Strong exceptional sequences provided by quivers

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Abstract

Let Q be a finite quiver without oriented cycles. Denote by $\mathcal{U} \to \mathcal{M}(Q)$ the fine moduli space of stable thin sincere representations of Q with respect to the canonical stability notion. We prove $\mathrm{Ext}^l_{\mathcal{M}(Q)}(\mathcal{U},\mathcal{U}) = 0$ for all $l > 0$ and compute the endomorphism algebra of the universal bundle $\mathcal{U}$. Moreover, we obtain a necessary and sufficient condition for when this algebra is isomorphic to the path algebra of the quiver Q. If so, then the bounded derived categories of finitely generated right kQ-modules and that of coherent sheaves on $\mathcal{M}(Q)$ are related via the full and faithful functor $- \otimes^{\mathbb{L}}_{kQ} \mathcal{U}$.

1 Introduction

(1.1) Let Q be a quiver (i.e. an oriented graph) without oriented cycles; denote by Q_0 the vertices and by Q_1 the arrows of Q. For a fixed dimension vector d, that is a map $d : Q_0 \to \mathbb{Z}_{\geq 0}$, we denote by $\mathbb{H}(d) := \{\theta : Q_0 \to \mathbb{R} \mid \sum_{q\in Q_0} \theta_q d_q = 0\}$ the vector space of the so-called weights with respect to d. We fix an algebraically closed field k. To each $\theta \in \mathbb{H}(d)$ there exists the moduli space $\mathcal{M}^\theta(Q,d)$ of θ-semistable k-representations of Q with dimension vector d (cf. [Ki]). This space is known to be projective and, in case θ is in general position and d is indivisible, also smooth. Moreover, if we restrict ourselves to thin sincere representations, that is $d_q = 1$ for all $q \in Q_0$, then $\mathcal{M}^\theta(Q)$ is also toric (cf. [Hi2]). In any case, each integral weight θ induces an ample line bundle $\mathcal{L}(\theta)$ on $\mathcal{M}^\theta(Q,d)$.
If θ is in general position and d indivisible, then $\mathcal{M}^\theta(Q,d)$ is, in addition, a fine moduli space admitting a universal bundle $\mathcal{U}$. The universal bundle splits into a direct sum of vector bundles $\mathcal{U} = \oplus_{q\in Q_0}\mathcal{U}_q$, and the summands $\mathcal{U}_q$ have rank d_q (cf. [Ki]). All known examples suggest that the universal bundles on those moduli spaces have no self-extensions, i.e. $\mathrm{Ext}^{\,l}_{\mathcal{M}^\theta(Q,d)}(\mathcal{U},\mathcal{U}) = 0$ for all $l > 0$. The issue of this paper is to prove this formula in special cases. The meaning of this property and its relation to tilting theory will be discussed in (1.4).
In this paper we restrict ourselves to thin sincere representations; the corresponding moduli spaces are called *toric quiver varieties*. Because $d = (1, \ldots, 1)$ is fixed, we will omit it in all notation introduced above. The direct summands of the universal bundle are line bundles, and they are characterized, up to a common twist, by the following property: For any arrow $\alpha \in Q_1$ pointing from p to q ($p, q \in Q_0$) the invertible sheaf $\mathcal{U}_p^{-1} \otimes \mathcal{U}_q$ corresponds to the divisor of all representations assigning the zero map to α. Furthermore, there exists a distinguished weight θ^c (see (1.2) and (2.7) for a definition and first properties). We denote the corresponding moduli space by $\mathcal{M}(Q)$.

(1.2) Polarized projective toric varieties may be constructed from lattice polytopes. If one wants to forget about the polarization, simply consider the inner normal fan of the polytope. In §2 we give a detailed description of the moduli space $\mathcal{M}^\theta(Q)$ of thin sincere representations via its "defining polytope" $\Delta(\theta)$. The easiest way to obtain $\Delta(\theta)$ from the quiver is to imagine Q as a one-way pipe system carrying liquid; a weight $\theta \in \mathbb{H}$ describes the input (possibly negative) into the system at each knot. Using this language, $\Delta(\theta)$ is simply given as the set of all possible flows

respecting both the direction of the pipes and the given input θ (see (2.3)).
Considering the opposite viewpoint, *each* flow through our pipe system requires a certain input, i.e. a weight. In particular, from the special flow that is constant 1 at each pipe we obtain a special, so-called canonical, weight θ^c. The corresponding $\Delta(\theta^c)$ is a reflexive polytope (in the sense of Batyrev, [Bat]), i.e. the moduli space $\mathcal{M}(Q)$ is Fano (Proposition (2.7)).
Fixing a weight θ in general position, i.e. $\mathcal{M}^\theta(Q)$ is smooth, flows and weights have still another meaning. Each flow defines an equivariant, with respect to the defining torus, effective divisor, and each weight θ' defines an element $\mathcal{L}(\theta')$ in the Picard group of $\mathcal{M}^\theta(Q)$. Assigning a flow its input weight corresponds to assigning a divisor its class in the Picard group (see (3.1)).

Example: 1) In the special case $\theta' := \theta$ this recovers our ample line bundle introduced before.
2) The line bundle $\mathcal{U}_p^{-1} \otimes \mathcal{U}_q$ corresponds to the weight with values 1 at p, -1 at q, and zero at all other points.

(1.3) Our first main result is Theorem (3.6) stating the lack of self-extensions of $\mathcal{U}$ on the moduli space $\mathcal{M}(Q)$ with respect to the canonical weight, i.e. $\mathrm{Ext}^l_{\mathcal{M}(Q)}(\mathcal{U},\mathcal{U}) = 0$ for all $l > 0$. This is proved by using a slightly generalized Kodaira vanishing argument which works for toric varieties, cf. Theorem (3.5). As a corollary of Theorem (3.6) we conclude that we obtain a full and faithful functor from the bounded derived category of finitely generated right modules over the endomorphism algebra $\mathcal{A}$ of $\mathcal{U}$ into the bounded derived category of coherent sheaves on the moduli space $\mathcal{M}(Q)$ (Theorem (4.4)). Moreover, in Theorem (4.3) we provide a criterion for $\mathcal{A} = \mathrm{End}_{\mathcal{M}(Q)}(\mathcal{U},\mathcal{U})$ to be isomorphic to the path algebra kQ of the quiver Q.
Combining both results we obtain the following relation between the derived categories of right kQ-modules and of coherent sheaves on $\mathcal{M}(Q)$, respectively.

Theorem: *Assume Q is a quiver lacking $(1,0)$- and (t,t)-walls (see (2.2) for an explanation). Then,*

$$- \otimes^{\mathbb{L}}_{kQ} \mathcal{U} : \mathcal{D}^b\Big(\textit{mod–}kQ\Big) \longrightarrow \mathcal{D}^b\Big(\mathit{Coh}(\mathcal{M}(Q))\Big)$$

is a full and faithful functor from the bounded derived category of finitely generated right kQ-modules into the bounded derived category of coherent sheaves on $\mathcal{M}(Q)$.

(1.4) The result above is closely related to tilting theory. Since the fundamental paper [Be], tilting theory has become a major tool in classifying vector bundles; a tilting sheaf induces an equivalence of bounded derived categories, as in the previous Theorem. To be precise we recall the definition of a tilting sheaf ([Bae]). A sheaf $\mathcal{T}$ on a smooth projective variety is called a *tilting sheaf* if

1) it has no higher self-extensions, that is $\mathrm{Ext}^l(\mathcal{T},\mathcal{T}) = 0$ for all $l > 0$,
2) the direct summands generate the bounded derived category, and
3) the endomorphism algebra $\mathcal{A}$ of $\mathcal{T}$ has finite global (homological) dimension.

Then, the functors $\mathbb{R}\mathrm{Hom}(\mathcal{T},-)$ and $- \otimes^{\mathbb{L}}_{\mathcal{A}} \mathcal{T}$ define mutually inverse equivalences of the bounded derived categories of coherent sheaves on the underlying variety of $\mathcal{T}$ and of the finitely generated right $\mathcal{A}$-modules, respectively. For constructions of tilting bundles and their relations to derived categories we refer to the following papers [Ka], [Be], [Ru], [Bo], and [Or]. For the similar notion of a tilting module we refer to [HR].
For our purpose, the notion of an exceptional sequence is more useful. Let $\mathcal{C}$ be any of the categories introduced above: the category of finitely generated right modules over a finite dimensional algebra, the category of coherent sheaves on a smooth projective variety, or one of its derived categories. Thus, $\mathcal{C}$ is either an abelian or a triangulated k-category. Each object in $\mathcal{C}$ has a unique, up to isomorphism and reordering, decomposition into indecomposable direct summands, i.e. $\mathcal{C}$ is a Krull-Schmidt category. Moreover, the extension groups are defined and globally bounded; they are finite-dimensional k-vector spaces. An object in $\mathcal{C}$ is called *exceptional* if it has no self-extensions and its endomorphism ring is k. A sequence $(\mathcal{E}_0,\ldots,\mathcal{E}_n)$ of objects in $\mathcal{C}$ is called *exceptional* if

1) each object $\mathcal{E}_i$ for $i = 0, \ldots, n$ is exceptional and
2) $\mathrm{Ext}^l(\mathcal{E}_j, \mathcal{E}_i) = 0$ for all $l \geq 0$, and $j > i$.

Such a sequence is called *strong exceptional* if, additionally,
3) $\mathrm{Ext}^l(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all $l > 0$ and all $i, j = 0, \ldots, n$.

Finally, it is called *full* if in addition to 1), 2) and 3)
4) the objects $\mathcal{E}_i$ for $i = 0, \ldots, n$ generate the bounded derived category.

Thus, each full strong exceptional sequence defines a tilting bundle $\oplus_{i=0}^n \mathcal{E}_i$, because the endomorphism algebra of $\oplus_{i=0}^n \mathcal{E}_i$ has global dimension at most n. Vice versa, each tilting bundle whose direct summands are line bundles gives rise to a strong exceptional sequence.
Using this language, our vanishing result Theorem (3.6) means that the direct summands of $\mathcal{U}$ form a strong exceptional sequence.

(1.5) In general, this sequence cannot be full. Assume the contrary; then the bounded derived categories in the previous theorem are equivalent. The first one is a derived category of a hereditary abelian category, whose structure is well-known ([Ri2]). In particular, the Serre functor (see [BK] for the definition) coincides with the Auslander-Reiten translation and fixes objects up to translation only in case the category is tame or just semi-simple (cf. [Hap] §1.4/5). On the other hand, the Serre functor in the bounded derived category of coherent sheaves fixes all skyscraper sheaves up to a translation. Consequently, an equivalence implies that $\mathcal{M}$ is a point or a projective line in case the algebra kQ is semi-simple or tame, respectively. It follows that Q is a point or the Kronecker quiver; the remaining tame cases may not appear (see [Ri1] Theorem p. 158).
Nevertheless, there is some hope that one may find a complement $\overline{\mathcal{U}}$ such that $\mathcal{U} \oplus \overline{\mathcal{U}}$ is a tilting bundle. At least a class of very particular examples of tilting bundles on toric quiver varieties is known ([Hi2], Theorem 3.9).

(1.6) For an introduction to quivers and path algebras we refer the reader to [Ri1] and [ARS]; the theory of localizations may be found in [S]. For an introduction to moduli spaces we mention [N] and for moduli of representations of quivers we refer to the work of King [Ki]. For results on triangulated categories we refer to [Hap] and [Har]. Our standard reference for toric geometry is [Ke]; for a short introduction to this area we also mention [F].
We would like to thank G. Hein, A. King, and A. Schofield for helpful discussions.

2 Moduli spaces of thin sincere representations

(2.1) Let Q be a connected quiver without oriented cycles; it consists of a set Q_0 of vertices, a set Q_1 of arrows, and two functions $s, t : Q_1 \to Q_0$ assigning to each arrow $\alpha \in Q_1$ its source $s(\alpha)$ and its target $t(\alpha)$. A representation of Q is a collection of finite dimensional k-vector spaces $x(q)$ for each vertex q together with a collection of linear maps $x(\alpha) : x(s(\alpha)) \to x(t(\alpha))$ for each arrow $\alpha \in Q_1$. The dimension vector $d = (d_q \mid q \in Q_0)$ of a representation x is defined by $d_q = \dim x(q)$. A representation is called *thin* if $\dim x(q) \leq 1$ for all $q \in Q_0$ and *sincere* if $\dim x(q) \geq 1$ for all $q \in Q_0$. In this paper we consider only thin sincere representations.
We denote by $\mathcal{R} = \oplus_{\alpha \in Q_1} k$ the space of all thin sincere representations, that is $x(q) = k$. By $G = \times_{q \in Q_0} k^*$ we denote the torus acting via conjugation on $\mathcal{R}$. The orbits of this action are exactly the isomorphism classes of thin sincere representations, i.e. their moduli space may be obtained via GIT. Doing so, we have to deal with the notion of stability with respect to a given weight (cf. [Ki]).

Definition: The elements of the real vector space $\mathbb{H} := \{\theta : Q_0 \to \mathbb{R} \mid \sum_{q \in Q_0} \theta_q = 0\}$ are called *weights* of the quiver Q.

Let $\theta \in I\!H$. A thin sincere representation x of Q is *θ-stable* (*-semistable*) if for each proper non-trivial subrepresentation $y \subset x$ we have $\sum_{q \in Q_0 | y(q) \neq 0} \theta_q < 0$ (≤ 0 respectively). Two semistable representations x and y are called *S-equivalent* with respect to θ if the factors of the stable Jordan-Hölder filtration coincide.
A subquiver $Q' \subseteq Q$ with $Q'_0 = Q_0$ is *θ-stable* (*θ-semistable*) if it has a θ-stable (θ-semistable) representation. Two quivers are *S-equivalent* with respect to θ if they admit θ-semistable representations of the same S-equivalence classes.
Finally, we denote by $\mathcal{T}(\theta)$ the set of all θ-semistable subtrees $T \subseteq Q$ with $T_0 = Q_0$ and by $Q_1(\theta)$ the set of all arrows α such that $Q \setminus \{\alpha\}$ is a θ-stable subquiver.

In other words, a representation x is θ-stable precisely when the subquiver $Q_0 \cup \{\alpha \in Q_1 \mid x(\alpha) \neq 0\}$ is θ-stable. Moreover, a subquiver Q' is θ-stable if and only if for all non-trivial proper subsets $S \subset Q_0$ which are closed under successors in Q' we have $\sum_{q \in S} \theta_q < 0$ (see [Hi1] §2 and [Hi2] Lemma 1.4). We also note that each S-equivalence class contains a unique minimal θ-semistable subquiver – just take the disjoint union of the support of the Jordan-Hölder factors.

(2.2) As already mentioned in the beginning, for any given weight θ the moduli space $\mathcal{M}^\theta(Q)$ exists; however, different weights θ may cause different moduli spaces. According to [Hi1] there is a chamber system in $I\!H$, and the type of $\mathcal{M}^\theta(Q)$ can only flip if θ crosses walls of the following type:

Definition: $W \subseteq I\!H$ is called a (t^+, t^-)*-wall* if

$$W = \Big\{\theta \in I\!H \mid \sum_{q \in Q_0^+} \theta_q = -\sum_{q \in Q_0^-} \theta_q = 0\Big\}$$

for some decomposition $Q_0 = Q_0^+ \sqcup Q_0^-$ such that the full subquivers Q^+ and Q^- are both connected and such there are exactly t^+ arrows pointing from Q_0^+ to Q_0^- and t^- arrows the other way around. We say that θ is in *general position* if θ does not lie on any wall and if the moduli space is not empty.

Assume θ is in general position, then $\mathcal{M}^\theta(Q)$ is smooth and has the (maximal) dimension $d = \#Q_1 - \#Q_0 + 1$, see [Hi1]. Moreover, for those weights, every semistable thin sincere representation is stable.

(2.3) To describe the toric structure of $\mathcal{M}^\theta(Q)$ we introduce the real vector space of flows defined as $I\!F := \{r : Q_1 \to I\!R\} = I\!R^{Q_1}$. A flow is called *regular* if it has only non-negative values, i.e. if it respects the direction of the pipes. For any $\alpha \in Q_1$ we denote by $f^\alpha \in I\!F$ the characteristic flow mapping α to 1 and keeping the remaining pipes dry. More generally, for each walk w without cycles in Q we define the characteristic flow f^w mapping an arrow $\alpha \in w$ to 1, an arrow β with $\beta^{-1} \in w$ to -1, and the remaining arrows to 0. This flow is regular if and only if w is a path, i.e. respects the orientation in Q.
There is a canonical linear map $\pi : I\!F \to I\!H$ describing the input of flows; if $\alpha \in Q_1$ points from p to q, then $\pi(f^\alpha)$ sends p and q onto 1 and -1, respectively. Thus

$$(\pi(r))_q = \sum_{s(\alpha)=q} r_\alpha - \sum_{t(\alpha)=q} r_\alpha.$$

This leads to the following definition:

Definition: The convex *polytope of flows* $\Delta(\theta)$ assigned to a weight θ is defined as the intersection

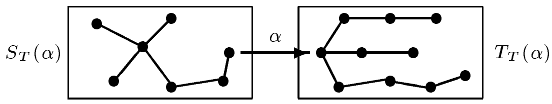
$$\Delta(\theta) := \pi^{-1}(\theta) \cap I\!R^{Q_1}_{\geq 0}.$$

This means that $\Delta(\theta)$ consists of exactly those regular flows respecting the prescribed input θ. Moreover, $\Delta(\theta)$ is compact since Q has no oriented cycles.
The vector spaces $I\!F$ and $I\!H$ contain the lattices $I\!F_{\mathbb{Z}}$ and $I\!H_{\mathbb{Z}}$ of integral flows and weights, respectively. For any integral weight $\theta : Q_0 \to \mathbb{Z}$ we define the affine lattice $M^\theta \subseteq I\!F$ as the fiber $I\!F_{\mathbb{Z}} \cap \pi^{-1}(\theta)$, i.e.

$$M^\theta = \Big\{ r \in \mathbb{Z}^{Q_1} \mid \sum_{s(\alpha)=q} r_\alpha - \sum_{t(\alpha)=q} r_\alpha = \theta_q \ \text{ for all } q \in Q_0 \Big\} .$$

Any element $r \in M^\theta$ provides obviously an isomorphism $(+r) : M := M^0 \xrightarrow{\sim} M^\theta$.

(2.4) The following lemma will be crucial for the understanding of our flow polytope as well as for proving the upcoming vanishing theorem in §3; it explicitly provides points of the lattices M^θ. Let $T \subseteq Q$ be an arbitrary maximal tree. Each arrow $\alpha \in T_1$ divides Q_0 into two disjoint subsets, the source $S_T(\alpha)$ and the target $T_T(\alpha)$.



Lemma: *Fix a maximal tree T and let $\theta \in I\!H$. For any flow ε there is a unique element $r = r^T \in \pi^{-1}(\theta) \subseteq I\!F$ satisfying $r_\alpha := \varepsilon(\alpha)$ for $\alpha \notin T_1$. Its remaining coordinates (i.e. for $\alpha \in T_1$) are given by*

$$r_\alpha \ - \ \varepsilon(\alpha) \ = \ \sum_{q \in S_T(\alpha)} \theta_q \ + \ \sum_{T_T(\alpha) \xrightarrow{\beta} S_T(\alpha)} \varepsilon(\beta) \ - \ \sum_{S_T(\alpha) \xrightarrow{\beta} T_T(\alpha)} \varepsilon(\beta) \, .$$

Proof: First, we should note that both ε and r are flows – the different notation for their coordinates ($\varepsilon(\alpha)$ and r_α, respectively) was chosen for psychological reasons only. Now, let r be *some* element of $I\!F$. If $\pi(r) = \theta$, then for any subdivision $Q_0 = Q_0^+ \sqcup Q_0^-$ we obtain

$$\sum_{s(\beta)\in Q_0^+, t(\beta)\in Q_0^-} r_\beta - \sum_{s(\beta)\in Q_0^-, t(\beta)\in Q_0^+} r_\beta = \sum_{q\in Q_0^+} \theta_q = - \sum_{q \in Q_0^-} \theta_q$$

just by summing up the M^θ-equations with $q \in Q_0^+$. The reverse implication is also true, even if we restrict ourselves to the special subdivisions provided by arrows $\alpha \in T_1$ via $Q_0^+ := S_T(\alpha)$ and $Q_1^+ := T_T(\alpha)$.
On the other hand, these subdivisions have the important property that α is the only arrow that belongs to both the tree T and to one of the index sets $\{\beta \mid s(\beta) \in Q_0^+, t(\beta) \in Q_0^-\}$ or $\{\beta \mid s(\beta) \in Q_0^-, t(\beta) \in Q_0^+\}$. In particular, by just taking care of this single exception, in the above equations we may always replace r_β by $\varepsilon(\beta)$. □

If the weight θ and the flow are integral, then so is r, i.e. $r \in M^\theta$.

(2.5) **Proposition:** *Let $\theta \in I\!H$ be an integral weight, then the polytope of flows $\Delta(\theta) \subseteq M^\theta$ is always a lattice polytope. The associated projective toric variety equals $\mathcal{M}^\theta(Q)$. Moreover, $\Delta(\theta)$ provides an ample, equivariant line bundle $\mathcal{L}(\theta)$ on $\mathcal{M}^\theta(Q)$.*

Proof: First, we establish a one-to-one correspondence between vertices of $\Delta(\theta)$ and S-equivalence classes of θ-semistable trees. Faces of $\Delta(\theta)$ in any dimension are obtained by forcing certain coordinates of $I\!F$ to be zero. In particular, vertices are points with a maximal set of vanishing coordinates. Let $r \in \Delta(\theta)$ be a vertex and denote its support by

$$\operatorname{supp} r := \{\alpha \in Q_1 \mid r_\alpha > 0\} \, .$$

If supp r contained any cycle of Q, then we could replace r by a different regular flow with the same weight and a smaller support. Hence, supp r is contained in maximal trees of Q. Moreover, we obtain that

- every maximal tree T containing supp r is θ-semistable, and
- those trees are stable if and only if supp $r = T_1$ (which determines the tree uniquely).

To prove these facts, take a proper subset $Q_0^+ \subset Q_0$ that is closed under successors in T; denoting $Q_0^- := Q_0 \setminus Q_0^+$ this means that there are no arrows pointing from Q_0^+ to Q_0^- in T. Hence, using the formula mentioned in the previous proof,

$$\sum_{q \in Q_0^+} \theta_q = - \sum_{[Q_0^- \xrightarrow{\beta} Q_0^+]} r_\beta \leq 0.$$

Conversely, let T be any maximal tree. The previous lemma tells us that there is exactly one $r \in \pi^{-1}(\theta) \subseteq I\!\!F$ such that supp $r \subseteq T_1$; it has integer coordinates $r_\alpha = \sum_{S_T(\alpha)} \theta_q$ for $\alpha \in T_1$. Moreover, if T is θ-semistable, then these numbers are non-negative, meaning that $r \in \Delta(\theta)$. Thus, it must be a vertex. Moreover, two different trees define the same vertex if and only if they are S-equivalent.

What does the inner normal fan $\Sigma(\theta)$ look like? Denote by C the matrix describing the incidences of our quiver; C consists of $\#Q_0$ rows and $\#Q_1$ columns, and for $q \in Q_0$, $\alpha \in Q_1$ we have

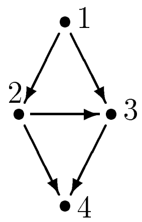
$$C_{q\alpha} := \begin{cases} +1 & \text{if } q = s(\alpha) \\ -1 & \text{if } q = t(\alpha) \\ 0 & \text{otherwise}. \end{cases}$$

The free abelian group $N := \mathbb{Z}^{Q_1}/(C\text{–rows})$ is dual to the lattice $M := M^0 = \big(C\text{–rows}\big)^\perp \subseteq I\!\!F$. Hence, writing $a^\alpha \in N$ for the images of the canonical vectors $e^\alpha \in \mathbb{Z}^{Q_1}$, we obtain

$$\Sigma(\theta) = \Big\{ \langle a^{\alpha_1}, \ldots, a^{\alpha_k} \rangle \,\Big|\, Q \setminus \{\alpha_1, \ldots, \alpha_k\} \text{ is the minimal element in some } S\text{-equivalence class of } \theta\text{-semistable subquivers} \Big\}.$$

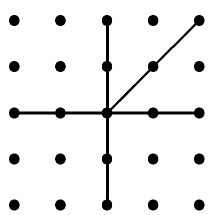
According to [Hi2], Theorem 1.7 and (3.3), this is exactly the fan defining the moduli space $\mathcal{M}^\theta(Q)$. □

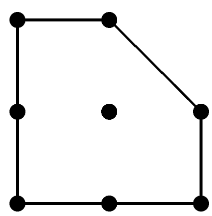
Example: In the quiver below the canonical weight is $\theta^c = (2, 1, -1, -2)$.



$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

The corresponding fan $\Sigma(\theta)$, and the polytope $\Delta(\theta^c)$ look like the following:





It is known that the toric variety of this fan is the blow up of the projective plane in two points, which is isomorphic to the blow up of the two-dimensional smooth quadric in one point.

(2.6) Equivariant (with respect to the torus action), invertible sheaves $\mathcal{L}$ on a toric variety $X(\Sigma)$ are completely determined by their order function $\operatorname{ord}\mathcal{L} : \Sigma^{(1)} \to \mathbb{Z}$ or its piecewise linear continuation $\operatorname{ord}\mathcal{L} : N_{\mathbb{R}} \to \mathbb{R}$. If $r(\sigma) \in M$ provides a local generator $x^{r(\sigma)}$ of $\mathcal{L}$ on $U(\sigma) \subseteq X(\Sigma)$, then $\operatorname{ord}\mathcal{L}(a)$ is defined as $\langle a, r(\sigma)\rangle$ if $a \in \sigma$ (cf. [Od]). Moreover, if $\mathcal{L}$ is an ample (or at least globally generated) invertible sheaf given by a lattice polytope $\Delta \subseteq M_{\mathbb{R}}$, then the local generators of $\mathcal{L}$ correspond to the vertices of Δ. In particular, $\operatorname{ord}\mathcal{L}(a) = \min\langle a, \Delta\rangle$. Shifting the polytope Δ by a vector $r \in M$ means to replace $\mathcal{L}$ by $x^r \cdot \mathcal{L}$. The corresponding order functions differ by the globally linear function $\langle \bullet, r\rangle$.

Lemma: *Let $r \in M^\theta$ be an arbitrary element. Then, the mapping $a^\alpha \mapsto -r_\alpha$ gives the order function of $\mathcal{L}(\theta)$ on $\mathcal{M}^\theta(Q)$. A different choice $r' \in M^\theta$ just changes the order function by the linear summand $\langle \bullet, r' - r\rangle$.*

Proof: We may use $r \in M^\theta$ to carry $\Delta(\theta)$ into the "right" lattice M (see the end of (2.3)). Then, the order function applied to $a^\alpha \in \Sigma(\theta)^{(1)} \subseteq N_{\mathbb{R}}$ is

$$\operatorname{ord}\mathcal{L}(\theta)(a^\alpha) = \min\langle a^\alpha, \Delta(\theta) - r\rangle = \min\langle e^\alpha, \Delta(\theta)\rangle - r_\alpha = -r_\alpha\,. \qquad \square$$

(2.7) Given the quiver Q, the canonical weight θ^c announced in the introduction is defined as the weight of the flow r^c that is constant 1 on every arrow. Explicitly, this means

$$\theta^c(q) := \#\{\text{arrows with source } q\} - \#\{\text{arrows with target } q\}\,.$$

The advantage of θ^c is the existence of a unique interior lattice point in the polytope $\Delta(\theta^c)$: it is again the flow $r^c = [1, \ldots, 1]$.

Proposition: *The polytope $\Delta(\theta^c)$ is reflexive (in the sense of [Bat]), its order function is -1 on the generators $a^\alpha \in \Sigma(\theta^c)^{(1)}$, and the ample divisor $\mathcal{L}(\theta^c)$ is anti-canonical.*

Proof: The three claims are synonymous, i.e. we just have to look at the order function of $\mathcal{L}(\theta^c)$. Applying Lemma (2.6) on $r^c = \underline{1}$ yields the result. $\square$

3 The cohomology of the universal bundle

(3.1) From now on we assume that θ is an integral weight in general position, i.e. $\mathcal{M}^\theta(Q)$ is a smooth variety. To each integral flow we associate a divisor as follows:

$$f^\alpha \in \mathbb{F}_{\mathbb{Z}} \mapsto D_\alpha := \{x \in \mathcal{M}^\theta(Q) \mid x_\alpha = 0\} = \begin{cases} \overline{\operatorname{orb}(\alpha)} & \text{if } a^\alpha \in \Sigma^{(1)} = Q_1(\theta) \\ \emptyset & \text{otherwise} \end{cases}$$

with $\overline{\operatorname{orb}(\alpha)}$ denoting the closed orbit corresponding to the one-dimensional cone α. One obtains surjective maps $\mathbb{F}_{\mathbb{Z}} \twoheadrightarrow \operatorname{Div}\mathcal{M}^\theta(Q)$ from the space of integral flows onto the equivariant divisors and, as a consequence, $\mathbb{H}_{\mathbb{Z}} \twoheadrightarrow \operatorname{Pic}\mathcal{M}^\theta(Q)$, $\theta' \mapsto \mathcal{L}(\theta')$ from the integral weights to the Picard group (see also [Hi2], Theorem 2.3). Applying the map $\pi : \mathbb{F}_{\mathbb{Z}} \twoheadrightarrow \mathbb{H}_{\mathbb{Z}}$ means to assign a divisor its class in the Picard group.
Copying the definition of (2.3), every weight θ' gives rise to

$$\overline{\Delta}(\theta') := \{r \in \pi^{-1}(\theta') \mid r_\alpha \geq 0 \text{ for } \alpha \in Q_1(\theta)\}\,.$$

Even if $\mathcal{L}(\theta')$ is not ample on $\mathcal{M}^\theta(Q)$, the polytope $\overline{\Delta}(\theta')$ may still be used to describe the global sections:

Proposition: *The lattice points of $\overline{\Delta}(\theta')$ provide a basis of the global sections of $\mathcal{L}(\theta')$. Moreover, if $Q_1(\theta') \subseteq Q_1(\theta)$, then both polytopes $\Delta(\theta')$ and $\overline{\Delta}(\theta')$ coincide.*

Proof: Given θ', we choose a flow $r \in M^{\theta'}$ providing the order function of a divisor in the class defined by θ'. The corresponding polytope of global sections is contained in $M = M^0$; via the isomorphism $(+r) : M^0 \to M^{\theta'}$ it is mapped onto $\overline{\Delta}(\theta')$.
If $Q_1(\theta') \subseteq Q_1(\theta)$, then $\overline{\Delta}(\theta')$ sits between $\Delta(\theta')$ and $\{r \in \pi^{-1}(\theta') \,|\, r_\alpha \geq 0 \text{ for } \alpha \in Q_1(\theta')\}$. On the other hand, Proposition (2.5) implies that the latter two polytopes are equal; its proof shows quite directly that the inequalities parametrized by $Q_1 \setminus Q_1(\theta')$ are redundant for the definition of $\Delta(\theta')$. □

(3.2) Since θ is in general position, there is a universal bundle $\mathcal{U}$ on $\mathcal{M}^\theta(Q)$; it splits into a direct sum $\mathcal{U} = \oplus_{q \in Q_0} \mathcal{U}_q$ of line bundles. The direct summands $\mathcal{U}_{p,q} := \mathcal{U}_p^{-1} \otimes \mathcal{U}_q$ of $\underline{End}(\mathcal{U})$ have the following shape: Choose a walk from p to q along (possibly reversed) arrows $\alpha_1^{\varepsilon(1)}, \ldots, \alpha_m^{\varepsilon(m)}$, i.e. $\alpha_1, \ldots, \alpha_m \in Q_1$ and $\varepsilon(i) \in \{\pm 1\}$. Then, denoting by $\mathcal{O}(\alpha) := \mathcal{O}(D_\alpha)$ the sheaf corresponding to the prime divisor D_α,

$$\mathcal{U}_{p,q} = \mathcal{U}_p^{-1} \otimes \mathcal{U}_q = \bigotimes_{i=1}^{m} \mathcal{O}(\alpha_i)^{\varepsilon(i)} .$$

In the Picard group of $\mathcal{M}^\theta(Q)$ this sheaf does not depend on the particular choice of the walk from p to q: Using the language of (3.1), the sheaves $\otimes_i \mathcal{O}(\alpha_i)^{\varepsilon(i)}$ are induced from the flows $\sum_i \varepsilon(i) \cdot f^{\alpha_i}$, which all have the same weight.

Notation: Setting $\varepsilon(\alpha^i) := \varepsilon(i)$ and $\varepsilon(\alpha) := 0$ for $\alpha \notin \{\alpha^1, \ldots, \alpha^m\}$ provides a function $\varepsilon : Q_1 \to \{1, -1, 0\}$ for every walk. This is the characteristic flow introduced in (2.3). Then, the sheaf $\mathcal{U}_{p,q}$ may be written as $\mathcal{U}_{p,q} = \mathcal{U}(\varepsilon) = \otimes_{\alpha \in Q_1} \mathcal{O}(\alpha)^{\varepsilon(\alpha)}$; the corresponding weight $\theta_{p,q} := \pi(\varepsilon)$ has value 1 in p, -1 in q, and 0 in all other vertices.

(3.3) Proposition:

(1) Let $\theta \in \mathbb{H}_{\mathbb{Z}}$ be an integral weight in general position. Then, the sheaves $\mathcal{U}_{p,q} \otimes \mathcal{L}(\theta)$ and $\mathcal{U}_{p,q}^{-1} \otimes \mathcal{L}(\theta)$ are generated by their global sections.

(2) If, additionally, $\theta = \theta^c$, then the polytopes $\overline{\Delta}(\theta^c \pm \theta_{p,q})$ (describing the global sections) have the same dimension as $\Delta(\theta^c)$.

Proof: Since $\mathcal{U}_{q,p} = \mathcal{U}_{p,q}^{-1}$, it is sufficient to consider the latter sheaf. The corresponding polytope $\overline{\Delta}(\theta - \theta_{p,q})$ may be studied in different level sets:

$$\begin{array}{rcl} \overline{\Delta}(\theta - \theta_{p,q}) & = & \{r \in \pi^{-1}(\theta - \theta_{p,q}) \,|\, r_\alpha \geq 0 \text{ for } \alpha \in Q_1(\theta)\} \\ & \cong & \{r \in \pi^{-1}(\theta) \,|\, r_\alpha \geq \varepsilon(\alpha) \text{ for } \alpha \in Q_1(\theta)\} . \end{array}$$

We will use the second description.

(1) The vertices of $\Delta(\theta)$ and thus also the top-dimensional cones of $\Sigma(\theta)$ are in a one-to-one correspondence with the θ-stable trees in Q. Let $T \in \mathcal{T}(\theta)$; the corresponding vertex Δ^T of $\Delta(\theta)$ provides a local generator of $\mathcal{L}(\theta)$. Since the Δ^T are characterized by the property $\Delta^T_\alpha = 0$ for $\alpha \notin T$, we obtain the local generators of $\mathcal{U}_{p,q}^{-1} \otimes \mathcal{L}(\theta)$ from the lattice points $r^T \in M^\theta$ assigned via Lemma (2.4) to the map $\varepsilon : Q_1 \to \mathbb{Z}$ describing a walk from p to q.
We have to show that these local generators r^T are regular on any open, affine subset $U_{T'} \subseteq \mathcal{M}^\theta(Q)$ corresponding to some possibly different tree $T' \in \mathcal{T}(\theta)$. That means, it remains to check that

$r^T \in r^{T'} + (\sigma_{T'})^\vee$ where $\sigma_{T'}$ denotes the cone corresponding to T', i.e. $\sigma_{T'}$ is spanned by those arrows *not* contained in T'.

Claim: Let $T \in \mathcal{T}(\theta)$ be a θ-stable tree, and let α be any arrow in Q. Then $r^T_\alpha \geq \varepsilon(\alpha)$ is true.

Before we prove that claim, we remark that it solves our problem, as, for any tree, we know for T' that $r^{T'}_\alpha = \varepsilon(\alpha)$ for $\alpha \notin T'$. Hence, the claim implies $r^T_\alpha \geq r^{T'}_\alpha$ for $\alpha \notin T'$. On the other hand, $(\sigma_{T'})^\vee$ is just given by the inequalities $r_\alpha \geq 0$ for those α.

The claim is trivial for $\alpha \notin T$; if $\alpha \in T$, we use the formula for $r_\alpha - \varepsilon(\alpha)$ presented in Lemma (2.4). First, as already used in (2.5), stability of T implies $\sum_{q \in S_T(\alpha)} \theta_q \geq 1$. Now, the point is to interpret the two remaining sums well: together they just count the number of arrows $\alpha_i^{\varepsilon(i)}$ in the walk from p to q pointing from $T_T(\alpha)$ to $S_T(\alpha)$ minus those from $S_T(\alpha)$ to $T_T(\alpha)$. In particular,

$$\sum_{T_T(\alpha) \stackrel{\beta}{\to} S_T(\alpha)} \varepsilon(\beta) \;-\; \sum_{S_T(\alpha) \stackrel{\beta}{\to} T_T(\alpha)} \varepsilon(\beta) \;=\; \begin{cases} -1 & \text{if } p \in S_T(\alpha) \text{ and } q \in T_T(\alpha) \\ 1 & \text{if } q \in S_T(\alpha) \text{ and } p \in T_T(\alpha) \\ 0 & \text{if } p,q \in S_T(\alpha) \text{ or } p,q \in T_T(\alpha)\,. \end{cases}$$

In any case, $r_\alpha - \varepsilon(\alpha)$ remains non-negative.

(2) For the second part, we consider $\theta = \theta^c$. Since ε has only -1, 0, or 1 as values, the canonical flow $r^c = \underline{1} \in \Delta(\theta^c)$ is also contained in $\overline{\Delta}(\theta^c - \theta_{p,q}) \subseteq \pi^{-1}(\theta^c)$. Assuming $\dim \overline{\Delta}(\theta^c - \theta_{p,q}) < \dim \Delta(\theta^c)$, this means that there exist arrows $\alpha^1, \ldots, \alpha^k \in Q_1(\theta^c)$ having the following two properties:

(i) The flow r^c satisfies the α^v-inequalities of $\overline{\Delta}(\theta^c - \theta_{p,q})$ sharp, i.e. $1 = \varepsilon(\alpha^v)$ for $v = 1, \ldots, k$.

(ii) It is possible to represent $0 \in N_{I\!R}$ as a positive linear combination of the vectors $a^{\alpha^v} \in N$, $v = 1, \ldots, k$. (Recall that a^α is the normal vector of the supporting hyperplane corresponding to the inequality "$r_\alpha \geq \varepsilon(\alpha)$".)

The first property means that, along the chosen walk ε from p to q, the arrows $\alpha^1, \ldots, \alpha^k$ have all the same direction. On the other hand, the second property implies that there is a decomposition $Q_0 = Q_0^+ \sqcup Q_0^-$ with $\alpha^1, \ldots, \alpha^k$ pointing from Q_0^+ to Q_0^- and being the only arrows connecting these two parts. This yields a contradiction. □

(3.4) Let Σ be a complete fan in some d-dimensional vector space $N_{I\!R}$ with lattice N. Denote by M the dual lattice. By [Ke], I/§3 we know that the cohomology groups of equivariant, invertible sheaves $\mathcal{L}$ are M-graded and how to calculate their summands as reduced topological cohomology groups of certain subsets of $N_{I\!R}$. With $r \in M$ and $A_r := \{a \in N_{I\!R} \mid \langle a, r\rangle < \operatorname{ord} \mathcal{L}(a)\}$ it follows that $H^l(X_\Sigma, \mathcal{L})_r = \widetilde{H}^{l-1}(A_r, k)$ for $l \geq 1$. We would like to use this method to prove a generalization of Kodaira-vanishing which holds for toric varieties. We restrict the subsets A_r in question to the $(d-1)$-dimensional unit sphere $S^{d-1} \subset N_{I\!R}$:

Lemma: *Let $\phi : N_{I\!R} \to I\!R$ be a continuous function which is linear on the cones of Σ. For $B_r := \{a \in S^{d-1} \subset N_{I\!R} \mid \langle a, r\rangle < \phi(a)\}$ we denote by $\Sigma_r \subseteq \Sigma$ the subfan consisting of all cones $\sigma \in \Sigma$ such that $\sigma \cap S^{d-1} \subseteq B_r$. Then, the sets B_r and $|\Sigma_r| \cap S^{d-1}$ are homotopy equivalent. Moreover, the assertion remains true if we replace "$<$" by "$\leq$" in the definition of B_r.*
($|\Sigma_r|$ denotes the union of all cones contained in Σ_r.)

Proof: If the fan Σ is simplicial (for instance for smooth varieties X_Σ), then it is possible to prove in one strike that $|\Sigma_r| \cap S^{d-1}$ is a deformation retract of B_r. In the general case, however, it seems to be necessary to project B_r successively down dimension by dimension. Using the following fact, each step can be worked out in the cones of Σ separately:
Let P be a compact convex polytope and H^+ an open or closed half space. Then $\partial P \cap H^+$ is a deformation retract of $P \cap H^+$.
We leave the proof of this obvious fact to the reader. □

(3.5) Proposition: (Kodaira-vanishing) *Let X_Σ be a complete toric variety with at most Gorenstein singularities. Assume that $\mathcal{L}$ is an equivariant line bundle which is generated by its global sections. Then, if the lattice polytope $\Delta \subseteq M_{\mathbb{R}}$ describing $H^0(X_\Sigma, \mathcal{L})$ (as explained in (2.6)) is full-dimensional, we have $H^l(X_\Sigma, \mathcal{L} \otimes \omega_X) = 0$ for $l \geq 1$.*

Proof: Denote by ψ the order function of $\mathcal{L} \otimes \omega_X$. The order function φ_K of the canonical divisor equals 1 on the skeleton $\Sigma^{(1)}$; "Gorenstein" means that φ_K is linear on the cones. Thus, we obtain by the previous Lemma that the sets $B_r := \{a \in S^{d-1} \subset N_{\mathbb{R}} \mid \langle a, r\rangle < \psi(a)\}$ and $C_r := \{a \in S^{d-1} \subset N_{\mathbb{R}} \mid \langle a, r\rangle \leq \psi(a) - \varphi_K(a)\}$ are homotopy equivalent. The first one computes the r-th graded piece of the desired cohomology, and the latter is contractible since $\psi - \varphi_K$ is the order function of $\mathcal{L}$, i.e. $(\psi - \varphi_K)(a) = \min\langle a, \Delta\rangle$. □

Remark: The assumption about the dimension of Δ means that $\mathcal{L}$ may not be obtained via pull back from some lower-dimensional variety.

(3.6) Theorem: *Assume θ^c is in general position. Then $\operatorname{Ext}^l_{\mathcal{M}(Q)}(\mathcal{U},\mathcal{U}) = 0$ for all $l > 0$.*

Proof: Recall that $\operatorname{Ext}^l_{\mathcal{M}(Q)}(\mathcal{U},\mathcal{U}) = H^l(\mathcal{M}(Q), \underline{End}\,\mathcal{U}) = \bigoplus_{p,q\in Q_0} H^l(\mathcal{M}(Q), \mathcal{U}_{p,q})$. Then, since $\mathcal{U}_{p,q} \otimes \omega_{\mathcal{M}}^{-1} \simeq \mathcal{U}_{p,q} \otimes \mathcal{L}(\theta^c)$ is globally generated with $\dim \overline{\Delta}(\theta_{p,q} + \theta^c) = \dim \Delta(\theta^c)$ (cf. Propositions (2.7) and (3.3)), the result follows from Kodaira vanishing. □

(3.7) The previous theorem asks for the canonical weight to be in general position. We would like to close this section with a criterion for this fact to hold. Moreover, we present a criterion for $Q_1 = Q_1(\theta^c)$.

Proposition: *The canonical weight θ^c is in general position if and only if there does not exist any (t,t)-wall. Moreover, $Q_1(\theta^c) = Q_1$ if and only if there are no $(1,0)$- or $(1,1)$-walls.*

Proof: Assume a (t^+, t^-)-wall is given. Then, $\sum_{q\in Q_0^+} \theta_q^c = t^+ - t^-$ by adding the M^θ-equations. Consequently, θ^c lies on this wall precisely when $t^+ - t^- = 0$. This proves the first claim.
For the second claim, we include the case of a $(1,1)$-wall in brackets. Let $Q_0 = Q_0^+ \sqcup Q_0^-$ be the subdivision defining the wall (see (2.2)), and denote by α the unique arrow with $s(\alpha) \in Q_0^+$ and $t(\alpha) \in Q_0^-$ [by β the unique arrow with $s(\beta) \in Q_0^-$ and $t(\beta) \in Q_0^+$]. We show that α is not in $Q_1(\theta^c)$ [α and β are not both in $Q_1(\theta^c)$]. The set Q_0^+ is closed under successors in $Q \setminus \{\alpha\}$, but $\sum_{q\in Q_0^+} \theta_q^c = 1$ [$= 0$]. Thus, $Q \setminus \{\alpha\}$ is not stable [both $Q \setminus \{\alpha\}$ and $Q \setminus \{\beta\}$ are not stable].
It remains to show the converse. Assume we have no $(1,0)$-wall and no $(1,1)$-wall. Thus, $t^+ \geq 2$ for each wall W. Assume further that W is a (t^+, t^-)-wall with $t^+ > t^-$. We define the open halfspace $W^+ := \{\theta \in \mathbb{H} \mid \sum_{q\in Q_0^+} \theta_q > 0\}$, i.e. $\theta^c \in W^+$. Using the wall crossing formula from [Hi2], Lemma 3.4, we obtain $Q_1(\theta^c) = \cup_{\theta\in\mathbb{H}} Q_1(\theta)$. It remains to show that $Q_1 = \cup_{\theta\in\mathbb{H}} Q_1(\theta)$. By assumption, for each $\alpha \in Q_1$ there exists a tree in $Q \setminus \{\alpha\}$. But for each tree T there exists a weight θ such that T is θ-stable ([Hi1], Proposition 2.5). This finishes the proof. □

4 The endomorphism algebra of the universal bundle

(4.1) In this section we always assume that θ is in general position, i.e. the universal bundle on $\mathcal{M}^\theta(Q)$ exists. We start this section with a result about the endomorphism algebra $\mathcal{A}$ of the universal bundle. This algebra is non-commutative and finite-dimensional; in order to formulate the statements in this section, we need some basic results about those algebras. We denote by $\operatorname{rad}(\mathcal{A})$ the radical of $\mathcal{A}$. It consists of all strongly nilpotent elements a of $\mathcal{A}$, that is $(a\mathcal{A})^n = 0$ for

n sufficiently large. Thus $\mathcal{A}$ is isomorphic to the quotient of the tensor algebra of the $\mathcal{A}/\mathrm{rad}(\mathcal{A})$-bimodule $\mathrm{rad}(\mathcal{A})/\mathrm{rad}^2(\mathcal{A})$ by some admissible ideal I

$$\mathcal{A} \simeq T_{(\mathcal{A}/\mathrm{rad}(\mathcal{A}))}(\mathrm{rad}(\mathcal{A})/\mathrm{rad}^2(\mathcal{A}))/I.$$

Recall that an ideal I is called *admissible* if

$$T^n_{(\mathcal{A}/\mathrm{rad}(\mathcal{A}))}(\mathrm{rad}(\mathcal{A})/\mathrm{rad}^2(\mathcal{A})) \subset I \subset T^2_{(\mathcal{A}/\mathrm{rad}(\mathcal{A}))}(\mathrm{rad}(\mathcal{A})/\mathrm{rad}^2(\mathcal{A}))$$

for some n. In case $\mathcal{A}$ is the path algebra of a quiver, the radical of $\mathcal{A}$ is the ideal generated by paths of length at least one.

A finite-dimensional algebra $\mathcal{A}$ is called *basic* if the semisimple quotient $\mathcal{A}/\mathrm{rad}(\mathcal{A})$ is a product of fields. Because we deal with basic algebras over an algebraically closed field, this semisimple quotient is a product of copies of the ground field. It turns out that each basic finite-dimensional algebra is isomorphic to the quotient of a path algebra of a finite quiver by some admissible ideal. Moreover, each finite-dimensional algebra is Morita equivalent to a basic finite-dimensional algebra, that is the module categories of both algebras are isomorphic. Thus, if we are interested in module categories, we may restrict ourselves to modules over basic algebras.
The endomorphism ring of $\mathcal{U}$ is basic precisely when $\mathcal{U}$ contains only pairwise non-isomorphic direct summands. Consequently, for $\mathrm{End}(\mathcal{U})$ to be isomorphic to the path algebra of Q, it is necessary that the direct summands $\mathcal{U}_q$ are pairwise non-isomorphic. In fact, in the theorem below we will see that the converse is also true.

(4.2) In any case it would be desirable to know $\mathrm{End}(\mathcal{U})$ and its Morita equivalent basic algebra. This leads to the following definitions (cf. [S]): Let Q be a quiver without oriented cycles. Thus, the path algebra kQ is finite-dimensional. For an arrow $\alpha \in Q_1$ we define the *localization* $kQ[\alpha^{-1}]$ by formally adjoining the inverse α^{-1} of the arrow α, i.e. $s(\alpha^{-1}) = t(\alpha)$, $t(\alpha^{-1}) = s(\alpha)$, and $\alpha^{-1}\alpha = e_{t(\alpha)}$, $\alpha\alpha^{-1} = e_{s(\alpha)}$ where e_q is the idempotent in kQ corresponding to the vertex q. In particular, α and $\alpha^{-1} \in kQ[\alpha^{-1}]$ are *not* in the radical. Consequently, $kQ[\alpha^{-1}]$ is not basic, because $kQ[\alpha^{-1}]/\mathrm{rad}(kQ[\alpha^{-1}])$ contains the two-by-two full matrix ring with basis $e_{s(\alpha)}, e_{t(\alpha)}, \alpha, \alpha^{-1}$.
We consider the *quotient quiver* $\overline{Q} := Q/(Q_1 \setminus Q_1(\theta))$ defined by killing the arrows from $Q_1 \setminus Q_1(\theta)$ while identifying their sources and targets, respectively. Each weight of Q provides in a canonical way a weight of $\overline{Q}$ – just add the values of the identified vertices. The corresponding moduli spaces are isomorphic. The localization $Q[\alpha^{-1} \mid \alpha \notin Q_1(\theta)]$ is Morita equivalent to $k\overline{Q}$. Moreover, this localization is finite-dimensional if and only if the quotient quiver $\overline{Q}$ contains no oriented cycle.

Lemma: *The line bundle $\mathcal{U}_q$ is isomorphic to the line bundle $\mathcal{U}_p$ if and only if there exists a walk from p to q consisting of arrows* not *in $Q_1(\theta)$.*

Proof: The class of the bundle $\mathcal{U}_p^{-1} \otimes \mathcal{U}_q$ is trivial if and only if the divisor $\sum_{\alpha\in w} D_\alpha - \sum_{\alpha^{-1}\in w} D_\alpha$ is linearly equivalent to zero for a walk w from p to q in Q. Moreover, this divisor is also equivalent to $\sum_{\alpha\in w\cap Q_1(\theta)} D_\alpha - \sum_{\alpha^{-1}\in w\cap Q_1(\theta)} D_\alpha$, because $D_\alpha \sim 0$ for $\alpha \notin Q_1(\theta)$. This divisor is linearly equivalent to 0 if and only if $w \cap Q_1(\theta)$ is a cycle in $Q/(Q_1 \setminus Q_1(\theta))$. This is true if and only if $w \cap Q_1(\theta) = \emptyset$. □

In particular, the indecomposable direct summands of the universal bundles on $\mathcal{M}^\theta(Q)$ and $\mathcal{M}^{\overline{\theta}}(\overline{Q})$ are isomorphic; in $\mathcal{U}(\overline{Q})$ we have just cancelled multiple summands.

(4.3) **Theorem:** *The endomorphism algebra $\mathcal{A}$ of the universal bundle $\mathcal{U}$ on $\mathcal{M}^\theta(Q)$ is isomorphic to the localization of the path algebra of the quiver by all arrows* not *in $Q_1(\theta)$. If $Q_1(\theta) = Q_1$, then $\mathcal{A}$ is isomorphic to the path algebra of the quiver Q.*

Proof: As explained in the previous remarks, we may assume that $Q_1(\theta) = Q_1$. By Proposition (3.1) we know that

$$\mathrm{Hom}_{\mathcal{M}^\theta(Q)}(\mathcal{U}_p,\mathcal{U}_q) = H^0(\mathcal{M}^\theta(Q),\mathcal{U}_p^{-1} \otimes \mathcal{U}_q) = k \cdot \Big\{\text{lattice points of } \Delta(\theta_{p,q})\Big\}$$

where $\theta_{p,q}$ is the weight introduced in (3.2). Since an integral flow in $\Delta(\theta_{p,q})$ has values in $\{0,1\}$, we obtain a bijection between the lattice points of $\Delta(\theta_{p,q})$ and the paths from p to q in Q; hence $\mathrm{End}_{\mathcal{M}^\theta(Q)}(\mathcal{U}) \simeq kQ$. □

(4.4) Let $\mathcal{U}$ be a vector bundle on a smooth projective algebraic variety $\mathcal{M}$. Let $\mathcal{A}$ be the endomorphism algebra of $\mathcal{U}$, which is finite-dimensional. Moreover, let $\mathcal{U} = \oplus_{q\in Q_0}\mathcal{U}_q$ be a decomposition into indecomposable direct summands. Then $\mathcal{A} = \oplus_{q\in Q_0} e_q\mathcal{A}$ is a decomposition of $\mathcal{A}$ into indecomposable projective right $\mathcal{A}$-modules. We denote by $K^b(\mathcal{U}_q \mid q \in Q_0)$ and $K^b(e_q\mathcal{A} \mid q \in Q_0)$ the homotopy category of bounded complexes $\{C^i\}$, where each C^i is a direct sum of copies of $\mathcal{U}_q$ or $e_q\mathcal{A}$, respectively. The functor induced by the map $\mathcal{U}_q \mapsto e_q\mathcal{A}$ is an equivalence between $K^b(\mathcal{U}_q \mid q \in Q_0)$ and $K^b(e_q\mathcal{A} \mid q \in Q_0)$ because the endomorphism algebra of $\mathcal{A}$ viewed as *right* $\mathcal{A}$-module is $\mathcal{A}$.

Theorem: *Assume Q is a quiver without any (t,t)-wall. Then the equivalence above induces a full and faithful functor*

$$\mathcal{D}^b\Big(\mathit{mod}\text{–}\mathcal{A}\Big) \longrightarrow \mathcal{D}^b\Big(\mathit{Coh}(\mathcal{M}(Q))\Big).$$

Proof: We define $p \le q$ if $\mathrm{Hom}(\mathcal{U}_p,\mathcal{U}_q) \neq 0$. This is a partial order on Q_0 because $\mathcal{U}_q$ is a line bundle for all $q \in Q_0$. Consequently, $\mathrm{End}(\mathcal{U})$ is a directed algebra (there is an order on Q_0 such that $\mathrm{Hom}_{\mathcal{A}}(e_p\mathcal{A}, e_q\mathcal{A}) = 0$ for $p > q$). A directed algebra is of finite global dimension, thus the bounded derived category $\mathcal{D}^b(\text{mod–}\mathcal{A})$ of finitely generated right $\mathcal{A}$-modules is equivalent to the bounded homotopy category $K^b(e_q\mathcal{A} \mid q \in Q_0)$ (cf. [Hap] §1, 3.3). Since $\mathcal{U}$ has no self-extension (Theorem (3.6)), the natural functor $K^b(\mathcal{U}_q \mid q \in Q_0) \to \mathcal{D}^b(\mathrm{Coh}(\mathcal{M}(Q)))$ is full and faithful. □

Proof of Theorem (1.3): If there exists no $(1,0)$-wall and no $(1,1)$-wall we have $\mathcal{A} \simeq kQ$ by Proposition (3.7) and Theorem (4.3). Thus, the result follows from Theorem (4.4) □

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