Infinitesimal Deformations and Obstructions for Toric Singularities

Klaus Altmann*
Institut für reine Mathematik, Humboldt-Universität zu Berlin
Ziegelstr. 13a, D-10099 Berlin, Germany.
E-mail: altmann@mathematik.hu-berlin.de

Abstract

The obstruction space T^2 and the cup product $T^1 \times T^1 \to T^2$ are computed for toric singularities.

Contents

1 Introduction

(1.1) For an affine scheme $Y = \text{Spec}\, A$, there are two important A-modules, T^1_Y and T^2_Y, carrying information about its deformation theory: T^1_Y describes the infinitesimal deformations, and T^2_Y contains the obstructions for extending deformations of Y to larger base spaces (cf. [KPR], [LiS], or [Pa]).

*This paper was written at M.I.T. and supported by a DAAD-fellowship.

In case Y admits a versal deformation, T^1_Y is the tangent space of the versal base space S. Moreover, if J denotes the ideal defining S as a closed subscheme of the affine space T^1_Y, the module $\left(J/_{m_{T^1}\,J}\right)^*$ can be canonically embedded into T^2_Y, i.e. $(T^2_Y)^*$-elements induce the equations defining S in T^1_Y.

The vector spaces T^i_Y come with a cup product $T^1_Y \times T^1_Y \to T^2_Y$. The associated quadratic form $T^1_Y \to T^2_Y$ describes the quadratic part of the elements of J, i.e. it can be used to get a better approximation of the versal base space S as regarding its tangent space only.

(1.2) In [Al 1] we have determined the vector space T^1_Y for affine toric varieties. The present paper can be regarded as its continuation - we will compute T^2_Y and the cup product.
These modules T^i_Y are canonically graded (induced from the character group of the torus). We will describe their homogeneous pieces as cohomology groups of certain complexes that are directly induced from the combinatorial structure of the rational, polyhedral cone defining our variety Y. (The paper [LdSl] has suggested that this should be possible at all.) The results can be found in §3.

Switching to another, quasiisomorphic complex provides a second formula for the vector spaces T^i_Y (cf. §6). We will use this particular version for describing these spaces and the cup product in the special case of three-dimensional toric Gorenstein singularities (cf. §7).

2 T^1, T^2, and the cup product (in general)

In this section we will give a brief reminder to the well known definitions of these objects. Moreover, we will use this opportunity to fix some notations.

(2.1) Let $Y \subseteq \mathbb{C}^{w+1}$ be given by equations $f_1, \dots, f_m$, i.e. its ring of regular functions equals

$$A = P/_I \quad \text{with} \quad \begin{array}{l} P = \mathbb{C}[z_0, \dots, z_w] \\ I = (f_1, \dots, f_m)\,. \end{array}$$

Then, using $d: {}^I/_{I^2} \to A^{w+1}$ $\;(d(f_i) := (\frac{\partial f_i}{\partial z_0}, \dots \frac{\partial f_i}{\partial z_w}))$, the vector space T^1_Y equals

$$T^1_Y = {\operatorname{Hom}_A({}^I/_{I^2}, A)} \Big/ {\operatorname{Hom}_A(A^{w+1}, A)}\;.$$

(2.2) Let $\mathcal{R} \subseteq P^m$ denote the P-module of relations between the equations $f_1, \dots, f_m$. It contains the so-called Koszul relations $\mathcal{R}_0 := \langle f_i\, e^j - f_j\, e^i \rangle$ as a

submodule.
Now, T^2_Y can be obtained as

$$T^2_Y = \left. \operatorname{Hom}_P(\mathcal{R}/_{\mathcal{R}_0}, A) \middle/ \operatorname{Hom}_P(P^m, A) \right. .$$

(2.3) Finally, it is well known (cf. [Ld], (5.1.5)) that the cup product $T^1 \times T^1 \to T^2$ can be defined in the following way:

(i) Starting with an $\varphi \in \operatorname{Hom}_A(I/_{I^2}, A)$, we lift the images of the f_i obtaining elements $\tilde{\varphi}(f_i) \in P$.

(ii) Given a relation $r \in \mathcal{R}$, the linear combination $\sum_i r_i\, \tilde{\varphi}(f_i)$ vanishes in A, i.e. it is contained in the ideal $I \subseteq P$. Denote by $\lambda(\varphi) \in P^m$ any set of coefficients such that

$$\sum_i r_i\, \tilde{\varphi}(f_i) + \sum_i \lambda_i(\varphi)\, f_i = 0 \quad \text{in } P.$$

(Of course, λ depends on r also.)

(iii) If $\varphi, \psi \in \operatorname{Hom}_A(I/_{I^2}, A)$ represent two elements of T^1_Y, then we define for each relation $r \in \mathcal{R}$

$$(\varphi \cup \psi)(r) := \sum_i \lambda_i(\varphi)\, \psi(f_i) + \sum_i \varphi(f_i)\, \lambda_i(\psi) \ \in A\,.$$

Remark: The definition of the cup product does not depend on the choices we made:

(a) Choosing a $\lambda'(\varphi)$ instead of $\lambda(\varphi)$ yields $\lambda'(\varphi) - \lambda(\varphi) \in \mathcal{R}$, i.e. in A we obtain the same result.

(b) Let $\tilde{\varphi}'(f_i)$ be different liftings to P. Then, the difference $\tilde{\varphi}'(f_i) - \tilde{\varphi}(f_i)$ is contained in I, i.e. it can be written as some linear combination

$$\tilde{\varphi}'(f_i) - \tilde{\varphi}(f_i) = \sum_j t_{ij}\, f_j\,.$$

Hence,

$$\sum_i r_i\, \tilde{\varphi}'(f_i) = \sum_i r_i\, \tilde{\varphi}(f_i) + \sum_{i,j} t_{ij}\, r_i\, f_j\,,$$

and we can define $\lambda'_j(\varphi) := \lambda_j(\varphi) - \sum_i t_{ij}\, r_i$ (corresponding to $\tilde{\varphi}'$ instead of $\tilde{\varphi}$). Then, we obtain for the cup product

$$(\varphi \cup \psi)'(r) - (\varphi \cup \psi)(r) = -\sum_i r_i \cdot \left(\sum_j t_{ij}\, \psi(f_j) \right)\,,$$

but this expression comes from some map $P^m \to A$.

3 T^1, T^2, and the cup product (for toric varieties)

(3.1) We start with fixing the usual notations when dealing with affine toric varieties (cf. [Ke] or [Od]):

- Let M, N be mutually dual free Abelian groups, we denote by $M_{I\!R}, N_{I\!R}$ the associated real vector spaces obtained by base change with $I\!R$.
- Let $\sigma = \langle a^1, \dots, a^N \rangle \subseteq N_{I\!R}$ be a rational, polyhedral cone with apex - given by its fundamental generators.
  $\sigma^\vee := \{r \in M_{I\!R} \mid \langle \sigma, r \rangle \geq 0\} \subseteq M_{I\!R}$ is called the dual cone. It induces a partial order on the lattice M via $[\, a \geq b$ iff $a - b \in \sigma^\vee \,]$.
- $A := \mathbb{C}[\sigma^\vee \cap M]$ denotes the semigroup algebra. It is the ring of regular functions of the toric variety $Y = \operatorname{Spec} A$ associated to σ.
- Denote by $E \subset \sigma^\vee \cap M$ the minimal set of generators of this semigroup ("the Hilbert basis"). E equals the set of all primitive (i.e. non-splittable) elements of $\sigma^\vee \cap M$. In particular, there is a surjection of semigroups $\pi : I\!N^E \longrightarrow\!\!\!\!\rightarrow \sigma^\vee \cap M$, and this fact translates into a closed embedding $Y \hookrightarrow \mathbb{C}^E$.
  To make the notations coherent with §2, assume that $E = \{r^0, \dots, r^w\}$ consists of $w + 1$ elements.

(3.2) To a fixed degree $R \in M$ we associate "thick facets" K_i^R of the dual cone

$$K_i^R := \{r \in \sigma^\vee \cap M \mid \langle a^i, r \rangle < \langle a^i, R \rangle\} \quad (i = 1, \dots, N).$$

Lemma:

(1) $\cup_i K_i^R = (\sigma^\vee \cap M) \setminus (R + \sigma^\vee)$.

(2) For each $r, s \in K_i^R$ there exists an $\ell \in K_i^R$ such that $\ell \geq r, s$. Moreover, if Y is smooth in codimension 2, the intersections $K_i^R \cap K_j^R$ (for 2-faces $\langle a^i, a^j \rangle < \sigma$) have the same property.

Proof: Part (i) is trivial; for (ii) cf. (3.7) of [Al 1]. □

Intersecting these sets with $E \subseteq \sigma^\vee \cap M$, we obtain the basic objects for describing the modules T_Y^i:

$$\begin{aligned} E_i^R &:= K_i^R \cap E = \{r \in E \mid \langle a^i, r \rangle < \langle a^i, R \rangle\}\,, \\ E_0^R &:= \bigcup_{i=1}^N E_i^R\,, \text{ and} \\ E_\tau^R &:= \bigcap_{a^i \in \tau} E_i^R \text{ for faces } \tau < \sigma\,. \end{aligned}$$

We obtain a complex $L(E^R)_\bullet$ of free Abelian groups via

$$L(E^R)_{-k} := \bigoplus_{\substack{\tau < \sigma \\ \dim \tau = k}} L(E^R_\tau)$$

with the usual differentials. ($L(\dots)$ denotes the free Abelian group of integral, linear dependencies.)
The most interesting part ($k \leq 2$) can be written explicitely as

$$L(E^R)_\bullet : \quad \dots \to \oplus_{\langle a^i, a^j\rangle < \sigma} L(E^R_i \cap E^R_j) \longrightarrow \oplus_i L(E^R_i) \longrightarrow L(E^R_0) \to 0\,.$$

(3.3) Theorem:

(1) $T^1_Y(-R) = H^0\left(L(E^R)^*_\bullet \otimes_{\mathbb{Z}} \mathbb{C}\right)$

(2) $T^2_Y(-R) \supseteq H^1\left(L(E^R)^*_\bullet \otimes_{\mathbb{Z}} \mathbb{C}\right)$

(3) Moreover, if Y is smooth in codimension 2 (i.e. if the 2-faces $\langle a^i, a^j\rangle < \sigma$ are spanned by a part of a $\mathbb{Z}$-basis of the lattice N), then

$$T^2_Y(-R) = H^1\left(L(E^R)^*_\bullet \otimes_{\mathbb{Z}} \mathbb{C}\right)\,.$$

(4) Module structure: If $x^s \in A$ (i.e. $s \in \sigma^\vee \cap M$), then $E^{R-s}_i \subseteq E^R_i$, hence $L(E^R)^*_\bullet \subseteq L(E^{R-s})^*_\bullet$. The induced map in cohomology corresponds to the multiplication with x^s in the A-modules T^1_Y and T^2_Y.

The first part was shown in [Al 1]; the formula for T^2 will be proved in §4. Then, the claim concerning the module structure will become clear automatically.

Remark: The assumption made in (3) can not be dropped:
Taking for Y a 2-dimensional cyclic quotient singularity given by some 2-dimensional cone σ, there are only two different sets E^R_1 and E^R_2 (for each $R \in M$). In particular, $H^1\left(L(E^R)^*_\bullet \otimes_{\mathbb{Z}} \mathbb{C}\right) = 0$.

(3.4) We want to describe the isomorphisms connecting the general T^i-formulas of (2.1) and (2.2) with the toric ones given in (3.3).

$Y \hookrightarrow \mathbb{C}^{w+1}$ is given by the equations

$$f_{ab} := \underline{z}^a - \underline{z}^b \quad (a, b \in \mathbb{N}^{w+1} \text{ with } \pi(a) = \pi(b) \text{ in } \sigma^\vee \cap M),$$

and it is easier to deal with this infinite set of equations (which generates the ideal I as a $\mathbb{C}$-vector space) instead of selecting a finite number of them in some non-canonical way. In particular, for m of (2.1) and (2.2) we take

$$m := \{(a, b) \in \mathbb{N}^{w+1} \times \mathbb{N}^{w+1} \mid \pi(a) = \pi(b)\}\,.$$

The general T^i-formulas mentioned in (2.1) and (2.2) remain true.

Theorem: *For a fixed element $R \in M$ let $\varphi : L(E)_{\mathbb{C}} \to \mathbb{C}$ represent an element of*

$$\left(L(E_0^R) \Big/ \sum_i L(E_i^R) \right)^* \otimes_{\mathbb{Z}} \mathbb{C} \cong T^1_Y(-R) \quad \textit{(cf. Theorem (3.3)(1)).}$$

Then, the A-linear map

$$\begin{array}{rcl} I/_{I^2} & \longrightarrow & A \\ \underline{z}^a - \underline{z}^b & \mapsto & \varphi(a-b) \cdot x^{\pi(a)-R} \end{array}$$

provides the same element via the formula (2.1).

Again, this Theorem follows from the paper [Al 1] - together with the commutative diagram of (4.3) in the present paper. (Cf. Remark (4.4).)

Remark: A simple, but nevertheless important check shows that the map $(\underline{z}^a - \underline{z}^b) \mapsto \varphi(a-b) \cdot x^{\pi(a)-R}$ goes into A, indeed:
Assume $\pi(a) - R \notin \sigma^\vee$. Then, there exists an index i such that $\langle a^i, \pi(a) - R \rangle < 0$. Denoting by "supp q" (for a $q \in \mathbb{R}^E$) the set of those $r \in E$ providing a non-vanishing entry q_r, we obtain

$$\operatorname{supp}(a-b) \subseteq \operatorname{supp} a \cup \operatorname{supp} b \subseteq E_i^R\,,$$

i.e. $\varphi(a-b) = 0$.

(3.5) The P-module $\mathcal{R} \subseteq P^m$ is generated by relations of two different types:

$$\begin{array}{rcll} r(a,b;c) & := & e^{a+c,\,b+c} - \underline{z}^c\, e^{a,b} & (a,b,c \in \mathbb{N}^{w+1};\ \pi(a) = \pi(b)) \quad \text{and} \\ s(a,b,c) & := & e^{b,c} - e^{a,c} + e^{a,b} & (a,b,c \in \mathbb{N}^{w+1};\ \pi(a) = \pi(b) = \pi(c))\,. \end{array}$$

($e^{\bullet,\bullet}$ denote the standard basis vectors of P^m.)

Theorem: *For a fixed element $R \in M$ let $\psi_i : L(E_i^R)_{\mathbb{C}} \to \mathbb{C}$ represent an element of*

$$\left(\frac{Ker\ \left(\oplus_i L(E_i^R) \longrightarrow L(E'^R)\right)}{Im\ \left(\oplus_{\langle a^i, a^j \rangle < \sigma} L(E_i^R \cap E_j^R) \to \oplus_i L(E_i^R)\right)} \right)^* \otimes_{\mathbb{Z}} \mathbb{C} \subseteq T^2_Y(-R) \quad \textit{(cf. (3.3)(2)).}$$

Then, the P-linear map

$$\begin{array}{rcl} \mathcal{R}/_{\mathcal{R}_0} & \longrightarrow & A \\ r(a,b;c) & \mapsto & \begin{cases} \psi_i(a-b)\, x^{\pi(a+c)-R} & \textit{for } \pi(a) \in K_i^R;\ \pi(a+c) \geq R \\ 0 & \textit{for } \pi(a) \geq R \textit{ or } \pi(a+c) \in \bigcup_i K_i^R \end{cases} \\ s(a,b,c) & \mapsto & 0 \end{array}$$

is correct defined, and via the formula (2.2) it induces the same element of T^2_Y.

For the proof we refer to §4. However, we may easily check the *correctness of the definition* of the P-linear map ${}^{\mathcal{R}}/_{\mathcal{R}_0} \to A$:

(i) If $\pi(a)$ is contained in two different sets K^R_i and K^R_j, then the two fundamental generators a^i and a^j can be connected by a sequence $a^{i_0}, \dots, a^{i_p}$, such that

- $a^{i_0} = a^i$, $a^{i_p} = a^j$,
- $a^{i_{v-1}}$ and a^{i_v} are the edges of some 2-face of σ ($v = 1, \dots, p$), and
- $\pi(a) \in K^R_{i_v}$ for $v = 0, \dots, p$.

Hence, $\operatorname{supp}(a-b) \subseteq E^R_{i_{v-1}} \cap E^R_{i_v}$ ($v = 1, \dots, p$), and we obtain

$$\psi_i(a-b) = \psi_{i_1}(a-b) = \dots = \psi_{i_{p-1}}(a-b) = \psi_j(a-b)\,.$$

(ii) There are three types of P-linear relations between the generators $r(\dots)$ and $s(\dots)$ of $\mathcal{R}$:

$$\begin{aligned} 0 &= \underline{z}^d\, r(a,b;c) - r(a,b;c+d) + r(a+c,b+c;d)\,, \\ 0 &= r(b,c;d) - r(a,c;d) + r(a,b;d) - s(a+d,b+d,c+d) + \underline{z}^d\, s(a,b,c)\,, \\ 0 &= s(b,c,d) - s(a,c,d) + s(a,b,d) - s(a,b,c)\,. \end{aligned}$$

Our map respects them all.

(iii) Finally, the typical element $(\underline{z}^a - \underline{z}^b)e^{cd} - (\underline{z}^c - \underline{z}^d)e^{ab} \in \mathcal{R}_0$ equals

$$-r(c,d;a)+r(c,d;b)+r(a,b;c)-r(a,b;d)-s(a+c,b+c,a+d)+s(b+c,a+d,b+d)\,.$$

It will be sent to 0, too.

(3.6) The cup product $T^1_Y \times T^1_Y \to T^2_Y$ respects the grading, i.e. it splits into pieces

$$T^1_Y(-R) \times T^1_Y(-S) \longrightarrow T^2_Y(-R-S) \quad (R,S \in M)\,.$$

To describe these maps in our combinatorial language, we choose a set-theoretical section $\Phi: M \to \mathbb{Z}^{w+1}$ of the $\mathbb{Z}$-linear map

$$\begin{aligned} \pi: \mathbb{Z}^{w+1} &\longrightarrow M \\ a &\mapsto \sum_v a_v\, r^v \end{aligned}$$

with the additional property $\Phi(\sigma^\vee \cap M) \subseteq \mathbb{N}^{w+1}$.

Let $q \in L(E) \subseteq \mathbb{Z}^{w+1}$ be an integral relation between the generators of the semigroup $\sigma^\vee \cap M$. We introduce the following notations:

- $q^+, q^- \in I\!N^{w+1}$ denote the positive and the negative part of q, respectively. (With other words: $q = q^+ - q^-$ and $\sum_v q_v^- \, q_v^+ = 0$.)
- $\bar{q} := \pi(q^+) = \sum_v q_v^+ \, r^v = \sum_v q_v^- \, r^v = \pi(q^-) \in M$.
- If $\varphi, \psi : L(E) \to Z\!\!Z$ are linear maps and $R, S \in M$, then we define

  $$t_{\varphi,\psi,R,S}(q) := \varphi(q) \cdot \psi\left(\Phi(\bar{q} - R) + \Phi(R) - q^-\right) + \psi(q) \cdot \varphi\left(\Phi(\bar{q} - S) + \Phi(S) - q^+\right) .$$

  (For an easier description in the most important special cases see the upcoming Proposition (3.7).)

Theorem: *Assume that Y is smooth in codimension 2.*
Let $R, S \in M$, and let $\varphi, \psi : L(E)_{C\!\!\!\!I} \to C\!\!\!\!I$ be linear maps vanishing on $\sum_i L(E_i^R)_{C\!\!\!\!I}$ and $\sum_i L(E_i^S)_{C\!\!\!\!I}$, respectively. In particular, they define elements $\varphi \in T_Y^1(-R)$, $\psi \in T_Y^1(-S)$ (which involves a slight abuse of notations).
Then, the cup product $\varphi \cup \psi \in T_Y^2(-R-S)$ is given (via (3.3)(3)) by the linear maps $(\varphi \cup \psi)_i : L(E_i^{R+S})_{C\!\!\!\!I} \to C\!\!\!\!I$ defined as follows:

(i) If $q \in L(E_i^{R+S})$ (i.e. $\langle a^i, \operatorname{supp} q \rangle < \langle a^i, R+S \rangle$) is an integral relation, then there exists a decomposition $q = \sum_k q^k$ such that

  - *$q^k \in L(E_i^{R+S})$, and moreover*
  - *$\langle a^i, \bar{q}^k \rangle < \langle a^i, R+S \rangle$.*

(ii) $(\varphi \cup \psi)_i \left(q \in L(E_i^{R+S})\right) := \sum_k t_{\varphi,\psi,R,S}(q^k)$.

First, we must show that the map $q \mapsto \sum_k t(q^k)$

- does not depend on the representation of q as a particular sum of q^k's (which implies linearity on $L(E_i^{R+S})$), and
- yields the same result on $L(E_i^{R+S} \cap E_j^{R+S})$ for i, j corresponding to edges a^i, a^j of some 2-face of σ.

The proof of these facts (cf. (5.4)) and of the entire theorem is contained in §5.

Remark: Replacing all the terms $\Phi(\bullet)$ in the t's of the previous formula for $(\varphi \cup \psi)_i(q)$ by arbitrary liftings from M to $Z\!\!Z^{w+1}$, the result in $T_Y^2(-R-S)$ will be unchanged as long as we obey the following two rules:

(i) Use always (for all q, q^k, and i) the *same liftings* of R and S to $Z\!\!Z^{w+1}$ (at the places of $\Phi(R)$ and $\Phi(S)$, respectively).

(ii) Elements of $\sigma^\vee \cap M$ always have to be lifted to $I\!N^{w+1}$.

Proof: Replacing $\Phi(R)$ by $\Phi(R)+d$ ($d \in L(E)$) at each occurence changes all maps $(\varphi \cup \psi)_i$ by the summand $\psi(d) \cdot \varphi(\bullet)$. However, this additional linear map comes from $L(E)^*$, hence it is trivial on $\operatorname{Ker}\big(\oplus_i L(E_i^{R+S}) \to L(E_0^{R+S}) \subseteq L(E)\big)$.

Let us look at the terms $\Phi(\bar{q}-R)$ in $t(q)$ now: Unless $\bar{q} \geq R$, the factor $\varphi(q)$ vanishes (cf. Remark (3.4)). On the other hand, the expression $t(q)$ is never used for those relations q satisfying $\bar{q} \geq R+S$ (cf. conditions for the q^k's). Hence, we may assume that

$$(\bar{q}-R) \geq 0 \;\text{ and, moreover, }\; (\bar{q}-R) \in \bigcup_i K_i^S\,.$$

Now, each two liftings of $(\bar{q}-R)$ to $I\!N^{w+1}$ differ by an element of $\operatorname{Ker}\psi$ only (apply the method of Remark (3.4) again), in particular, they cause the same result for $t(q)$. □

(3.7) **Proposition:** *In the special case of $R \geq S \geq 0$ we can choose liftings $\Phi(R) \geq \Phi(S) \geq 0$ in $I\!N^{w+1}$. Then, there exists an easier description for $t(q)$:*

(i) Unless $\bar{q} \geq R$, we have $t(q) = 0$.

(ii) In case of $\bar{q} \geq R$ we may assume that $q^+ \geq \Phi(R)$ is true in $I\!N^{w+1}$. (The general q's are differences of those ones.) Then, t can be computed as the product $t(q) = \varphi(q)\,\psi(q)$.

Proof: (i) As used many times, the property $\bar{q} \in \bigcup_i E_i^R$ implies $\varphi(q) = 0$. Now, we can distinguish between two cases:
Case 1: $\bar{q} \in \bigcup_i E_i^S$. We obtain $\psi(q) = 0$, in particular, both summands of $t(q)$ vanish.
Case 2: $\bar{q} \geq S$. Then, $\bar{q}-S$, $S \in \sigma^\vee \cap M$, and Φ lifts these elements to $I\!N^{w+1}$. Now, the condition $\bar{q} \in \bigcup_i E_i^R$ implies that $\varphi\left(\Phi(\bar{q}-S) + \Phi(S) - q^+\right) = 0$.

(ii) We can choose $\Phi(\bar{q}-R) := q^+ - \Phi(R)$ and $\Phi(\bar{q}-S) := q^+ - \Phi(S)$. Then, the claim follows straight forward. □

4 Proof of the T^2-formula

(4.1) We will use the sheaf $\Omega^1_Y = \Omega^1_{A|\mathbb{C}}$ of Kähler differentials for computing the modules T^i_Y. The maps

$$\alpha_i : \operatorname{Ext}^i_A\left(\Omega^1_Y \big/ \operatorname{tors}(\Omega^1_Y)\,,\, A\right) \hookrightarrow \operatorname{Ext}^i_A\left(\Omega^1_Y,\, A\right) \cong T^i_Y \quad (i = 1,2)$$

are injective. Moreover, they are isomorphisms for

- $i = 1$, since Y is normal, and for
- $i = 2$, if Y is smooth in codimension 2.

(4.2) As in [Al 1], we build a special A-free resolution (one step further now)

$$\mathcal{E} \xrightarrow{d_E} \mathcal{D} \xrightarrow{d_D} \mathcal{C} \xrightarrow{d_C} \mathcal{B} \xrightarrow{d_B} \Omega^1_Y \big/ \mathrm{tors}\,(\Omega^1_Y) \to 0\,.$$

With $L^2(E) := L(L(E))$, $L^3(E) := L(L^2(E))$, and

$$\mathrm{supp}^2\xi := \bigcup_{q\in supp\,\xi} \mathrm{supp}\,q \quad (\xi \in L^2(E)), \quad \mathrm{supp}^3\omega := \bigcup_{\xi\in supp\,\omega} \mathrm{supp}^2\xi \quad (\omega \in L^3(E)),$$

the A-modules involved in this resolution are defined as follows:

$$\mathcal{B} := \oplus_{r\in E} A\cdot B(r), \qquad \mathcal{C} := \oplus_{\substack{q\in L(E)\\ \ell\geq supp\,q}} A\cdot C(q;\ell),$$

$$\mathcal{D} := \Big(\oplus_{\substack{q\in L(E)\\ \eta\geq\ell\geq supp\,q}} A\cdot D(q;\ell,\eta)\Big) \oplus \Big(\oplus_{\substack{\xi\in L^2(E)\\ \eta\geq supp^2\xi}} A\cdot D(\xi;\eta)\Big), \quad \text{and}$$

$$\mathcal{E} := \Big(\oplus_{\substack{q\in L(E)\\ \mu\geq\eta\geq\ell\geq supp\,q}} A\cdot E(q;\ell,\eta,\mu)\Big) \oplus \Big(\oplus_{\substack{\xi\in L^2(E)\\ \mu\geq\eta\geq supp^2\xi}} A\cdot E(\xi;\eta,\mu)\Big) \oplus \Big(\oplus_{\substack{\omega\in L^3(E)\\ \mu\geq supp^3\omega}} A\cdot E(\omega;\mu)\Big)$$

(B, C, D, and E are just symbols). The differentials equal

$$\begin{array}{rccl}
d_B: & B(r) & \mapsto & d\,x^r\,; \\
d_C: & C(q;\ell) & \mapsto & \sum_{r\in E} q_r\, x^{\ell-r}\cdot B(r)\,; \\
d_D: & D(q;\ell,\eta) & \mapsto & C(q;\eta) - x^{\eta-\ell}\cdot C(q;\ell)\,, \\
d_D: & D(\xi;\eta) & \mapsto & \sum_{q\in L(E)} \xi_q\cdot C(q;\eta)\,; \qquad \text{and} \\
d_E: & E(q;\ell,\eta,\mu) & \mapsto & D(q;\eta,\mu) - D(q;\ell,\mu) + x^{\mu-\eta}\cdot D(q;\ell,\eta)\,, \\
d_E: & E(\xi;\eta,\mu) & \mapsto & D(\xi;\mu) - x^{\mu-\eta}\cdot D(\xi;\eta) - \sum_{q\in L(E)} \xi_q\cdot D(q;\eta,\mu)\,, \\
d_E: & E(\omega;\mu) & \mapsto & \sum_{\xi\in L^2(E)} \omega_\xi\cdot D(\xi;\mu)\,.
\end{array}$$

Looking at these maps, we see that the complex is M-graded: The degree of each of the elements B, C, D, or E can be obtained by taking the last of its parameters (r, ℓ, η, or μ, respectively).

Remark: If one prefered a resolution with free A-modules of finite rank (as it was used in [Al 1]), the following replacements would be necessary:

(i) Define succesively $F \subseteq L(E)$, $G \subseteq L(F) \subseteq L^2(E)$, and $H \subseteq L(G) \subseteq L^2(F) \subseteq L^3(E)$ as the finite sets of normalized, minimal relations between elements of E, F, or G, respectively. Then, use them instead of $L^i(E)$ $(i = 1, 2, 3)$.

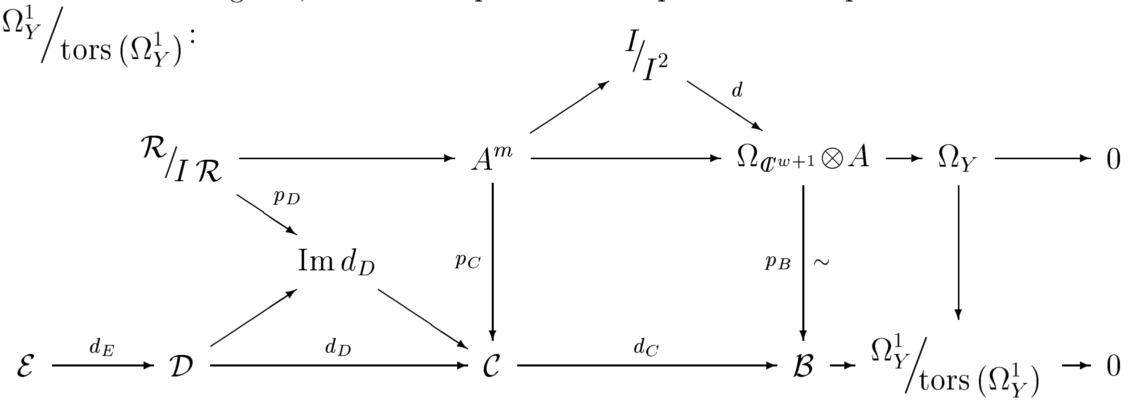
(ii) Let ℓ, η, and μ run through finite systems generating (via the $(\sigma^\vee \cap M)$-action) all possible elements meeting the desired inequalities.

The disadvantages of this treatment are a more comlplicated description of the resolution, on the one hand, and difficulties to obtain the commutative diagram (4.3), on the other hand.

(4.3) Combining the two exact sequences

$$\mathcal{R}/_{I\,\mathcal{R}} \longrightarrow A^m \longrightarrow I/_{I^2} \to 0 \quad \text{and} \quad I/_{I^2} \longrightarrow \Omega^1_{\mathbb{C}^{w+1}} \otimes A \longrightarrow \Omega^1_Y \to 0\,,$$

we get a complex (not exact at the place of A^m) involving Ω^1_Y. In the following commutative diagram, we will compare this complex with the previous resolution of $\Omega^1_Y/_{\text{tors}\,(\Omega^1_Y)}$:



Let us explain the three labeled vertical maps:

(B) $p_B : dz_r \mapsto B(r)$ is an isomorphism between two free A-modules of rank $w+1$.

(C) $p_C : e^{ab} \mapsto C(a-b; \pi(a))$. In particular, the image of this map is spanned by those $C(q, \ell)$ meeting $\ell \geq \bar{q}$ (which is stronger than just $\ell \geq \operatorname{supp} q$).

(D) Finally, p_D arises as pull back of p_C to $\mathcal{R}/_{I\,\mathcal{R}}$. It can be described by $r(a, b; c) \mapsto D(a-b; \pi(a), \pi(a+c))$ and $s(a, b, c) \mapsto D(\xi; \pi(a))$ (ξ denotes the relation $\xi = [(b-c) - (a-c) + (a-b) = 0]$).

Remark: Starting with the typical $\mathcal{R}_0$-element mentioned in (3.5)(iii), the previous description of the map p_D yields 0 (even in $\mathcal{D}$).

(4.4) By (4.1) we get the A-modules T^i_Y by computing the cohomology of the complex dual to that of (4.2).

As in [Al 1], denote by G one of the capital letters B, C, D, or E. Then, an element ψ of the dual free module $(\bigoplus_G \mathbb{C}[\check{\sigma}\cap M]\cdot G)^*$ can be described by giving elements $g(x)\in\mathbb{C}[\check{\sigma}\cap M]$ to be the images of the generators G (g stands for b, c, d, or e, respectively).

For ψ to be homogeneous of degree $-R\in M$, $g(x)$ has to be a monomial of degree

$$\deg g(x) = -R + \deg G.$$

In particular, the corresponding complex coefficient $g\in\mathbb{C}$ (i.e. $g(x)=g\cdot x^{-R+\deg G}$) admits the property that

$$g\neq 0 \quad \text{implies} \quad -R+\deg G\geq 0 \quad (\text{i.e. } -R+\deg G\in\check{\sigma}).$$

Remark: Using these notations, Theorem (3.3)(1) was proved in [Al 1] by showing that

$$\begin{array}{ccc} \left(L(E_0^R) \Big/ \sum_i L(E_i^R) \right)^* \otimes_{\mathbb{Z}} \mathbb{C} & \longrightarrow & \operatorname{Ker}(\mathcal{C}^*_{-R}\to\mathcal{D}^*_{-R}) \Big/ \operatorname{Im}(\mathcal{B}^*_{-R}\to\mathcal{C}^*_{-R}) \\ \varphi & \mapsto & [\ldots, c(q;\ell) := \varphi(q), \ldots] \end{array}$$

is an isomorphism of vector spaces.
Moreover, looking at the diagram of (4.3), $e^{ab}\in A^m$ maps to both $\underline{z}^a-\underline{z}^b\in {}^I\!/_{I^2}$ and $C(a-b;\pi(a))\in\mathcal{C}$. In particular, we can verify Theorem (3.4): Each $\varphi: L(E)_{\mathbb{C}}\to\mathbb{C}$ induces the same element of T^1_Y as its associated A-linear map

$$\begin{array}{ccc} {}^I\!/_{I^2} & \longrightarrow & A \\ \underline{z}^a-\underline{z}^b & \mapsto & \varphi(a-b)\cdot x^{\pi(a)-R}, \end{array}$$

on the other hand, induce the same element of $T^1_Y(-R)$.

(4.5) For computing $T^2_Y(-R)$, the interesting part of the dualized complex $(4.2)^*$ in degree $-R$ equals the complex of $\mathbb{C}$-vector spaces

$$\mathcal{C}^*_{-R} \xrightarrow{d^*_D} \mathcal{D}^*_{-R} \xrightarrow{d^*_E} \mathcal{E}^*_{-R}$$

with coordinates $\underline{c}$, $\underline{d}$, and $\underline{e}$, respectively:

$$\begin{array}{rcl} \mathcal{C}^*_{-R} & = & \{\underline{c(q;\ell)} \mid c(q;\ell)=0 \text{ for } \ell-R\notin\check{\sigma}\} \\ \mathcal{D}^*_{-R} & = & \{[\underline{d(q;\ell,\eta)}, \underline{d(\xi;\eta)}] \mid \begin{array}{rcl} d(q;\ell,\eta) & = & 0 \;\text{ for }\; \eta-R\notin\check{\sigma}, \text{ and} \\ d(\xi;\eta) & = & 0 \;\text{ for }\; \eta-R\notin\check{\sigma}\} \end{array} \\ \mathcal{E}^*_{-R} & = & \{[\underline{e(q;\ell,\eta,\mu)}, \underline{e(\xi;\eta,\mu)}, \underline{e(\omega;\mu)}] \mid \text{each coordinate vanishes for } \mu-R\notin\check{\sigma}\}. \end{array}$$

The differentials d_D^* and d_E^* can be described by

$$\begin{array}{lcl} d(q;\ell,\eta) & = & c(q;\eta)-c(q;\ell), \\ d(\xi;\eta) & = & \sum\limits_{q\in L(E)} \xi_q\cdot c(q;\eta), \qquad\qquad \text{and} \\ e(q;\ell,\eta,\mu) & = & d(q;\eta,\mu)-d(q;\ell,\mu)+d(q;\ell,\eta), \\ e(\xi;\eta,\mu) & = & d(\xi;\mu)-d(\xi;\eta)-\sum_{q\in F}\xi_q\cdot d(q;\eta,\mu), \\ e(\omega;\mu) & = & \sum\limits_{\xi\in L^2(E)} \omega_\xi\cdot d(\xi;\mu). \end{array}$$

Denote $V:=\operatorname{Ker} d_E^*\subseteq \mathcal{D}^*_{-R}$ and $W:=\operatorname{Im} d_D^*\subseteq V$, i.e.

$$\begin{array}{lcll} V & = & \{[\underline{d(q;\ell,\eta)};\,\underline{d(\xi;\eta)}]\,| & q\in L(E),\ \eta\geq\ell\geq\operatorname{supp} q \text{ in } M; \\ & & & \xi\in L^2(E),\ \eta\geq\operatorname{supp}^2\xi; \\ & & & d(q;\ell,\eta)=d(\xi;\eta)=0 \text{ for } \eta-R\notin\check\sigma, \\ & & & d(q;\ell,\mu)=d(q;\ell,\eta)+d(q;\eta,\mu)\ (\mu\geq\eta\geq\ell\geq\operatorname{supp} q), \\ & & & d(\xi;\mu)=d(\xi;\eta)+\sum_q\xi_q\cdot d(q;\eta,\mu)\ (\mu\geq\eta\geq\operatorname{supp}^2\xi), \\ & & & \sum_{\xi\in L^2(E)}\omega_\xi\, d(\xi;\mu)=0 \text{ for } \omega\in L^3(E) \text{ with } \mu\geq\operatorname{supp}^3\omega\,\}, \\ W & = & \{[\underline{d(q;\ell,\eta)};\,\underline{d(\xi;\eta)}]\,| & \text{there are } c(q;\ell)\text{'s with } c(q,\ell)=0 \text{ for } \ell-R\notin\check\sigma, \\ & & & \qquad d(q;\ell,\eta)=c(q;\eta)-c(q;\ell), \\ & & & \qquad d(\xi;\eta)=\sum_q\xi_q\cdot c(q;\eta)\,\}. \end{array}$$

By construction, we obtain

$$V\big/W = \operatorname{Ext}^i_A\left(\Omega^1_Y\big/\operatorname{tors}(\Omega^1_Y)\,,\,A\right)(-R)\subseteq T^2_Y(-R)$$

(which is an isomorphism, if Y is smooth in codimension 2).

(4.6) Let us define the much easier vector spaces

$$\begin{array}{lcl} V_1 & := & \{[\underline{x_i(q)}_{(q\in L(E_i^R))}]\,|\ x_i(q)=x_j(q) \text{ for } \bullet\,\langle a^i,a^j\rangle<\sigma \text{ is a 2-face and } \bullet\, q\in L(E_i^R\cap E_j^R)\,, \\ & & \qquad \xi\in L^2(E_i^R) \text{ implies } \sum_q\xi_q\cdot x_i(q)=0\,\} \ \text{ and} \\ W_1 & := & \{[\underline{x(q)}_{(q\in\cup_i L(E_i^R))}]\,|\ \xi\in L(\bigcup_i L(E_i^R)) \text{ implies } \sum_q\xi_q\cdot x(q)=0\,\}. \end{array}$$

Lemma: *The linear map $V_1\to V$ defined by*

$$\begin{array}{rcl} d(q;\ell,\eta) & := & \left\{\begin{array}{lll} x_i(q) & \textit{for} & \ell\in K_i^R,\ \eta\geq R \\ 0 & \textit{for} & \ell\geq R \ \textit{ or } \ \eta\in\bigcup_i K_i^R\,; \end{array}\right. \\ d(\xi;\eta) & := & 0 \end{array}$$

induces an injective map

$$V_1\big/W_1 \hookrightarrow V\big/W\ .$$

If Y is smooth in codimension 2, it will be an isomorphism.

Proof: 1) The map $V_1 \to V$ is *correct defined*: On the one hand, an argument as used in (3.5)(i) shows that $\ell \in K_i^R \cap K_j^R$ would imply $x_i(q) = x_j(q)$. On the other hand, the image of $[x_i(q)]_{q\in L(E_i^R)}$ meets all conditions in the definition of V.

2) W_1 maps to W (take $c(q,\ell) := x(q)$ for $\ell \geq R$ and $c(q,\ell) := 0$ otherwise).

3) The map between the two factor spaces is *injective*: Assume for $[x_i(q)]_{q\in L(E_i^R)}$ that there exist elements $c(q,\ell)$, such that

$$\begin{aligned} c(q;\ell) &= 0 \;\text{ for } \ell \in \bigcup_i K_i^R\,, \\ x_i(q) &= c(q;\eta) - c(q;\ell) \;\text{ for } \eta \geq \ell,\ \ell \in K_i^R,\ \eta \geq R\,, \\ 0 &= c(q;\eta) - c(q;\ell) \;\text{ for } \eta \geq \ell \text{ and } [\ell \geq R \text{ or } \eta \in \bigcup_i K_i^R]\,,\ \text{and} \\ 0 &= \sum_q \xi_q \cdot c(q;\eta) \;\text{ for } \eta \geq \operatorname{supp}^2\xi\,. \end{aligned}$$

In particular, $x_i(q)$ do not depend on i, and these elements have the property

$$\sum_q \xi_q \cdot x_\bullet(q) = 0 \;\text{ for } \xi \in L(\bigcup_i L(E_i^R)).$$

4) If Y is smooth in codimension 2, the map is *surjective* :
Given an element $[d(q;\ell,\eta),\, d(\xi;\eta)] \in V$, there exist complex numbers $c(q;\eta)$ such that:

(i) $d(\xi;\eta) = \sum_q \xi_q \cdot c(q;\eta)$,

(ii) $c(q;\eta) = 0$ for $\eta \notin R + \sigma^\vee$ (i.e. $\eta \in \bigcup_i K_i^R$) .

(Do this separately for each η and distinguish between the cases $\eta \in R + \sigma^\vee$ and $\eta \notin R + \sigma^\vee$.)
In particular, $[c(q;\eta) - c(q;\ell),\, d(\xi;\eta)] \in W$. Hence, we have seen that we may assume $d(\xi;\eta) = 0$.

Let us choose some sufficiently high degree $\ell^* \geq E$. Then,

$$x_i(q) := d(q;\ell,\eta) - d(q;\ell^*,\eta)$$

(with $\ell \in K_i^R$, $\ell \geq \operatorname{supp} q$ (cf. Lemma (3.2)(2)), and $\eta \geq \ell, \ell^*, R$) defines some preimage:

(i) It is independent from the choice of η: Using a different η' generates the difference $d(q;\eta,\eta') - d(q;\eta,\eta')$.

(ii) It is independent from $\ell \in K_i^R$: Choosing another $\ell' \in K_i^R$ with $\ell' \geq \ell$ would add the summand $d(q;\ell,\ell')$, which is 0; for the general case use Lemma (3.2)(2).

(iii) If $\langle a^i, a^j\rangle < \sigma$ is a 2-face with $\operatorname{supp} q \subseteq L(E_i^R) \cap L(E_j^R)$, then by Lemma (3.2)(2) we can choose an $\ell \in K_i^R \cap K_j^R$ achieving $x_i(q) = x_j(q)$.

(iv) For $\xi \in L^2(E_i^R)$ we have

$$\sum_q \xi_q \cdot d(q;\ell,\eta) = \sum_q \xi_q \cdot d(q;\ell^*,\eta) = 0\,,$$

and this gives the corresponding relation for the $x_i(q)'$s.

(v) Finally, if we apply to $[\underline{x_i(q)}] \in V_1$ the linear map $V_1 \to V$, the result differs from $[d(q;\ell,\eta), 0] \in V$ by the W-element built from

$$c(q;\ell) := \begin{cases} d(q;\ell,\eta) - d(q;\ell^*,\eta) & \text{if } \ell \geq R \\ 0 & \text{otherwise}\,. \end{cases}$$

□

(4.7) Now, it is easy to complete the proofs for Theorem (3.3) (part 2 and 3) and Theorem (3.5):

First, for a tuple $[\underline{x_i(q)}]_{q\in L(E_i^R)}$, the condition

$$\xi \in L^2(E_i^R) \text{ implies } \sum_q \xi_q \cdot x_i(q) = 0$$

is equivalent to the fact the components $x_i(q)$ are induced by elements $x_i \in L(E_i^R)^*_{\mathbb{C}}$. The other condition for elements of V_1 just says that for 2-faces $\langle a^i, a^j\rangle < \sigma$ there is $x_i = x_j$ on $L(E_i^R \cap E_j^R)_{\mathbb{C}} = L(E_i^R)_{\mathbb{C}} \cap L(E_j^R)_{\mathbb{C}}$. In particular, we obtain

$$V_1 = \operatorname{Ker}\left(\oplus_i L(E_i^R)^*_{\mathbb{C}} \to \oplus_{\langle a^i,a^j\rangle<\sigma} L(E_i^R \cap E_j^R)^*_{\mathbb{C}}\right)\,.$$

In the same way we get

$$W_1 = \left(\sum_i L(E_i^R)_{\mathbb{C}}\right)^*\,,$$

and our T^2-formula is proven.

Finally, if $\psi_i : L(E_i^R)_{\mathbb{C}} \to \mathbb{C}$ are linear maps defining an element of V_1, they induce the following A-linear map on $\mathcal{D}$ (even on $\operatorname{Im} d_D$):

$$\begin{array}{rcl} D(q;\ell,\eta) & \mapsto & \begin{cases} \psi_i(q)\cdot x^{\eta-R} & \text{for } \ \ell \in K_i^R,\ \ \eta \geq R \\ 0 & \text{for } \ \ell \geq R \ \text{ or } \ \eta \in \bigcup_i K_i^R \end{cases} \\ D(\xi;\eta) & \mapsto & 0\,. \end{array}$$

Now, looking at the diagram of (4.3), this translates exactly into the claim of Theorem (3.5).

5 Proof of the cup product formula

(5.1) Fix an $R \in M$, and let $\varphi \in L(E)^*_{\mathbb{C}}$ represent an element (also denoted by φ) of $T^1_Y(-R)$. Using the notations of (2.3), (3.4), and (3.6) we can take

$$\tilde{\varphi}(f_{\alpha\beta}) := \varphi(\alpha - \beta) \cdot \underline{z}^{\Phi(\pi(\alpha)-R)}$$

for the auxiliary P-elements needed to compute the $\lambda(\varphi)$'s (cf. Theorem (3.4)).

Now, we have to distinguish between the two several types of relations generating the P-module $\mathcal{R} \subseteq P^m$:

(r) Regarding the relation $r(a,b;c)$ we obtain

$$\begin{aligned} \sum_{(\alpha,\beta)\in m} r(a,b;c)_{\alpha\beta} \cdot \tilde{\varphi}(f_{\alpha\beta}) &= \tilde{\varphi}(f_{a+c,b+c}) - \underline{z}^c\, \tilde{\varphi}(f_{ab}) \\ &= \varphi(a-b) \cdot \left(\underline{z}^{\Phi(\pi(a+c)-R)} - \underline{z}^{c+\Phi(\pi(a)-R)}\right) \\ &= \varphi(a-b) \cdot f_{\Phi(\pi(a+c)-R),\, c+\Phi(\pi(a)-R)}\,. \end{aligned}$$

In particular,

$$\lambda^{r(a,b;c)}_{\alpha\beta}(\varphi) = \begin{cases} \varphi(a-b) & \text{for } [\alpha,\beta] = [c+\Phi(\pi(a)-R),\, \Phi(\pi(a+c)-R)] \\ 0 & \text{otherwise}\,. \end{cases}$$

(s) The corresponding result for the relation $s(a,b,c)$ is much nicer:

$$\begin{aligned} \sum_{(\alpha,\beta)\in m} s(a,b,c)_{\alpha\beta} \cdot \tilde{\varphi}(f_{\alpha\beta}) &= \tilde{\varphi}(f_{bc}) - \tilde{\varphi}(f_{ac}) + \tilde{\varphi}(f_{ab}) \\ &= [\varphi(b-c) - \varphi(a-c) + \varphi(a-b)] \cdot \underline{z}^{\Phi(\pi(a)-R)} \\ &= 0\,. \end{aligned}$$

In particular, $\lambda^{s(a,b,c)}(\varphi) = 0$.

(5.2) Now, let R, S, φ, and ψ as in the assumption of Theorem (3.6). Using formula (2.3)(iii), our previous computations yield $(\varphi \cup \psi)(s(a,b,c)) = 0$ and

$$\begin{aligned} (\varphi\cup\psi)(r(a,b;c)) &= \textstyle\sum_{\alpha,\beta} \lambda^{r(a,b;c)}_{\alpha\beta}(\varphi)\cdot\psi(f_{\alpha\beta}) + \sum_{\alpha,\beta}\varphi(f_{\alpha\beta})\cdot\lambda^{r(a,b;c)}_{\alpha\beta}(\psi) \\ &= \varphi(a-b)\cdot\psi\left(c+\Phi(\pi(a)-R)-\Phi(\pi(a+c)-R)\right)\cdot x^{\pi(c+\Phi(\pi(a)-R))-S} + \\ &\quad +\psi(a-b)\cdot\varphi\left(c+\Phi(\pi(a)-S)-\Phi(\pi(a+c)-S)\right)\cdot x^{\pi(c+\Phi(\pi(a)-S))-R} \\ &= [\varphi(a-b)\cdot\psi\left(c+\Phi(\pi(a)-R)-\Phi(\pi(a+c)-R)\right) + \\ &\quad +\psi(a-b)\cdot\varphi\left(c+\Phi(\pi(a)-S)-\Phi(\pi(a+c)-S)\right)]\cdot x^{\pi(a+c)-R-S}\,. \end{aligned}$$

Remark: Unless $\pi(a+c) \geq R+S$, both summand in the brackets will vanish. For instance, on the one hand, $\pi(a) \in \bigcup_i K_i^R$ would cause $\varphi(a-b) = 0$, and, on the other hand, $\pi(a) - R \geq 0$ and $\pi(c + \Phi(\pi(a) - R)) \in \bigcup_i K_i^S$ imply $\psi(c + \Phi(\pi(a) - R) - \Phi(\pi(a+c) - R)) = 0$.

To apply Theorem (3.5) we would like to remove the argument c from the big coefficient. This will be done by adding a suitable coboundary T.

(5.3) Let us start with defining for $(\alpha, \beta) \in m$

$$\begin{aligned} t(\alpha,\beta) := \; & \varphi(\alpha-\beta) \cdot \psi\left(\Phi(\pi(\alpha) - R) + \Phi(R) - \beta\right) + \\ & + \psi(\alpha-\beta) \cdot \varphi\left(\Phi(\pi(\alpha) - S) + \Phi(S) - \alpha\right). \end{aligned}$$

(This expression is related to $t_{\varphi,\psi,R,S}$ from (3.6) by $t(q) = t(q^+, q^-)$.)

Lemma: *Let $\alpha, \beta, \gamma \in \mathbb{N}^E$ with $\pi(\alpha) = \pi(\beta) = \pi(\gamma)$.*

(1) $t(\alpha, \beta) = t(\alpha - \beta)$ as long as $\pi(\alpha) \in \bigcup_i K_i^{R+S}$.

(2) $t(\beta, \gamma) - t(\alpha, \gamma) + t(\alpha, \beta) = 0$.

Proof: (1) It is enough to show that $t(\alpha + r, \beta + r) = t(\alpha, \beta)$ for $r \in \mathbb{N}^E$, $\pi(\alpha + r) \in \bigcup_i K_i^{R+S}$. But the difference of these two terms has exactly the shape of the coefficient of $x^{\pi(a+c)-R-S}$ in (5.2). In particular, the argument given in the previous remark applies again.

(2) By extending φ and ψ to linear maps $\mathbb{C}^E \to \mathbb{C}$, we obtain

$$\begin{aligned} t(\alpha,\beta) = \; & [\varphi(\alpha-\beta)\, \psi\left(\Phi(\pi(\alpha) - R) + \Phi(R)\right) + \psi(\alpha-\beta)\, \varphi\left(\Phi(\pi(\alpha) - S) + \Phi(S)\right)] + \\ & + [\varphi(\beta)\, \psi(\beta) - \varphi(\alpha)\, \psi(\alpha)]. \end{aligned}$$

Now, since $\pi(\alpha) = \pi(\beta) = \pi(\gamma)$, both types of summands add up to 0 separately in $t(\beta, \gamma) - t(\alpha, \gamma) + t(\alpha, \beta)$. □

Remark: The previous lemma does not imply that $t(q)$ is $\mathbb{Z}$-linear in q. The assumption for $\pi(\alpha)$ made in (1) is really essential.

Now, we obtain a P-linear map $T \in \operatorname{Hom}(P^m, A)$ by

$$T : e^{\alpha\beta} \mapsto \begin{cases} t(\alpha,\beta)\, x^{\pi(\alpha)-R-S} & \text{for } \pi(\alpha) \geq R+S \\ 0 & \text{otherwise}. \end{cases}$$

Pulling back T to $\mathcal{R} \subseteq P^m$ yields (in case of $\pi(a+c) \geq R+S$)

$$\begin{aligned} T(r(a,b;c)) &= \begin{cases} [t(a+c, b+c) - t(a,b)] \cdot x^{\pi(a+c)-R-S} & \text{for } \pi(a) \geq R+S \\ t(a+c, b+c) \cdot x^{\pi(a+c)-R-S} & \text{otherwise} \end{cases} \\ &= \begin{cases} -(\varphi \cup \psi)(r(a,b;c)) & \text{for } \pi(a) \geq R+S \\ t(a,b)\, x^{\pi(a+c)-R-S} - (\varphi \cup \psi)(r(a,b;c)) & \text{otherwise} \end{cases} \end{aligned}$$

and $T(s(a,b,c)) = 0$ (by (2) of the previous lemma).

On the other hand, T yields a trivial element of $T^2_Y(-R-S)$, i.e. inside this group we may replace $\varphi \cup \psi$ by $(\varphi \cup \psi) + T$ to obtain

$$\begin{aligned}(\varphi \cup \psi)(r(a,b;c)) &= \begin{cases} t(a,b) \cdot x^{\pi(a+c)-R-S} & \text{for } \pi(a) \in \bigcup_i K_i^{R+S};\ \pi(a+c) \geq R+S \\ 0 & \text{otherwise}, \end{cases} \\ (\varphi \cup \psi)(s(a,b,c)) &= 0\,.\end{aligned}$$

Having Theorem (3.5) in mind, this formula for $\varphi \cup \psi$ is exactly what we were looking for:
Given an $r(a,b;c)$ with $\pi(a) \in K_i^{R+S}$, let us compute $(\varphi \cup \psi)_i(q := a-b)$ following the recipe of (i), (ii) of Theorem (3.6): We do not need to split $q = a - b$ into a sum $q = \sum_k q^k$ - the element q itself already satisfies the condition

$$\langle a^i, \bar{q} \rangle \leq \langle a^i, \pi(a) \rangle < \langle a^i, R+S \rangle.$$

In particular, with $(\varphi \cup \psi)_i(a-b) = t(a-b) = t(a,b)$ we obtain the right result - if the recipe is assumed to be correct.

(5.4) We will fill this remaining gap now, i.e. we will show that

(a) each $q \in L(E_i^{R+S})$ admits a decomposition $q = \sum_k q^k$ with the desired properties,

(b) $\sum_k q^k = 0$ (with $\bar{q}^k \in K_i^{R+S}$) implies $\sum_k t(q^k) = 0$, and

(c) for adjacent a^i, a^j the relations $q \in L(E_i^{R+S} \cap E_j^{R+S})$ admit a decomposition $q = \sum_k q^k$ that works for both i and j.

(In particular, this answers the questions rised right after stating the theorem in (3.6).)

Let us fix an element $i \in \{1, \ldots, N\}$. Since $\sigma^\vee \cap M$ contains elements r with $\langle a^i, r \rangle = 1$, some of them must be contained in the generating set E, too. We choose one of these elements and call it $r(i)$.
Now, to each $r \in E$ we associate some relation $p(r) \in L(E)$ via

$$p(r) := e^r - \langle a^i, r \rangle \cdot e^{r(i)} + [\text{suitable element of } \mathbb{Z}^{E \cap (a^i)^\perp}]\,.$$

The two essential properties of these special relations are

(i) $\langle a^i, \bar{p}(r) \rangle = \langle a^i, r \rangle$, and

(ii) if $q \in L(E)$ is any relation, then q and $\sum_{r \in E} q_r \cdot p(r)$ differ by some element of $L(E \cap (a^i)^\perp)$ only.

In particular, this proves (a). For (b) we start with the following

Claim: Let $q^k \in L(E)$ be relations such that $\sum_k \langle a^i, \bar{q}^k \rangle < \langle a^i, R+S \rangle$. Then, $\sum_k t(q^k) = t(\sum_k q^k)$.

Proof: We can restrict ourselves to the case of two summands, q^1 and q^2. Then, by Lemma (5.3),

$$\begin{aligned} t(q^1)+t(q^2) &= t\left((q^1)^+,(q^1)^-\right)+t\left((q^2)^+,(q^2)^-\right) \\ &= t\left((q^1)^+ + (q^2)^+,(q^1)^- + (q^2)^+\right)+t\left((q^2)^+ + (q^1)^-,(q^2)^- + (q^1)^-\right) \\ &= t\left((q^1)^+ + (q^2)^+,(q^2)^- + (q^1)^-\right) \\ &= t(q^1+q^2)\,. \qquad \square \end{aligned}$$

In particular, if $\sum_k q^k = 0$ (with $\bar{q}^k \in K_i^{R+S}$), then this applies for the special decompositions

$$q^k = \sum_r q_r^k \cdot p(r) + q^{0,k} \quad (q^{0,k} \in L(E \cap (a^i)^\perp))$$

of the summands q^k themselves. We obtain

$$\sum_{q_r^k>0} q_r^k \cdot t\left(p(r)\right) + t(q^{0,k}) = t\left(\sum_{q_r^k>0} q_r^k\, p(r) + q^{0,k}\right) =: t(q^{1,k})$$

and

$$\sum_{q_r^k<0} q_r^k \cdot t\left(p(r)\right) = t\left(\sum_{q_r^k<0} q_r^k\, p(r)\right) =: t(q^{2,k})\,.$$

Up to elements of $E\cap(a^i)^\perp$, the relations $q^{1,k}$ and $q^{2,k}$ are connected by the common

$$(q^{1,k})^- = -q^{1,k}_{r(i)} \cdot e^{r(i)} = \langle a^i, \bar{q}^k \rangle \cdot e^{r(i)} = q^{2,k}_{r(i)} \cdot e^{r(i)} = (q^{2,k})^+\,.$$

Hence, Lemma (5.3) yields

$$\sum_r q_r^k \cdot t\left(p(r)\right) + t(q^{0,k}) = t(q^{1,k}) + t(q^{2,k}) = t\left(q^{1,k}+q^{2,k}\right) = t(q^k)\,,$$

and we conclude

$$\begin{aligned} \sum_k t(q^k) &= \sum_k \left(\sum_r q_r^k \cdot t\left(p(r)\right) + t(q^{0,k})\right) \\ &= \sum_r \left(\sum_k q_r^k\right) t\left(p(r)\right) + t\left(\sum_k q^{0,k}\right) \quad \text{(cf. previous claim)} \\ &= 0 + t\left(\sum_k q^k - \sum_{k,r} q_r^k\, p(r)\right) \\ &= 0\,. \end{aligned}$$

Finally, only (c) is left. Let a^i, a^j be two adjacent edges of σ. We adapt the construction of the elementary relations $p(r)$. Instead of the $r(i)$'s, we will use elements $r(i,j) \in E$ characterized by the property

$$\langle a^i, r(i,j) \rangle = 1\,,\ \langle a^j, r(i,j) \rangle = 0\,.$$

(Those elements exist, since Y is assumed to be smooth in codimension 2.)
Now, we define

$$p(r) := e^r - \langle a^i, r \rangle \cdot e^{r(i,j)} - \langle a^j, r \rangle \cdot e^{r(j,i)} + [\text{suitable element of } \mathbb{Z}^{E \cap (a^i)^\perp \cap (a^j)^\perp}]\,.$$

These special $p(r)$'s meet the usual properties (i) and (ii) - but for the two different indices i and j at the same time. In particular, if $q \in L(E)$ is any relation, then q and $\sum_{r \in E} q_r \cdot p(r)$ differ by some element of $L(E \cap (a^i)^\perp \cap (a^j)^\perp)$ only.

6 An alternative to the complex $L(E^R)_\bullet$

(6.1) Let $R \in M$ be fixed for the whole §6. The complex $L(E^R)_\bullet$ introduced in (3.2) fits naturally into the exact sequence

$$0 \to L(E^R)_\bullet \longrightarrow (\mathbb{Z}^{E^R})_\bullet \longrightarrow \operatorname{span}(E^R)_\bullet \to 0$$

of complexes built in the same way as $L(E^R)_\bullet$, i.e.

$$(\mathbb{Z}^{E^R})_{-k} := \oplus_{\substack{\tau<\sigma \\ dim\,\tau=k}} \mathbb{Z}^{E^R_\tau} \qquad \text{and} \qquad \operatorname{span}(E^R)_{-k} := \oplus_{\substack{\tau<\sigma \\ dim\,\tau=k}} \operatorname{span}(E^R_\tau)\,.$$

Lemma: *The complex $(\mathbb{Z}^{E^R})_\bullet$ is exact.*

Proof: The complex $(\mathbb{Z}^{E^R})_\bullet$ can be decomposed into a direct sum

$$(\mathbb{Z}^{E^R})_\bullet = \bigoplus_{r \in M} (\mathbb{Z}^{E^R})(r)_\bullet$$

showing the contribution of each $r \in M$. The complexes occuring as summands are defined as

$$\begin{aligned} (\mathbb{Z}^{E^R})(r)_{-k} &:= \oplus_{\substack{\tau<\sigma \\ dim\,\tau=k}} \left\{ \begin{array}{ll} \mathbb{Z} = \mathbb{Z}^{\{r\}} & \text{for } r \in E^R_\tau \\ 0 & \text{otherwise} \end{array} \right\} \\ &= \mathbb{Z}^{\#\{\tau \,|\, dim\,\tau=k;\, r \in E^R_\tau\}}\,. \end{aligned}$$

Denote by H^+ the halfspace

$$H^+ := \{a \in N_{\mathbb{R}} \,|\, \langle a, r \rangle < \langle a, R \rangle\} \subseteq N_{\mathbb{R}}.$$

Then, for $\tau \neq 0$, the fact that $r \in E_\tau^R$ is equivalent to $\tau \setminus \{0\} \subseteq H^+$. On the other hand, $r \in E_0^R$ corresponds to the condition $\sigma \cap H^+ \neq \emptyset$.
In particular, $(\mathbb{Z}^{E^R})(r)_\bullet$, shifted by one place, equals the complex for computing the reduced homolgy of the topological space $\cup\{\tau \mid \tau \setminus \{0\} \subseteq H^+\} \subseteq \sigma$ cut by some affine hyperplane. Since this space is contractable, the complex is exact. □

Corollary: *The complexes $L(E^R)^*_\bullet$ and $span(E^R)^*_\bullet[1]$ are quasiisomorphic. In particular, under the usual assumptions (cf. Theorem (3.3)), we obtain*

$$T^i_Y(-R) = H^i\left(span(E^R)^*_\bullet \otimes_{\mathbb{Z}} \mathbb{C}\right).$$

(6.2) We define the $\mathbb{R}$-vector spaces

$$\begin{aligned} V_i^R &:= \operatorname{span}_{\mathbb{R}}(E_i^R) = \begin{cases} 0 & \text{for } \langle a^i, R\rangle \leq 0 \\ [a^i = 0] \subseteq M_{\mathbb{R}} & \text{for } \langle a^i, R\rangle = 1 \\ M_{\mathbb{R}} = \mathbb{R}^n & \text{for } \langle a^i, R\rangle \geq 2 \end{cases} \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, \ldots, N), \quad \text{and} \\ V_\tau^R &:= \cap_{a^i \in \tau} V_i^R \supseteq \operatorname{span}_{\mathbb{R}}(E_\tau^R) \quad (\text{for faces } \tau < \sigma). \end{aligned}$$

Proposition: *With $\mathcal{V}^R_{-k} := \oplus_{\substack{\tau<\sigma \\ \dim \tau = k}} V_\tau^R$ we obtain a complex $\mathcal{V}^R_\bullet \supseteq span_{\mathbb{R}}(E^R)_\bullet$. Moreover, if Y is smooth in codimension k, then both complexes are equal at $\geq (-k)$.*

Proof: $V_\tau^R = \operatorname{span}_{\mathbb{R}}(E_\tau^R)$ is true for smooth cones $\tau < \sigma$ (cf.(3.7) of [Al 1]). □

Corollary:

(1) If Y is smooth in codimension 2, then $T^1_Y(-R) = H^1\left((\mathcal{V}^R_\bullet)^* \otimes_{\mathbb{R}} \mathbb{C}\right)$.

(2) If Y is smooth in codimension 3, then $T^2_Y(-R) = H^2\left((\mathcal{V}^R_\bullet)^* \otimes_{\mathbb{R}} \mathbb{C}\right)$.

The formula (1) for T^1_Y (with a more boring proof) was already obtained in (4.4) of [Al 1].

7 3-dimensional Gorenstein singularities

(7.1) We want to apply the previous results for the special case of an isolated, 3-dimensional, toric Gorenstein singularity. We start with fixing the notations.

Let $Q = \mathrm{conv}(a^1, \ldots, a^N) \subseteq I\!\!R^2$ be a lattice polygon with primitive edges

$$d^i := a^{i+1} - a^i \in Z\!\!\!Z^2 .$$

Embedding $I\!\!R^2$ as the affine hyperplane $[a_3 = 1]$ into $N_{I\!\!R} := I\!\!R^3$, we can define the cone

$$\sigma := \mathrm{Cone}(Q) \subseteq N_{I\!\!R} .$$

The fundamental generators of σ equal the vectors $(a^1, 1), \ldots, (a^N, 1)$, which we will also denote by $a^1, \ldots, a^N$, respectively.

The vector space $M_{I\!\!R}$ contains a special element $R^* := [0, 0; 1]$:

- $\langle \bullet, R^* \rangle = 1$ defines the affine hyperplane containing Q,
- $\langle \bullet, R^* \rangle = 0$ describes the vectorspace containing the edges d^i of Q.

The structure of the dual cone $\sigma^\vee$ can be described as follows:

- $[c; \eta] \in M_{I\!\!R}$ is contained in $\sigma^\vee$, iff $\langle Q, -c \rangle \leq \eta$.
- $[c; \eta] \in \partial\sigma^\vee$ iff there exists some i with $\langle a^i, -c \rangle = \eta$.
- The set E contains R^*. However, $E \setminus \{R^*\} \subseteq \partial\sigma^\vee$.

Remark: The toric variety Y built by the cone σ is 3-dimensional, Gorenstein, and regular outside its 0-dimensional orbit. Moreover, all those singularities can be obtained that way.

(7.2) Let V denote the $(N-2)$-dimensional $I\!\!R$-vector space

$$V := \{(t_1, \ldots, t_N) \mid \sum_i t_i \, d^i = 0\} \subseteq I\!\!R^N .$$

The non-negative tuples among the $\underline{t} \in V$ describe the set of Minkowski summands $Q_{\underline{t}}$ of positive multiples of the polygon Q. (t_i is the scalar by which d^i has to be multiplied to get the i-th edge of $Q_{\underline{t}}$.)

We consider the bilinear map

$$\begin{array}{ccccl} V & \times & I\!\!R^E & \stackrel{\Psi}{\longrightarrow} & I\!\!R \\ \underline{t} & , & [c;\eta] \in E & \mapsto & \left\{ \begin{array}{ll} 0 & \text{if } c = 0 \quad (\text{i.e. } [c;\eta] = R^*) \\ \sum_{v=1}^{i-1} t_v \cdot \langle d^v, -c \rangle & \text{if } \langle a^i, -c \rangle = \eta \, . \end{array} \right. \end{array}$$

Assuming both a^1 and the associated vertices of all Minkowski summands $Q_{\underline{t}}$ to coincide with $0 \in I\!\!R^2$, the map Ψ detects the maximal values of the linear functions c on these summands

$$\Psi(\underline{t}, [c; \eta]) = \mathrm{Max}\left(\langle a, -c \rangle \mid a \in Q_{\underline{t}}\right) .$$

In particular, $\Psi(\underline{1}, [c;\eta]) = \eta$, i.e. Ψ induces a map

$$\Psi : \quad {}^{V}\!\big/_{\mathbb{R}\cdot\underline{1}} \times L_{\mathbb{R}}(E) \longrightarrow \mathbb{R}\,.$$

The results of [Al 2] and [Sm] imply that Ψ provides an isomorphism

$${}^{V_{\mathbb{C}}}\!\big/_{\mathbb{C}\cdot\underline{1}} \stackrel{\sim}{\longrightarrow} \left({}^{L(E_0^{R^*})}\Big/_{\sum_i L(E_i^{R^*})} \right)^* \otimes_{\mathbb{Z}} \mathbb{C} \cong T^1_Y(-R^*) = T^1_Y\,.$$

In particular, $\dim T^1_Y = N-3$.

(7.3) Let $R \in M$. Combining the general results of §6 with the fact

$$\bigcap_i V_i^R = \operatorname{Ker}\left[\oplus_i (V_i^R \cap V_{i+1}^R) \longrightarrow \oplus_i V_i^R\right],$$

we obtain the handsome formula

$$T^2_Y(-R) = \left[{}^{\bigcap_i (\operatorname{span}_{\mathbb{C}} E_i^R)}\Big/_{\operatorname{span}_{\mathbb{C}}(\bigcap_i E_i^R)} \right]^*.$$

T^1_Y is concentrated in the degree $-R^*$. Hence, for computing T^2_Y, the degrees $-kR^*$ ($k \geq 2$) are the most interesting (but not only) ones. In this special case, the vector spaces $V_i^{kR^*}$ equal $M_{\mathbb{R}}$, i.e.

$$T^2_Y(-kR^*) = \left[{}^{M_{\mathbb{C}}}\Big/_{\operatorname{span}_{\mathbb{C}}(\bigcap_i E_i^{kR^*})} \right]^* \subseteq \left[{}^{M_{\mathbb{C}}}\!\big/_{\mathbb{C}\cdot R^*} \right]^* = \operatorname{span}_{\mathbb{C}}(d^1, \dots, d^N) \subseteq N_{\mathbb{C}}\,.$$

Proposition: *For $c \in \mathbb{R}^2$ denote by*

$$d(c) := \mathit{Max}\,(\langle a^i, c\rangle \mid i = 1, \dots, N) - \mathit{Min}\,(\langle a^i, c\rangle \mid i = 1, \dots, N)$$

the diameter of Q in c-direction. If

$$k_1 := \mathop{\mathit{Min}}_{c \in \mathbb{Z}^2 \setminus 0} d(c) \quad \mathit{and} \quad k_2 := \mathop{\mathit{Min}}_{\substack{c,c' \in \mathbb{Z}^2 \\ \mathit{lin.\,indept.}}} \mathit{Max}\,[d(c), d(c')]\,,$$

then $\quad \dim T^2_Y(-kR^*) = 2 \quad$ *for* $\quad 2 \leq k \leq k_1$,
$\quad \dim T^2_Y(-kR^*) = 1 \quad$ *for* $\quad k_1 + 1 \leq k \leq k_2$, *and*
$\quad \dim T^2_Y(-kR^*) = 0 \quad$ *for* $\quad k_2 + 1 \leq k$.

Proof: We have to determine the dimension of $\operatorname{span}_{\mathbb{C}}\left(\bigcap_i E_i^{kR^*}\right)\Big/\mathbb{C}\cdot R^*$. Computing modulo R^* simply means to forget the η in $[c;\eta]\in M$. Hence, we are done by the following observation for each $c\in\mathbb{Z}^2\setminus 0$:

$$\begin{aligned}\exists\eta\in\mathbb{Z}:\ [c,\eta]\in\textstyle\bigcap_i K_i^{kR^*} &\iff \exists\eta\in\mathbb{Z}:\ (k-1)R^*\geq[c;\eta]\geq 0\\ &\iff d(c)\leq k-1\,.\end{aligned}$$

□

Corollary: *Unless $Y=\mathbb{C}^3$ or $Y=$ cone over $\mathbb{P}^1\times\mathbb{P}^1$, we have*

$$T_Y^2(-2R^*)=\operatorname{span}_{\mathbb{C}}(d^1,\ldots,d^N),$$

i.e. $\dim T_Y^2(-2R^*)=2$.

(7.4) Proposition: *Using both the isomorphism $V_{\mathbb{C}}\Big/\mathbb{C}\cdot\underline{1}\stackrel{\sim}{\to}T_Y^1$ and the injection $T_Y^2(-2R^*)\hookrightarrow span_{\mathbb{C}}(d^1,\ldots,d^N)$, the cup product $T_Y^1\times T_Y^1\to T_Y^2$ equals the bilinear map*

$$\begin{array}{ccccc}V_{\mathbb{C}}\Big/\mathbb{C}\cdot\underline{1} & \times & V_{\mathbb{C}}\Big/\mathbb{C}\cdot\underline{1} & \longrightarrow & \operatorname{span}_{\mathbb{C}}(d^1,\ldots,d^N)\\ \underline{s} & , & \underline{t} & \mapsto & \sum_i s_i\,t_i\,d^i\,.\end{array}$$

Proof: *Step 1:* To apply Theorem (3.6) we will combine the isomorphisms for T_Y^2 presented in §6 and (7.3). Actually, we will describe the dual map by associating to each $r\in M$ an element $[q^1(r),\ldots,q^N(r)]\in\oplus_i L(E_i^{2R^*})$.
First, for every $i=1,\ldots,N$, we have to write $r\in M=(\operatorname{span}E_i^{2R^*})\cap(\operatorname{span}E_{i+1}^{2R^*})$ as a linear combination of elements from $E_i^{2R^*}\cap E_{i+1}^{2R^*}$. This set contains a $\mathbb{Z}$-basis for M consisting of

- $r^i :=$ primitive element of $\sigma^\vee\cap(a^i)^\perp\cap(a^{i+1})^\perp$,
- R^*, and
- $r(i):=r(i,i+1)$ (cf. notation at the end of (5.4)), i.e. $\langle a^i,r(i)\rangle=1$ and $\langle a^{i+1},r(i)\rangle=0$.

In particular, we can write

$$r=g^i(r)\cdot r^i+\langle a^{i+1},r\rangle\cdot R^*+\big(\langle a^i,r\rangle-\langle a^{i+1},r\rangle\big)\cdot r(i)$$

with some integer $g^i(r)\in\mathbb{Z}$.

Now, we have to apply the differential in the complex $(\mathbb{Z}^{E^{2R^*}})_\bullet$, i.e. we map the previous expression via the map

$$\oplus_i\mathbb{Z}^{E_i^{2R^*}\cap E_{i+1}^{2R^*}}\longrightarrow\oplus_i\mathbb{Z}^{E_i^{2R^*}}\,.$$

The result is (for every i) the element of $L(E_i^{2R^*})$

$$g^i(r)\,e^{r^i} - g^{i-1}(r)\,e^{r^{i-1}} + \langle a^i - a^{i+1}, r\rangle \cdot e^{r(i)} - \langle a^{i-1} - a^i, r\rangle \cdot e^{r(i-1)} +$$
$$+\langle a^{i+1} - a^i, r\rangle \cdot e^{R^*} = \langle d^i, r\rangle \cdot \left(e^{R^*} - e^{r(i)}\right) + [(a^i)^\perp\text{-summands}] =: q^i(r)\,.$$

Step 2: Defining

$$q^i := e^{R^*} - e^{r(i)} + [(a^i)^\perp\text{-summands}] \in L(E_i^{2R^*}) \quad (i = 1, \ldots, N)\,,$$

we use Theorem (3.6) and Proposition (3.7) to obtain

$$(\underline{s} \cup \underline{t})_i \left(q^i(r)\right) = \langle d^i, r\rangle \cdot t_{\Psi(\underline{s},\bullet),\Psi(\underline{t},\bullet),R^*,R^*}(q^i) = \Psi(\underline{s}, q^i) \cdot \Psi(\underline{t}, q^i)\,.$$

To compute those two factors, we take a closer look at the q^i's. Let

$$q^i = e^{R^*} - e^{r(i)} + \sum_v \lambda_v^i\, e^{[c^v;\eta^v]}\,,$$

and the sum is taken over those v's meeting the condition $\langle a^i, -c^v\rangle = \eta^v$. Then, by definition of Ψ in (7.2),

$$\Psi(\underline{s}, q^i) = \sum_{j=1}^{(i+1)-1} s_j\,\langle d^j, r(i)\rangle - \sum_v \lambda_v^i \cdot \left(\sum_{j=1}^{i-1} s_j\,\langle d^j, c^v\rangle\right)\,.$$

On the other hand, we know that q^i is a relation, i.e. the equation

$$R^* - r(i) + \sum_v \lambda_v^i [c^v; \eta^v] = 0$$

is true in M. Hence,

$$\begin{aligned} \Psi(\underline{s}, q^i) &= \textstyle\sum_{j=1}^{i} s_j\,\langle d^j, r(i)\rangle - \sum_{j=1}^{i-1} s_j \langle d^j, r(i)\rangle \\ &= s_i \cdot \langle d^i, r(i)\rangle \\ &= -s_i\,. \end{aligned}$$

□

$T_Y^1 \subseteq \mathbb{C}^N$ is the tangent space of the versal base space S of our singularity Y. It is given by the linear equation $\sum_i t_i \cdot d^i = 0$.
On the other hand, the cup product $T_Y^1 \times T_Y^1 \to T_Y^2$ shows the quadratic part of the equations defining $S \subseteq \mathbb{C}^N$. By the previous proposition, it equals $\sum_i t_i^2 \cdot d^i$.

These facts suggest the form of all equations of $S \subseteq \mathbb{C}^N$. In [Al 3] we have, indeed, shown that S is given by the equations

$$\sum_{i=1}^{N} t_i^k \cdot d^i = 0 \quad (k \geq 1)\,.$$

References

[Al 1] Altmann, K.: Computation of the vector space T^1 for affine toric varieties. J. Pure Appl. Algebra **95** (1994), 239-259.

[Al 2] Altmann, K.: Minkowski sums and homogeneous deformations of toric varieties. Tôhoku Math. J. **47** (1995), 151-184.

[Al 3] Altmann, K.: The versal Deformation of an isolated, toric Gorenstein Singularity.
Preprint M.I.T., February 1994; preprint 59 "Europäisches Singularitäten-projekt"; e-print alg-geom/9403004.

[Ke] Kempf, G., Knudsen, F., Mumford, D., Saint-Donat, B.: Toroidal Embeddings I.
Lecture Notes in Mathematics **339**, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[KPR] Kurke, H., Pfister, G. Roczen, M.: Henselsche Ringe und algebraische Geometrie.
Berlin: Deutscher Verlag der Wissenschaften 1975.

[Ld] Laudal, O.A.: Formal Moduli of Algebraic Structures.
Lecture Notes in Mathematics **754** (1979), Springer-Verlag.

[LdSl] Laudal, O.A., Sletsjœ, A.: Cohomology of groups, monoids and their algebras.
Preprint, University of Oslo.

[LiS] Lichtenbaum, S., Schlessinger, M.: The cotangent complex of a morphism.
Trans. AMS **128** (1967), 41-70.

[Od] Oda, T.: Convex bodies and algebraic geometry.
Ergebnisse der Mathematik und ihrer Grenzgebiete (3/15), Springer-Verlag, 1988.

[Pa] Palamodov, V.P.: Deformations of complex spaces.
Russian Math. Surveys **31** (1976), 129-197.

[Sm] Smilansky, Z.: Decomposability of Polytopes and Polyhedra.
Geometriae Dedicata. **24** (1987), 29-49.