# Computation of the vector space $T^1$ for affine toric varieties

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**Abstract**

We compute the vector space $T^1$ of first order infinitesimal deformations for affine toric varieties $X_\sigma$. The result is given by two different formulas which use combinatorial data of the given cone $\sigma$ only.
One of these formulas suggests that there is a relation between deformations of affine toric varieties on the one hand and splitting of certain polyhedra into a Minkowski sum on the other hand.

As an application, the result is used to investigate when three-dimensional affine toric varieties are rigid.

## Contents



# 1 Introduction

**(1.1)** Let X be an algebraic variety (affine, complete or a germ) over $\mathbb{C}$. Then, the corresponding deformation functor is defined as

$$\mathrm{Def}_X((T,0)) := \{\tilde{X} \to T \text{ (flat) together with an isomorphism between the special fibre and } X\} \quad \text{(modulo equivalence)}.$$

If $X$ admits only isolated singularities (i.e. $X$ is a smooth manifold except in finitely many points), there exists the semiuniversal deformation $[\mathcal{X} \to S] \in \mathrm{Def}_X(S)$ that is characterized by the following property:

> Given an arbitrary deformation $[\tilde{X} \to T] \in \mathrm{Def}_X(T)$, there exists a map $T \to S$ (uniquely determined on the level of tangent spaces) such that this deformation is induced by the semiuniversal one. That means, up to equivalence, we have a fibre product diagramme $\begin{array}{ccc} \tilde{X} & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ T & \to & S \end{array}$ compatible with the isomorphisms between the special fibres and $X$.

Now, the idea is to obtain information about $X$ by describing the base space $S$ of its semiuniversal deformation. Numbers such as the dimension, the embedding dimension or the number of irreducible components of $S$ at 0 provide interesting numerical invariants of $X$.

**(1.2)** Let us denote the ring $\mathbb{C}[t]/_{t^2}$ by the symbol $\mathbb{C}[\varepsilon]$ (i.e. the relation $\varepsilon^2 = 0$ is included). Then, the non-reduced space Spec $\mathbb{C}[\varepsilon]$ is a fat point containing not only information about the point but also about the tangent direction starting from this point. Therefore, the tangent-space $T_{S,0}$ of $S$ in 0 can be computed as

$$T_{S,0} = \{\mathrm{Spec}\ \mathbb{C}[\varepsilon] \to (S,0)\} = \mathrm{Def}_X(\mathrm{Spec}\ \mathbb{C}[\varepsilon]).$$

This vector space is an important invariant of the space $X$. It is denoted by $T^1_X$, and its dimension is equal to the embedding dimension of $S$.
For defining $T^1_X := \mathrm{Def}_X(\mathrm{Spec}\ \mathbb{C}[\varepsilon])$ it is not necessary that $X$ admits only isolated singularities, and we will drop this assumption from now on. (But then, $T^1_X$ may be infinite-dimensional.)

**(1.3)** There are two extremal cases for $X$:
If $X$ is a *"local variety"* (i.e. a germ or an affine variety), then the whole interest is concentrated on the singular points of $X$ ($T^1_X = 0$ for smooth $X$).
One may compute $T^1_X$ using the well-developed machinery of cotangential complexes of $X$ which provides $T^1_X$ as a certain cohomology group (cf. André, Quillen,[LSch] or [KPR], ch. 5).

In the *global case*, even the case of a *smooth varity* $X$ is not without interest: Deformations of $X$ are always locally trivial, and computing $T^1_X$ means to study how these local pieces are patched together. The result is $T^1_X = H^1(X, \Theta_X)$ ($\Theta_X$ denotes the tangent sheaf on $X$).

In general, both extremal cases are combined by using the corresponding local-global spectral sequence for $T^1_X$.

**(1.4)** In this paper, we investigate the case of an affine variety $X$. Let us mention three "down to earth"-methods to compute $T^1_X$.
We always use the following notations:

- $X = \text{Spec } A$, i.e. $A = \Gamma(X, \mathcal{O}_X)$ is the ring of regular functions;
- $X \hookrightarrow \mathbb{C}^n$ is given by the corresponding surjection $\mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow A$;
- $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ denotes the ideal of the embedding $X \hookrightarrow \mathbb{C}^n$.

**(1.4.1)** There are exact sequences of $\mathcal{O}_X$-sheaves

$$0 \to \Theta_X \to \Theta_{\mathbb{C}^n} \otimes \mathcal{O}_X \to \mathcal{N}_{X|\mathbb{C}^n} \to T^1_X \to 0,$$

or of $A$-modules

$$\begin{array}{ccccccc} 0 \to & \text{Der}_{\mathbb{C}}(A,A) & \longrightarrow & A^n & \longrightarrow & \text{Hom}_A({}^I/_{I^2}, A) & \to T^1_X \to 0 \\ & D & \mapsto & (D(x_1),\ldots,D(x_n)) & & & \\ & & & e_i & \mapsto & [f \in I \mapsto \frac{\partial f}{\partial x_i}] & \end{array}$$

, respectively.

($\Theta_X = \text{Der}_{\mathbb{C}}(A,A)$ denotes the tangent sheaf, i.e. the derivations from $A$ to $A$;
$\Theta_{\mathbb{C}^n} = \bigoplus_{i=1}^{n} \mathbb{C}[x_1, \ldots, x_n] \cdot \frac{\partial}{\partial x_i} \cong \mathbb{C}[x_1, \ldots, x_n]^n$;
$\mathcal{N}_{X|\mathbb{C}^n} = \text{Hom}_A({}^I/_{I^2}, A)$ denotes the normal sheaf of $X \hookrightarrow \mathbb{C}^n$.)
(cf. [KPR], ch. 5)

**Example:** If $X = [f = 0]$ is a hypersurface, then $\text{Hom}_A({}^f/_{f^2}, A) \cong A \ \ (\varphi \mapsto \varphi(f))$, and $T^1_X$ can be given as

$$T^1_X = A \Big/ \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) = \mathbb{C}[x_1, \ldots, x_n] \Big/ \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

**(1.4.2)** Let $A$ be a normal ring. Then, $T^1_X$ is computable as $T^1_X = \text{Ext}^1_A(\Omega^1_{A|\mathbb{C}}, A)$ ($\Omega^1_{A|\mathbb{C}}$ = sheaf of Kähler differentials, i.e. $\Theta_X = \text{Hom}(\Omega^1_{A|\mathbb{C}}, A)$).
(cf. [KPR], ch. 5)

**(1.4.3)** Let $X = \text{Spec } A$ be a normal surface singularity with the only singular point $0 \in X$. Then, $T^1_X$ is the kernel of the map

$$H^1(X \setminus 0, \Theta_{X \setminus 0}) \to H^1(X \setminus 0, \Theta_{\mathbb{C}^n}|_{X \setminus 0}).$$

(cf. [Sch], §1)

**Remark:** The embedding $T^1_X \subseteq H^1(X\backslash 0, \Theta_{X\backslash 0})$ corresponds to the restriction map $\text{Def}_X \to \text{Def}_{X\backslash 0}$.

**(1.5)** The aim of this paper is to compute $T^1_X$ for affine toric varieties by using the method (1.4.2). The result is given in the Theorems (2.3) and (4.4).
(In [LS] a very abstract theory that deals with cohomology groups of semigroup algebras was developed. Nevertheless, direct formulas for computing $T^1_X$ were obtained in the cyclic quotient case ($\dim X = 2$) only.)

**Acknowledgements:** I should like to thank Duco van Straten who pointed out to me a mistake resulting from using the "wrong differentials".
I am also grateful to Bernd Sturmfels and the MSI for inviting me to Cornell. During very interesting and stimulating discussions, the idea of the relation between deformations of toric varieties and the decompositions of certain polyhedra into Minkowski sums was formed.
Finally, I wish to thank the referee for many useful comments and Ms. Dobers and Ms. Kerger for typing (with LaTeX) and correcting the manuscript .

## 2 The first $T^1$-formula for affine toric varieties

**(2.1)** For the notion of toric varieties cf. [Ke], [Oda] or [Da]. We only want to fix the following notations:
Let $\sigma \subseteq \mathbb{R}^n$ be a rational, polyhedral cone with vertex in 0;
let $\check{\sigma} := \{r \in \mathbb{R}^n |\; \langle a, r\rangle \geq 0 \text{ for all } a \in \sigma\}$ be the dual cone.
The corresponding lattices are denoted by $N \subseteq \mathbb{R}^n$ and $M \subseteq \mathbb{R}^n$, respectively.
Then, the affine toric variety $X_\sigma$ is defined as

$$X_\sigma := \text{Spec } \mathbb{C}[\check{\sigma} \cap M].$$

**Remark:** As generally known, the affine toric varieties are exactly those affine, normal varieties admitting a $(\mathbb{C}^*)^n$-action with an open, dense orbit.

**Example:** The two-dimensional cyclic quotient singularities are exactly the two-dimensional affine toric varieties. If $X(n, q)$ denotes the quotient by the $\mathbb{Z}/_{n\mathbb{Z}}$-action $\xi \mapsto \begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix}$, then $X(n, q)$ is given by the cone $\sigma = \langle (1, 0); (-q, n)\rangle$.

We will use these singularities as an example to illustrate how the general formula for $T^1$ works.

**(2.2)** Let $E$ be the minimal set that generates the semigroup $\check{\sigma} \cap M$. That means, $E$ is finite and consists exactly of the irreducible elements of $\check{\sigma}\cap M$:

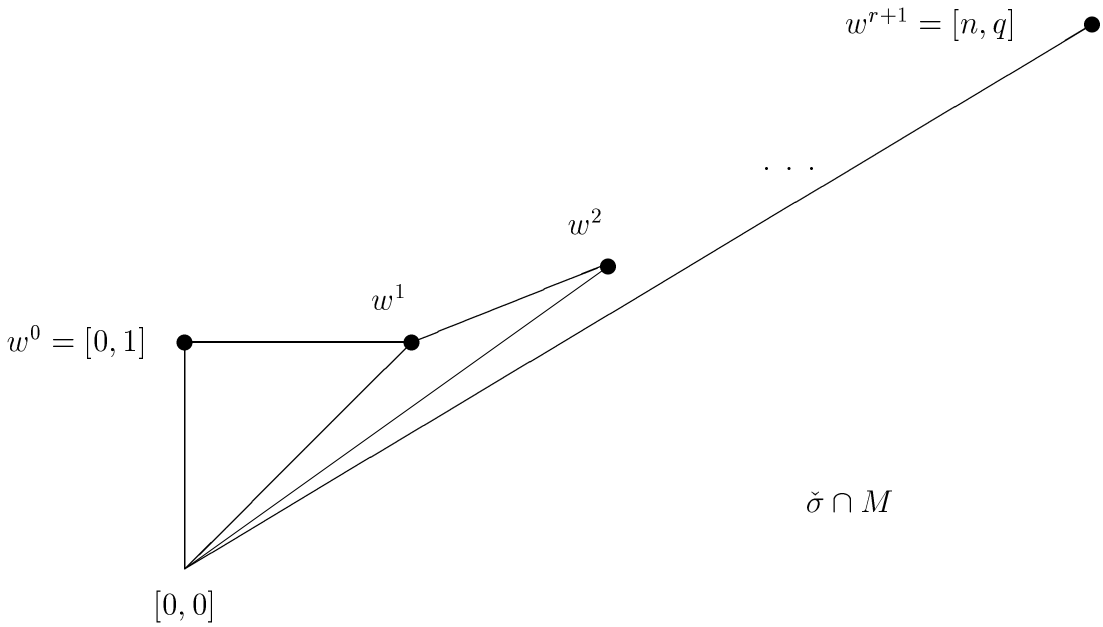
$$E = \{r \in \check{\sigma}\cap M \mid \quad 1)\ r \neq 0 \text{ and } \quad 2)\ r = r^1 + r^2 \ \ (r^i \in \check{\sigma}\cap M) \text{ always implies } r^1 = 0 \text{ or } r^2 = 0\}.$$

**Example:** In case of $X = X(n,q)$ we can develop $\frac{n}{n-q}$ into a continued fraction

$$\frac{n}{n-q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{a_3 \ddots - \cfrac{1}{a_r}}} \qquad (a_i \geq 2).$$

Then, $E$ is given as the set $E = \{w^0, \ldots, w^{r+1}\}$ with elements $w^i \in \mathbb{Z}^2$ and

- $w^0 = [0,1]$, $w^1 = [1,1]$, $w^{r+1} = [n,q]$;
- $w^{i-1} + w^{i+1} = a_i \cdot w^i$ $(i = 1, \ldots, r)$. (cf. [Ch])

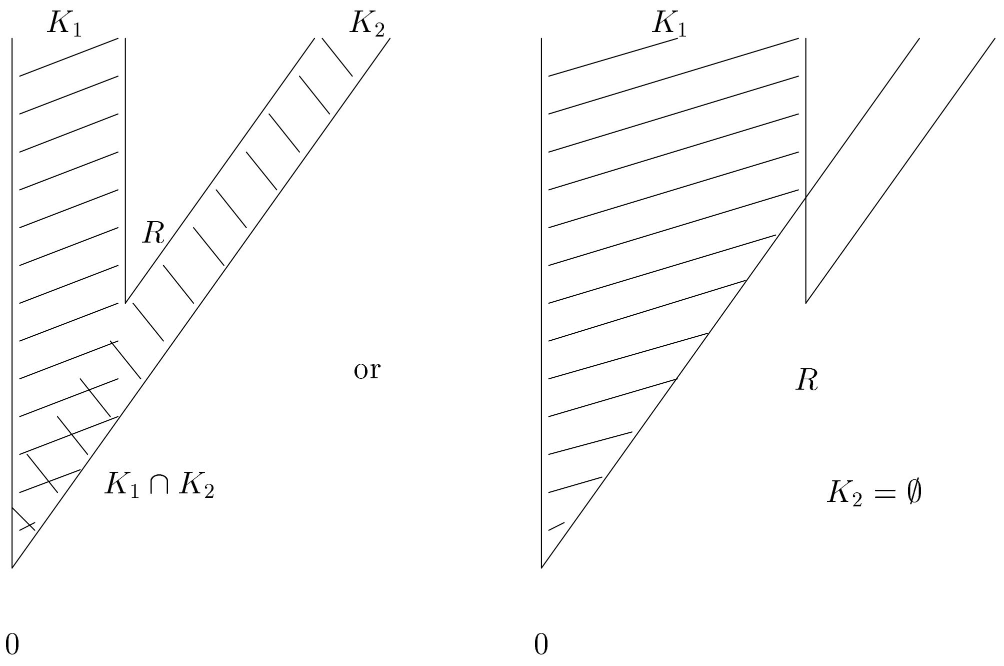



**(2.3)** $T^1_X$ is a $\mathbb{C}[\check{\sigma}\cap M]$-module which also admits an $M$-grading. Therefore, we choose and fix an arbitrary element $R \in M$ – the corresponding task will be to compute the (always finite dimensional) homogeneous piece $T^1_X(-R)$.
If $\sigma \subseteq \mathbb{R}^n$ is given by its fundamental generators $\sigma = \langle a^1, \ldots, a^N \rangle$ (i.e. $a^i$ are the

edges of $\sigma$, and the dual cone is given by $\check{\sigma} = \{r \in I\!R^n | \langle a^i, r\rangle \geq 0 \text{ for } i = 1, \dots, N\}$), then we define the following sets:

$$\begin{aligned} K_i &:= \{r \in \check{\sigma} \cap M \mid (0 \leq)\langle a^i, r\rangle < \langle a^i, R\rangle\} \quad (i = 1, \dots N); \\ E_i &:= E \cap K_i \quad (i = 1, \dots, N); \\ E' &:= \bigcup_{i=1}^{N} E_i. \end{aligned}$$

*Attention:* Though it is not visible in the notation: The sets $K_i, E_i$ and $E'$ depend on the special choice of the lattice point $R \in M$!




**Theorem:**

$$T^1_X(-R) = \left( L(E') \Big/ \sum_{i=1}^{N} L(E_i) \right)^* \otimes_{I\!R} \mathbb{C}.$$

*(• If $F \subseteq I\!R^n$ is an arbitrary subset, we denote by $L(F)$ the vector space of all linear dependences between elements of $F$.*
*• Take the canonical embeddings $L(E_i) \hookrightarrow L(E')$.)*

**(2.4)** Let us explain this theorem for cyclic quotient singularities $X = X(n, q)$. There are four different cases for the multidegree $R \in M \cong \mathbb{Z}^2$:

(i) $R = w^1$ (or analogously $R = w^r$)
We obtain $E_1 = \{w^0\}$ and $E_2 = \{w^2, \ldots, w^{r+1}\}$, and the theorem yields $T^1(-R) \cong \begin{cases} \mathbb{C} & \text{for } r \geq 2 \\ 0 & \text{for } r = 1. \end{cases}$

(ii) $R = w^i$ $(2 \leq i \leq r-1)$
We obtain $E_1 = \{w^0, \ldots, w^{i-1}\}$ and $E_2 = \{w^{i+1}, \ldots, w^{r+1}\}$, and the theorem yields $T^1(-R) \cong \mathbb{C}^2$.

(iii) $R = \ell \cdot w^i$ $(1 \leq i \leq r;\ 2 \leq \ell < a_i)$
We obtain $E_1 = \{w^0, \ldots, w^i\}$ and $E_2 = \{w^i, \ldots, w^{r+1}\}$, and the theorem yields $T^1(-R) \cong \mathbb{C}$.

(iv) For the remaining $R \in M$, either $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$ or $\#(E_1 \cap E_2) \geq 2$. In all these cases the theorem yields $T^1(-R) = 0$.

**(2.5)** In [Sm] Smilansky has proved the following result concerning the decomposition of a polyhedron $P$ into a Minkowski sum:
If $P^*$ denotes the dual polyhedron (vertices of $P^*$ correspond to facets of $P$), and if $F^* < P^*$ runs through all faces that arise as dual ones of proper, compact faces of $P$, then

$$\lambda(P) := \dim \left( D(P^*) \Big/ \sum D(F^*) \right) + 1$$

($D(\ldots)$ := vector space of the affine dependences between the vertices of the argument)

is equal to the dimension of the summand set of $P$.
In particular, $P$ is decomposable into a non-trivial Minkowski sum if and only if $\lambda(P) = 1$.

As Bernd Sturmfels pointed out to me, the $\lambda(P)$-formula looks very similar to the $T^1_X$-formula written down in Theorem (2.3). This reflects the fact that there is a correspondence between the decomposition of certain polyhedra into a Minkowski sum and the deformation of affine toric varieties. (This has already been proved for one parameter deformations and will appear in a forthcoming paper.)

**Example:** (cf. (5.4) of [Al])
Let $H \subseteq \mathbb{R}^2$ be the hexagon $H := \operatorname{conv}\{(0,0); (1,0); (2,1); (2,2); (1,2); (0,1)\}$; putting $H$ into the affine hyperplane $[z = 1] \subseteq \mathbb{R}^3$, we obtain a cone $\sigma := \operatorname{Cone}(H) \subseteq \mathbb{R}^3$.
$X_\sigma$ is a three-dimensional singularity (cone over a Del Pezzo surface) with $\dim_{\mathbb{C}} T^1_X = 3$. The whole vector space is concentrated in the single degree $R_0 = [0, 0, 1]$ that defines the $[z = 1]$-plane containing $H$:

$$\sigma = \langle a^1, a^2, \ldots, a^6 \rangle; \quad \check{\sigma} = \langle r^1, r^2, \ldots, r^6 \rangle.$$

Then, the semigroup $\check{\sigma} \cap M$ is generated by $\{R_0; r^1, \ldots, r^6\}$, and we obtain $E_i = \{r^i, r^{i+1}\}$ ($i = 1, \ldots, 6$; $r^7 := r^1$). These six sets consist of linear independent vectors only, i.e.

$$\begin{aligned} T^1_X(-R_0) &= L(r^1, r^2, \ldots, r^6)^* \otimes_{\mathbb{R}} \mathbb{C} \\ &= \mathbb{C}^6 \Big/ \{(\langle a, r^1 \rangle, \ldots, \langle a, r^6 \rangle) \;/\; a \in N_{\mathbb{R}}\} \\ &= \text{Div } X_{\check{\sigma}} \big/ \text{Cart } X_{\check{\sigma}} \otimes_{\mathbb{Z}} \mathbb{C}. \end{aligned}$$

(Div $X_{\check{\sigma}}$ and Cart $X_{\check{\sigma}}$ denote the sets of Weil and Cartier (i.e. principal) divisors, respectively. Divisors are determined by their so-called order functions (cf. [Ke]), hence, by giving six integers as values of $r^1, \ldots, r^6$.)

The base space of the semi-universal deformation consists of two irreducible components of dimension 1 and 2, respectively.

These two components correspond to the following Minkowski decompositions of the hexagon $H$:

(i) $H = \text{conv}\{(0,0); (1,1); (0,1)\} + \text{conv}\{(0,0); (1,0); (1,1)\}$ and

(ii) $H = \text{conv}\{(0,0); (1,0)\} + \text{conv}\{(0,0); (0,1)\} + \text{conv}\{(0,0); (1,1)\}$.

# 3 Proof of Theorem (2.3)

**(3.1)** Let $E \subseteq \check{\sigma} \cap M$ be an arbitrary finite set that generates $\check{\sigma} \cap M$ as a semigroup (minimality is not necessary); let $K_i$ and $E_i$ (for a fixed degree $R \in M$) as defined in (2.3).
As mentioned in (1.4.2) we have $T^1_X = \text{Ext}^1_{\mathbb{C}[\check{\sigma} \cap M]}(\Omega^1_X, \mathbb{C}[\check{\sigma} \cap M])$, and the $\mathbb{C}[\check{\sigma} \cap M]$-module $\Omega^1_X$ can be computed modulo its torsion by the following map (cf. [Da]):

$$\begin{array}{rcll} \Omega^1_X & \stackrel{\Phi}{\longrightarrow} & V \otimes_{\mathbb{C}} \mathbb{C}[\check{\sigma} \cap M] & \qquad (V := M \otimes_{\mathbb{Z}} \mathbb{C}) \\ dx^r & \longmapsto & r \otimes x^r. & \end{array}$$

That means, $Q := \text{Im}\,\Phi = \Omega^1_X \Big/ \text{tors}(\Omega^1_X)$ is the submodule of $V \otimes_{\mathbb{C}} \mathbb{C}[\check{\sigma} \cap M]$ that is generated by the elements $B(r) := r \otimes x^r$ ($r \in E$). On the other hand, $\Phi$ is injective (i.e. $Q = \Omega^1_X$) outside a closed subset of $X$ of codimension 2 ($X_\sigma$ is normal), hence

$$T^1_X = \text{Ext}^1_{\mathbb{C}[\check{\sigma} \cap M]}(Q, \mathbb{C}[\check{\sigma} \cap M]).$$

**(3.2)** We want to build a $\mathbb{C}[\check{\sigma}\cap M]$-free resolution of the module $Q$; hence, our first step has to be to determine the kernel of the surjection
$\mathcal{B} := \bigoplus_{r\in E} \mathbb{C}[\check{\sigma}\cap M]\cdot B(r) \longrightarrow\!\!\!\!\to Q$.
As we are in a graded situation, it will suffice to regard the $M$-homogeneous pieces of the kernel. They consist of the following elements:

If $q \in L(E) \subseteq \mathbb{R}^E$ is a relation within $E$, i.e. $\sum_{r\in E} q_r \cdot r = 0$, then we define

$$C(q;\ell) := \sum_{r\in E} q_r \cdot x^{\ell-r} B(r)$$

(for those $\ell \in M$ such that $\ell \geq r$ for all $r$ with $q_r \neq 0$).
(The partial ordering "$\geq$" on $M$ is defined with respect to the cone $\check{\sigma}$. That means $\ell \geq r :\Leftrightarrow \ell - r \in \check{\sigma}$.)

By restricting to the set $F \subseteq \mathbb{R}^E$ of minimal relations $q$ (up to a constant factor they are uniquely determined by the set of elements $r \in E$ that are involved) we obtain a finite set of generators of the kernel. (For each $q \in F$ only finitely many elements $\ell$ are needed.)
*Attention:* Since we are looking for generators of a certain $\mathbb{C}[\check{\sigma}\cap M]$-module, it is not enough for $F$ to consist of a linear basis of $L(E)$ only.

**(3.3)** Analogously we determine the kernel of the map

$$\mathcal{C} := \bigoplus_{q\in F;\ell} \mathbb{C}[\check{\sigma}\cap M]\cdot C(q;\ell) \to \bigoplus_{r\in E} \mathbb{C}[\check{\sigma}\cap M]\cdot B(r) = \mathcal{B}.$$

Now, there are two kinds of homogeneous generators:

(i) $D(q;\ell_1,\ell_2;\eta) := x^{\eta-\ell_1} C(q;\ell_1) - x^{\eta-\ell_2} C(q;\ell_2)$ (for $\eta \geq \ell_1;\ell_2$)

(ii) If $\xi \in \mathbb{R}^F$ is a (minimal) relation within $F$, i.e. $\sum_{q\in F} \xi_q \cdot q = 0$, then we define

$$D(\xi;\underline{\ell};\eta) \;:=\; \sum_{q\in F} \xi_q \cdot x^{\eta-\ell_q} \cdot C(q,\ell_q)$$

(for those $\eta \in M$ such that $\eta \geq \ell_q$ for all $q$ with $\xi_q \neq 0$).

**(3.4)** By defining $B$, $C$ and $D$ to be homogeneous of the right degree (it is always visible in the last argument: $r$, $\ell$ and $\eta$, respectively), we obtain an $M$-graded complex of free $\mathbb{C}[\check{\sigma} \cap M]$-modules

$$\begin{array}{ccccc} \mathcal{D} & \xrightarrow{d_C} & \mathcal{C} & \xrightarrow{d_B} & \mathcal{B} \\ \| & & \| & & \| \\ \bigoplus_D \mathbb{C}[\check{\sigma} \cap M] \cdot D & \xrightarrow{d_C} & \bigoplus_C \mathbb{C}[\check{\sigma} \cap M] \cdot C & \xrightarrow{d_B} & \bigoplus_B \mathbb{C}[\check{\sigma} \cap M] \cdot B, \end{array}$$

which is a resolution of $Q$.
Now, we can compute $T^1_X$ as the homology of the dualized complex.

**(3.5)** Denote by $G$ one of the capital letters $B$, $C$ or $D$. Then, an element $\varphi$ of the dual free module $(\bigoplus_G \mathbb{C}[\check{\sigma} \cap M] \cdot G)^*$ can be described by considering elements $g(x) \in \mathbb{C}[\check{\sigma} \cap M]$ to be the images of the generators $G$ ($g$ stands for $b$, $c$, or $d$, respectively).

For $\varphi$ to be homogeneous of degree $-R \in M$, $g(x)$ will have to be a monomial of degree

$$\deg g(x) = -R + \deg G.$$

In particular, the corresponding complex coefficient $g \in \mathbb{C}$ (i.e. $g(x) = g \cdot x^{-R+\deg G}$) admits the property that

$$g \neq 0 \quad \text{implies} \quad -R + \deg G \geq 0 \quad (\text{i.e.} \quad -R + \deg G \in \check{\sigma}).$$

Therefore, the homogeneous part of degree $-R$ of the dualized complex (3.4)* is equal to the complex

$$\mathcal{B}^*_{-R} \xrightarrow{d_B^*} \mathcal{C}^*_{-R} \xrightarrow{d_C^*} \mathcal{D}^*_{-R}$$

of $\mathbb{C}$-vector spaces with coordinates $\underline{b}$, $\underline{c}$, and $\underline{d}$, respectively:

$$\begin{array}{rcl} \mathcal{B}^*_{-R} & = & \{\underline{b(r)} \in \mathbb{C}^E \mid b(r) = 0 \text{ for } r - R \notin \check{\sigma}\} \\ \mathcal{C}^*_{-R} & = & \{\underline{c(q;\ell)}_{q\in F,\ell} \mid c(q;\ell) = 0 \text{ for } \ell - R \notin \check{\sigma}\} \\ \mathcal{D}^*_{-R} & = & \{[\underline{d(q;\ell_1,\ell_2;\eta)}, \underline{d(\xi;\underline{\ell};\eta)}] \mid \begin{array}{rcl} d(q;\ell_1,\ell_2;\eta) & = & 0 \;\text{ for }\; \eta - R \notin \check{\sigma}, \text{ and} \\ d(\xi;\underline{\ell};\eta) & = & 0 \;\text{ for }\; \eta - R \notin \check{\sigma}\}. \end{array} \end{array}$$

The differentials $d_B^*$ and $d_C^*$ can be described by

$$\begin{array}{rcl} c(q;\ell) & = & \sum_{r\in E} q_r \cdot b(r) \qquad \text{and} \\ d(q;\ell_1,\ell_2;\eta) & = & c(q;\ell_1) - c(q;\ell_2), \\ d(\xi;\underline{\ell};\eta) & = & \sum_{q\in F} \xi_q \cdot c(q,\ell_q). \end{array}$$

**(3.6)** To compute the homology we start describing the kernel $V$ of the second differential. Setting $\underline{d} = 0$, the first observation we make is that $c(q) := c(q;\ell)$ does not depend on $\ell$:

$$\tilde{V} := \mathbb{C}^F = \{(\ldots, c(q), \cdots) \mid \; q \in F\}.$$

Now, we can interpret the remaining conditions "$d(\xi; \underline{\ell}; \eta) = 0$" and "$c(q;\ell) = 0$ for $\ell - R \notin \check{\sigma}$" to obtain

$$V = \{\underline{c(q)} \in \tilde{V} \quad | \quad \begin{array}{l} \bullet \text{ The complex numbers } c(q) \text{ inherit all linear} \\ \quad \text{dependences from the original vectors } q \in F. \\ \quad \text{(Equivalently: The linear equations} \\ \quad \sum_{r \in E} q_r \cdot x_r = c(q) \quad (q \in F) \text{ admit a common} \\ \quad \text{solution } \underline{x} = (\ldots, x_r, \ldots) \in \mathbb{C}^E.) \\ \bullet \text{ The existence of an } \ell \in M \text{ with} \\ \quad \bullet\ \ell \geq r \text{ for } q_r \neq 0 \text{ and} \\ \quad \bullet\ \ell - R \notin \check{\sigma} \\ \quad \text{implies } c(q) = 0. \} . \end{array}$$

The image $W$ of the first differential is easy to describe:

$$\begin{array}{rcl} \tilde{W} & := & \mathbb{C}^E = \{(\ldots, b(r), \ldots) |\ r \in E\} \\ W & = & \{\underline{b(r)} \in \tilde{W} |\ b(r) = 0 \text{ for } r - R \notin \check{\sigma}\}. \end{array}$$

Finally, the embedding $W \hookrightarrow V$ is given by the formula

$$c(q) = \sum_{r \in E} q_r \cdot b(r),$$

and we obtain $T^1_X(-R) = {}^V\!/_W$.

**(3.7)** Now, it is time for involving the sets $K_i$. By definition

$$\bigcup_i K_i = (\check{\sigma} \cap M) \setminus (R + (\check{\sigma} \cap M)),$$

i.e. for $r$ occurring in the definition of $W$, the condition "$r - R \notin \check{\sigma}$" is equivalent to "$\exists\, i : r \in K_i$".
For reformulating the conditions in the definition of $V$, too, we have to look closer at the sets $K_i$:

**Lemma:** (properties of the sets $K_i$)

1) $\langle a^i, R \rangle \leq 0$ iff $K_i = \emptyset$;
   $\langle a^i, R \rangle = 1$ iff $K_i = [a^i\text{-face of } \check{\sigma}] \cap M$;
   $\langle a^i, R \rangle \geq 2$ iff $K_i$ contains an interior point of $\check{\sigma}$.

2) $K_i$ is closed for descent, i.e. $r \in K_i$ and $0 \leq s \leq r$ imply $s \in K_i$.

3) Let $\langle a^1, \ldots, a^k \rangle < \sigma$ be a smooth face of $\sigma$ (i.e. $\{a^1, \ldots, a^k\}$ is a part of a $\mathbb{Z}$-basis of the lattice $N$).
Then, for elements $r_1, \ldots, r_m \in \check{\sigma} \cap M$ the conditions

(i) $r_1, \ldots, r_m \in K_1 \cap \ldots \cap K_k$ and

(ii) $\exists\, \ell \geq r_1, \ldots, r_m : \ell \in K_1 \cap \ldots \cap K_k$

are equivalent.

**Proof:** The only non-trivial part is the third one. It will be sufficient to prove it for $m = 2$. Let $r_1, r_2 \in K_1 \cap \ldots \cap K_k$ be given.
*Step 1:* Since $\langle a^1, \ldots, a^k \rangle < \sigma$ is a smooth face, there exist elements $s^1, \ldots, s^k \in \check{\sigma} \cap M$ such that

$$\langle a^i, s^j \rangle = \delta_{ij} \qquad (1 \leq i, j \leq k).$$

*Step 2:* We can assume that $\langle a^i, r_1 \rangle = \langle a^i, r_2 \rangle$ for $i = 1, \ldots, k$.
(In fact, let $\langle a^i, r_1 \rangle - \langle a^i, r_2 \rangle = g_i \geq 0$. Then, $r_2$ can be corrected by $r_2 := r_2 + g_i \cdot s^i$.)

*Step 3:* Let $s \in M$ be an interior point of the dual face $\langle a^1, \ldots, a^k \rangle^* = (a^1)^\perp \cap \ldots \cap (a^k)^\perp \cap \check{\sigma}$ of the cone $\check{\sigma}$, i.e.

$$\begin{array}{rcll} \langle a^i, s \rangle & = & 0 & \text{for} \quad i = 1, \ldots, k \quad \text{and} \\ \langle a^v, s \rangle & > & 0 & \text{for} \quad v \notin \{1, \ldots, k\}. \end{array}$$

Hence,

$$\begin{array}{rcll} \langle a^i, r_1 - r_2 + g \cdot s \rangle & = & 0 & \text{for} \quad i = 1, \ldots, k \quad \text{and} \\ \langle a^v, r_1 - r_2 + g \cdot s \rangle & \geq & 0 & \text{for} \quad v \notin \{1, \ldots, k\} \quad \text{and} \quad g \gg 0, \end{array}$$

and we can define $\ell := r_1 + g \cdot s = r_2 + (r_1 - r_2 + g \cdot s)$. □

We will need two special cases of the third part of the previous lemma:

**Corollary:** Let $\sigma = \langle a^1, \ldots, a^N \rangle$; $r_1, \ldots, r_m \in \check{\sigma} \cap M$.

a) For $i = 1, \ldots, N$ we obtain that

$$r_1, \ldots, r_m \in K_i \quad \text{iff} \quad \exists\, \ell \geq r_1, \ldots, r_m : \ell \in K_i.$$

b) Let $X_\sigma$ be smooth in codimension 2; let $\langle a^i, a^j \rangle < \sigma$ be a two-dimensional face of the cone $\sigma$.
Then, we obtain

$$r_1, \ldots, r_m \in K_i \cap K_j \quad \text{iff} \quad \exists\, \ell \geq r_1, \ldots, r_m : \ell \in K_i \cap K_j.$$

Now, we can use (a) to reformulate the second condition contained in the definition of the vector space $V$ (cf. (3.6)).
Altogether we obtain the following situation:

$$\begin{array}{rcl} V &=& \{\underline{c(q)} \;\;|\;\; \bullet \text{ The linear equations } \sum_{r\in E} q_r\cdot x_r = c(q) \;\; (q\in F) \\ && \qquad\quad \text{admit a common solution } \underline{x}. \\ && \qquad\; \bullet \text{ For each } q \text{ the condition } \{r|\; q_r\neq 0\}\subseteq E_i \\ && \qquad\quad (\text{for some } i) \text{ implies } c(q)=0.\}; \\ W &=& \{\underline{b(r)} \;\;|\;\; b(r)=0 \text{ for } r\in E'\}; \\ T^1_X(-R) &=& {}^V\!/_W \quad (W\subseteq V \text{ is embedded by } c(q)=\sum_{r\in E} q_r\cdot b(r)). \end{array}$$

**(3.8)** Let $\Phi$ be the following map:

$$\begin{array}{rcl} \Phi: V & \longrightarrow & L_{\mathbb{C}}(E)^* \\ \underline{c(q)} & \longmapsto & \left[ Q=\sum_{q\in F}\xi_q\cdot q \mapsto \sum \xi_q\cdot c(q)\right]. \end{array}$$

(The first condition contained in the definition of $V$ guarantees that $\Phi$ is defined correctly. It does not depend on the special representation of $Q$ as a linear combination of elements $q\in F$.)

**(3.8.1)** *$Im\,\Phi\subseteq L_{\mathbb{C}}(E_i)^{\perp}$ for $i=1,\ldots,N$ :*
Let $\underline{c(q)}\in V$ and $Q\in L_{\mathbb{C}}(E_i)$. Then, since $F\subseteq L_{\mathbb{C}}(E)$ is more than a linear basis (cf. (3.2)), $Q$ can be written as a linear combination of elements $q\in F\cap L_{\mathbb{C}}(E_i)$. We obtain

$$\begin{array}{rcl} \langle\Phi(\underline{c(q)}),Q\rangle &=& \langle\Phi(\underline{c(q)}), \sum_{q\in F\cap L(E_i)}\xi_q\cdot q\rangle \\ &=& \sum_{q\in F\cap L(E_i)}\xi_q\cdot c(q)=\sum_q \xi_q\cdot 0 = 0. \end{array}$$

**(3.8.2)** *$\Phi: V\to\bigcap_{i=1}^N L_{\mathbb{C}}(E_i)^{\perp}$ $(\subseteq L_{\mathbb{C}}(E)^*)$ is surjective:*
Let $\varphi: L_{\mathbb{C}}(E)\to\mathbb{C}$ be a linear map such that $\varphi|_{L_{\mathbb{C}}(E_i)}=0$ for $i=1,\ldots,N$. Then, $c(q):=\varphi(q)$ define an element of $V$ that maps onto $\varphi$.

**(3.8.3)** Finally, the restriction to $E'\subseteq E$ provides the surjective maps

$$\Phi': V \stackrel{\Phi}{\twoheadrightarrow} \left( L_{\mathbb{C}}(E) \Big/ \sum_i L_{\mathbb{C}}(E_i)\right)^* \twoheadrightarrow \left( L_{\mathbb{C}}(E') \Big/ \sum_i L_{\mathbb{C}}(E_i)\right)^* .$$

It is clear that Ker $\Phi'$ contains the vector space $W$. Let us look at the reversed inclusion:
Given an element $\underline{c(q)} \in \operatorname{Ker} \Phi'$ we obtain the following property:

If $Q = \sum_{q\in F} \xi_q \cdot q$ fulfils $Q_r = 0$ for $r \notin E'$, then $\sum_{q\in F} \xi_q \cdot c(q) = 0$.

But this means that the equations $\sum_{r\in E\setminus E'} q_r \cdot x_r = c(q)$ $(q \in F)$ admit a common solution $\underline{x}$, i.e. $\underline{c(q)}$ is contained in $W$.

Now, the proof of Theorem (2.3) is complete.

# 4 The second $T^1$-formula

**(4.1)** As in (2.3) we fix the degree $R \in M$.
We are looking for a different $T^1_X(-R)$-formula that does not involve the semigroup structure of $\check{\sigma} \cap M$ (by using $E$) but only the structure of the real cone $\check{\sigma}$.
Let us start with defining those vector spaces that play the crucial role in this formula:

$$\begin{aligned} V_i &:= \operatorname{span}_{\mathbb{R}}(E_i) = \begin{cases} 0 & \text{for } \langle a^i, R\rangle \le 0 \\ [a^i = 0] \subseteq M_{\mathbb{R}} & \text{for } \langle a^i, R\rangle = 1 \\ M_{\mathbb{R}} = \mathbb{R}^n & \text{for } \langle a^i, R\rangle \ge 2 \end{cases} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \ldots, N); \\ V_{ij} &:= \operatorname{span}_{\mathbb{R}}(E_i \cap E_j) \subseteq V_i \cap V_j \qquad\qquad (i, j = 1, \ldots, N). \end{aligned}$$

**Remark:** It is not necessary to have any information about the lattice points inside $\check{\sigma}$ (or about $E \subseteq \check{\sigma} \cap M$) for defining the vector spaces $V_i$. However, this statement is not true for the vector spaces $V_{ij}$.

**(4.2)** **Proposition:**

(i) Let $\pi : V_1 \oplus \ldots \oplus V_N \longrightarrow\!\!\!\!\!\rightarrow V_1 + \ldots + V_N$ be the canonical map, and let $V_{ij} \subseteq V_i \cap V_j$ be embedded as subspaces of $\operatorname{Ker} \pi \subseteq V_1 \oplus \ldots \oplus V_N$. Then,

$$T^1_X(-R) = \left( \operatorname{Ker} \pi \Big/ \sum_{i,j} V_{ij} \right)^* \otimes_{\mathbb{R}} \mathbb{C}.$$

(ii) For computing $\sum_{i,j} V_{ij}$ it is possible to restrict the sum to those pairs $(i, j)$ such that $\langle a^i, a^j \rangle < \sigma$ is a two-dimensional face ("edge") of $\sigma$.

**Proof:** (i) We will use Theorem (2.3), hence, the first step of the proof is to define a map

$$\psi : L(E') \to \text{Ker}\,\pi \Big/ \sum_{i,j} V_{ij} \,.$$

For $q \in L(E') \subseteq I\!R^{E'}$ there exists a (not unique) decomposition

$$q = \sum_{i=1}^{N} q^i \quad \text{with} \quad q^i \in I\!R^{E_i} \subseteq I\!R^{E'}.$$

Then, $v_i := \sum_{r \in E'} q^i_r \cdot r \in V_i$ are vectors with

$$\sum_{i=1}^{N} v_i = \sum_{i=1}^{N} \sum_{r \in E'} q^i_r \cdot r = \sum_{r \in E'} \Big( \sum_{i=1}^{N} q^i_r \Big) \cdot r = \sum_{r \in E'} q_r \cdot r = 0,$$

and we define

$$\psi(q) := (v_1, \ldots, v_N).$$

To investigate the behaviour of the vectors $v_1, \ldots, v_N$ by choosing different decompositions $q = \sum\limits_{i=1}^{N} q^i = \sum\limits_{i=1}^{N} \tilde{q}^i$, we use induction on the number of pairs $(i, r)$ such that $q^i_r \neq \tilde{q}^i_r$:
Let $q^i_r$, $\tilde{q}^i_r$ be different numbers ($\lambda := q^i_r - \tilde{q}^i_r$). Then, there exists an index $j \neq i$ such that $q^j_r$ and $\tilde{q}^j_r$ are different too, and we can define another decomposition $(\bar{q}^1, \ldots, \bar{q}^N)$ by

$$\begin{array}{rcl} \bar{q}^i_r & := & \tilde{q}^i_r + \lambda = q^i_r, \\ \bar{q}^j_r & := & \tilde{q}^j_r - \lambda, \;\; \text{and} \\ \bar{q}^{\bullet}_{\bullet} & := & \tilde{q}^{\bullet}_{\bullet} \;\; \text{for the remaining positions.} \end{array}$$

On the one hand, $(\bar{q}^1, \ldots, \bar{q}^N)$ sits "between" $(q^1, \ldots, q^N)$ and $(\tilde{q}^1, \ldots, \tilde{q}^N)$, and on the other hand, the difference $\psi(q = \sum \bar{q}^i) - \psi(q = \sum \tilde{q}^i)$ is equal to the tuple $\Big(0, \ldots, 0, \underset{[i]}{\lambda \cdot r}, 0, \ldots, 0, \underset{[j]}{-\lambda \cdot r}, 0, \ldots, 0\Big)$. But this element corresponds to $\lambda \cdot r \in V_{ij}$ via the embedding mentioned in the proposition. In particular, $\psi$ is defined correctly.

Since $\sum\limits_i L(E_i) \subseteq \text{Ker}\,\psi$ we obtain a map

$$\psi : L(E') \Big/ \sum_i L(E_i) \longrightarrow \text{Ker}\,\pi \Big/ \sum_{i,j} V_{ij} \,.$$

To construct the inverse of $\psi$, let $(v_1, \ldots, v_N) \in \operatorname{Ker} \pi$ be given. Each $v_i \in V_i = \operatorname{span}(E_i)$ can be written as

$$v_i = \sum_{r \in E_i} q_r^i \cdot r \qquad \text{(for some coefficients } q_r^i\text{)},$$

hence, $q := \sum_{i=1}^{N} q^i \in L(E')$ is a pre-image of $(v_1, \ldots, v_N)$. (Different decompositions of $v_i$ yield elements $q$ and $q'$ which are equal modulo $\sum_i L(E_i)$.)

(ii) Let an element $r \in E_i \cap E_j \subseteq K_i \cap K_j$ be given. If we define the following open half space

$$H^+ := \{a \in \mathbb{R}^n | \; \langle a, R - r \rangle > 0\},$$

then $a^i$ and $a^j$ are contained in $H^+$. Hence, there exists a chain of fundamental generators of $\sigma$ $\quad a^i = a^{i_0}, \ldots, a^{i_k} = a^j$ such that

- $a^{i_v} \in H^+$ (i.e. $r \in E_{i_v}$) and
- $\langle a^{i_v}, a^{i_{v+1}} \rangle < \sigma$ is always an edge. □

**Example:** In the case of two-dimensional cyclic quotients $X = X(n,q)$ we obtain $T^1_X(-R) = \left( {}^{V_1 \cap V_2}/_{V_{12}} \right)^* \otimes_{\mathbb{R}} \mathbb{C}$.

To give an explicit description of the vector spaces $V_1$, $V_2$, and $V_{12}$, we use the terminology of (2.4):

(i) $V_1 = \mathbb{R} \cdot w^0$; $V_2 = \begin{cases} \mathbb{R}^2 & \text{for } r \geq 2 \\ \mathbb{R} \cdot w^{r+1} & \text{for } r = 1 \end{cases}$ ; $\quad V_{12} = 0$

(ii) $V_1 = V_2 = \mathbb{R}^2$; $\; V_{12} = 0$

(iii) $V_1 = V_2 = \mathbb{R}^2$; $\; V_{12} = \mathbb{R} \cdot w^i$

(iv) $V_1 = V_{12}$ or $V_2 = V_{12}$ or $V_{12} = \mathbb{R}^2$.

**(4.3)** Let $\tau = \langle a^1, \ldots, a^k \rangle < \sigma$ be an arbitrary face of $\sigma$. Then, the corresponding lattices are denoted by

$$\begin{array}{rcll} N_\tau & \subseteq & N & \text{and} \\ M_\tau & = & {}^{M}/_{\tau^\perp}. & \end{array}$$

As usual, we fix a degree $R \in M \stackrel{\pi}{\longrightarrow\!\!\!\!\to} M_\tau$.

**Lemma:**

1) $E^0 := E \cap \tau^\perp$ generates $(\check{\sigma} \cap \tau^\perp) \cap M$ as a semigroup. In particular, $\mathrm{span}(E^0) = \tau^\perp$.

2) As soon as they are not empty, the sets $E_i$ $(i = 1, \ldots, k)$ contain $E^0$.

3) Let $r \in E$. Then, $\pi(r)$ splits into a sum of elements of $E^{(\tau)}$.

4) Each element $\bar{r} \in E^{(\tau)}$ can be lifted to $E$.

5) Let $r \in \check{\sigma} \cap M$. Then, $\pi(r) \in \check{\tau} \cap M_\tau$, and for $i = 1, \ldots, k$, the conditions "$r \in K_i$" and "$\pi(r) \in K_i^{(\tau)}$" are equivalent.
In particular, (3) and (4) will remain valid if we replace $E$ and $E^{(\tau)}$ by $E_i$ (or $E_i \cap E_j$) and $E_i^{(\tau)}$ (or $E_i^{(\tau)} \cap E_j^{(\tau)}$), respectively.

**Proof:** Most of the claims are quite clear - we only want to prove (4):

Let $r \in M$ be an arbitrary lift of $\bar{r}$ to the lattice $M$.
By $r := r + g \cdot s$ $(g \gg 0;\ s \in \mathrm{int}(\tau^\perp \cap \check{\sigma}))$ we may asume that $r \in \check{\sigma} \cap M$, hence, $r$ can be split into a sum

$$r = r_1 + \ldots + r_m \qquad (r_j \in E).$$

On the $M_\tau$-level this means that we have found a decomposition

$$\bar{r} = \pi(r) = \pi(r_1) + \ldots + \pi(r_m).$$

Then, since $\bar{r} \in E^{(\tau)}$ is irreducible, there has to exist a summand $r_j$ such that $\bar{r} = \pi(r_j)$, i.e. $r_j \in E$ is the desired lift of $\bar{r}$.
(The remaining summands $r_v$ are contained in $E^0$.) □

**Corollary:** Let $i, j \in \{1, \ldots, k\}$. Then, if $V_i, V_j \neq 0$ (i.e. $\langle a^i, R\rangle$, $\langle a^j, R\rangle \geq 1$), we will obtain

$$\begin{aligned} V_i &= \pi^{-1}(V_i^{(\tau)}) \qquad \text{and} \\ V_{ij} &= \pi^{-1}(V_{ij}^{(\tau)}). \end{aligned}$$

**(4.4)** **Proposition:** Let $T^1_{X_\sigma}$ be a finite-dimensional vector space. Then, all proper faces $\tau < \sigma$ are rigid (i.e. $T^1_{X_\tau} = 0$).

**Proof:** $X_\sigma$ contains $X_\tau \times (\mathbb{C}^*)^{\dim\sigma - \dim\tau}$ as an open, dense subset, hence, the claim of the proposition is not very surprising. Nevertheless, we will give a proof in terms

of our $T^1$-formula; we use the terminology of (4.3).
Let $R \in M$ such that $T^1_{X_\tau}(-\bar{R}) \neq 0$ ($\bar{R} := \pi(R)$). Then, for

$$R_g := R - g \cdot s \quad (g \gg 0; s \in \mathrm{int}(\tau^\perp \cap \check{\sigma}))$$

we obtain the following properties:

- $V_j = 0$ if $j \notin \{1, \dots, k\}$;
- $$\begin{array}{ccc} V_1 \oplus \dots \oplus V_k \Big/ \sum V_{ij} & \stackrel{\psi}{\twoheadrightarrow} & V_1 + \dots + V_k \\ \downarrow & & \downarrow \\ V_1^{(\tau)} \oplus \dots \oplus V_k^{(\tau)} \Big/ \sum V_{ij}^{(\tau)} & \stackrel{\psi^{(\tau)}}{\twoheadrightarrow} & V_1^{(\tau)} + \dots + V_k^{(\tau)} \end{array}$$

  is a commutative diagramme involving surjections only.

If $p = (p_1, \dots, p_k) \in V_1 \oplus \dots \oplus V_k \Big/ \sum V_{ij}$ is an arbitrary pre-image of a non-trivial element of $T^1_{X_\tau}(-\bar{R}) = \mathrm{Ker}\ \psi^{(\tau)}$, then $\psi(p)$ will be contained in $\tau^\perp$. Hence, we can modify $p$ by $p' := (p_1 - \psi(p), p_2, \dots, p_k)$ to obtain a non-trivial element of $T^1_{X_\sigma}(-R_g)$.

That means that $T^1_{X_\sigma}$ would have non-trivial homogeneous summands in infinitely many degrees $R_g$ ($g \gg 0$), i.e. $\dim_{\mathbb{C}} T^1_{X_\sigma} = \infty$. □

**Theorem:** *Let* $\dim X_\sigma \geq 3$.

1) *If* $T^1_X$ *is a finite-dimensional vector space,* $X$ *will be smooth in codimension 2.*

2) *If this is the case (i.e. if all "edges" of* $\sigma$ *are smooth), we will obtain the formula*

$$T^1_X(-R) = \mathrm{Ker} \left[ V_1 \oplus \dots \oplus V_N \Big/ \sum_{\{a^i, a^j\} < \sigma} V_i \cap V_j \longrightarrow\!\!\!\!\!\rightarrow V_1 + \dots + V_N \right]^* .$$

**Proof:** (1) We apply the previous proposition to the two-dimensional faces $\{a^i, a^j\} < \sigma$. The corresponding toric varieties $X_\tau$ are two-dimensional cyclic quotient singularities, hence, rigidity implies smoothness.

(2) Looking at proposition (4.2), the only point that remains to be checked is $V_{ij} = V_i \cap V_j$ ($\{a^i, a^j\} < \sigma$ being a face provided).
By Corollary (4.3) this question can be investigated on the face being fixed instead of $\sigma$. $X_{\{a^i, a^j\}}$ was assumed to be smooth, hence, $V_{ij} = V_i \cap V_j$ becomes trivial at this level (cf. the example in (4.2)). □

# 5 Applications and Examples

**(5.1)** Let $\sigma = \langle a^1, \dots, a^n \rangle$ be an $n$-dimensional simplicial cone that is smooth in codimension 2. Then, each pair of "vertices" yields an "edge" $\langle a^i, a^j \rangle < \sigma$, hence,

$$T^1_{X_\sigma}(-R) = \operatorname{Ker}\left[ V_1 \oplus \dots \oplus V_n \Big/ \sum_{i,j} V_i \cap V_j \longrightarrow\!\!\!\!\rightarrow V_1 + \dots + V_n \right]^* .$$

If $V_1, \dots, V_m$ denote the non-trivial vector spaces (equal to $[a^i = 0] \subseteq I\!\!R^n$ or $I\!\!R^n$), we will obtain the important equality

$$V_i \cap \Big( \sum_{j \geq i+1} V_j \Big) = \sum_{j \geq i+1} (V_i \cap V_j) \quad (i = 1, \dots, m-1).$$

This means that, by induction, all elements of $\operatorname{Ker}[V_1 \oplus \dots \oplus V_m \longrightarrow\!\!\!\!\rightarrow V_1 + \dots + V_m]$ can be put into the vector space $\sum_{i,j} V_i \cap V_j$.
Hence, $T^1_{X_\sigma}(-R) = 0$ for each degree $R \in M$, i.e $X_\sigma$ is rigid.

**Remark:** Simplicial cones $\sigma$ yield $n$-dimensional cyclic quotient singularities. Therefore, the rigidity of these toric varieties is a special case of the well-known result of Schlessiger (cf. [Sch]) concerning arbitrary isolated quotient singularities in dimension $\geq 3$.

**(5.2)** Let $\sigma = \langle a^0, \dots, a^m; b^0, \dots, b^n \rangle$ $(m, n \geq 1)$ be an $(m+n+1)$-dimensional cone such that the fundamental generators admit the relation

$$\sum_{i=0}^{m} \lambda_i a^i = \sum_{j=0}^{n} \mu_j b^j \qquad (\lambda_i, \mu_j > 0).$$

Then, the dual cone $\check{\sigma}$ is the cone over the product of two simplices $\Delta^m \times \Delta^n$, i.e. the fundamental generators can be listed as pairs $(i,j)$ $(i = 0, \dots, m;\ j = 0, \dots, n)$. The relation between $\sigma$ and $\check{\sigma}$ is given by

$$\begin{aligned} \langle a^p, (i,j) \rangle = 0 \quad &\text{iff} \quad p \neq i \quad \text{and} \\ \langle b^q, (i,j) \rangle = 0 \quad &\text{iff} \quad q \neq j. \end{aligned}$$

Assume that $X_\sigma$ is smooth in codimension 2.

*Case 1: Let $m, n \geq 2$.*
Then, each pair of "vertices" yields an "edge" of $\sigma$, hence,

$$T^1_{X_\sigma}(-R) = \operatorname{Ker}\left[ {\scriptstyle (\bigoplus_i V_i) \oplus (\bigoplus_j W_j)} \Big/ {\scriptstyle \sum (V_{i_1} \cap V_{i_2}) + \sum (V_i \cap W_j) + \sum (W_{j_1} \cap W_{j_2})} \longrightarrow\!\!\!\!\rightarrow (\sum_i V_i) + (\sum_j W_j) \right]^* .$$

As in (5.1) we will regard only those vector space $V_i$, $W_j$ that are non-trivial. Our task will be to put them into a new, linear order $P_1, \ldots, P_N$ such that

$$P_\mu \cap \Big( \sum_{\nu \geq \mu+1} P_\nu \Big) = \sum_{\nu \geq \mu+1} (P_\mu \cap P_\nu).$$

Well, this is very easy: For $N \geq 3$, the only condition is to put two vector spaces of the same type ($V$ or $W$) to the last two places $P_{N-1}$ and $P_N$. The case $N \leq 2$ is trivial, anyway.
Therefore, $T^1_{X_\sigma}(-R) = 0$ for each degree $R \in M$, i.e. $X_\sigma$ is rigid.

*Case 2: Let $m \geq 2$; $n = 1$ ($\sigma$ is the cone over an $(m+1)$-dimensional double simplex; $\check{\sigma}$ is the cone over a prism).*
Then, $\{b^0, b^1\}$ is the only pair of vertices that does not correspond to a (two-dimensional) face of $\sigma$.
We can proceed in the usual way (to reorder the non-trivial $V_i$, $W_j$ into a chain $P_1, \ldots, P_N$), but we have to achieve the stronger formula

$$P_\mu \cap \Big( \sum_{\nu \geq \mu+1} P_\nu \Big) = \sum_{\substack{\nu \geq \mu+1 \\ \{P_\mu, P_\nu\} \neq \{W_0, W_1\}}} (P_\mu \cap P_\nu).$$

If there are at least two non-vanishing vector spaces $V_i$, $V_j$, this will be possible (put them to the last two places $P_{N-1}$, $P_N$). If there is at least one vector space $V_i$ that is equal to $I\!R^{m+2}$, we will also succeed (put $V_i$ to the place $P_N$).
Hence, only two cases are left to be a candidat for $T^1(-R) \neq 0$:
a) $V_0 = \ldots = V_m = 0$; $W_0, W_1 \neq 0$.
This situation implies

$$\begin{array}{rcll} \langle a^i, R \rangle & \leq & 0 & (i = 0, \ldots, m) \quad \text{and} \\ \langle b^j, R \rangle & \geq & 1 & (j = 0, 1), \end{array}$$

i.e. we obtain a contradiction to the assumption $\sum_{i=0}^{m} \lambda_i a^i = \sum_{j=0}^{1} \mu_j b^j$ $(\lambda_i, \mu_j > 0)$.
b) $V_{i_0} = [a^{i_0} = 0] \subseteq I\!R^n$; $V_i = 0$ $(i \neq i_0)$; $W_0, W_1 \neq 0$.
The two vector spaces $V_{i_0} \cap W_0$; $V_{i_0} \cap W_1 \subseteq V_{i_0} \oplus W_0 \oplus W_1$ have 0 as the intersection, hence, to compute $T^1_{X_\sigma}(-R)$ it will be suffice to count the dimensions:

$$\begin{aligned} \dim_{\mathbb{C}} T^1_{X_\sigma}(-R) &= \dim_{I\!R}(V_{i_0} \oplus W_0 \oplus W_1) - \dim_{I\!R}(V_{i_0} + W_0 + W_1) \\ &\quad - \dim_{I\!R}(V_{i_0} \cap W_0) - \dim_{I\!R}(V_{i_0} \cap W_1) \\ &= ([m+1] + \dim W_0 + \dim W_1) - (m+2) \\ &\quad -(\dim W_0 - 1) - (\dim W_1 - 1) \\ &= 1. \end{aligned}$$

On the other hand, this situation occurs iff

$$\begin{array}{rcll} \langle a^{i_0}, R\rangle &=& 1; & \\ \langle a^i, R\rangle &\le& 0 & (i=0,\dots,m;\ i\neq i_0); \\ \langle b^j, R\rangle &\ge& 1 & (j=0,1). \end{array}$$

In particular, if $X_\sigma$ is not rigid, the following condition will be implied:

$$\lambda_{i_0} \ge \lambda_{i_0} + \sum_{i\neq i_0} \lambda_i \cdot \langle a^i, R\rangle = \sum_{i=0}^{m} \lambda_i \cdot \langle a^i, R\rangle = \sum_{j=0}^{1} \mu_j \langle b^j, R\rangle \ge \sum_{j=0}^{1} \mu_j,$$

i.e. $\lambda_{i_0} \ge \mu_0 + \mu_1$.

*Case 3: $m = n = 1$ ( $\sigma$ and $\check{\sigma}$ are cones over a quadrangle).*
This case is included in the investigation of the three-dimensional toric varieties (cf. (5.3)).

**Remark:** In a discussion with Ancus Röhr, we have observed that the class of cones $\sigma = \langle a^0, \dots, a^m; b^0, \dots, b^n\rangle$ considered in this section contains the general determinantal singularities (2-minors of a $(m+1)\times(n+1)$ matrix with general entries). They correspond to those cones $\sigma$ such that

(i) $\lambda_0 = \dots = \lambda_m = \mu_0 = \dots = \mu_n = 1$ and

(ii) each system of $m+n+1$ vectors of $\{a^0, \dots, a^m;\ b^0, \dots, b^n\}$ provides a $\mathbb{Z}$-basis of the lattice $N \cong \mathbb{Z}^{m+n+1}$.

In particular, we can see that the general determinantal singularities of dimension $\ge 4$ are rigid.

**(5.3)** *Three-dimensional affine toric varieties*
Let $\sigma = \langle a^1, \dots, a^N\rangle$ be a three-dimensional cone that is smooth in codimension 2 (i.e. $X_\sigma$ admits an isolated singularity). Since $\langle a^i, a^{i+1}\rangle < \sigma$ ($i \in \mathbb{Z}/N\mathbb{Z}$) are the edges of $\sigma$, we obtain

$$T^1_{X_\sigma}(-R) = \operatorname{Ker}\left[ V_1 \oplus \dots \oplus V_N \Big/ \sum_i (V_i \cap V_{i+1}) \longrightarrow\!\!\!\!\rightarrow V_1 + \dots + V_N \right]^* .$$

As in the second case of (5.2), we try to count dimensions to compute $T^1$. The sum $\sum_i (V_i \cap V_{i+1})$ is not a direct one, but we can measure the difference by the exact sequence

$$0 \to \bigcap_i V_i \to \bigoplus_i (V_i \cap V_{i+1}) \to \sum_i (V_i \cap V_{i+1}) \to 0.$$

Hence,

$$\dim_{\mathbb{C}} T^1_{X_\sigma}(-R) = \sum_i \dim V_i - \sum_i \dim(V_i \cap V_{i+1}) + \dim \bigcap_i V_i - \dim(\sum_i V_i).$$

To distinguish between some cases, let $V_1, \ldots, V_k \neq 0$ and $V_{k+1} = \ldots = V_N = 0$. (The vertices that admit trivial vector spaces form a connected chain.)

*Case 1:* $k \leq 2$.
Then, $T^1_{X_\sigma}(-R) = 0$.

*Case 2:* $3 \leq k \leq N-1$.

$$\begin{aligned} \dim_{\mathbb{C}} T^1_{X_\sigma}(-R) &= \sum_{i=1}^{k} \dim V_i - \sum_{i=1}^{k-1} (\dim V_i + \dim V_{i+1} - 3) + 0 - 3 \\ &= 3(k-2) - \sum_{i=2}^{k-1} \dim V_i. \end{aligned}$$

In particular, $T^1_{X_\sigma}(-R) = 0$ iff $\dim V_i = 3$ for $i = 2, \ldots, k-1$.

*Case 3:* $k = N$.

$$\begin{aligned} \dim_{\mathbb{C}} T^1_{X_\sigma}(-R) &= \sum_{i \in \mathbb{Z}/N\mathbb{Z}} \dim V_i - \sum_{i \in \mathbb{Z}/N\mathbb{Z}} (\dim V_i + \dim V_{i+1} - 3) + \dim \bigcap_i V_i - 3 \\ &= 3(N-1) - \sum_{i=1}^{N} \dim V_i + \dim \bigcap_i V_i \quad ; \\ \dim \bigcap_i V_i &= \begin{cases} 3 & \text{for} \quad \dim V_1 = \ldots = \dim V_N = 3 \\ 2 & \text{for} \quad \#\{i \mid \dim V_i = 2\} = 1 \\ 1 & \text{for} \quad \#\{i \mid \dim V_i = 2\} = 2 \\ 0 & \text{for} \quad \#\{i \mid \dim V_i = 2\} \geq 3. \end{cases} \end{aligned}$$

In particular, $T^1_{X_\sigma}(-R) = 0$ iff $\#\{i \mid \dim V_i = 2\} \leq 3$.
Otherwise, we obtain $\dim_{\mathbb{C}} T^1_{X_\sigma}(-R) = 3(N-1) - \sum_{i=1}^{N} \dim V_i$.

**Summary:**

1) $T^1_{X_\sigma}$ consists of homogeneous pieces $T^1_{X_\sigma}(-R)$ of two different types:
   (I) $R \notin \operatorname{int} \check{\sigma}$.
   $$\dim_{\mathbb{C}} T^1_{X_\sigma}(-R) = \#\{i \in {}^{\mathbb{Z}}\!/_{N\mathbb{Z}} \mid \; \langle a^{i-1}, R\rangle \geq 1;\; \langle a^i, R\rangle = 1;\; \langle a^{i+1}, R\rangle \geq 1\}.$$
   (II) $R \in \operatorname{int} \check{\sigma}$.
   $$\dim_{\mathbb{C}} T^1_{X_\sigma}(-R) = \operatorname{Max}\left\{0;\; \#\{i \in {}^{\mathbb{Z}}\!/_{N\mathbb{Z}} \mid \; \langle a^i, R\rangle = 1\} - 3\right\}.$$

2) In particular, $X_\sigma$ is not rigid iff there exists an $R \in M$ such that

$$\begin{array}{llll} \text{(i)} & \exists i: & \langle a^{i-1}, R\rangle & \geq 1, \\ & & \langle a^i, R\rangle & = 1, \\ & & \langle a^{i+1}, R\rangle & \geq 1, \\ & \exists j: & \langle a^j, R\rangle & \leq 0, \qquad \text{or} \\ \text{(ii)} & \multicolumn{3}{l}{\exists i_v \;\; (v = 1,2,3,4) : \langle a^{i_v}, R\rangle = 1.} \end{array}$$

**Example:** Let $(\alpha_1, -\alpha_2, \alpha_3, -\alpha_4) \in \mathbb{Z}^4$ be a primitive lattice point with $\alpha_i > 0$ $(i = 1,2,3,4)$. Then, ${}^{\mathbb{Z}^4}\!/_{\mathbb{Z}\cdot(\alpha_1,-\alpha_2,\alpha_3,-\alpha_4)} \cong \mathbb{Z}^3$, i.e. we obtain an exact sequence

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z}^4 & \longrightarrow & \mathbb{Z}^3 & \longrightarrow 0 \\ & 1 & \mapsto & (\alpha_1,-\alpha_2,\alpha_3,-\alpha_4) & & & \\ & & & e^i & \longmapsto & a^i & . \end{array}$$

Now, the cone $\sigma := \langle a^1, a^2, a^3, a^4\rangle \subseteq \mathbb{R}^3$ admits the following properties:

1) $\alpha_1 a^1 + \alpha_3 a^3 = \alpha_2 a^2 + \alpha_4 a^4$
   (the real relation between the fundamental generators).

2) There exists an $R \in M$ such that $\langle a^i, R\rangle = g_i$, iff $\alpha_1 g_1 + \alpha_3 g_3 = \alpha_2 g_2 + \alpha_4 g_4$. (The fundamental generators $a^i \in \mathbb{Z}^3$ are in the most general position that is possible when (1) holds.)

*Proof:* Consider the dual of the above sequence!

We want to describe $T^1_{X_\sigma}$ for this cone:

(i) $X_\sigma$ admits infinitesimal deformations of type (I) iff

$$\begin{array}{lll} \alpha_1 & \in & \alpha_2 + \alpha_4 + \mathbb{N}\alpha_2 + \mathbb{N}\alpha_4 + \mathbb{N}\alpha_3 \quad \text{or} \\ \alpha_3 & \in & \alpha_2 + \alpha_4 + \mathbb{N}\alpha_2 + \mathbb{N}\alpha_4 + \mathbb{N}\alpha_1 \quad \text{or} \\ \alpha_2 & \in & \alpha_1 + \alpha_3 + \mathbb{N}\alpha_1 + \mathbb{N}\alpha_3 + \mathbb{N}\alpha_4 \quad \text{or} \\ \alpha_4 & \in & \alpha_1 + \alpha_3 + \mathbb{N}\alpha_1 + \mathbb{N}\alpha_3 + \mathbb{N}\alpha_2. \end{array}$$

   These conditions imply $\alpha_1 \geq \alpha_2 + \alpha_4$, $\alpha_3 \geq \alpha_2 + \alpha_4$, $\alpha_2 \geq \alpha_1 + \alpha_3$, and $\alpha_4 \geq \alpha_1 + \alpha_3$, respectively.

(ii) $X_\sigma$ admits infinitesimal deformations of type (II), iff $\alpha_1 + \alpha_3 = \alpha_2 + \alpha_4$.
   (If this condition is satisfied, the whole space $T^1_{X_\sigma}$ will be one-dimensional.)

## References


[Al] Altmann, K.: Deformations of affine torus varieties.
Heft Nr. 137 der Schriftenreihe des Forschungsschwerpunktes "Komplexe Mannigfaltigkeiten"; Kaiserslautern 1991.

[Ch] Christophersen, J.A.: Obstruction spaces for rational singularities and deformations of cyclic quotients.
Thesis, University of Oslo, 1989/90.

[Da] Danilov, V.I.: The Geometry of Toric Varieties.
Russian Math. Surveys **33** (1978), 97-154.

[Ke] Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B.: Toroidal embeddings I.
Lect. Notes Math., vol. 339. Berlin-Heidelberg-New York: Springer 1973.

[KPR] Kurke, H., Pfister, G. Roczen, M.: Henselsche Ringe und algebraische Geometrie.
Berlin: Deutscher Verlag der Wissenschaften 1975.

[LS] Laudal, O.A., Sletsjøe, A.: Cohomology of groups, monoids and their algebras.
Preprint Oslo 1983.

[LSch] Lichtenbaum, S., Schlessinger, M.: The cotangent complex of a morphism.
Trans. AMS **128** (1967), 41-70.

[Oda] Oda, T.: Convex Bodies and Algebraic Geometry.
Berlin-Heidelberg-New York: Springer 1988.

[Sch] Schlessinger, M.: Rigidity of quotient singularities.
Invent. math. **14** (1971), 17-26.

[Sm] Smilansky, Z.: Decomposability of Polytopes and Polyhedra.
Geometriae Dedicata. **24** (1987), 29-49.