X' equals the scheme theoretical image of the map $X \to \mathbb{C}^w \times S$ induced by the elements $[c^1, \underline{\eta}^*(c^1)], \ldots, [c^w, \underline{\eta}^*(c^w)] \in \Gamma$. And, in return, X is the normalization of X'.

6.3 A flat family

(6.3.1) Theorem (5.3.1) remains valid with just one exception: $\bar{X} \subseteq \mathbb{C}^w \times \mathbb{C}^N$ is the scheme theoretical image of X as well as of X'. On the other hand, if we start with $\bar{\pi} : \bar{X} \to \bar{S}$, then base change via $S \to \bar{S}$ brings back $X' \to S$ instead of $X \to S$.
The spaces $\mathcal{M} \subseteq V(Q)$ and $\bar{\mathcal{M}} \subseteq V(Q)/\underline{1}$ are still defined by the equations $g_{\varepsilon,k}(t)$ (cf. section (2.2)) assigned to compact two-faces $\varepsilon < Q$. (However, as it was explained in (6.2.1), we are using a modified version of the vector space $V(Q)$ through the whole chapter.)

(6.3.2) Replace X by X'.

(6.3.3) Replace η_0 and $\underline{\eta}$ by η_0^* and $\underline{\eta}^*$, respectively.

(6.3.4) Replacing $A(X)$ by $A(X')$, we obtain an isomorphism $A(\bar{X}) \otimes_{A(\bar{S})} A(S) \xrightarrow{\sim} A(X')$, and these rings are contained in $A(X)$.

(6.3.5) There are no additional changes necessary.

6.4 The Kodaira-Spencer map

(6.4.1) The condition for an element $[c, \eta_0^*(c)]$ to be contained in E_j^R looks different in the non-Gorenstein case:

$$E_j^R = \{[c^v, \eta_0^*(c^v)] \mid \langle \bar{a}^j, -c^v \rangle > \eta_0^*(c^v) - 1\}\,.$$

(In particular, $[c, \eta_0^*(c)]$ is always contained in $E^R_{\bar{a}(c)}$.) This leads to the following description of $\underline{\eta}^*(c)$ generalizing its definition from (6.2.6):

Lemma: *Assume that* $[c, \eta_0^*(c)]$ *is contained in* E_j^R. *Then,*

$$\underline{\eta}^*(c) = \langle \bar{a}^j_\bullet, -c \rangle + (\eta_0^*(c) + \langle \bar{a}^j, c \rangle) \cdot e[\bar{a}^j]\,.$$

($\bar{a}^j_\bullet$ *denotes the map assigning* $\underline{t} \in V(Q)$ *the vertex* $\bar{a}^j_{\underline{t}}$ *of the (generalized) Minkowski summand* $Q_{\underline{t}}$*.)*

Proof: If $\bar{a}^j \in \mathbb{L}$, then the condition $\langle \bar{a}^j, -c \rangle > \eta_0^*(c) - 1$ is equivalent to $\langle \bar{a}^j, -c \rangle = \eta_0(c) = \eta_0^*(c)$. Hence, the second summand in our formula vanishes, and we are done.
On the other hand, if $\bar{a}^j \notin \mathbb{L}$, then there is not any lattice point contained in the strip $\langle \bar{a}^j, c \rangle \geq \langle \bullet, c \rangle > \langle \bar{a}(c), c \rangle$. In particular, every edge on the path from $\bar{a}^j$ to $\bar{a}(c)$ (decreasing the c-value at each step) belongs to the "component" induced by $\bar{a}^j$ (cf. (6.2.5)). Now, our formula follows from the definition of $\underline{\eta}^*(c)$. □

Similarly to (5.4.1) we consider the bilinear map

$$\begin{array}{rccccc} \Phi: & {}^{V_{\mathbb{Z}}}\!/_{(1,\ldots,1)} & \times & L(\cup_j E_j^R) & \longrightarrow & \mathbb{Z} \\ & \underline{t} & , & q & \mapsto & \sum_{v,i} t_i\, q_v\, \eta_i^*(c^v)\,, \end{array}$$

and we obtain $\Phi(\underline{t}, q) = 0$ for $q \in L(E_j^R)$ again. (Indeed,

$$\begin{array}{rcl} \Phi(\underline{t}, q) & = & \sum_v q_v \,\langle \underline{t}, \underline{\eta}^*(c^v) \rangle \\ & = & \sum_v q_v \cdot \Big(\langle \bar{a}^j_{\underline{t}}, -c^v \rangle + (\eta_0^*(c^v) - \langle \bar{a}^j, -c^v \rangle) \cdot t_{\bar{a}^j} \Big) \\ & = & \langle \bar{a}^j_{\underline{t}}, -\sum_v q_v c^v \rangle + \big(\sum_v q_v\, \eta_0^*(c^v) - \langle \bar{a}^j, \sum_v q_v c^v \rangle \big) \cdot t_{\bar{a}^j} \;=\; 0\,.) \end{array}$$

(6.4.2) Theorem (5.4.2) (stating that Φ, on the one hand, describes the Kodaira-Spencer map and, on the other hand, yields exactly the isomorphism built in section (3.4)) remains true in the non-Gorenstein case. Even the proof of the first part transfers completely (with the usual replacements).
To prove the second statement we start as in (3.4.1): Take a $q \in L(E_0^R)$, decompose it into $q = \sum_j q^j$ ($q^j \in \mathbb{Z}^{E_j}$) and consider $v^j := -\sum_v q_v^j\, [c^v, \eta_0^*(c^v)]$. Now,

$$\begin{array}{rcl} \Phi(\underline{t}, q) & = & \sum_j \Phi(\underline{t}, q^j) \;=\; \sum_{j,v} q_v^j \cdot \langle \underline{t}, \underline{\eta}^*(c^v) \rangle \\ & = & \sum_{j,v} \langle \bar{a}^j_{\underline{t}}, -c^v \rangle + \sum_{j,v} q_v^j \cdot [\eta_0^*(c^v) + \langle \bar{a}^j, c^v \rangle] \cdot t_{\bar{a}^j} \\ & = & \sum_j \langle (\bar{a}^j_{\underline{t}}, 0),\, v^j \rangle - \langle (\bar{a}^j, 1),\, v^j \rangle \cdot t_{\bar{a}^j}\,. \end{array}$$

Hence, in the sense of (3.4.1), our $\Phi(\underline{t},\bullet) \in L(E_0^R)^*$ corresponds to the elements $b^j := (\bar{a}^j_{\underline{t}}, 0) - (\bar{a}^j, 1)\, t_{\bar{a}^j} \in N_{I\!R}$ defining some map $\oplus_j \operatorname{span}_{I\!R} E_j^R \to I\!R$.
On the other hand, it is exactly those elements b^j bringing back the original $\underline{t} \in V(Q)$ via the construction of (3.4.4): Recalling that $\bar{a}^j$ is just an abbreviation for $(\bar{a}^j, 1)$, we obtain $\bar{b}^j = (\bar{a}^j_{\underline{t}}, 0)$ and $s_j = t_{\bar{a}^j}$.

In particular, the Kodaira-Spencer map $T_{\bar{\mathcal{M}},0} \to T^1_Y(-R)$ is always an isomorphism (even in the non-Gorenstein case).

6.5 The obstruction map

(6.5.1) According to the description of $V^\perp$ in (6.2.5), the generators $g_{\underline{d},k}(\underline{t} - t_1)$ ($\underline{d} \in V^\perp \cap Z\!\!Z^N, k \geq 1$) of $\mathcal{J}$ split into the following two types:

- $g_{\varepsilon,k}(\underline{t} - t_1)$ for compact two-faces $\varepsilon < Q$ and
- $t_i - t_j$, if the edges d^i, d^j contain a common non-lattice vertex of Q.

We can get rid of the latter type by the usual recipe: Do not consider coordinates t_i corresponding to single edges, but to entire "components". This point of view is related to the following fact: $t_i - t_j \in \mathcal{J}_1 \subseteq \tilde{\mathcal{J}}$, i.e. $t_i - t_j$ vanishes in $W = \mathcal{J}/\tilde{\mathcal{J}}$ anyway.

Theorem (5.5.1) still holds - but the corollary now just states that we have found the versal deformation of degree $-R$.

(6.5.2) We use the definitions of $\underline{\eta}^c(c)$ and $\underline{\eta}^c(c^v)$ given in (5.5.2)(ii). Additionally, we need (similarly to (6.2.6)(2))

$$\underline{\eta}^{*c}(c) := \underline{\eta}^c(c) + [\eta_0^*(c) - \eta_0(c)] \cdot e[\bar{a}(c)] \quad \text{and} \quad \underline{\eta}^{*c}(c^v) := \underline{\eta}^c(c^v) + [\eta_0^*(c^v) - \eta_0(c^v)] \cdot e[\bar{a}(c^v)]\,.$$

The coefficients $p_v^c \in I\!N$ introduced in (iii) are now defined via $[c,\, \eta_0^*(c)] = \sum_v p_v^c\, [c^v,\, \eta_0^*(c^v)]$ for $c \in (Q^\infty)^\vee \cap I\!L^*$. Similarly, add stars to every η_0 or $\underline{\eta}$ occurring in the sequel.

(6.5.3) Add stars to the η's.

(6.5.4) The lemma remains valid. In the corollary, it should be $k_0 = \alpha + \sum_v (a_v + r_v)\, \eta_0^*(c^v) - \eta^*(\tilde{c})$ now. On the other hand, it holds

$$\underline{d} = \sum_v (a_v - b_v) \cdot (\underline{\tilde{\eta}}(c^v) - \underline{\eta}(c^v)) = \sum_v (a_v - b_v) \cdot (\underline{\tilde{\eta}}^*(c^v) - \underline{\eta}^*(c^v))\,,$$

since the $e[\bar{a}(c^v)]$-terms kill each other.

(6.5.5) For $c \in I\!L^*$ and paths $\rho \in Z\!\!Z^N$ it is not in general true that $\underline{d}(\rho, c) = [\langle \rho_1\, d^1, c\rangle, \ldots, \langle \rho_N\, d^N, c\rangle] \in I\!R^N$ is contained in $Z\!\!Z^N$. However, if ρ governs the walk between two lattice vertices, and if $\underline{d}(\rho, c)$ will be regarded modulo $t_i - t_j$ (if d^i, d^j contain a common non-lattice vertex), then it is. (Proceed as in the proof of Lemma (6.2.6).) In particular, the property $\underline{d}(\rho, c) \in Z\!\!Z^N$ still holds for closed paths.

Looking at Remark (5.5.5), we just take the sum in (1) over all *compact* two-faces $\varepsilon < Q$. The map remains surjective, since the generators $g_{\underline{d},k}$ of type $t_i - t_j$ vanish in W. Part (2) remains true for arbitrary vertices $\bar{a}, \bar{b} \in Q$, even for $|\langle \bar{a}, c\rangle - \langle \bar{b}, c\rangle| < k$.

Now, we use the sets

$$E_j^{kR} = \{[c^v, \eta_0^*(c^v)] \,|\, \langle \bar{a}^j, c^v\rangle + \eta_0^*(c^v) < k\} \cup \{R\} \quad (k \geq 2)$$

again. The maps $\psi_j^{(k)} : L(E_j^{kR}) \to W_k$ are defined as in the Gorenstein case. Even the lemma remains valid, i.e. we obtain a linear map $\psi^{(k)} : T_Y^2(-kR) \to W_k$.

(6.5.6) Just make the usual changes. It may be helpful to keep in mind that, for instance, the statements $\sum_v a_v\, \eta_0^*(c^v) - \eta_0(c) + (\alpha - k) \geq 0$ and $\sum_v a_v\, \eta_0^*(c^v) - \eta_0^*(c) + (\alpha - k) \geq 0$ are equivalent (since $a_v, \alpha, k \in \mathbb{N}$).

(6.5.7) Finally, one shows surjectivity of $\psi^{(k)}$ by $\psi^{(k)} \circ p^k = \sum_{\varepsilon < Q} g^{(k)}(\underline{\varepsilon}, \bullet)$ just as in the Gorenstein case.

6.6 Regular functions as variables

Finally, we give a conjecture how the equations of the versal base space could look like, if all negative degrees are involved at the same time. The idea is to replace ordinary variables for the base space by variable Laurent series: A former variable in $T^1(-R)$ occurs as the (variable) x^R-coefficient of some of these Laurent series. (Actually, since just negative degrees will be involved, our variables will already be contained in $\mathbb{C}[\sigma^\vee \cap M]$, i.e. equal (variable) regular functions on Y.)

Definition-Proposition: Let $\sigma = \langle a^1, \ldots, a^M\rangle$. If $D := \sum_j \overline{\text{orb}(a^j)}$ denotes the negative of the canonical divisor on Y, then the map

$$\begin{array}{rcl} N & \longrightarrow & \theta_Y(-\log D) \\ a & \mapsto & \partial_a := [x^r \mapsto \langle a, r\rangle \cdot x^r] \end{array}$$

induces an isomorphism $N \otimes_{\mathbb{Z}} \mathcal{O}_Y \xrightarrow{\sim} \theta_Y(-\log D)$.

Now, we introduce for every two-dimensional face $\langle a^j, a^k\rangle < \sigma$ some variable $f_{jk}(\underline{x}) \in \mathbb{C}[\sigma^\vee \cap M]$.

Conjecture: *The equations for the negative part of the versal base space of Y are*

- $[\partial_{a^k} - id](f_{jk} - f_{kl}) = 0$, *whenever $\langle a^j, a^k\rangle, \langle a^k, a^l\rangle < \sigma$ are two-dimensional faces, and*
- $\sum_{v=1}^k f^q_{a^{j_{v-1}}, a^{j_v}}(\underline{x}) \cdot [a^{j_{v-1}} - a^{j_v}] = 0$ *($q \geq 1$) for every three-dimensional face $\varepsilon = \langle a^{j_1}, \ldots, a^{j_k}\rangle < \sigma$ (with $j_0 := j_k$).*

(In the previous sections we have proved, that these are indeed the right equations, if restricted to each degree $-R$ ($R \in \sigma^\vee \cap M$) separately.)

Chapter 7

Toric deformations

7.1 Toric deformations

(7.1.1) From chapter 5 we know that (at least for isolated toric Gorenstein singularities) the total spaces over irreducible components of the versal base space are toric again. Hence, it pays to turn the whole situation upside down: Even if Y is not smooth in codimension two, we ask for deformations of Y with toric total space.

Definition: A deformation of Y, i.e. a flat map $f : X \to S$ with an isomorphism $Y \stackrel{\sim}{\to} f^{-1}(0)$ ($0 \in S$), is said to be *toric*, if

(i) X is an affine, toric variety,

(ii) $i : Y \stackrel{\sim}{\to} f^{-1}(0) \hookrightarrow X$ is a morphism in the category of toric varieties, i.e. it induces an algebraic group homomorphism $T_Y \hookrightarrow T_X$ between the embedded tori which makes i equivariant, and

(iii) i sends the closed T_Y-orbit in Y isomorphically onto the closed T_X-orbit in X.

Remark: The rather technical condition (iii) could be replaced by the weaker (and perhaps more natural) one that asks for mapping the closed T_Y-orbit in Y *into* the closed T_X-orbit in X. This would not essentially change the notion of a toric deformation, but the actual version of (iii) makes the theory more convenient. In most applications the closed orbits under the torus action are points, anyway. Since we are actually interested in the germs of Y, X, and S only, the condition (iii) arises quite naturally then.

Example: Both families $Y_t \to \mathbb{C}^3$ and $Y_s \to \mathbb{C}$ of (1.1.3) are toric deformations of the cone over the rational normal curve of degree four.

(7.1.2) For X and Y as before we introduce the following notation:

- Let $\tilde{M}$, $\tilde{N}$ be mutually dual lattices; let $\tilde{\sigma}$ be a rational polyhedral cone in $\tilde{N}_{\mathbb{R}} := \tilde{N} \otimes_{\mathbb{Z}} \mathbb{R}$ that does not contain any linear subspace. These objects are used to build $X = \text{Spec}\ \mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]$.
- Analogously, we use M, N, and σ to obtain $Y = \text{Spec}\ \mathbb{C}[\sigma^\vee \cap M]$.

Let Y and $f : X \to S$ meet the conditions (i) and (ii) of Definition (7.1.1). Then, the equivariant closed embedding $i : Y \hookrightarrow X$ corresponds to an embedding

$$i : N \hookrightarrow \tilde{N}$$

on the level of lattices. The dual map i^* induces a surjection

$$i^* : \tilde{\sigma}^\vee \cap \tilde{M} \twoheadrightarrow \sigma^\vee \cap M$$

even between the semigroups. The kernel $L := \ker i^* \subseteq \tilde{M}$ is a sublattice (say of dimension n), and we obtain

$$N = \tilde{N} \cap L^\perp ; \quad \sigma = \tilde{\sigma} \cap L^\perp \subseteq N_{\mathbb{R}} .$$

Lemma: *The condition (iii) of Definition (7.1.1) is equivalent to* $\tilde{\sigma}^\vee \cap L_{\mathbb{R}} = \{0\}$.

Proof: The condition that i maps the closed orbit of Y into the closed orbit of X can be written as

$$i^*(\tilde{\sigma}^\vee \setminus \tilde{\sigma}^\perp) \subseteq \sigma^\vee \setminus \sigma^\perp.$$

Moreover, to obtain equality of both orbits, the map i^* has to induce a bijection $\tilde{\sigma}^\perp \cap \tilde{M} \xrightarrow{\sim} \sigma^\perp \cap M$, which is equivalent to the injectivity of i^* on $\tilde{\sigma}^\perp$. Now, these two conditions can be translated into

$$\tilde{\sigma}^\vee \cap (L + \tilde{\sigma}^\perp) = \tilde{\sigma}^\perp \ \ (\text{i.e. } \tilde{\sigma}^\vee \cap L \subseteq \tilde{\sigma}^\perp) \quad \text{and} \quad \tilde{\sigma}^\perp \cap L = \{0\},$$

which is equivalent to $\tilde{\sigma}^\vee \cap L = \{0\}$. □

(7.1.3) Proposition: *Let $f : X \to S$ be a toric deformation of Y. Then, the germ $(S, 0)$ is smooth, and the ideal $I := \ker \left(\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}] \to \mathbb{C}[\sigma^\vee \cap M] \right)$ defining $Y \subseteq X$ can be generated by n binomials $x^{u^1} - x^{v^1}, \dots, x^{u^n} - x^{v^n} \in \mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]$ ($u^i, v^i \in \tilde{\sigma}^\vee \cap \tilde{M}$; $u^i - v^i \in L$). In particular, they form a binomial regular sequence, and Y is a relative complete intersection in X.*

Proof: *Step 1:* A deformation diagram

$$\begin{array}{ccc} Y & \stackrel{i}{\hookrightarrow} & X \\ \downarrow & \otimes & \downarrow f \\ \{0\} & \hookrightarrow & S \end{array}$$

induces a deformation of the corresponding torus T_Y:

$$\begin{array}{ccc} T_Y & \hookrightarrow & T_X \\ \downarrow & \otimes & \downarrow f \\ \{0\} & \hookrightarrow & S \ . \end{array}$$

T_Y and T_X are smooth. Hence, T_X splits locally into a product $T_X \cong T_Y \times S$, and the germ $(S, 0)$ has to be smooth, too.

Step 2: On the level of local rings (at the general points of the closed orbits and at the special point of S, respectively) we obtain the following diagram:

$$\begin{array}{ccccc} \mathcal{O}_{X,0} & \stackrel{i^*}{\twoheadrightarrow} & \mathcal{O}_{Y,0} & & \\ \uparrow\text{flat} & \otimes & \uparrow & & \\ \mathcal{O}_{S,0} & \twoheadrightarrow & \mathcal{O}_{S,0}/m_{S,0} & = & \mathbb{C}. \end{array}$$

Therefore, $I \cdot \mathcal{O}_{X,0} = m_{S,0} \cdot \mathcal{O}_{X,0}$ is generated by n elements $g_1, \dots, g_n$ ($(S, 0)$ is smooth), and by the Nakayama lemma we can choose these generators among the elements of the form $x^u - x^v$ ($u, v \in \tilde{\sigma}^\vee \cap \tilde{M}$; $u - v \in L$).

Step 3: Let $\tilde{I} := (g_1, \dots, g_n) \subseteq \mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]$. Then, $\tilde{I} \subseteq I$ are ideals in $\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]$ meeting the following conditions:

(i) $\tilde{I}$ and I are homogeneous with respect to the M-grading;

(ii) $\tilde{I} = I$ in the local ring $\mathcal{O}_{X,0}$.

We want to show $\tilde{I} = I$. For that purpose, let $g \in I$ be an arbitrary M-homogeneous element. By (ii) there exists an $h \in \mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]$ with $h \cdot g \in \tilde{I}$ and

$$h \notin m_0 := \bigoplus_{\substack{u \in \tilde{\sigma}^\vee \cap \tilde{M} \\ u \notin \tilde{\sigma}^\perp}} \mathbb{C} \cdot x^u$$

(i.e. h contains a term of $\mathbb{C}[\tilde{\sigma}^\perp \cap \tilde{M}]$); by (i) we can additionally assume h to be M-homogeneous. Since $\tilde{\sigma}^\vee \cap L_{\mathbb{R}} = \{0\}$, this implies that h is a monomial of $\mathbb{C}[\tilde{\sigma}^\perp \cap \tilde{M}]$, i.e. h is invertible. □

Remark: The n vectors $u^1 - v^1, \ldots, u^n - v^n$ built from the exponents of the binomial regular sequence generating I are free generators of the sublattice $L \subseteq \tilde{M}$.

Proof: Let $L' := \operatorname{span}_{\mathbb{Z}}(u^1 - v^1, \ldots, u^n - v^n) \subseteq L$. In particular, the ideal I is homogeneous under the $\tilde{M}/L'$-grading of $\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]$.
Now, for each $l \in L$ there are $u, v \in \tilde{\sigma}^\vee \cap \tilde{M}$ such that $l = u - v$. The monomials x^u and x^v map onto equal functions in $\Gamma(Y, \mathcal{O}_Y)$, hence $x^u - x^v \in I$. Since I does not contain monomials at all, $x^u - x^v$ has to be $\tilde{M}/L'$-homogeneous itself, i.e. $u - v \in L'$. □

Definition: Those binomial regular sequences defining a relative complete intersection between affine toric varieties satisfying (7.1.1)(ii) and (iii) (as in the previous proposition) are said to be *toric regular sequences*. (They give rise to a so-called format of Y; cf. [Rö].)

(7.1.4) Christophersen [Ch 2] developed the notion of relative deformations.

Definition: If $Y \hookrightarrow X$ is a relative complete intersection (i.e. given by a regular sequence in X), then a *relative deformation* (of Y in X) over S is given by a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \hookrightarrow & X \times S \\ \downarrow f & & \downarrow \mathrm{pr}_S \\ S & = & S \end{array}$$

(f is flat) and an isomorphism $Y \xrightarrow{\sim} f^{-1}(0)$ compatible with the embeddings into X. Two relative deformations are said to be equivalent, if they are so as abstract deformations (without regarding the embeddings).

Relative deformations can be obtained by arbitrary perturbations of the regular sequence. In particular, they form a smooth subfunctor $\mathrm{Def}_{Y \hookrightarrow X}$ of the deformation functor Def_Y. If Y admits a versal deformation with base space S_Y, then the versal relative deformation of $Y \hookrightarrow X$ equals the subfamily over some smooth subscheme $S_{Y \hookrightarrow X} \subseteq S_Y$.
Christophersen has worked out that $S_{Y \hookrightarrow X}$ should be a good candidate for components of the reduced base space $(S_Y)_{\mathrm{red}}$.

(7.1.5) Let $f : X \to S$ be a toric deformation of Y. Then, the diagram

$$\begin{array}{ccc} X & \overset{(\mathrm{id}, f)}{\hookrightarrow} & X \times S \\ \downarrow f & & \downarrow \mathrm{pr}_S \\ S & = & S \end{array}$$

and Proposition (7.1.3) show that $f : X \to S$ is a relative deformation for the embedding $Y \hookrightarrow X$ given by the toric regular sequence $g := (x^{u^1} - x^{v^1}, \ldots, x^{u^n} - x^{v^n})$. In particular, it is induced by some morphism $S \to S_{Y \hookrightarrow X}$. If, moreover, f equals the restriction of the versal family of Y to some component S of $(S_Y)_{\mathrm{red}}$, then S and $S_{Y \hookrightarrow X}$ are equal as germs.
On the other hand, the regular sequence g itself provides a special toric deformation by regarding the (flat) map $g : X \to \mathbb{C}^n$. f and g need not be equivalent in general. However, similar to the case of f, the deformation g is induced by some map $\mathbb{C}^n \to S_{Y \hookrightarrow X}$, i.e. S and $\mathbb{C}^n$ map at least into the same component of $(S_Y)_{\mathrm{red}}$. If, moreover, $S = S_{Y \hookrightarrow X}$ (for instance, if f is a component of the versal deformation), then $\mathbb{C}^n \to S_{Y \hookrightarrow X} = S$ is an isomorphism, i.e. $f = g$.

7.2 Constructing homogeneous, toric regular sequences

(7.2.1) The previous section provides motivation for looking for those pairs (Y, X) of affine toric varieties such that Y is a relative complete intersection in X, given by a toric regular sequence g.
One possibility for doing so is fixing the "big" space X and searching for those binomial regular sequences yielding a toric variety as zero set. This was the principal approach in [Al 2]. However, starting with Y and looking for toric X to map into, is a completely different story. In this section we will solve this problem, if the sequences are additionally assumed to be homogeneous:

Definition: If $g = (x^{u^1} - x^{v^1}, \ldots, x^{u^n} - x^{v^n})$ is a toric regular sequence defining $Y \hookrightarrow X$, then the common images $\bar{u}^i \in M$ of $u^i, v^i \in \tilde{M}$ are called the *degrees* of g. The sequence g (and its associated toric deformation) is said to be *homogeneous of degree* R, if $R = \bar{u}^1 = \ldots = \bar{u}^n$. (Obviously, if $m = 1$, then this will always be the case.)

Lemma: *Up to $\mathbb{Z}$-linear transformations, a homogeneous, toric regular sequence g has the shape $g = (x^{u^1} - x^{u^0}, \ldots, x^{u^n} - x^{u^0})$, i.e. $u^0 = v^1 = \ldots = v^n$.*

Proof: Choose an arbitrary $u^0 \in \{u^1, \ldots, u^n, v^1, \ldots, v^n\}$. Then, since $u^i - u^0, v^i - u^0 \in L$, the $2n$ binomials $x^{u^i} - x^{u^0}, x^{v^i} - x^{u^0}$ $(i = 1, \ldots, n)$ are contained in I. Moreover, since $x^{u^i} - x^{v^i} = (x^{u^i} - x^{u^0}) - (x^{v^i} - x^{u^0})$, they generate this ideal, and we can choose n among them still doing so (cf. the proof of Proposition (7.1.3)).
It remains to show that this changing of generators could be done using a $\mathbb{Z}$-linear transformation only. We regard the following, more general situation: Let $x^u - x^v$ be contained in an ideal I generated by binomials $x^{u^1} - x^{v^1}, \ldots, x^{u^n} - x^{v^n}$, and assume that all exponents $u, v, u^i, v^i \in \tilde{M}$ map onto a single $R \in M$. Using the natural M-grading, all these binomials are homogeneous of degree R. Hence, representing $x^u - x^v$ as a $\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]$-linear combination of the generators of I can be done by using homogeneous coefficients of degree $0 \in M$ only. Since $\tilde{\sigma}^\vee \cap L = \{0\}$, they have to be constants. Moreover, it is obvious that they can be taken even from $\mathbb{Z}$ then. □

Remark: If $g = (x^{u^1} - x^{u^0}, \ldots, x^{u^n} - x^{u^0})$, then

$$L = \sum_{i=1}^{n} \mathbb{Z} \cdot (u^i - u^0) = \sum_{i,j=0}^{n} \mathbb{Z} \cdot (u^i - u^j) = \ker \left(\deg : \bigoplus_{i=0}^{n} \mathbb{Z} \cdot u^i \twoheadrightarrow \mathbb{Z} \right)$$

$(\deg(u^i) := 1)$. The elements $u^0, \ldots, u^n$ are linearly independent in $\tilde{M}_{\mathbb{R}}$.

(7.2.2) Now, we approach our main issue. We will produce homogeneous toric regular sequences (yielding pairs (Y, X)) from so-called deformation data.

Definition: Let $(\boldsymbol{A}, \mathbb{L})$ be a pair of a real vector space $\boldsymbol{A}$ and a lattice $\mathbb{L} \subseteq \boldsymbol{A}$. A *deformation datum* of size n is a tuple $(R_0, \ldots, R_n;\, C;\, p)$ admitting the following properties:

(i) $C \subseteq \boldsymbol{A}$ is a rational polyhedral cone with apex in $0 \in \boldsymbol{A}$, and $p \geq 1$ is a natural number.

(ii) $R_0, \ldots, R_n \subseteq \boldsymbol{A}$ are rational polyhedra with C as their common cone of unbounded directions, i.e. $R_i = \bar{R}_i + C$ for suitable compact polytopes $\bar{R}_i$, $i = 0, \ldots, n$. (It is possible to choose these polytopes in a canonical way by taking the convex hull of the vertices of R_i.)

(7.2.3) For a given polyhedron $P \subseteq \boldsymbol{A}$ and a non-trivial linear form $t \in (P^\infty)^\vee \subseteq \boldsymbol{A}^*$ (i.e. $t \neq 0$, t bounded below on P) we denote by $F(P,t)$ the face of P that is defined by t being minimal on it.

Definition: A deformation datum $(R_0, \ldots, R_n;\, C;\, p)$ is said to be *admissible*, if it meets the following conditions:

Case 1: $p = 1$. For each $t \in C^\vee \subseteq \boldsymbol{A}^*$ at least n of the $n+1$ faces $F(R_i,t)$ of R_i $(i = 0, \ldots, n)$ contain lattice points.

Case 2: $p \geq 2$. $R_1, \ldots, R_n$ are lattice polyhedra, i.e. they admit only lattice points as vertices.

The following alternative description will be useful:

Lemma: *A deformation datum $(R_0, \ldots, R_n;\, C;\, p)$ is admissible, if and only if for each $t \in \mathbb{L}^* \cap C^\vee$ the values of t on at least n of the $n+1$ faces $F(R_i,t)$ of R_i $(i = 0, \ldots, n)$ (or exactly on the faces $F(R_1,t), \ldots, F(R_n,t)$, if $p \geq 2$) are integers.*

Proof: Linear forms $t \in \mathbb{L}^* \cap C^\vee$ yield integers as values on lattice points. Hence, admissible deformation data always admit the property described in the lemma. To obtain the opposite implication, we proceed in three steps:

Step 1: Perturbing the linear form $t \in C^\vee$ slightly (inside the cone $C^\vee$), the corresponding faces $F(R_i,t)$ of R_i will at most be replaced by smaller ones. Hence, it will be sufficient to regard only those t such that each $F(R_i,t)$ is a vertex of R_i. Moreover, if convenient, it will be possible to replace t by suitable linear forms $t' \in C^\vee$ close to t again; they provide the same vertex as t.

Step 2: Claim: Let $b^0, b^1 \in \boldsymbol{A} \setminus \mathbb{L}$, and let $t \in C^\vee \setminus \{0\}$. Then, there exists a linear form $t' \in (\mathbb{L}^* \cap C^\vee) \setminus \{0\}$ such that

- $t'/\|t'\|$ and $t/\|t\|$ are arbitrarily close to each other, and
- $\langle b^0, t'\rangle, \langle b^1, t'\rangle \notin \mathbb{Z}$.

Proof: First we try to meet the latter condition. Choose $t^0, t^1 \in \mathbb{L}^*$ having no integer value on b^0, b^1, respectively. If there exists a t^j among them such that both $\langle b^0, t^j\rangle, \langle b^1, t^j\rangle \notin \mathbb{Z}$, then take $t' := t^j$. Otherwise, we know that $\langle b^0, t^1\rangle, \langle b^1, t^0\rangle \in \mathbb{Z}$ and $\langle b^0, t^0\rangle, \langle b^1, t^1\rangle \notin \mathbb{Z}$. Hence, $t' := t^1 + t^2$ has the desired property.
Now, we have to improve our linear form t' to obtain the additional property $t' \in C^\vee$. If $s \in \mathbb{L}^* \cap (\operatorname{int} C^\vee)$, and if $N \in \mathbb{N} \setminus \{0\}$ such that $N \cdot b^0, N \cdot b^1$ are contained in the lattice $\mathbb{L}$, then $\langle b^0, N \cdot s\rangle, \langle b^1, N \cdot s\rangle \in \mathbb{Z}$. Each element of $\mathbb{L}^*$ can be put into the cone $C^\vee$ by adding a sufficiently large multiple of s. Hence, we substitute $t' := t' + (kN) \cdot s$ (with $k \gg 0$).
Finally, we can add sufficiently large multiples of t (which are contained in the lattice and yield integers as values on b^0, b^1) to t'. This operation ensures that the directions $t'/\|t'\|$ and $t/\|t\|$ are arbitrarily close to each other.

Step 3: If a deformation datum $(R_0, \ldots, R_n;\ C;\ p)$ is not admissible, then there exists an element $t \in C^\vee$ such that two of the faces $F(R_i, t)$ $(i = 0, \ldots, n)$ equal vertices (denoted by b^0, b^1) that are not contained in the lattice (*Case* $p \geq 2$*: one* of the faces $F(R_i, t)$ $(i = 1, \ldots, n)$, denoted by $b^0 = b^1$).
Let t' be a linear form as constructed in the second step; suppose that t and t' define the same vertices of $R_0, \ldots, R_n$ (including b^0 and b^1). Then, t' violates the conditions of the lemma. □

(7.2.4) To a given deformation datum $(R_0, \ldots, R_n;\ C;\ p)$ we associate the following objects:

(7.2.4.1) Define the polyhedron Q to be the Minkowski sum

$$Q := R_0 + \ldots + R_n = C + (\bar{R}_0 + \ldots + \bar{R}_n) \subseteq \boldsymbol{A}.$$

We embed the whole space as an affine hyperplane into a higher-dimensional space:

- $N_{\mathbb{R}} := \boldsymbol{A} \times \mathbb{R}$ is a vector space containing the lattice $N := \mathbb{L} \times \mathbb{Z}$ $(M_{\mathbb{R}} := N_{\mathbb{R}}^*,\ M := N^*)$;
- $\psi_1 : \boldsymbol{A} \hookrightarrow N_{\mathbb{R}}\ ; \quad a \mapsto (a, p^{-1})$.

In particular, $\psi_1(Q)$ is a polyhedron in $N_{\mathbb{R}}$. Denoting by

$$\psi : \boldsymbol{A} \hookrightarrow N_{\mathbb{R}}\ ; \quad a \mapsto (a, 0)$$

the associated linear embedding, we can define $Y :=$ Spec $\mathbb{C}[\sigma^\vee \cap M]$ as the affine toric variety that is given by the cone

$$\sigma := \overline{\mathbb{R}_{\geq 0} \cdot \psi_1(Q)} = \psi(C) \cup\ \mathbb{R}_{\geq 0} \cdot \psi_1(Q)\,.$$

(7.2.4.2) To define $\tilde{\sigma}$ and X, we put the polyhedra $R_0, \ldots, R_n$ into parallel affine planes of a vector space that is large enough.

- $\tilde{N}_{\mathbb{R}} := \boldsymbol{A} \times \mathbb{R}^{n+1}$, $\tilde{N} := \mathbb{L} \times \mathbb{Z}^{n+1}$; $\tilde{M}_{\mathbb{R}} := \tilde{N}_{\mathbb{R}}^*$, $\tilde{M} := \tilde{N}^*$.
  Denote by $\Phi : \tilde{N}_{\mathbb{R}} \longrightarrow \mathbb{R}^{n+1}$ the projection onto the second factor.
- $\phi_i : \boldsymbol{A} \hookrightarrow \tilde{N}_{\mathbb{R}}\,; \quad a \mapsto \begin{cases} (a, p^{-1}e^0) & \text{for } i = 0 \\ (a, e^i) & \text{for } i = 1, \ldots, n \end{cases}$
  $(e^0, \ldots, e^n$ denotes the standard basis of $\mathbb{Z}^{n+1})$.
  On the homogeneous level, the affine maps $\phi_0, \ldots, \phi_n$ correspond to the trivial embedding $\phi : \boldsymbol{A} \hookrightarrow \tilde{N}_{\mathbb{R}}$ $(a \mapsto (a, 0))$.

Now, we denote by P the convex hull

$$P := \text{conv}\left(\bigcup_{i=0}^{n} \phi_i(R_i)\right) = \phi(C) + \text{conv}\left(\bigcup_{i=0}^{n} \phi_i(\bar{R}_i)\right) \subseteq \tilde{N}_{\mathbb{R}}$$

and define $X :=$ Spec $\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]$ as the affine toric variety given by the cone

$$\tilde{\sigma} := \overline{\mathbb{R}_{\geq 0} \cdot P} = \phi(C) \cup\ \mathbb{R}_{\geq 0} \cdot P.$$

(7.2.4.3) If $\text{pr}_i : \mathbb{R}^{n+1} \to \mathbb{R}$ denotes the projection onto the i-th factor, we can define linear maps $u^0, \ldots, u^n : \tilde{N} \to \mathbb{Z}$ by

$$u^i := \begin{cases} p \cdot (\text{pr}_0 \circ \Phi) & \text{for } i = 0 \\ \text{pr}_i \circ \Phi & \text{for } i = 1, \ldots, n. \end{cases}$$

By construction, these maps correspond to elements $u^i \in \tilde{\sigma}^\vee \cap \tilde{M}$.

(7.2.4.4) N can be considered a sublattice of $\tilde{N}$ via the inclusion map

$$N \hookrightarrow \tilde{N}\,, \quad (a;1) \mapsto (a;1,p,\ldots,p).$$

This embedding admits the following properties:

(i) $N = \tilde{N} \cap \bigcap_{i,j}(u^i - u^j)^\perp = \tilde{N} \cap \bigcap_{i=1}^n (u^i - u^0)^\perp$

(ii) $\sigma = \tilde{\sigma} \cap N_{\mathbb{R}}$.

In particular, we obtain a map $Y \to X$ which sends Y into the special fiber of the morphism $X \to \mathbb{C}^n$ defined by the regular functions $x^{u^1} - x^{u^0}, \ldots, x^{u^n} - x^{u^0} \in \mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]$.

(7.2.5) **Theorem:** *Starting with an admissible deformation datum $(R_0,\ldots,R_n;\, C;\, p)$, the previous construction provides a pair (Y,X) of affine toric varieties such that $Y \subseteq X$ is given by a homogeneous toric regular sequence $x^{u^1} - x^{u^0}, \ldots, x^{u^n} - x^{u^0}$. (Looking at (7.2.4.4), Y equals the special fiber of $X \to \mathbb{C}^n$.) Moreover, all those pairs (Y,X) arise that way.*

The proof is contained in section 7.3.

Remark: Up to isomorphisms, the construction (7.2.4) will yield the same result, if the polyhedra R_i from the deformation datum $(R_0,\ldots,R_n;\, C;\, p)$ are shifted via vectors from $\mathbb{L}$ (for $i \geq 1$) or even from $p^{-1}\mathbb{L}$ (for $i = 0$).

(7.2.6) Eventually, let us switch to the natural viewpoint that an affine toric variety $Y = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ is given, and we are asking for its (homogeneous) toric deformations.
Fixing some degree $R \in \sigma^\vee \cap M$ corresponds to the choice of an affine cross cut Q of the cone $\sigma \subseteq N_{\mathbb{R}}$. Then, the previous theorem tells us that homogeneous toric deformations of Y arise from certain subdivisions of Q into a Minkowski sum. More precisely, we have to proceed as follows:

(i) Define the vector space $\mathbb{A}_0 := \{a \in N_{\mathbb{R}} \mid \langle a, R\rangle = 0\}$, which contains the lattice $\mathbb{L}_0 := \mathbb{A}_0 \cap N$.

(ii) Let p be the greatest common divisor of the coordinates of R, i.e. $p^{-1}R$ is a primitive element of M.

(iii) Define the affine space $\mathbb{A} := \{a \in N_{\mathbb{R}} \mid \langle a, R\rangle = 1\}$. Fixing some point $0 \in \mathbb{A} \cap p^{-1}N$, we obtain a sublattice via $\mathbb{L} := 0 + \mathbb{L}_0$. (In case $p = 1$, $\mathbb{L}$ equals $\mathbb{A} \cap N$.) Moreover, we can use the point 0 to identify $(\mathbb{A}, \mathbb{L})$ with the pair $(\mathbb{A}_0, \mathbb{L}_0)$ providing a linear structure which was assumed in (7.2.2).

(iv) Let $C := \sigma \cap \mathbb{A}_0$ and $Q := \sigma \cap \mathbb{A}$. Then, by Theorem (7.2.5), homogeneous regular sequences of degree R (in some larger affine toric variety X) correspond to admissible splittings of Q into a Minkowski sum $Q = R_0 + \ldots + R_n$.

Remark: If Y_σ is smooth in codimension two, then (according to Remark (3.4.3)) we may assume that $p = 1$. This case has been discussed in chapter 6, and we should convince ourselves that Theorem (7.2.5) fits into the results obtained there:

Of course, splitting Q into $Q = R_0 + \ldots + R_n$ corresponds to a decomposition of $\underline{1} \in C(Q) \subseteq V(Q)$ into a sum $\underline{1} = \underline{t}^0 + \ldots + \underline{t}^n$ (inside $C(Q)$). However, the crucial point is the admissibility condition; it is now equivalent to the fact that the summands $\underline{t}^i$ $(i = 0,\ldots,n)$ belong to the *lattice of the smaller vector space* $V'(Q) \subseteq V(Q)$ defined in (6.2.1). Hence, using an adapted version of (5.6.1) to describe the reduced structure of the versal base space $\bar{\mathcal{M}}$ in degree $-R$ (as constructed in chapter 6), we obtain:

The irreducible components of $\bar{\mathcal{M}}_{\text{red}}$ indeed correspond to the maximal, admissible decompositions of Q into Minkowski summands.

7.3 Proof of the previous theorem

(7.3.1) Let an admissible deformation datum $(R_0, \ldots, R_n;\, C;\, p)$ be given. We have to show that the associated data $\tilde{\sigma}$, σ and $u^0, \ldots, u^n$ (cf. (7.2.4)) provide indeed a toric regular sequence.

(7.3.1.1) $\tilde{\sigma}^\vee \cap \tilde{M} \longrightarrow \sigma^\vee \cap M$ *is surjective:*
Let $s \in \tilde{M}$ such that

$$\langle (b^0 + \ldots + b^n;\; p^{-1}, 1, \ldots, 1),\; s\rangle \geq 0 \qquad \text{for } b^i \in R_i,\; i = 0, \ldots, n \quad (\text{i.e. } b^0 + \ldots + b^n \in Q)\,;$$

it means that s maps onto an element $\bar{s} \in \sigma^\vee \cap M$. We have to show that $\bar{s}$ can be lifted to $\tilde{\sigma}^\vee \cap \tilde{M}$. Projecting s to the $\mathbb{A}^*$−component, we obtain an element $t = s|_{\mathbb{A}} \in \mathbb{L}^* \cap C^\vee$. Since our deformation element is admissible, we may assume that, on the faces $F(R_1, t), \ldots, F(R_n, t)$, the linear form t provides integers only. Hence, even on the embedded polyhedra $\phi_1(R_1), \ldots, \phi_n(R_n) \subseteq \tilde{N}_{\mathbb{R}}$, the minimal value of s is contained in $\mathbb{Z}$. Denote these values by k_i and suppose that they occur at points $(\tilde{b}^i, e^i) \in \phi_i(R_i) \subseteq \Phi^{-1}(e^i) \subseteq \tilde{N}_{\mathbb{R}}$ $(i = 1, \ldots, n)$.
Modifying s by $s := s - \sum_{i=1}^n k_i \cdot u^i + (\sum_{i=1}^n k_i) \cdot u^0$, we can assume that s is *non-negative* on $\phi_1(R_1), \ldots, \phi_n(R_n) \subseteq \tilde{N}_{\mathbb{R}}$ and, moreover, $\langle (\tilde{b}^i, e^i),\; s\rangle = 0$ for $i = 1, \ldots, n$. Now, if $b^0 \in R_0$ (embedded as $(b^0,\, p^{-1}e^0) \in \phi_0(R_0) \subseteq \tilde{N}_{\mathbb{R}}$) is given, we obtain

$$\left\langle (b^0, p^{-1}e^0)\,,\; s\right\rangle = \left\langle \left(b^0 + \sum_{i=1}^n \tilde{b}^i,\;\; p^{-1}e^0 + \sum_{i=1}^n e^i\right),\; s\right\rangle \geq 0.$$

Hence, s is contained in $\tilde{\sigma}^\vee$.

(7.3.1.2) $\tilde{\sigma}^\vee \cap L_{\mathbb{R}} = \{0\}$ $\;(L_{\mathbb{R}} := \ker(\tilde{M}_{\mathbb{R}} \longrightarrow\!\!\!\!\!\rightarrow M_{\mathbb{R}}))$:
The cone $\tilde{\sigma}$ contains points from $\Phi^{-1}(e^0), \ldots, \Phi^{-1}(e^n)$. In particular, a non-trivial element of $L_{\mathbb{R}} = \ker\left(\deg : \bigoplus_{i=0}^n \mathbb{R} \cdot u^i \longrightarrow\!\!\!\!\!\rightarrow \mathbb{R}\right)$ cannot be contained in $\tilde{\sigma}^\vee$.

(7.3.1.3) *We show that* $I = \ker\left(\mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}] \longrightarrow \mathbb{C}[\sigma^\vee \cap M]\right)$ *is generated by* $x^{u^1} - x^{u^0}, \ldots, x^{u^n} - x^{u^0}$*:*
Let $x^u - x^v \in I$ be any binomial element of I, i.e. $u, v \in \tilde{\sigma}^\vee \cap \tilde{M}$, $u - v \in L$. Hence,

$$u - v = \sum_{i=0}^n g_i \cdot u^i \quad (g_i \in \mathbb{Z},\; \sum_i g_i = 0)\,.$$

Defining

$$s := u - \sum_i g_i^+ \cdot u^i = v - \sum_i g_i^- \cdot u^i \in \tilde{M} \quad (g_i^+, g_i^- \in \mathbb{N},\; g_i^+ - g_i^- = g_i,\; g_i^+ g_i^- = 0)\,,$$

this linear form will equal u or v, if it is restricted to the polyhedra $\phi_i(R_i)$ with $g_i \leq 0$ or $g_i \geq 0$, respectively. In particular, s is non-negative at P, i.e. $s \in \tilde{\sigma}^\vee \cap \tilde{M}$. Now, expressing $x^u - x^v$ as a linear combination of the generators $x^{u^i} - x^{u^0}$ is straightforward.

(7.3.2) It remains to prove that all homogeneous, toric regular sequences can be obtained from deformation data. Hence, in the remainder of section (7.3) we assume that we are given such a sequence $x^{u^1} - x^{u^0}, \dots, x^{u^n} - x^{u^0}$ defining a relative complete intersection $Y \hookrightarrow X$ as in (7.1.1) and (7.1.2).

Lemma: *Let $u, v \in \tilde{\sigma}^\vee \cap \tilde{M}$ such that $u - v \in L \setminus \{0\}$. Then, there are two different indices $i, j \in \{0, \dots, n\}$ satisfying $u - u^i, v - u^j \in \tilde{\sigma}^\vee$.*

Proof: Since $x^u - x^v \in I$, there must be some equation

$$x^u - x^v = \sum_\mu c_\mu \, x^{t^\mu} \, (x^{u^{i(\mu)}} - x^{u^0}) \qquad (c_\mu \in \mathbb{C};\ t^\mu \in \tilde{\sigma}^\vee \cap \tilde{M}).$$

In particular, both exponents u and v have to occur somewhere on the right hand side providing the existence of the desired u^i, u^j. Moreover, if $i = j$, then we could apply that procedure to $u' := u - u^i$ and $v' := v - u^i$ again. This recursion eventually stops. □

(7.3.3) **Lemma:** *Denote by $R \in M = \tilde{M}/L$ the common image of the elements $u^0, \dots, u^n$ via the surjection $\tilde{M} \twoheadrightarrow M$. We obtain:*

(1) R is not trivial on $\sigma \subseteq N_{\mathbb{R}}$.

(2) Let $R = p \cdot R'$ ($p \in \mathbb{N}$, $R' \in M$ primitive). Then, without loss of generality, u^0 is also divisible by p and can be written as $u^0 = p \cdot \tilde{u}^0$. Moreover, the elements $\tilde{u}^0, u^1, \dots, u^n$ equal a part of a $\mathbb{Z}$-basis of the lattice $\tilde{M}$.

Proof: *Step 1:* If R were trivial on σ, not only R but also $-R$ would belong to $\sigma^\vee$. In particular, there would be an $r \in \tilde{\sigma}^\vee$ (lifting $-R$) such that

$$r + u^i \in \tilde{\sigma}^\vee \cap L = \{0\}, \quad \text{i.e. } -u^i \in \tilde{\sigma}^\vee \text{ for } i = 0, \dots, n.$$

Hence, the linearly independent vectors $u^0 - u^i$ would have to be contained in $\tilde{\sigma}^\vee \cap L = \{0\}$, which is impossible.

Step 2: Since $\tilde{\sigma}^\vee \cap \tilde{M} \twoheadrightarrow \sigma^\vee \cap M$ is surjective, the element $R' \in \sigma^\vee \cap M$ can be lifted to some $r \in \tilde{\sigma}^\vee \cap \tilde{M}$. Then, $p \cdot r$ differs from $u^0, \dots, u^n$ by $\mathbb{L}$-elements only, and we can apply the previous lemma. There has to be an index $i \in \{0, \dots, n\}$ such that $p \cdot r - u^i$ is contained in $\tilde{\sigma}^\vee$. On the other hand, $p \cdot r - u^i$ is obviously contained in the lattice L, and we obtain $p \cdot r - u^i \in \tilde{\sigma}^\vee \cap L = \{0\}$. We may assume that $i = 0$.

Step 3: We will show that there are lattice elements $b^i \in \tilde{N}$ for $i = 0, \dots, n$ such that

$$\begin{aligned} \langle b^i, u^i \rangle &= \begin{cases} p & \text{for } i = 0 \\ 1 & \text{for } i = 1, \dots, n \end{cases} \qquad \text{and} \\ \langle b^i, u^j \rangle &= 0 \quad \text{for } j \neq i. \end{aligned}$$

Since, for a fixed i, $(u^i - u^j)_{j \in \{0,\dots,n\} \setminus \{i\}}$ is a basis of the lattice $L \subseteq \tilde{M}$, it is possible to choose an element $b^i \in \tilde{N}$ such that $\langle b^i, u^i - u^j \rangle = 1$, i.e.

$$\langle b^i, u^i \rangle = \langle b^i, u^j \rangle + 1 \quad (\text{for } j \neq i).$$

$R' \in M$ is primitive, hence, there is a $b \in N = L^\perp \cap \tilde{N}$ such that $\langle b, R' \rangle = 1$. Therefore,

$$\langle b, u^j \rangle = p \quad \text{for } j = 0, \dots, n.$$

Case 1: $i \geq 1$. The equation $\langle b^i, u^i\rangle = \langle b^i, u^j\rangle + 1$ $(j \neq i)$ implies

$$\langle b^i, u^j\rangle + 1 = \langle b^i, u^i\rangle = \langle b^i, u^0\rangle + 1 = \langle b^i, p\cdot \tilde{u}^0\rangle + 1\,,$$

in particular, $p|\langle b^i, u^j\rangle$ for $j \neq i$. Hence, there is a $k \in \mathbb{Z}$ such that the improved $b^i := b^i + k\cdot b$ additionally yields $\langle b^i, u^j\rangle = 0$ $(j\neq i)$.

Case 2: $i = 0$. Here, we have to use the modification $b^0 := p\cdot b^0 + k\cdot b$ (with suitable $k\in\mathbb{Z}$) to obtain the equations $\langle b^0, u^j\rangle = 0$ $(j \geq 1)$. □

(7.3.4) **Lemma:** *Let $u \in \tilde{M}$ be such that, for each index $i = 0,\dots,n$, it can be pushed into $\tilde{\sigma}^\vee$ without using u^i:*

$$u + \sum_{j\neq i} \lambda^i_j \cdot u^j \in \tilde{\sigma}^\vee \quad (\textit{for some } \lambda^i_j \in \mathbb{Z}).$$

Then, u itself is contained in $\tilde{\sigma}^\vee$.

Proof: We will proceed by induction on $\sum_j \lambda^i_j$. To do so we first have to modify the presumption of the lemma slightly: For $u \in \tilde{M}$ suppose that

$$u + \sum_j \lambda^i_j \cdot u^j \in \tilde{\sigma}^\vee \quad \text{with } \lambda^i_j \in \mathbb{Z},\ \lambda^i_i \leq 0, \text{ and } \sum_j \lambda^i_j \text{ is constant in } i.$$

(The latter fact can be obtained by increasing some of the coefficients λ^i_j with $j \neq i$.)
Now, on the one hand, the sum $\sum_j \lambda^i_j$ is bounded below. (Look at the vector space $M_{\mathbb{R}}$: Subtracting R sufficiently often from a given point leads out of the cone $\sigma^\vee$.)
On the other hand, as soon as the elements $u + \sum_j \lambda^i_j \cdot u^j$ are not mutually equal (for different i), we can apply (7.3.2) to all these elements. We obtain $u + \sum_j (\lambda^i_j)' \cdot u^j \in \tilde{\sigma}^\vee$ with $(\lambda^i_j)' = \lambda^i_j$ or $\lambda^i_j - 1$ and $\sum_{j=0}^n (\lambda^i_j)' = \sum_{j=0}^n \lambda^i_j - 1$.
Therefore, only the case that the coefficients λ^i_j do *not* depend on i is left. This implies $\lambda^i_j = \lambda^j_j \leq 0$; hence, with $u + \sum_j \lambda^i_j \cdot u^j$, u has to be contained in $\tilde{\sigma}^\vee$, too. □

Corollary: *We can replace the fact $L \cap \tilde{\sigma}^\vee = \{0\}$ by an even stronger statement:*
If any linear combination $\sum_{i=0}^n \lambda_i \cdot u^i$ (with integer coefficients $\lambda_i \in \mathbb{Z}$) is contained in $\tilde{\sigma}^\vee \cap \tilde{M}$, then each of the coefficients will be non-negative.

Proof: Assume that $\lambda_i < 0$. Then, the element $u := \lambda_i \cdot u^i$ admits the following properties:

$$u + \sum_{j\neq i} \lambda_j \cdot u^j \in \tilde{\sigma}^\vee \quad \text{and} \quad u - \lambda_i \cdot u^i \in \tilde{\sigma}^\vee.$$

In particular, u fulfills the assumption of the previous lemma. We obtain $-u^i \in \tilde{\sigma}^\vee$, but this cannot be true (cf. Step 1 in the proof of Lemma (7.3.3)). □

(7.3.5) Now, we start with the direct construction of the deformation datum that will induce the given toric regular sequence. Let

$$\begin{aligned} (\mathbb{A}, \mathbb{L}) &:= \Big[(u^0)^\perp \cap \ldots \cap (u^n)^\perp,\ (u^0)^\perp \cap \ldots \cap (u^n)^\perp \cap \tilde{N}\Big] \quad \text{and} \\ C &:= (u^0)^\perp \cap \ldots \cap (u^n)^\perp \cap \tilde{\sigma} \subseteq \mathbb{A}. \end{aligned}$$

Fixing some point in each of the sets

$$\begin{aligned} &\{a \in p^{-1}\tilde{N} \mid \langle a, u^0\rangle = 1,\ \langle a, u^j\rangle = 0 \text{ for } j \neq 0\} \quad \text{and} \\ &\{a \in \tilde{N} \mid \langle a, u^i\rangle = 1,\ \langle a, u^j\rangle = 0 \text{ for } j \neq i\} \quad (\text{for } i = 1,\dots,n), \end{aligned}$$

we obtain $n+1$ different affine embeddings $\mathbb{A} \hookrightarrow \tilde{N}_{\mathbb{R}}$ which induce isomorphisms

$$\begin{array}{rcll} \mathbb{A} & \xrightarrow{\phi_i} & \{a \in \tilde{N}_{\mathbb{R}} \mid \langle a, u^i \rangle = 1,\ \langle a, u^j \rangle = 0 \text{ for } j \neq i\} \quad (i \geq 0) & \text{and} \\ \mathbb{L} & \xrightarrow{\phi_i} & \{a \in \tilde{N} \mid \langle a, u^i \rangle = 1,\ \langle a, u^j \rangle = 0 \text{ for } j \neq i\} \quad (\text{only for } i \geq 1 \text{ or } p = 1). & \end{array}$$

Hence, with

$$R_i := \phi_i^{-1}\left(\tilde{\sigma} \cap \{a \in \tilde{N}_{\mathbb{R}} \mid \langle a, u^i \rangle = 1,\ \langle a, u^j \rangle = 0 \text{ for } j \neq i\}\right)$$

we have got $n+1$ polyhedra $R_0, \ldots, R_n \subseteq \mathbb{A}$ admitting C as their cone of unbounded directions (i.e. $R_i = C + \text{compact set}$).

Lemma:

(1) The polyhedra $R_0, \ldots, R_n$ are not empty.

(2) $\tilde{\sigma} = C \cup \mathbb{R}_{\geq 0} \cdot \operatorname{conv}\left(\bigcup_{i=0}^n \phi_i(R_i)\right)$.

Proof: (1) We define some auxiliary cones in $\tilde{N}_{\mathbb{R}}$:

$$\begin{array}{rcl} C_i & := & (u^0)^{\perp} \cap \ldots \cap \left(\hat{u^i}\right)^{\perp} \cap \ldots \cap (u^n)^{\perp} \cap \tilde{\sigma} \\ & = & C \cup \mathbb{R}_{\geq 0} \cdot \phi_i(R_i) \quad (\text{for } i = 0, \ldots, n). \end{array}$$

Then, the claim $R_i \neq \emptyset$ is equivalent to the fact that the cone C is properly contained in C_i. We will switch to the dual level showing $C_i^{\vee} \subset C^{\vee}$:
The dual cone $C^{\vee} \subseteq \tilde{M}_{\mathbb{R}}$ can be obtained as the pull back of $C^{\vee}$ regarded as a subset of $\tilde{M}_{\mathbb{R}}/\operatorname{span}_{\mathbb{R}}(u^0, \ldots, u^n)$. Moreover, the latter one equals the image of $\tilde{\sigma}^{\vee}$ via the canonical projection. Looking at $C_i^{\vee}$ in the same way (take $\operatorname{span}_{\mathbb{R}}(u^0, \ldots, \hat{u^i}, \ldots, u^n)$ instead of $\operatorname{span}_{\mathbb{R}}(u^0, \ldots, u^n)$), we obtain

$$C_i^{\vee} + \mathbb{R} \cdot u^i = C^{\vee}.$$

Hence, it suffices to check that $-u^i \notin C_i^{\vee}$:
The opposite would mean that there is an $r \in \tilde{\sigma}^{\vee}$ such that r and $-u^i$ differ by an element of $\operatorname{span}_{\mathbb{R}}(u^0, \ldots, \hat{u^i}, \ldots, u^n)$ only, i.e.

$$r + u^i = \sum_{j \neq i} \lambda_j \cdot u^j \quad \text{in } \tilde{M}_{\mathbb{R}}.$$

Enlarging the coefficients λ_j, we may assume that they are integers, but then the fact $-u^i + \sum_{j \neq i} \lambda_j \cdot u^j \in \tilde{\sigma}^{\vee}$ contradicts the previous corollary.

(2) The second part of the lemma is equivalent to the fact $\tilde{\sigma} = \sum_{i=0}^n C_i$ (or to $\tilde{\sigma}^{\vee} = \bigcap_{i=0}^n C_i^{\vee}$). However, since

$$C_i^{\vee} = \tilde{\sigma}^{\vee} + \sum_{j \neq i} \mathbb{R} \cdot u^j \quad (\text{obtained in the first part of the proof}),$$

this was already shown in Lemma (7.3.4). □

(7.3.6) We finish the proof of Theorem (7.2.5) by showing that the deformation datum just constructed is admissible:

Step 1: Let $t \in \mathbb{L}^* \cap C^{\vee}$, and we assume (without loss of generality) that it has no integers as values on the faces $F(R_0, t)$ and $F(R_1, t)$ (or only on $F(R_1, t)$ in case of $p \geq 2$).

Let $b^0, \ldots, b^n \in \tilde{N}_{\mathbb{R}}$ be points of the faces $F(R_0,t), \ldots, F(R_n,t)$ in the embedded polyhedra $\phi_0(R_0), \ldots, \phi_n(R_n)$. Then, $t \in \mathbb{L}^*$ can be lifted to $T \in \tilde{M}$ such that

$$\langle b^0, T\rangle ; \langle b^1, T\rangle \notin \mathbb{Z} \quad \text{(even in case } p \geq 2).$$

(For $i \geq 1$ or $p = 1$, the value of t on $F(R_i,t)$ equals $\langle b^i, T\rangle$ in $\mathbb{Q}/\mathbb{Z}$. In case $i = 0, p \geq 2$, the value of $\langle b^0, T\rangle$ is determined up to $p^{-1}\mathbb{Z}$ only. In particular, we can choose T in such a way that this value is not an integer.)

Step 2: Denote by $q_0, q_1 \in \mathbb{Q}$ the numbers $q_i := \langle b^i, T\rangle$. Then, there is an integer $k \geq 1$ such that $[kq_0 + kq_1] = [kq_0] + [kq_1] + 1$. ($[\ldots]$ denotes the integral part, and k can always be obtained as one of the two possibilities $k = 1$ or $k =$ (common denominator of q_0, q_1) $- 1$.) Therefore, we obtain

$$[\langle b^0, T\rangle + \langle b^1, T\rangle] = [\langle b^0, T\rangle] + [\langle b^1, T\rangle] + 1\,,$$

if t and T are replaced by kt and kT, respectively. (This operation does not change the faces $F(R_i, t)$ of R_i.)

Step 3: Now, we modify our T by adding elements of $\operatorname{span}_{\mathbb{Z}}(u^0, \ldots, u^n)$; it remains a lifting of t, and the values of $\langle b^i, T\rangle$ do not change in $\mathbb{Q}/\mathbb{Z}$. It is possible to obtain the following situation:

$$\langle b^0, T\rangle \in [-1, 0) \quad \text{and} \quad \langle b^i, T\rangle \in [0, 1) \quad (\text{for } i = 1, \ldots, n),$$

i.e. T is not contained in the cone $\tilde{\sigma}^\vee$ and even cannot be put into $\tilde{\sigma}^\vee$ by adding vectors of the sublattice $L \subseteq \tilde{M}$ only.
On the other hand, the result of the second step yields that

$$\langle \sum_{i=0}^{n} b^i, T\rangle \geq 0.$$

This means that T is non-negative on the cone $\sigma = C \cup \mathbb{R}_{\geq 0} \cdot (\phi_0(R_0) + \ldots \phi_n(R_n))$. Hence, we have found an element of $\sigma^\vee \cap M$ that cannot be lifted to $\tilde{\sigma}^\vee \cap \tilde{M}$. This is a contradiction to the fact that $\tilde{\sigma}$ and $u^0, \ldots, u^n$ define a toric regular sequence.

7.4 The Kodaira-Spencer map

(7.4.1) Let $(R_0, \ldots, R_n;\ C;\ p)$ be an admissible deformation datum ($R_i, C \subseteq \mathbb{A}$). Then, in (7.2.4) we have defined cones $\sigma \subseteq \tilde{\sigma} \subseteq \tilde{N}_{\mathbb{R}}$ and elements $u^0, \ldots, u^n \in \tilde{\sigma}^\vee \cap \tilde{M}$ such that the map

$$\operatorname{Spec} \mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}] \xrightarrow{g} \mathbb{C}^n \qquad \text{(defined by the regular functions } x^{u^1} - x^{u^0}, \ldots, x^{u^n} - x^{u^0} \in \mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}])$$

yields a deformation of the special fiber $Y = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. Now, we will compute the Kodaira-Spencer map

$$\varrho : \mathbb{C}^n = T_0\, \mathbb{C}^n \longrightarrow T^1_Y$$

associated to this deformation.

(7.4.2) The cone σ was defined as the cone over the polyhedron $Q = R_0 + \ldots + R_n$ embedded into the affine hyperplane $\{(\bullet, p^{-1})\} \subseteq \mathbb{A} \times \mathbb{R} = N_{\mathbb{R}}$. Hence, the elements of E can be written as

$$r^v = [c^v, \eta^v] \quad \text{with} \quad c^v \in \mathbb{L}^* \cap C^\vee,\ \eta^v \in \mathbb{Z}, \quad \text{and} \quad \langle a, -p\, c^v\rangle \leq \eta^v \text{ for } a \in Q \quad (\text{since } \langle (a, p^{-1}), [c^v, \eta^v]\rangle \geq 0).$$

Let $\{\tilde{r}^0,\dots,\tilde{r}^w\}$ be an arbitrary lift of E to $\tilde{\sigma}^\vee\cap\tilde{M}\subseteq\tilde{M}=L\!\!L^*\times Z\!\!\!Z^{n+1}$, i.e.

$$\tilde{r}^v=[c^v;\eta_0^v,\dots,\eta_n^v]\quad\text{with}\quad \eta_i^v\in Z\!\!\!Z,\ \eta_0^v+p\,\eta_1^v+\ldots+p\,\eta_n^v=\eta^v\quad\text{and}$$
$$\langle a,-p\,c^v\rangle\le\begin{cases}\eta_0^v & \text{for } a\in R_0\\ p\,\eta_i^v & \text{for } a\in R_i\ (i\ge 1).\end{cases}$$

Remark: Those integers η_i^v exist, because the given deformation datum is admissible. In n out of the $n+1$ possibilities $i=0,\dots,n$ the linear form c^v has integer value on $F(R_i,c^v)$, and we can take this value for $-\eta_i^v$.

Theorem:

(i) The Kodaira-Spencer map sends the whole space $C\!\!\!\!I^n$ into the homogeneous summand $T^1_Y(-R)$. (The lattice point $R\in\sigma^\vee\cap M$ was defined as either the common image $[0,p]$ of $u^0,\dots,u^n$, or by the fact that Q is contained in the affine hyperplane $[R=1]$ of $N_{I\!\!R}$.)

(ii) Using the T^1_Y-formula of (3.2.4), the Kodaira-Spencer map equals

$$\varrho: C\!\!\!\!I^n\longrightarrow\left(L(\cup_{j=1}^M E_j^R)\Big/\sum_{j=1}^M L(E_j^R)\right)^*\otimes_{I\!\!R}C\!\!\!\!I$$

induced by the bilinear map $I\!\!R^n\times L(E)\to I\!\!R$ that is described by the matrix

$$\begin{pmatrix}\eta_1^0 & \dots & \eta_1^w\\ \vdots & & \vdots\\ \eta_n^0 & \dots & \eta_n^w\end{pmatrix}.$$

Remark: The previous theorem justifies the notions "degree" or "homogeneous" built up in (7.2.1). On the other hand, it shows the limits of the concept of toric deformations; only "strictly negative" degrees (i.e. $-R$ with $R\in\sigma^\vee\cap M$) occur in the image of the Kodaira-Spencer map.

(7.4.3) **Proof:** *Step 1: We show that the maps $\varrho(e^i):L(E)\to I\!\!R$ $\quad(q=(q_v)_{v=0,\dots,w}\mapsto\sum_{v=0}^w\eta_i^v\,q_v)$ are trivial on each $L(E_j^R)$.*
First, we notice that

$$E_j^R\ne\emptyset\Leftrightarrow a^j\notin C\Leftrightarrow\langle a^j,R\rangle>0\Leftrightarrow a^j\text{ corresponds to a vertex }\bar{a}^j\text{ of }Q.$$

Let $a^j=\langle a^j,R\rangle\cdot(\bar{a}^j,p^{-1})$ be one of the fundamental generators of σ that meets these conditions. Then, $\bar{a}^j$ splits into a sum $\bar{a}^j=(\bar{a}^j)_0+\ldots+(\bar{a}^j)_n$, and $(\bar{a}^j)_i\in R_i$ are vertices defined all by the same hyperplane $t\in L\!\!L^*\cap C^\vee$ as $\bar{a}^j\in Q$. Since our deformation datum is admissible, n of these vertices (say $(\bar{a}^j)_1,\dots,(\bar{a}^j)_n$) have to be contained in the lattice $L\!\!L$.
For an element $q\in L(E)$ the property "$q\in L(E_j^R)$" means that the components q_v are allowed to be non-trivial at most for $\langle(\bar{a}^j,p^{-1}),[c^v,\eta^v]\rangle<\langle(\bar{a}^j,p^{-1}),R\rangle$. Since $R=[0,p]$, this condition is equivalent to

$$\langle\bar{a}^j,-p\,c^v\rangle>\eta^v-p.$$

Restricting ourselves to those indices v, we obtain

$$\begin{aligned}\eta^v-p &< \langle\bar{a}^j,-p\,c^v\rangle=\langle(\bar{a}^j)_0,-p\,c^v\rangle+\langle(\bar{a}^j)_1,-p\,c^v\rangle+\ldots+\langle(\bar{a}^j)_n,-p\,c^v\rangle\\ &\le \eta_0^v+p\,\eta_1^v+\ldots+p\,\eta_n^v=\eta^v.\end{aligned}$$

The numbers $\langle(\bar{a}^j)_i,-p\,c^v\rangle$ $(i\ge 1)$ are contained in $pZ\!\!\!Z$; hence

$$\eta_i^v=\langle(\bar{a}^j)_i,-c^v\rangle\quad(\text{for } i=1,\dots,n)\quad\text{and}\quad\eta_0^v=\eta^v+\sum_{i=1}^n\langle(\bar{a}^j)_i,p\,c^v\rangle.$$

Therefore, if $q \in L(E_j^R)$, the equation $\sum_v q_v \cdot [c^v, \eta^v] = 0$ implies $\sum_v q_v\, \eta_i^v = 0$ for $i = 0, \ldots, n$.

Step 2: To compute the image of the i-th canonical unit vector $e^i \in \mathbb{C}^n$ via the Kodaira-Spencer map, we may assume without loss of generality that $i = 1$ and consider the ring

$$\tilde{A} := \mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}] \Big/ \Big((x^{u^1} - x^{u^0})^2, x^{u^2} - x^{u^0}, \ldots, x^{u^n} - x^{u^0}\Big).$$

Via $\varepsilon \mapsto (x^{u^1} - x^{u^0})$ this is a flat $\mathbb{C}[\varepsilon]/\varepsilon^2$-algebra and, moreover, $A := \tilde{A} \otimes \mathbb{C}[\varepsilon]/\varepsilon$ equals the ring $\mathbb{C}[\sigma^\vee \cap M]$. Spec $\tilde{A}$ is the infinitesimal deformation of $Y = \text{Spec}\, A$ obtained by restricting our toric deformation to the tangent vector $e^1 \in \mathbb{C}^n = T_0\, \mathbb{C}^n$, and the corresponding class in T^1_Y is induced from the A-linear map $\psi : I/I^2 \to A$ described by

$$\psi : P(z_0, \ldots, z_w) \mapsto \left[\frac{P(x^{\tilde{r}^0}, \ldots, x^{\tilde{r}^w})}{x^{u^1} - x^{u^0}} \text{ (computed in } \tilde{A})\right] \in A.$$

We compute ψ for $P(\underline{z}) = \underline{z}^a - \underline{z}^b$ (as already mentioned in (3.2.6), those binomials with $(a, b) \in m \subseteq \mathbb{N}^{w+1} \times \mathbb{N}^{w+1}$ generate the ideal I). Inside the semigroup $\tilde{\sigma}^\vee \cap \tilde{M}$ it is possible to write

$$\sum_{v=0}^{w} (a_v - b_v) \cdot \tilde{r}^v = \sum_{i=0}^{n} g_i \cdot u^i \quad (g_i \in \mathbb{Z},\ \sum_i g_i = 0).$$

Claim: $\psi(\underline{z}^a - \underline{z}^b) = g_1 \cdot x^{(\sum a_v r^v) - R} \in A.$

Proof: Similar to (7.3.1.3) we know that

$$s := \sum_{v=0}^{w} a_v \cdot \tilde{r}^v - \sum_{i=0}^{n} g_i^+ \cdot u^i = \sum_{v=0}^{w} b_v \cdot \tilde{r}^v - \sum_{i=0}^{n} g_i^- \cdot u^i$$

is contained in $\tilde{\sigma}^\vee \cap \tilde{M}$. Hence,

$$\begin{aligned} P(x^{\tilde{r}^0}, \ldots, x^{\tilde{r}^w}) &= x^{\sum a_v \tilde{r}^v} - x^{\sum b_v \tilde{r}^v} = x^s \cdot \left(x^{\sum g_i^+ u^i} - x^{\sum g_i^- u^i}\right) \\ &= x^s \cdot \left(\prod_{i=0}^{n} (x^{u^i})^{g_i^+} - \prod_{i=0}^{n} (x^{u^i})^{g_i^-}\right) \quad (\text{in the ring } \mathbb{C}[\tilde{\sigma}^\vee \cap \tilde{M}]). \end{aligned}$$

Assuming (w.l.o.g.) $g_1 \geq 0$, we obtain in the ring $\tilde{A}$:

$$\begin{aligned} P(x^{\tilde{r}^0}, \ldots, x^{\tilde{r}^w}) &= x^s \cdot \left(x^{u^0}\right)^{(\sum g_i^+) - g_1^+} \cdot \left((x^{u^1})^{g_1^+} - (x^{u^0})^{g_1^+}\right) \\ &= \left[x^{u^1} - x^{u^0}\right] \cdot x^s \cdot \left(x^{u^0}\right)^{(\sum g_i^+) - g_1^+} \cdot \left(\textstyle\sum_{\mu=1}^{g_1^+} (x^{u^1})^{g_1^+ - \mu} (x^{u^0})^{\mu - 1}\right). \end{aligned}$$

In particular,

$$\psi(\underline{z}^a - \underline{z}^b) = x^{\bar{s}} \cdot \left(x^R\right)^{(\sum g_i^+) - g_1^+} \cdot \left(g_1 \cdot (x^R)^{g_1^+ - 1}\right) = g_1 \cdot x^{(\sum a_v r^v) - R} \in A.$$

Step 3: To finish the proof, we have to involve the η's. Since

$$\begin{aligned} \sum_{v=0}^{w} (a_v - b_v)\, \tilde{r}^v &= \sum_{v=0}^{w} (a_v - b_v) \cdot [c^v; \eta_0^v, \eta_1^v, \ldots, \eta_n^v] \\ &= \left(0; \sum_{v=0}^{w} \eta_0^v\, (a_v - b_v), \sum_{v=0}^{w} \eta_1^v\, (a_v - b_v), \ldots, \sum_{v=0}^{w} \eta_n^v\, (a_v - b_v)\right) \\ &= p^{-1} \left(\sum_{v=0}^{w} \eta_0^v\, (a_v - b_v)\right) \cdot u^0 + \sum_{i=1}^{n} \left(\sum_{v=0}^{w} \eta_i^v\, (a_v - b_v)\right) \cdot u^i\,, \end{aligned}$$

we obtain $g_1 = \sum_v \eta_1^v\,(a_v - b_v)$. Hence,

$$\psi(\underline{z}^a - \underline{z}^b) = \sum_v \eta_1^v\,(a_v - b_v)\cdot x^{(\sum a_v r^v)-R} = \varrho(e^1)(a-b)\cdot x^{(\sum a_v r^v)-R},$$

and we are done by Theorem (3.2.6). □

(7.4.4) Finally, we want to present the Kodaira-Spencer map using the formula

$$T_Y^1(-R) = \ker\left[V_{\mathbb{C}}(Q)\oplus W_{\mathbb{C}}(Q) \longrightarrow\!\!\!\!\rightarrow (\sum_{d^{jk}} U_{jk})^*\right] \Big/ \mathbb{C}\cdot(\underline{1},\underline{1})$$

from (3.4.3). Already knowing the result in the Gorenstein case (cf. (5.4.2)) or even for arbitrary Y that are smooth in codimension two, it will be no surprise that ϱ yields the Minkowski summands $R_i \in V(Q)/\mathbb{R}\cdot\underline{1}$ as images of the standard vectors $e^i \in \mathbb{C}^n$. However, it might be interesting to see what the $W(Q)$-part looks like.

Theorem: *The Kodaira-Spencer map*

$$\varrho : \mathbb{C}^n \longrightarrow T_Y^1(-R) \subseteq V_{\mathbb{C}}(Q)\oplus W_{\mathbb{C}}(Q)/_{\mathbb{C}\cdot(\underline{1},\underline{1})}$$

sends e^i onto the pair $[R_i,\,\underline{s}^i] \in V(Q)\oplus W(Q)$ $(i=1,\dots,n)$ with

$$s_j^i = \begin{cases} 0 & \text{if the vertex } (\bar a^j)_i \text{ of } R_i \text{ belongs to the lattice} \\ 1 & \text{if } (\bar a^j)_i \text{ is not a lattice point} \end{cases} \qquad (\text{if } p=1),$$

or $\underline{s}^i := \underline{0}$, if $p \geq 2$.

Remark: Defining $e^0 := -(e^1+\ldots+e^n)$, we obtain $\varrho(e^0) = [R_0,\,\underline{s}^0]$ with $\underline{s}^0$ similar to $\underline{s}^i$ in the theorem (for $p=1$), but $\underline{s}^0 = \underline{1}$ for $p \geq 2$.

Proof: Fixing some $i \in \{1,\dots,n\}$, Theorem (7.4.2) tells that $\varrho(e^i)$ maps a relation $q \in L(E_0^R)$ onto $\sum_v q_v\,\eta_i^v$. As usual (cf. (3.4.1)), we can translate this fact into

$$\begin{aligned} \varrho(e^i):\ (v^j) \in \oplus_j \operatorname{span} E_j^R \ (\text{with } \textstyle\sum_j v^j = 0) \ &\mapsto\ (q^j) \in \oplus_j \mathbb{Z}^{E_j^R}\ (\text{with } \textstyle\sum_v q_v^j r^v = -v^j) \\ &\mapsto\ \textstyle\sum_j (\sum_v q_v^j\,\eta_i^v) \end{aligned}$$

meaning that $\varrho(e^i)$ is given by the linear maps $b^j : \operatorname{span} E_j^R \to \mathbb{Z}$ sending v^j onto $\sum_v q_v^j\,\eta_i^v$.
On the other hand, by the first step of the proof in (7.4.3), we know that the condition $r^v = [c^v, \eta^v] \in E_j^R$ implies

$$\eta_i^v = \begin{cases} -\langle (\bar a^j)_i, c^v\rangle & \text{if } p \geq 2 \text{ or } (\bar a^j)_i \in \mathbb{L}\ (p=1) \\ \langle \bar a^j, c^v\rangle - \langle (\bar a^j)_i, c^v\rangle + \eta^v & \text{if } p=1 \text{ and } (\bar a^j)_i \notin \mathbb{L}. \end{cases}$$

In the first case, we obtain

$$b^j(v^j) = -\sum_v q_v^j\,\langle (\bar a^j)_i, c^v\rangle = \langle (\bar a^j)_i, -\sum_v q_v^j c^v\rangle = \langle ((\bar a^j)_i, 0),\, v^j\rangle,$$

i.e. $b^j = ((\bar a^j)_i, 0) \in N_{\mathbb{R}}$. Hence, in the language of (3.4.4), we have $s_j = -\langle b^j, R\rangle = 0$, and the elements $\bar b^j = b^j$ form the vertices of the Minkowski summand R_i.
Similarly, we obtain in the second case $(p=1,\ (\bar a^j)_i \notin \mathbb{L})$

$$b^j = -(\bar a^j, 0) + ((\bar a^j)_i, 0) - (\underline{0}, 1).$$

Hence, $s_j = -\langle b^j, R\rangle = 1$ and $\bar b^j = b^j + (\bar a^j, 1) = ((\bar a^j)_i, 0)$. □

7.5 Examples

(7.5.1) See the examples (5.6.4) and (5.7.3).

(7.5.2) The cone over the rational normal curve of degree four (cf. Example (1.1.3)) is a two-dimensional cyclic quotient singularity. These singularities are toric, and this special one is given by the cone $\sigma := \langle(1,0);(-1,4)\rangle \subseteq \mathbb{R}^2$.
The dual cone equals $\sigma^\vee = \langle[0,1],[4,1]\rangle$, T^1 is four-dimensional, and the homogeneous pieces $T^1(-R)$ are non-trivial only for $R = [1,1],[2,1],[3,1]$.
Now, we cut σ with the affine hyperplanes corresponding to these values:

(i) $\mathbb{A} := [\langle\bullet,[1,1]\rangle = 1]$; $\mathbb{L} := \mathbb{A} \cap \mathbb{Z}^2$ (in particular $p = 1$).
To get an isomorphism $(\mathbb{A},\mathbb{L}) \cong (\mathbb{R},\mathbb{Z})$ we have to choose and fix an origin (contained in $\mathbb{L}$) and a $\mathbb{Z}$-basis of $\mathbb{L}$:

$$(\mathbb{A},\mathbb{L}) = [(0,1) + \mathbb{R}\cdot(-1,1),\ (0,1) + \mathbb{Z}\cdot(-1,1)]\,.$$

In particular, $Q := \sigma \cap \mathbb{A} = [(1,0),(-1/3,\,4/3)]$ corresponds to $[-1,\,1/3] \subseteq \mathbb{R}$, and this interval admits only one admissible decomposition into a Minkowski sum:

$$[-1,\,1/3] = [-1,0] + [0,\,1/3]\,.$$

(ii) Analogously, we consider $\mathbb{A} := [\langle\bullet,[3,1]\rangle = 1] = (0,1) + \mathbb{R}\cdot(-1,3) \quad (p=1)$.
Then, $Q = [-1/3,1]$ splits into $[-1/3,1] = [-1/3,0] + [0,1]$.

(iii) Let $\mathbb{A} := [\langle\bullet,[2,1]\rangle = 1] = (0,1) + \mathbb{R}\cdot(-1,2) \quad (p=1)$.
Then, $Q = [-1/2,\,1/2]$ admits two different decompositions:

$$[-1/2,\,1/2] = [-1/2,0] + [0,\,1/2] = \{1/2\} + [-1,0]\,.$$

The decompositions (i), (ii) and the first one of (iii) provide the (three-dimensional) Artin component in the versal deformation of our singularity; the remaining decomposition of (iii) yields the other (one-dimensional) component. The corresponding families equal $Y_t \to \mathbb{C}^3$ and $Y_s \to \mathbb{C}$ from (1.1.3), respectively.

(7.5.3) Finally, we want to determine those two-dimensional cyclic quotient singularities that correspond to rational intervals of length one in $\mathbb{R}$. (Considering admissible Minkowski decompositions, those intervals seem to be very interesting; they admit the funny decomposition into $[0,1]$ and an interval of length zero.)
For a given parameter $p \in \mathbb{N}$, each positive rational number can be written as a quotient $x/(pd)$ ($x,d \in \mathbb{N}$ with $\gcd(x,d) = 1$). Then, the interval $Q = [x/(pd),\, x/(pd)+1]$ yields the cone $\sigma = \langle(x/(pd),\,1/p),\,(x/(pd)+1,\,1/p)\rangle = \langle(x,d),\,(x+pd,d)\rangle$.

(i) $\begin{vmatrix} x+pd & d \\ x & d \end{vmatrix} = xd + pd^2 - xd = pd^2.$
This is the order of the cyclic group acting on $\mathbb{C}^2$.

(ii) We will regard x as an (invertible) element of $\mathbb{Z}/d\mathbb{Z}$. Then, we obtain

$$(pdx^{-1} - 1)\cdot(x+pd,d) + (x,d) = (p^2d^2x^{-1},\, pd^2x^{-1}),$$

i.e. both components are divisible by pd^2.

Hence, Q corresponds to the cyclic quotient singularity $X(pd^2, pdx^{-1} - 1)$ which equals A_{p-1} for $d = 1$ and which is called a T-singularity for $d \geq 2$. Those singularities form the fundamental bricks for building P-resolutions (cf. §3 of [KoSh]).
The canonical decomposition $[x/(pd),\, x/(pd) + 1] = \{x/(pd)\} + [0,1]$ corresponds to the one-parameter deformation presenting $X(pd^2, pdx^{-1} - 1)$ as a hypersurface in a three-dimensional cyclic quotient singularity (of type $(d; x, d-x)$).

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