

Polyhedral divisors and algebraic torus actions

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Received: 29 October 2003 / Revised version: 15 June 2005

Published online: 26 December 2005 – © Springer-Verlag 2005

Abstract. We provide a complete description of normal affine varieties with effective algebraic torus action in terms of what we call proper polyhedral divisors on semiprojective varieties. Our approach extends classical cone constructions of Dolgachev, Demazure and Pinkham to the multigraded case, and it comprises the theory of affine toric varieties.

Mathematics Subject Classification (2000): 14L24, 14L30, 14M25, 13A50

Introduction

We present a complete description of n -dimensional, normal, affine varieties with an effective action of a k -dimensional algebraic torus in terms of “proper polyhedral divisors” living on semiprojective varieties of dimension $n - k$. Our approach comprises two well known theories: on the one hand, for varieties with an almost transitive torus action ($k = n$), our description specializes to the theory of affine toric varieties [9], and on the other, for \mathbb{C}^* -actions ($k = 1$), we recover classical constructions of generalized affine cones of Dolgachev [6], Demazure [5] and Pinkham [22].

Besides the special cases $k = 1$ and $k = n$, also the case $k = n - 1$ is studied by other authors, even for not necessarily affine T -varieties X . In the last chapter of [17], the combinatorial methods for toroidal varieties developed in this book are applied to study torus actions of codimension one; see [29] for a comparison of this approach and ours. In [28], Timashev presents a general theory of reductive group actions of complexity one on normal algebraic varieties. Specializing his language of hypercones to the case of a torus action, he obtains a picture quite

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* partially supported by MSRI Berkeley, CA and SFB 647 of the DFG

** partially supported by SWP 1094 of the DFG

similar to ours, see Example 4.1 in loc. cit. Moreover, there is recent work by Flenner and Zaidenberg on affine \mathbb{K}^* -surfaces [8], which fits into our framework, see Example 3.5. Finally, the analogous setting is also studied in symplectic geometry, see for example the treatment in [16] using the moment map.

Let us outline the main results of the present paper. Let Y be a normal semi-projective variety, where “semiprojective” merely means that Y is projective over some affine variety. In order to introduce the notion of a proper polyhedral divisor on Y , consider a linear combination

$$\mathfrak{D} = \sum \Delta_i \otimes D_i$$

where the D_i are prime divisors on Y , the coefficients Δ_i are convex polyhedra in a rational vector space $N_{\mathbb{Q}} = \mathbb{Q} \otimes N$ with a free finitely generated abelian group N , and all Δ_i have a common pointed cone $\sigma \subset N_{\mathbb{Q}}$ as their tail cone (see Section 2 for the precise definitions).

Let $M := \text{Hom}(N, \mathbb{Z})$ be the dual of N , and write $\sigma^\vee \subset M_{\mathbb{Q}}$ for the dual cone. Then the above \mathfrak{D} defines an evaluation map into the group of rational Weil divisors on Y :

$$\sigma^\vee \rightarrow \text{Div}(Y), \quad u \mapsto \mathfrak{D}(u) := \sum_{v \in \Delta_i} \min\langle u, v \rangle D_i.$$

We say that \mathfrak{D} is a *proper polyhedral divisor* if any evaluation $\mathfrak{D}(u)$ is a semiample rational Cartier divisor, being big whenever u belongs to the relative interior of the cone σ^\vee .

The evaluation map $u \mapsto \mathfrak{D}(u)$ turns out to be piecewise linear and convex in the sense that the difference $\mathfrak{D}(u + u') - (\mathfrak{D}(u) + \mathfrak{D}(u'))$ is always effective. This convexity property enables us to define a graded algebra of global sections:

$$A := \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{O}(\mathfrak{D}(u))).$$

As we will prove, among other things, in Theorem 3.1, this ring is normal and finitely generated. Thus, it gives rise to a normal affine variety $X := \text{Spec}(A)$, and the M -grading of A defines an effective action of the torus $T := \text{Spec}(\mathbb{C}[M])$ on X .

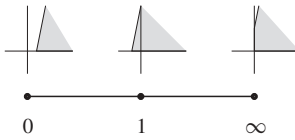
Example. Let $Y = \mathbb{P}^1$ and $N = \mathbb{Z}^2$. The vectors $(1, 0)$ and $(1, 12)$ generate a pointed convex cone σ in $N_{\mathbb{Q}} = \mathbb{Q}^2$, and we consider the polyhedra

$$\Delta_0 = \left(\frac{1}{3}, 0\right) + \sigma, \quad \Delta_1 = \left(-\frac{1}{4}, 0\right) + \sigma, \quad \Delta_\infty = (\{0\} \times [0, 1]) + \sigma.$$

Attaching these polyhedra as coefficients to the points $0, 1, \infty$ on the projective line, we obtain a proper polyhedral divisor

$$\mathfrak{D}^{E_6} = \Delta_0 \otimes \{0\} + \Delta_1 \otimes \{1\} + \Delta_\infty \otimes \{\infty\}.$$

In this situation, we may even represent our proper polyhedral divisor by a little picture as follows:



As we shall see in Section 11, the proper polyhedral divisor \mathfrak{D}^{E_6} describes the affine threefold $X = V(z_1^3 + z_2^4 + z_3 z_4)$ in \mathbb{C}^4 with the action of $T = (\mathbb{C}^*)^2$ given by

$$t \cdot z = (t_1^4 z_1, t_1^3 z_2, t_2 z_3, t_1^{12} t_2^{-1} z_4).$$

Assigning to the pp-divisor \mathfrak{D} the affine T -variety X , as indicated, turns out to be functorial. Moreover, a canonical construction, based on the chamber structure of the set of GIT-quotients of X , shows that in fact every normal affine variety with effective torus action arises from a proper polyhedral divisor. These results can be summarized as follows, see Theorems 3.4 and Proposition 8.6.

Theorem. *The assignment $\mathfrak{D} \mapsto X$ defines an essentially surjective faithful covariant functor from the category of proper polyhedral divisors on semiprojective varieties to the category of normal affine varieties with effective torus action.*

After localizing the category of proper polyhedral divisors by the maps coming from (birational) modifications of the semiprojective base varieties, we even arrive at an equivalence of categories, compare Corollary 8.14. In particular, these results allow the determination of when two proper polyhedral divisors define (equivariantly) isomorphic varieties.

As an application, in Section 10, we provide a description of the collection of T -orbits of an affine T -variety X in terms of its defining pp-divisor \mathfrak{D} , and we indicate how to read local orbit data from \mathfrak{D} . Moreover, we indicate in Section 11 a recipe for the computation of the pp-divisor of a given affine variety with torus action.

In a subsequent paper, we will deal with *non-affine* T -varieties X . Then, the coefficients of the former polyhedral divisors on Y will turn into polyhedral complexes.

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1. Tailed polyhedra

In this section, we introduce and discuss the groups of tailed polyhedra, which will serve later as the group of coefficients for our polyhedral divisors. While setting the definitions and statements, we also fix our notation from convex geometry, and we recall some basic facts needed later. For further background on convex geometry we refer to standard text books, like [10] and [25].

From here on, N denotes a *lattice*, i.e. a finitely generated free abelian group. The rational vector space associated to N is denoted by $N_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} N$, and, given a linear map $F: N \rightarrow N'$, we also write $F: N_{\mathbb{Q}} \rightarrow N'_{\mathbb{Q}}$ for the induced map of rational vector spaces. The dual lattice of N is $M := \text{Hom}(N, \mathbb{Z})$, and we denote the corresponding pairing by

$$M \times N \rightarrow \mathbb{Z}, \quad (u, v) \mapsto \langle u, v \rangle.$$

By a *polyhedron* in $N_{\mathbb{Q}}$, we mean a convex polyhedron, i.e. the intersection of finitely many closed affine half spaces in $N_{\mathbb{Q}}$. If Δ' is a face of a polyhedron Δ in $N_{\mathbb{Q}}$, then we write $\Delta' \preceq \Delta$. For a polyhedron Δ in $N_{\mathbb{Q}}$, we denote by $\text{relint}(\Delta)$ its *relative interior*, i.e. the set obtained by removing all proper faces from Δ .

For us, a *cone* in $N_{\mathbb{Q}}$ is always a convex, polyhedral cone, i.e. the intersection of finitely many closed linear half spaces in $N_{\mathbb{Q}}$. The *dual cone* σ^{\vee} of a cone σ in $N_{\mathbb{Q}}$ lives in the dual vector space $M_{\mathbb{Q}}$ and consists of all linear forms of $M_{\mathbb{Q}}$ that are nonnegative along σ . A cone is *pointed* if it does not contain any line.

The set of all polyhedra in $N_{\mathbb{Q}}$ comes with a natural abelian semigroup structure: one defines the *Minkowski sum* of two polyhedra Δ_1 and Δ_2 in $N_{\mathbb{Q}}$ to be the polyhedron

$$\Delta_1 + \Delta_2 = \{v_1 + v_2; v_i \in \Delta_i\}.$$

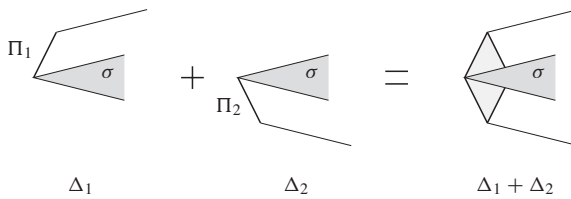
Any polyhedron Δ in $N_{\mathbb{Q}}$ allows a Minkowski sum decomposition $\Delta = \Pi + \sigma$ where $\Pi \subset N_{\mathbb{Q}}$ is a *polytope*, i.e. the convex hull of finitely many points, and $\sigma \subset N_{\mathbb{Q}}$ is a cone. In this decomposition, the *tail cone* σ is unique; in the literature it is also called the *recession cone* of Δ and is given by

$$\sigma = \{v \in N_{\mathbb{Q}}; v' + tv \in \Delta \text{ for all } v' \in \Delta, t \in \mathbb{Q}_{\geq 0}\}.$$

Definition 1.1. Let σ be a pointed cone in $N_{\mathbb{Q}}$.

- (i) By a σ -tailed polyhedron (or σ -polyhedron, in short) in $N_{\mathbb{Q}}$, we mean a polyhedron Δ in $N_{\mathbb{Q}}$ having the cone σ as its tail cone. We denote the set of all σ -polyhedra in $N_{\mathbb{Q}}$ by $\text{Pol}_{\sigma}^{+}(N_{\mathbb{Q}})$.
- (ii) We call $\Delta \in \text{Pol}_{\sigma}^{+}(N_{\mathbb{Q}})$ integral if $\Delta = \Pi + \sigma$ holds with a polytope $\Pi \subset N_{\mathbb{Q}}$ having its vertices in N . We denote the set of all integral σ -polyhedra in $N_{\mathbb{Q}}$ by $\text{Pol}_{\sigma}^{+}(N)$.

Note that the Minkowski sum $\Delta_1 + \Delta_2$ of two σ -polyhedra Δ_1 and Δ_2 in $N_{\mathbb{Q}}$ is again a σ -polyhedron in $N_{\mathbb{Q}}$. Thus, together with Minkowski addition, $\text{Pol}_{\sigma}^{+}(N_{\mathbb{Q}})$ is an abelian monoid; its neutral element is $\sigma \in \text{Pol}_{\sigma}^{+}(N_{\mathbb{Q}})$, and $\text{Pol}_{\sigma}^{+}(N) \subset \text{Pol}_{\sigma}^{+}(N_{\mathbb{Q}})$ is a submonoid.



Minkowski addition of two σ -polyhedra $\Delta_i = \Pi_i + \sigma$ in \mathbb{Q}^2 .

Definition 1.2. Let σ be a pointed cone in $N_{\mathbb{Q}}$.

- (i) The group of σ -polyhedra is the Grothendieck group of $\text{Pol}_{\sigma}^{+}(N_{\mathbb{Q}})$; we denote it by $\text{Pol}_{\sigma}(N_{\mathbb{Q}})$.
- (ii) The group of integral σ -polyhedra is the Grothendieck group of $\text{Pol}_{\sigma}^{+}(N)$; we denote it by $\text{Pol}_{\sigma}(N)$.

The key to basic properties of these groups is a version of the general correspondence between convex sets and so-called support functions [25, Theorem 13.2] adapted to the setting of tailed polyhedra. In order to state this adapted version, we firstly have to recall further notions from convex geometry.

A *quasifan* Λ in $M_{\mathbb{Q}}$ is a finite collection of cones in $M_{\mathbb{Q}}$ with the following properties, compare [23, 1.2]: for any $\lambda \in \Lambda$, all the faces $\lambda' \preceq \lambda$ belong to Λ , and, for any two $\lambda_i \in \Lambda$, the intersection $\lambda_1 \cap \lambda_2$ is a face of each λ_i . The *support* of a quasifan is the union of its cones. A quasifan is called a *fan* if all its cones are pointed.

To every polyhedron Δ in $N_{\mathbb{Q}}$, one associates its *normal quasifan* $\Lambda(\Delta)$ in $M_{\mathbb{Q}}$; the faces $F \preceq \Delta$ are in order reversing bijection with the cones of $\Lambda(\Delta)$ via

$$F \mapsto \lambda(F) := \{u \in M_{\mathbb{Q}}; \langle u, v - v' \rangle \geq 0 \text{ for all } v \in \Delta, v' \in F\}.$$

It is a basic observation that the normal quasifan $\Lambda(\Delta_1 + \Delta_2)$ of a Minkowski sum is supported on the intersection of the supports of the normal quasifans $\Lambda(\Delta_1)$ and $\Lambda(\Delta_2)$ and, moreover, equals the coarsest common refinement of both.

Lemma 1.3. *Let $\sigma \subset N_{\mathbb{Q}}$ be a pointed cone, and let $\Delta \in \text{Pol}_{\sigma}^+(N_{\mathbb{Q}})$. Then the normal quasifan $\Lambda(\Delta)$ has the dual cone $\sigma^{\vee} \subset M_{\mathbb{Q}}$ as its support.*

Proof. For every face $F \preceq \Delta$, the set $\{v - v'; v \in \Delta, v' \in F\}$ contains the tail cone σ . Dualizing yields that the cone of $\Lambda(\Delta)$ corresponding to F is contained in σ^{\vee} . Conversely, every $u \in \sigma^{\vee}$ attains its minimum along some face of $F \preceq \Delta$, and hence belongs to a cone of $\Lambda(\Delta)$. □

Since we require σ to be pointed, σ^{\vee} is of full dimension. The Lemma thus implies that for any σ -polyhedron Δ , the maximal cones of $\Lambda(\Delta)$ are of full dimension, and hence, the minimal faces of Δ are vertices, i.e. are of dimension zero. The vertices of Δ are vertices of any polytope Π with $\Delta = \Pi + \sigma$, and we may canonically write $\Delta = \Pi_0 + \sigma$, where Π_0 is the convex hull of the vertices of Δ .

Next, we have to recall the definition of the *support function* associated with a convex set Δ in $N_{\mathbb{Q}}$; this is the map given by

$$h_{\Delta} : M_{\mathbb{Q}} \rightarrow \mathbb{Q} \cup \{-\infty\}, \quad u \mapsto \inf_{v \in \Delta} \langle u, v \rangle.$$

The *domain* of this function is the subset of $M_{\mathbb{Q}}$ where it takes values in \mathbb{Q} . Here are the basic properties of the support function of a σ -polyhedron.

Lemma 1.4. *Let σ be a pointed cone in $N_{\mathbb{Q}}$, let $\Delta \in \text{Pol}_{\sigma}^+(N_{\mathbb{Q}})$, and let h_{Δ} be the corresponding support function.*

- (i) *The function h_{Δ} has the dual cone σ^{\vee} as its domain, and it is linear on each cone of the normal quasifan $\Lambda(\Delta)$.*
- (ii) *The function h_{Δ} is convex, that means that for any two vectors $u_1, u_2 \in \sigma^{\vee}$ we have*

$$h_{\Delta}(u_1) + h_{\Delta}(u_2) \leq h_{\Delta}(u_1 + u_2).$$

Moreover, strict inequality holds if and only if the vectors $u_1, u_2 \in \sigma^{\vee}$ do not belong to the same maximal cone of $\Lambda(\Delta)$.

Proof. The statements are standard in the case that Δ is a polytope of full dimension, see [20, Appendix A]; the simple proofs given there are easily adapted to our setting. □

As usual, we say that a function $h : M_{\mathbb{Q}} \rightarrow \mathbb{Q} \cup \{-\infty\}$ with a cone $\omega \subset M_{\mathbb{Q}}$ as its domain is *piecewise linear* if there is a quasifan Λ having ω as its support such that h is linear on the cones of Λ . We denote the set of convex piecewise linear functions on $M_{\mathbb{Q}}$ having a given cone ω as its domain by $\text{CPL}_{\mathbb{Q}}(\omega)$. Together with pointwise addition, $\text{CPL}_{\mathbb{Q}}(\omega)$ is an abelian monoid.

Proposition 1.5. *Let σ be a pointed cone in $N_{\mathbb{Q}}$. Then the map $\text{Pol}_{\sigma}^+(N_{\mathbb{Q}}) \rightarrow \text{CPL}_{\mathbb{Q}}(\sigma^{\vee}), \Delta \mapsto h_{\Delta}$ is an isomorphism of abelian semigroups.*

Proof. According to Lemma 1.4, the map is well-defined, and it is easily checked to be a monoid homomorphism. Moreover, the assignment

$$h \mapsto \Delta_h := \{v \in N_{\mathbb{Q}}; \langle u, v \rangle \geq h(u) \text{ for all } u \in \sigma^{\vee}\}$$

associates to any $h \in \text{CPL}_{\mathbb{Q}}(\sigma^{\vee})$ a σ -polyhedron, and it is directly checked that this gives the inverse homomorphism. \square

As announced, we now apply this observation to provide basic properties of the groups of σ -polyhedra.

Proposition 1.6. *Let σ be a pointed cone in $N_{\mathbb{Q}}$. Then $\text{Pol}_{\sigma}^{+}(N_{\mathbb{Q}})$ and $\text{Pol}_{\sigma}^{+}(N)$ are abelian monoids with cancellation law. Their respective groups of units are*

$$\text{Pol}_{\sigma}^{+}(N_{\mathbb{Q}})^{*} = \{v + \sigma; v \in N_{\mathbb{Q}}\}, \quad \text{Pol}_{\sigma}^{+}(N)^{*} = \{v + \sigma; v \in N\}.$$

Proof. Clearly, $\text{CPL}_{\mathbb{Q}}(\sigma^{\vee})$ is an abelian monoid with cancellation law. By Proposition 1.5, the same holds for $\text{Pol}_{\sigma}^{+}(N_{\mathbb{Q}})$ and the submonoid $\text{Pol}_{\sigma}^{+}(N) \subset \text{Pol}_{\sigma}^{+}(N_{\mathbb{Q}})$. Moreover, the polyhedra $v + \sigma$ correspond to the linear functions $u \mapsto \langle u, v \rangle$, which are invertible in $\text{CPL}_{\mathbb{Q}}(\sigma^{\vee})$. Since the negative of a nonlinear convex function can never be convex, the assertion follows. \square

Proposition 1.7. *Let σ be a pointed cone in $N_{\mathbb{Q}}$. Then we have the following statements for the associated groups of σ -polyhedra:*

- (i) *There is a commutative diagram of canonical, injective homomorphisms of monoids:*

$$\begin{array}{ccc} \text{Pol}_{\sigma}^{+}(N) & \longrightarrow & \text{Pol}_{\sigma}^{+}(N_{\mathbb{Q}}) \\ \downarrow & & \downarrow \\ \text{Pol}_{\sigma}(N) & \longrightarrow & \text{Pol}_{\sigma}(N_{\mathbb{Q}}) \end{array}$$

- (ii) *The multiplication of elements $\Delta \in \text{Pol}_{\sigma}^{+}(N_{\mathbb{Q}})$ by positive rational numbers $\alpha \in \mathbb{Q}_{>0}$, defined as*

$$\alpha \Delta := \{\alpha v; v \in \Delta\},$$

uniquely extends to a scalar multiplication $\mathbb{Q} \times \text{Pol}_{\sigma}(N_{\mathbb{Q}}) \rightarrow \text{Pol}_{\sigma}(N_{\mathbb{Q}})$ making $\text{Pol}_{\sigma}(N_{\mathbb{Q}})$ into a rational vector space.

- (iii) *The group $\text{Pol}_{\sigma}(N)$ of integral σ -polyhedra is free abelian, and we have a canonical isomorphism*

$$\text{Pol}_{\sigma}(N_{\mathbb{Q}}) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \text{Pol}_{\sigma}(N).$$

- (iv) *For every element $u \in \sigma^{\vee}$, there is a unique linear evaluation functional $\text{eval}_u: \text{Pol}_{\sigma}(N_{\mathbb{Q}}) \rightarrow \mathbb{Q}$ satisfying*

$$\text{eval}_u(\Delta) = \min_{v \in \Delta} \langle u, v \rangle, \quad \text{if } \Delta \in \text{Pol}_{\sigma}^{+}(N_{\mathbb{Q}}).$$

- (v) Two elements $\Delta_1, \Delta_2 \in \text{Pol}_\sigma(N_{\mathbb{Q}})$ coincide if and only if $\text{eval}_u(\Delta_1) = \text{eval}_u(\Delta_2)$ holds for all $u \in \sigma^\vee$.
- (vi) An element $\Delta \in \text{Pol}_\sigma(N_{\mathbb{Q}})$ belongs to $\text{Pol}_\sigma(N)$ if and only if for every $u \in \sigma^\vee \cap M$, the evaluation $\text{eval}_u(\Delta)$ is an integer.

Proof. For assertion (i), note that by Proposition 1.6, the monoids $\text{Pol}_\sigma^+(N)$ and $\text{Pol}_\sigma^+(N_{\mathbb{Q}})$ embed into their Grothendieck groups. The rest of the assertion is a consequence of functoriality of the Grothendieck group. Similarly, existence and uniqueness of the scalar multiplication in assertion (ii) can be established via functoriality of the Grothendieck group.

For assertion (iii), note that the map $\text{Pol}_\sigma^+(N_{\mathbb{Q}}) \rightarrow \text{CPL}_{\mathbb{Q}}(\sigma^\vee)$ of Proposition 1.5 sends the elements of $\text{Pol}_\sigma^+(N)$ to functions having integer values on $\sigma^\vee \cap M$. Thus, we may view $\text{Pol}_\sigma(N)$ as a subgroup of the abelian group of all integer-valued functions on the countable set $\sigma^\vee \cap M$. Countable subgroups of this group are free abelian, see for example [26, Satz 1]. This applies to $\text{Pol}_\sigma(N)$. The claimed isomorphism is then easily obtained by considering a \mathbb{Z} -basis for $\text{Pol}_\sigma(N)$.

On $\text{Pol}_\sigma^+(N_{\mathbb{Q}})$, the existence of the evaluation functional asserted in (iv) is due to Proposition 1.5; in fact, we have $\text{eval}_u(\Delta) = h_\Delta(u)$. The unique continuation to $\text{Pol}_\sigma(N_{\mathbb{Q}})$, is, once more, a consequence of the universal property of the Grothendieck group.

To verify the “if” part of assertion (v), write $\Delta_i = \Delta_i^+ - \Delta_i^-$ with two σ -polyhedra Δ_i^+ and Δ_i^- . Then the sums $\Delta_1 + \Delta_1^- + \Delta_2^-$ as well as $\Delta_2 + \Delta_1^- + \Delta_2^-$ are σ -polyhedra, and all their evaluations coincide. Thus, Proposition 1.5 says that these two σ -polyhedra coincide. The assertion follows.

For the “if” part of assertion (vi), it suffices to consider σ -polyhedra Δ , because any element of $\text{Pol}_\sigma(N_{\mathbb{Q}})$ can be shifted into $\text{Pol}_\sigma^+(N_{\mathbb{Q}})$ by adding an integral σ -polyhedron. For any vertex $v \in \Delta$, the linear forms $u \in \sigma^\vee \cap M$ attaining their minimum over Δ in v generate M as a lattice, because the cone of $\Lambda(\Delta)$ corresponding to v is of full dimension. Hence, the vertices of Δ belong to N if all evaluations eval_u , where $u \in \sigma^\vee \cap M$, are integral on Δ . □

2. Polyhedral divisors

In this section, we introduce the language of polyhedral divisors. The idea is to allow not only integral or rational numbers as coefficients of a divisor, but more generally, integral or arbitrary tailed polyhedra. The essential points of this section are the definition of proper polyhedral divisors (pp-divisors) and an interpretation of this notion in terms of convex piecewise linear maps, see 2.7 and 2.11.

Here, and moreover in the entire paper, the words algebraic variety refer to an integral scheme of finite type over a variety over an algebraically closed field \mathbb{K} of characteristic zero (though we expect to hold the results as well in positive

characteristics, with basically the same proofs). By a point, we always mean a closed point, and $\mathbb{K}(Y)$ denotes the function field of Y .

The following class of varieties will be of special importance for us; it comprises the affine as well as the projective ones, compare also [12].

Definition 2.1. *An algebraic variety Y is said to be semiprojective if its \mathbb{K} -algebra of global functions $A_0 := \Gamma(Y, \mathcal{O})$ is finitely generated, and Y is projective over $Y_0 := \text{Spec}(A_0)$.*

The groups of Weil and Cartier divisors on a normal algebraic variety Y are denoted by $\text{Div}(Y)$ and $\text{CaDiv}(Y)$, and the corresponding vector spaces of rational divisors are denoted by $\text{Div}_{\mathbb{Q}}(Y)$ and $\text{CaDiv}_{\mathbb{Q}}(Y)$. Since Y is normal, we have the inclusions $\text{CaDiv}(Y) \subset \text{Div}(Y)$, and $\text{CaDiv}_{\mathbb{Q}}(Y) \subset \text{Div}_{\mathbb{Q}}(Y)$.

Let us briefly recall the basic notions around divisors used later. The sheaf of sections $\mathcal{O}(D)$ of a rational Weil divisor D on a normal algebraic variety Y is, similar to the usual case, defined via

$$\Gamma(V, \mathcal{O}(D)) := \{f \in \mathbb{K}(Y); \text{div}(f|_V) + D|_V \geq 0\} = \Gamma(V, \mathcal{O}(\lfloor D \rfloor)),$$

where $V \subset Y$ is open and $\lfloor D \rfloor$ denotes the round-down divisor of D . For a section $f \in \Gamma(Y, \mathcal{O}(D)) \subset \mathbb{K}(Y)$ of a rational Weil divisor D on a normal algebraic variety Y , we define its zero set and its non-vanishing locus as

$$Z(f) := \text{Supp}(\text{div}(f) + D), \quad Y_f := Y \setminus Z(f).$$

Moreover, $D \in \text{CaDiv}_{\mathbb{Q}}(Y)$ is called *semiample* if it admits a basepoint-free multiple, i.e. for some $n \in \mathbb{Z}_{>0}$ the sets Y_f , where $f \in \Gamma(Y, \mathcal{O}(nD))$, cover Y . We also need a straightforward generalization of the concept of a big divisor on a projective variety, compare [18, Lemma. 2.60].

Definition 2.2. *We say that a divisor $D \in \text{CaDiv}_{\mathbb{Q}}(Y)$ on a variety Y is big if for some $n \in \mathbb{Z}_{>0}$ there is a section $f \in \Gamma(Y, \mathcal{O}(nD))$ with an affine non-vanishing locus Y_f .*

Now we turn to divisors with tailed polyhedra as coefficients. Here are the first definitions.

Definition 2.3. *Let Y be a normal algebraic variety; let N be a lattice, and let $\sigma \subset N_{\mathbb{Q}}$ be a pointed cone.*

- (i) *The groups of rational polyhedral Weil divisors and rational polyhedral Cartier divisors of Y with respect to $\sigma \subset N_{\mathbb{Q}}$ are*

$$\begin{aligned} \text{Div}_{\mathbb{Q}}(Y, \sigma) &:= \text{Pol}_{\sigma}(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \text{Div}(Y), \\ \text{CaDiv}_{\mathbb{Q}}(Y, \sigma) &:= \text{Pol}_{\sigma}(N_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \text{CaDiv}(Y). \end{aligned}$$

- (i) *The groups of integral polyhedral Weil divisors and integral polyhedral Cartier divisors of Y with respect to $\sigma \subset N_{\mathbb{Q}}$ are*

$$\begin{aligned} \text{Div}(Y, \sigma) &:= \text{Pol}_{\sigma}(N) \otimes_{\mathbb{Z}} \text{Div}(Y), \\ \text{CaDiv}(Y, \sigma) &:= \text{Pol}_{\sigma}(N) \otimes_{\mathbb{Z}} \text{CaDiv}(Y). \end{aligned}$$

From here on, when we speak of divisors, or polyhedral divisors we mean rational ones; if we want to consider integral divisors, then this is explicitly stated. Here is a list of first properties of the groups of polyhedral divisors.

Proposition 2.4. *Let Y be a normal algebraic variety; let N be a lattice, and let $\sigma \subset N_{\mathbb{Q}}$ be a pointed cone.*

- (i) *$\text{Div}_{\mathbb{Q}}(Y, \sigma)$ and $\text{CaDiv}_{\mathbb{Q}}(Y, \sigma)$ are rational vector spaces, and $\text{Div}(Y, \sigma)$ and $\text{CaDiv}(Y, \sigma)$ are free abelian groups.*
- (ii) *There is commutative diagram of canonical injections*

$$\begin{array}{ccc} \text{CaDiv}(Y, \sigma) & \longrightarrow & \text{Div}(Y, \sigma) \\ \downarrow & & \downarrow \\ \text{CaDiv}_{\mathbb{Q}}(Y, \sigma) & \longrightarrow & \text{Div}_{\mathbb{Q}}(Y, \sigma). \end{array}$$

Moreover, we have canonical isomorphisms

$$\begin{aligned} \text{Div}_{\mathbb{Q}}(Y, \sigma) &\cong \mathbb{Q} \otimes_{\mathbb{Z}} \text{Div}(Y, \sigma), \\ \text{CaDiv}_{\mathbb{Q}}(Y, \sigma) &\cong \mathbb{Q} \otimes_{\mathbb{Z}} \text{CaDiv}(Y, \sigma). \end{aligned}$$

- (iii) *For every element $u \in \sigma^{\vee}$, there is a well defined linear evaluation functional*

$$\begin{aligned} \text{Div}_{\mathbb{Q}}(Y, \sigma) &\rightarrow \text{Div}_{\mathbb{Q}}(Y), \\ \mathfrak{D} = \sum \Delta_i \otimes D_i &\mapsto \mathfrak{D}(u) := \sum \text{eval}_u(\Delta_i) D_i. \end{aligned}$$

- (iv) *Two polyhedral divisors $\mathfrak{D}_1, \mathfrak{D}_2 \in \text{Div}_{\mathbb{Q}}(Y, \sigma)$ coincide if and only if we have $\mathfrak{D}_1(u) = \mathfrak{D}_2(u)$ for all $u \in \sigma^{\vee}$.*
- (v) *A polyhedral divisor $\mathfrak{D} \in \text{Div}_{\mathbb{Q}}(Y, \sigma)$ is integral if and only if all its evaluations $\mathfrak{D}(u)$, where $u \in \sigma^{\vee} \cap M$, are integral divisors.*
- (vi) *A polyhedral divisor $\mathfrak{D} \in \text{Div}_{\mathbb{Q}}(Y, \sigma)$ is Cartier if and only if all its evaluations $\mathfrak{D}(u)$, where $u \in \sigma^{\vee}$, are Cartier divisors.*

Proof. The first three assertions are immediate consequences of Proposition 1.7. For the last three assertions, note first that any $\mathfrak{D} \in \text{Div}_{\mathbb{Q}}(Y, \sigma)$ allows a representation

$$\mathfrak{D} = \sum \Delta_i \otimes D_i = \sum (\Delta_i^+ - \Delta_i^-) \otimes D_i$$

with prime divisors D_i , and coefficients $\Delta_i \in \text{Pol}(N_{\mathbb{Q}})$ which are of the form $\Delta_i = \Delta_i^+ - \Delta_i^-$ with σ -polyhedra $\Delta_i^+, \Delta_i^- \in \text{Pol}_{\sigma}^+(N_{\mathbb{Q}})$.

Combining this observation with Proposition 1.7 (v) gives assertion (iv). Moreover, if all evaluations $\mathcal{D}(u)$, where $u \in \sigma^\vee \cap M$, are integral divisors, then Proposition 1.7 (vi) yields that the coefficients Δ_i are integral. This merely means that \mathcal{D} is an integral polyhedral divisor.

Now, suppose that all evaluations of \mathcal{D} are Cartier divisors. Consider the vector space $W \subset \text{Div}_{\mathbb{Q}}(Y)$ generated by the prime divisors D_i and the vector subspace $W_0 \subset W$ generated by the evaluations $\mathcal{D}(u)$, where $u \in \sigma^\vee \cap M$. Then, with respect to a basis E_1, \dots, E_k of W_0 , we may write

$$\mathcal{D}(u) = \sum g_j^+(u)E_j - g_j^-(u)E_j,$$

where the g_j^+ and the g_j^- are nonnegative linear combinations of the support functions corresponding to Δ_i^+ and Δ_i^- . In particular, g_j^+ and g_j^- are piecewise linear, convex functions. Thus, by assertion (iv), we have a representation

$$\mathcal{D} = \sum (\Gamma_j^+ - \Gamma_j^-) \otimes E_j,$$

where Γ_j^+ and Γ_j^- are the σ -polyhedra having the functions g_j^+ and g_j^- as their support functions, respectively. Since the E_j are Cartier divisors, this shows that \mathcal{D} is a polyhedral Cartier divisor. □

In the spirit of the last two statements of this proposition, we may introduce further notions for polyhedral divisors.

Definition 2.5. *Let Y be a normal algebraic variety; let N be a lattice, and let $\sigma \subset N_{\mathbb{Q}}$ be a pointed cone.*

- (i) *We call a polyhedral divisor $\mathcal{D} \in \text{Div}_{\mathbb{Q}}(Y, \sigma)$ effective (written $\mathcal{D} \geq 0$), if all evaluations $\mathcal{D}(u)$, where $u \in \sigma^\vee$, are effective divisors.*
- (ii) *We call a polyhedral divisor $\mathcal{D} \in \text{CaDiv}_{\mathbb{Q}}(Y, \sigma)$ semiample if all evaluations $\mathcal{D}(u)$, where $u \in \sigma^\vee$, are semiample divisors.*

Example 2.6. Let Y be any normal variety, $N := \mathbb{Z}$, and $\sigma := \mathbb{Q}_{\geq 0}$. Then we have a canonical isomorphism

$$\text{Div}_{\mathbb{Q}}(Y) \rightarrow \text{Div}_{\mathbb{Q}}(Y, \sigma), \quad \sum \alpha_i D_i \mapsto \sum (\alpha_i + \sigma) \otimes D_i.$$

Integral (effective, Cartier, semiample) divisors correspond to integral (effective, Cartier, semiample) polyhedral divisors. The inverse isomorphism is given by

$$\text{Div}_{\mathbb{Q}}(Y, \sigma) \rightarrow \text{Div}_{\mathbb{Q}}(Y), \quad \mathcal{D} \mapsto \mathcal{D}(1).$$

We come to the central definition of the paper; we introduce the class of proper polyhedral divisors.

Definition 2.7. Let Y be a normal algebraic variety; let N be a lattice, and let $\sigma \subset N_{\mathbb{Q}}$ be a pointed cone. A proper polyhedral divisor (abbreviated pp-divisor) on Y with respect to $\sigma \subset N_{\mathbb{Q}}$ is a semiample polyhedral divisor $\mathfrak{D} \in \text{CaDiv}_{\mathbb{Q}}(Y, \sigma)$ such that

- (i) there is a representation $\mathfrak{D} = \sum \Delta_i \otimes D_i$ with σ -polyhedra $\Delta_i \in \text{Pol}_{\sigma}^+(N_{\mathbb{Q}})$ and effective divisors $D_i \in \text{Div}(Y)$,
- (ii) for every $u \in \text{relint}(\sigma^{\vee})$, the evaluation $\mathfrak{D}(u) \in \text{CaDiv}_{\mathbb{Q}}(Y)$ is a big divisor on Y .

Clearly, the sum of two pp-divisors with respect to a given $\sigma \subset N_{\mathbb{Q}}$ is again a pp-divisor with respect to $\sigma \subset N_{\mathbb{Q}}$. Thus, these polyhedral divisors form a semigroup. Our notation is the following.

Definition 2.8. Let Y be a normal algebraic variety; let N be a lattice, and let $\sigma \subset N_{\mathbb{Q}}$ be a pointed cone. The semigroup of all pp-divisors on Y with respect to $\sigma \subset N_{\mathbb{Q}}$ is denoted by $\text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$.

We show now that the pp-divisors $\mathfrak{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ correspond to certain convex piecewise linear maps $\sigma^{\vee} \rightarrow \text{CaDiv}_{\mathbb{Q}}(Y)$. The precise definition of these maps is the following.

Definition 2.9. Let Y be a normal variety; let M be a lattice, and let $\omega \subset M_{\mathbb{Q}}$ be a cone of full dimension. We say that a map $\mathfrak{h}: \omega \rightarrow \text{CaDiv}_{\mathbb{Q}}(Y)$ is

- (i) convex if $\mathfrak{h}(u) + \mathfrak{h}(u') \leq \mathfrak{h}(u + u')$ holds for any two elements $u, u' \in \omega$,
- (ii) piecewise linear if there is a quasifan Λ in $M_{\mathbb{Q}}$ having ω as its support such that \mathfrak{h} is linear on the cones of Λ ,
- (iii) strictly semiample if $\mathfrak{h}(u)$ is always semiample and, for $u \in \text{relint}(\omega)$, it is even big.

The sum of two convex, piecewise linear, strictly semiample maps is again of this type. We use the following notation:

Definition 2.10. Let Y be a normal variety; let M be a lattice, and let $\omega \subset M_{\mathbb{Q}}$ be a cone of full dimension. The semigroup of all convex, piecewise linear, strictly semiample maps $\mathfrak{h}: \omega \rightarrow \text{CaDiv}_{\mathbb{Q}}(Y)$ is denoted by $\text{CPL}_{\mathbb{Q}}(Y, \omega)$.

With these definitions, we are ready to state the analogue of Proposition 1.5 for proper polyhedral divisors.

Proposition 2.11. Let Y be a normal variety, N a lattice, and $\sigma \subset N_{\mathbb{Q}}$ a pointed cone. Then there is a canonical isomorphism of semigroups:

$$\text{PPDiv}_{\mathbb{Q}}(Y, \sigma) \rightarrow \text{CPL}_{\mathbb{Q}}(Y, \sigma^{\vee}), \quad \mathfrak{D} \mapsto [u \mapsto \mathfrak{D}(u)].$$

Under this isomorphism, the integral polyhedral divisors correspond to those maps sending $\sigma^{\vee} \cap M$ to $\text{CaDiv}(Y)$.

Proof. By Proposition 2.4, the assignment is a well defined injective homomorphism. Thus, we only need to verify that any convex piecewise linear map $\mathfrak{h}: \sigma^\vee \rightarrow \text{CaDiv}_{\mathbb{Q}}(Y)$, in the sense of Definition 2.9, arises from a polyhedral divisor. Since σ^\vee is polyhedral, there occur only finitely many prime divisors D_1, \dots, D_r in the images $\mathfrak{h}(u)$, where $u \in \sigma^\vee$. Thus, we may write

$$\mathfrak{h}(u) = \sum_{i=1}^r h_i(u)D_i,$$

where every $h_i: \sigma^\vee \rightarrow \mathbb{Q}$ is a \mathbb{Q} -valued, convex, piecewise linear function in the usual sense. According to Proposition 1.5, each of the functions h_i is of the form $u \mapsto \text{eval}_u(\Delta_i)$ with a σ -polyhedron $\Delta_i \subset N_{\mathbb{Q}}$. Consequently, the sum of all $\Delta_i \otimes D_i$ is a polyhedral divisor defining the map $\mathfrak{h}: \sigma^\vee \rightarrow \text{CaDiv}_{\mathbb{Q}}(Y)$. \square

This observation allows us to switch freely between pp-divisors and convex, piecewise linear, strictly semiample maps. In particular, we denote these objects by the same symbol, preferably by the gothic letter \mathfrak{D} .

Example 2.12. Let Y be a smooth projective curve; let N be a lattice, and let $\sigma \subset N_{\mathbb{Q}}$ be a pointed cone. To any polyhedral divisor on Y with respect to $\sigma \subset N_{\mathbb{Q}}$, we associate its *polyhedral degree* by setting

$$\text{deg}(\sum \Delta_i \otimes D_i) := \sum \text{deg}(D_i)\Delta_i \in \text{Pol}_{\sigma}(N_{\mathbb{Q}}).$$

This does not depend on the representation of a given $\mathfrak{D} \in \text{CaDiv}_{\mathbb{Q}}(Y, \sigma)$, and $\text{eval}_u(\text{deg}(\mathfrak{D}))$ equals $\text{deg}(\mathfrak{D}(u))$ for any $u \in \sigma^\vee$. We will figure out, in terms of the degree, when a given $\mathfrak{D} \in \text{CaDiv}_{\mathbb{Q}}(Y, \sigma)$ is a pp-divisor.

First, recall that on the curve Y , a divisor is big if and only if it has positive degree, and a divisor is semiample if and only if it is big or some multiple of it is a principal divisor. Consequently, $\mathfrak{D} \in \text{CaDiv}_{\mathbb{Q}}(Y, \sigma)$ is a pp-divisor if and only if the following holds:

- (i) $\mathfrak{D} = \sum \Delta_i \otimes \{y_i\}$, with $y_i \in Y$ pairwise disjoint and $\Delta_i \in \text{Pol}_{\sigma}^+(N_{\mathbb{Q}})$,
- (ii) the σ -polyhedron $\text{deg}(\mathfrak{D})$ is a proper subset of the cone σ ,
- (iii) if $\text{eval}_u(\text{deg}(\mathfrak{D})) = 0$, then $u \in \partial\sigma^\vee$ and a multiple of $\mathfrak{D}(u)$ is principal.

Note that the first of these conditions is a reformulation of Condition 2.7 (i). Moreover, the last two conditions are satisfied if $\text{deg}(\mathfrak{D})$ is contained in the relative interior of σ .

3. pp-divisors and torus actions

In this section, we formulate the first results of this paper. They show that the affine normal varieties with an effective algebraic torus action arise from pp-divisors on normal semiprojective varieties.

Let us briefly fix our notation around torus actions and also recall a little background. An (algebraic) torus is an affine algebraic group $T = \text{Spec}(\mathbb{K}[M])$, where M is a lattice, and $\mathbb{K}[M]$ denotes the associated group algebra. For an element $u \in M$, we denote, as usual, the corresponding character by $\chi^u: T \rightarrow \mathbb{K}^*$.

If a torus $T = \text{Spec}(\mathbb{K}[M])$ acts on a variety X , then we always assume that this action is given by a morphism

$$T \times X \rightarrow X, \quad (t, x) \mapsto t \cdot x,$$

and we also speak about the T -variety X . A *semiinvariant* with respect to the character $\chi^u: T \rightarrow \mathbb{K}^*$ is a function $f \in \Gamma(X, \mathcal{O})$ satisfying

$$f(t \cdot x) = \chi^u(t) f(x) \quad \text{for all } (t, x) \in T \times X.$$

We write $\Gamma(X, \mathcal{O})_u$ for the vector space of semiinvariants with respect to χ^u , and $\Gamma(X, \mathcal{O})^T$ for the algebra of invariants, i.e. the semiinvariants with respect to χ^0 . The action of T on X is called *effective* if the neutral element of T is the only element acting trivially on X .

A morphism $\pi: X \rightarrow Y$ is called a *good quotient* for a T -action on X if it is affine, T -invariant, i.e. constant on T -orbits, and the pullback map $\pi^*: \mathcal{O}_Y \rightarrow \pi_*(\mathcal{O}_X)^T$ is an isomorphism. If a good quotient exists, then it is unique up to an isomorphism, and the quotient space is frequently denoted as $X//T$.

The possible actions of a torus $T = \text{Spec}(\mathbb{K}[M])$ on an affine variety $X = \text{Spec}(A)$ correspond to M -gradings of the algebra A : given a T -action, the homogeneous part $A_u \subset A$ for $u \in M$ consists precisely of the semiinvariants with respect to the character $\chi^u: T \rightarrow \mathbb{K}^*$.

The *weight monoid* S of a T -action on $X = \text{Spec}(A)$ consists of all $u \in M$ with $A_u \neq \{0\}$, and the *weight cone* is the (convex, polyhedral) cone $\omega \subset M_{\mathbb{Q}}$ generated by S . We will usually denote the M -grading of A as

$$A = \bigoplus_{u \in \omega \cap M} A_u.$$

Let us present the first result. Fix a normal semiprojective variety Y , a lattice N , a pointed cone $\sigma \subset N_{\mathbb{Q}}$, and a pp-divisor $\mathcal{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$. Then these data define a sheaf of multigraded algebras on Y : the convexity property 2.9 (i) of the map $u \mapsto \mathcal{D}(u)$ ensures the existence of canonical multiplication maps

$$\mathcal{O}(\mathcal{D}(u)) \otimes \mathcal{O}(\mathcal{D}(u')) \rightarrow \mathcal{O}(\mathcal{D}(u + u')),$$

and thus, the sheaves $\mathcal{O}(\mathcal{D}(u))$, where $u \in \sigma^{\vee} \cap M$, can be put together to an \mathcal{O}_Y -algebra \mathcal{A} , graded by the monoid $\sigma^{\vee} \cap M$. Now we take the relative spectrum $\tilde{X} := \text{Spec}_Y(\mathcal{A})$. Here are the basic properties of this construction.

Theorem 3.1. *Let Y be a normal semiprojective variety, N a lattice, $\sigma \subset N_{\mathbb{Q}}$ a pointed cone, and $M := \text{Hom}(N, \mathbb{Z})$. Given a pp-divisor $\mathfrak{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$, consider the \mathcal{O}_Y -algebra*

$$\mathcal{A} := \bigoplus_{u \in \sigma^{\vee} \cap M} \mathcal{A}_u := \bigoplus_{u \in \sigma^{\vee} \cap M} \mathcal{O}(\mathfrak{D}(u)),$$

the algebraic torus $T := \text{Spec}(\mathbb{K}[M])$, and the relative spectrum $\tilde{X} = \text{Spec}_Y(\mathcal{A})$. Then the following statements hold.

- (i) *The scheme \tilde{X} is a normal algebraic variety of dimension $\dim(Y) + \dim(T)$, and the grading of \mathcal{A} defines an effective torus action $T \times \tilde{X} \rightarrow \tilde{X}$ having the canonical map $\pi : \tilde{X} \rightarrow Y$ as a good quotient.*
- (ii) *The ring of global sections $A := \Gamma(\tilde{X}, \mathcal{O}) = \Gamma(Y, \mathcal{A})$ is a finitely generated M -graded normal \mathbb{K} -algebra, and we have a proper, birational T -equivariant contraction morphism $\tilde{X} \rightarrow X$ with $X := \text{Spec}(A)$.*
- (iii) *Let $u \in \sigma^{\vee} \cap M$ and $f \in A_u$. Then we have $\pi(\tilde{X}_f) = Y_f$. In particular, if Y_f is affine, then so is \tilde{X}_f , and the canonical map $\tilde{X}_f \rightarrow X_f$ is an isomorphism. Moreover, even for non-affine Y_f , we have*

$$\Gamma(Y_f, \mathcal{A}) = \bigoplus_{u \in \sigma^{\vee} \cap M} (A_f)_u.$$

The proof will be given in Section 4. As the following two examples show, the result extends both the construction of affine toric varieties, see e.g. [9], and the Dolgachev-Demazure-Pinkham construction of good \mathbb{K}^* -actions, see [6], [5] and [22]:

Example 3.2 (Affine Toric Varieties). Let N be a lattice and $\sigma \subset N_{\mathbb{Q}}$ a cone. Then the associated affine toric variety X_{σ} is defined as the spectrum of the semigroup algebra $\mathbb{K}[\sigma^{\vee} \cap M]$ where M is the dual lattice of N , and σ^{\vee} is the dual cone of σ . We can also obtain X_{σ} as the $\tilde{X} = X$ of a pp-divisor: Take $Y := \text{Spec}(\mathbb{K})$, and let $\mathfrak{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ be the trivial divisor.

Example 3.3 (Good \mathbb{K}^ -Actions).* Let Y be a normal projective variety, and let D be an ample rational Cartier divisor on Y . These data give rise to a pp-divisor: take $N := \mathbb{Z}$; let $\sigma \subset \mathbb{Q}$ be the positive ray, and consider $\mathfrak{D} := (1 + \sigma) \otimes D$. Then \mathfrak{D} corresponds to the map

$$\sigma^{\vee} \rightarrow \text{CaDiv}_{\mathbb{Q}}(Y), \quad u \mapsto uD.$$

If D is an integral Cartier divisor, then $\tilde{X} = \text{Spec}_Y(\mathcal{A})$ is the total space of a line bundle, and $\tilde{X} \rightarrow X$ is the \mathbb{K}^* -equivariant contraction of the zero section. Thus, the affine variety X is an affine cone over Y . For D being a rational divisor, X is usually called a generalized cone over Y .

In our second result, we go the other way around. We show that every normal affine variety X with an effective torus action arises from a pp-divisor $\mathfrak{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ on some normal semiprojective variety Y in the sense of the preceding construction.

Theorem 3.4. *Let X be a normal affine variety and suppose that $T = \text{Spec}(\mathbb{K}[M])$ acts effectively on X with weight cone $\omega \subset M_{\mathbb{Q}}$. Then there exists a normal semiprojective variety Y and a pp-divisor $\mathfrak{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \omega^{\vee})$ such that we have an isomorphism of graded algebras:*

$$\Gamma(X, \mathcal{O}) \cong \bigoplus_{u \in \omega \cap M} \Gamma(Y, \mathcal{O}(\mathfrak{D}(u))).$$

For the proof, we refer to Section 6. Our construction of the semiprojective variety Y and the pp-divisor \mathfrak{D} is basically canonical. It relies on the chamber structure of the collection of all GIT-quotients of X that arise from possible linearizations of the trivial bundle.

We conclude this section with a further example. We indicate how to recover the Flenner–Zaidenberg description [8] of normal affine \mathbb{K}^* -surfaces from Theorems 3.1 and 3.4:

Example 3.5 (Normal affine \mathbb{K}^ -surfaces).* Any normal affine surface X with effective \mathbb{K}^* -action arises from a pp-divisor $\mathfrak{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ with a normal and hence smooth curve Y and a pointed cone $\sigma \subset \mathbb{Q}$ where the lattice is \mathbb{Z} .

The curve Y is either affine or projective. In the latter case, we have $\sigma \neq \{0\}$, because otherwise the convexity property $\mathfrak{D}(1) + \mathfrak{D}(-1) \leq \mathfrak{D}(0) = 0$ would contradict strict semiampleness of \mathfrak{D} . Thus, up to switching the action by $t \mapsto t^{-1}$, there are three cases:

Elliptic case. The curve Y is projective and $\sigma = \mathbb{Q}_{\geq 0}$ holds. Then the \mathbb{K}^* -action on X is good, i.e. it has an (isolated) attractive fixed point. Moreover, \mathfrak{D} is of the form

$$\mathfrak{D} = \sum_{i=1}^r [v_i, \infty) \otimes \{y_i\},$$

with $y_i \in Y$ and $v_i \in \mathbb{Q}$ such that $v_1 + \dots + v_r > 0$. Note that in this case, \mathfrak{D} is determined by its evaluation $D := \mathfrak{D}(1)$, namely $\mathfrak{D} = [1, \infty) \otimes D$.

Parabolic case. The curve Y is affine, and $\sigma = \mathbb{Q}_{\geq 0}$ holds. Then the \mathbb{K}^* -action has an attractive fixed point curve, isomorphic to Y . Moreover,

$$\mathfrak{D} = \sum_{i=1}^r [v_i, \infty) \otimes \{y_i\},$$

with $y_i \in Y$, but no condition on the numbers $v_1, \dots, v_r \in \mathbb{Q}$. Again, \mathfrak{D} is determined by its evaluation $D := \mathfrak{D}(1)$.

Hyperbolic case. The curve Y is affine, and $\sigma = \{0\}$ holds. Then the generic \mathbb{K}^* -orbit is closed. For the pp-divisor \mathfrak{D} , we obtain a representation

$$\mathfrak{D} = \sum_{i=1}^r [v_i^-, v_i^+] \otimes \{y_i\},$$

with $y_i \in Y$, and $v_i^- \leq v_i^+$. Note that \mathfrak{D} is determined by $D^- := \mathfrak{D}(-1)$ and $D^+ := \mathfrak{D}(1)$. This pair satisfies $D^- + D^+ \leq 0$, and we have

$$\mathfrak{D} = \{1\} \otimes D^+ - [0, 1] \otimes (D^- + D^+).$$

4. Proof of Theorem 3.1

This section is devoted to proving Theorem 3.1. Note that parts of the assertions (i) and (ii) are well known for the case that Y_0 is a point and $u \mapsto \mathfrak{D}(u)$ is linear, see for example [30, Thm. 4.2] and [13, Lemma 2.8].

We start with a basic observation concerning multigraded rings which will also be used apart from the proof of Theorem 3.1.

Lemma 4.1. *Let M be a lattice, and let A be a finitely generated M -graded \mathbb{K} -algebra. Then, every (convex, polyhedral) cone $\omega \subset M_{\mathbb{Q}}$ defines a finitely generated \mathbb{K} -algebra*

$$A_{(\omega)} := \bigoplus_{u \in \omega \cap M} A_u.$$

Proof. Let f_1, \dots, f_r be homogeneous generators of A , and let $u_i \in M$ denote the degree of f_i . Consider the linear map $F: \mathbb{Z}^r \rightarrow M$ sending the i -th canonical basis vector to u_i . Then, $\gamma := F^{-1}(\omega) \cap \mathbb{Q}_{\geq 0}^r$ is a pointed, polyhedral cone. Let $H \subset \gamma$ be the Hilbert Basis of the semigroup $\gamma \cap \mathbb{Z}^r$. Then, $A_{(\omega)}$ is generated by the elements $f_1^{m_1} \cdot \dots \cdot f_r^{m_r}$, where $(m_1, \dots, m_r) \in H$. □

Proof of Theorem 3.1. The proof is subdivided into six steps. Successively weakening the hypotheses, we prove in the first three steps that \tilde{X} and X are in fact varieties, that $\tilde{X} \rightarrow X$ is a proper morphism and that $\dim(\tilde{X}) = \dim(Y) + \dim(T)$ holds. In step four, we show that T acts effectively with $\pi: \tilde{X} \rightarrow Y$ as a good quotient. Step five is devoted to proving assertion (iii), and in step six, we prove normality of \tilde{X} and X .

Step 1. Assume that $\sigma^\vee \subset M_{\mathbb{Q}}$ is a regular cone, i.e. it is mapped to $\mathbb{Q}_{\geq 0}^r$ under a suitable isomorphism $M \cong \mathbb{Z}^r$. Moreover, assume $u \mapsto \mathfrak{D}(u)$ to be linear with integral, basepoint free Cartier divisors $D_i := \mathfrak{D}(e_i)$ where e_1, \dots, e_r are the primitive generators of σ^\vee . In this case, $\tilde{X} \rightarrow Y$ is a rank r vector bundle, and we only have to show that $\Gamma(Y, \mathcal{A})$ is finitely generated and that $\tilde{X} \rightarrow X$ is proper.

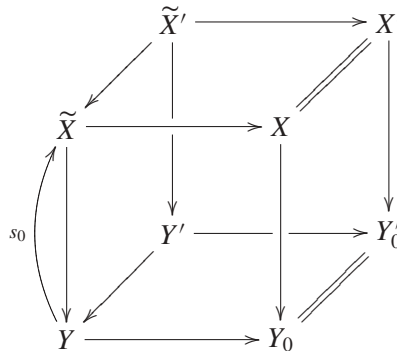
If $r = 1$ with an ample Cartier divisor D_1 , then we are in the classical setup, cf. Example 3.3. For D_1 being just basepoint free, we can reduce to the classical case by contracting Y via a morphism $Y \rightarrow \bar{Y}$ with connected fibers such that D_1 is the pull back of an ample Cartier divisor. For general r , we “coarsen” the grading of the \mathcal{O}_Y -algebra \mathcal{A} : for $u \in \mathbb{N}^r$, denote $|u| := u_1 + \dots + u_r$ and set

$$\mathcal{B} := \bigoplus_{k \in \mathbb{N}} \mathcal{B}_k, \quad \text{where } \mathcal{B}_k := \bigoplus_{|u|=k} \mathcal{A}_u.$$

Consider the corresponding projective space bundle $Y' := \text{Proj}_Y(\mathcal{B})$ with its projection $\varphi: Y' \rightarrow Y$ and $\mathcal{L}' := \mathcal{O}_{Y'}(1)$, which means that $\varphi_*(\mathcal{L}'^{\otimes k}) = \mathcal{B}_k$, compare [11, p. 162]. Then we obtain an $\mathcal{O}_{Y'}$ -algebra and an associated variety:

$$\mathcal{A}' := \bigoplus_{k \in \mathbb{N}} \mathcal{L}'^{\otimes k}, \quad \tilde{X}' := \text{Spec}_{Y'}(\mathcal{A}').$$

Note that \tilde{X}' is obtained from the rank r vector bundle \tilde{X} over Y by blowing up the zero section $s_0: Y \hookrightarrow \tilde{X}$. In summary, everything fits nicely into the following commutative diagram:



The equality $X' = X$ of the spectra of the respective rings of global sections follows from

$$\bigoplus_{k \in \mathbb{N}} \Gamma(Y', \mathcal{L}'^{\otimes k}) = \bigoplus_{k \in \mathbb{N}} \Gamma(Y, \varphi_* \mathcal{L}'^{\otimes k}) = \bigoplus_{k \in \mathbb{N}} \Gamma(Y, \mathcal{B}_k) = \bigoplus_{u \in \mathbb{N}^r} \Gamma(Y, \mathcal{A}_u).$$

In order to reduce our problem to the case $r = 1$, we have to ascertain that \mathcal{L}' is basepoint free. Since $\pi(\tilde{X}'_f) = Y'_f$ holds for all homogeneous $f \in \Gamma(Y', \mathcal{A}')$, it suffices to show that any given $x \in \tilde{X}' \setminus Y'$ admits such an f of degree $n \in \mathbb{N}_{>0}$ with $f(x) \neq 0$. For the latter, consider the canonical projections

$$\tilde{X} \rightarrow \tilde{X}_i := \text{Spec}_Y(\mathcal{A}_i), \quad \text{where } \mathcal{A}_i := \bigoplus_{m \in \mathbb{N}} \mathcal{O}(mD_i).$$

Since $\tilde{X}' \setminus Y'$ equals $\tilde{X} \setminus Y$, at least one of these maps does not send $x \in \tilde{X} \setminus Y$ to the zero section of \tilde{X}_i . By semiampleness of D_i , there is a homogeneous $f \in \Gamma(Y, \mathcal{A}_i)$ of nontrivial degree me_i with $f(x) \neq 0$.

Step 2. Assume that $\sigma^\vee \subset M_{\mathbb{Q}}$ is a simplicial cone, i.e. it is generated by linearly independent vectors, and that $u \mapsto \mathfrak{D}(u)$ is still a linear map.

We will first show that \tilde{X} is a variety. For this we only have to verify that \mathcal{A} is locally of finite type over \mathcal{O}_Y . Choose a sublattice $L \subset M$ of finite index such that L, σ^\vee and $u \mapsto \mathfrak{D}(u)$ restricted to $\sigma^\vee \cap L$ match the assumptions of the previous step. By linearity, we may extend the assignment $u \mapsto \mathfrak{D}(u)$ from $\sigma^\vee \cap M$ to M . This gives further \mathcal{O}_Y -algebras

$$\mathcal{A}_L^{\text{grp}} := \bigoplus_{u \in L} \mathcal{A}_u, \quad \mathcal{A}_M^{\text{grp}} := \bigoplus_{u \in M} \mathcal{A}_u.$$

Locally, $\mathcal{A}_L^{\text{grp}}$ looks like $\mathcal{O}_Y \otimes \mathbb{K}[L]$; hence, it is locally of finite type. Choosing representatives u_1, \dots, u_k of M/L and finitely many local generators $g_{ij} \in \mathcal{A}_{u_i}$, we see that $\mathcal{A}_M^{\text{grp}}$ is locally finitely generated as an $\mathcal{A}_L^{\text{grp}}$ -module; hence, it also is locally of finite type as an \mathcal{O}_Y -algebra. Finally, we notice that the inclusion $\mathcal{A} \subset \mathcal{A}_M^{\text{grp}}$ fits exactly into the situation of Lemma 4.1. Hence, \mathcal{A} is a locally finitely generated \mathcal{O}_Y -algebra.

Write for the moment $\mathcal{A}_M := \mathcal{A}$, and, analogously, let \mathcal{A}_L denote the \mathcal{O}_Y -algebra associated to \mathfrak{D} restricted to $\sigma^\vee \cap L$. Then the canonical morphism $\tilde{X}_M \rightarrow \tilde{X}_L$ of the corresponding relative spectra is a finite map of varieties; in fact, it is the quotient for the action of the finite group $\text{Spec}(\mathbb{K}[M/L])$ on \tilde{X} given by the grading.

From the preceding step, we know that \tilde{X}_L is proper over the affine variety $X_L := \text{Spec}(\Gamma(Y, \mathcal{A}_L))$. Thus, the affine scheme $X_M := \text{Spec}(\Gamma(\tilde{X}_M, \mathcal{O}))$ gives a commutative diagram

$$\begin{array}{ccc} \tilde{X}_M & \xrightarrow{\text{finite}} & \tilde{X}_L \\ \text{proper} \downarrow & & \downarrow \text{proper} \\ X_M & \xrightarrow{\text{finite}} & X_L \end{array}$$

where the lower row is finite because $\kappa: \tilde{X}_M \rightarrow X_L$ is proper, and thus, $\Gamma(X_M, \mathcal{O}) = \Gamma(\tilde{X}_M, \mathcal{O}) = \Gamma(X_L, \kappa_*(\mathcal{O}_{\tilde{X}_M}))$ is finite over $\Gamma(X_L, \mathcal{O})$.

Step 3. Let \mathfrak{D} be general.

We may subdivide σ^\vee by a simplicial fan Λ such that \mathfrak{D} is linear on each of the maximal cones $\lambda_1, \dots, \lambda_s$ of Λ . Then, by Step 2, we know about the corresponding proper maps $\tilde{X}_i \rightarrow X_i$. The embedding of the cones into the fan yields birational projections $\tilde{X} \rightarrow \tilde{X}_i$ and $X \rightarrow X_i$, which in turn lead to closed embeddings

$$\tilde{X} \hookrightarrow \tilde{X}_1 \times_Y \dots \times_Y \tilde{X}_s, \quad X \hookrightarrow X_1 \times_{Y_0} \dots \times_{Y_0} X_s.$$

Step 4. The grading of \mathcal{A} defines an effective torus action $T \times \tilde{X} \rightarrow \tilde{X}$ having the canonical map $\tilde{X} \rightarrow Y$ as a good quotient.

For any affine $V \subset Y$, the grading of $\Gamma(V, \mathcal{A})$ defines a T -action on $\text{Spec}(\Gamma(V, \mathcal{A}))$. This is compatible with gluing, and thus we obtain a T -action on \tilde{X} . By construction, $\tilde{X} \rightarrow Y$ is affine, and $\mathcal{O}_Y = \mathcal{A}_0$ is the sheaf of invariants. Hence, $\tilde{X} \rightarrow Y$ is a good quotient for the T -action. The fact that T acts effectively is seen as follows. Since the algebra \mathcal{A} admits locally nontrivial sections in any degree $u \in \sigma^\vee \cap M$, the weight monoid of the T -action generates M as a group. Consequently, the T -action has free orbits, and hence is effective.

Step 5. Let $f \in A_u$. Here we will prove the third part of the theorem. Note that by Condition 2.9 (iii) of the map $u \mapsto \mathfrak{D}(u)$, this will imply birationality of $\tilde{X} \rightarrow X$.

In the situation of Step 1, the equality $\pi(\tilde{X}_f) = Y_f$ is obvious (and was already used there). Suppose we are in the setting of Step 2. There we introduced a finite map $\tilde{X}_M \rightarrow \tilde{X}_L$ where $\tilde{X}_M = \tilde{X}$. This map fits into the commutative diagram

$$\begin{array}{ccc}
 \tilde{X}_M & \xrightarrow{\quad} & \tilde{X}_L \\
 \pi_M \searrow & & \swarrow \pi_L \\
 & Y &
 \end{array}$$

where we denote by π_M and π_L the respective canonical projections. Then $f^k \in \Gamma(\tilde{X}_L, \mathcal{O})$ holds for some positive power of $f \in \Gamma(\tilde{X}_M, \mathcal{O})$, and, clearly, $\pi_M((\tilde{X}_M)_f)$ equals $\pi_L((\tilde{X}_L)_{f^k})$. This reduces the problem to the setting of Step 1.

Thus, we are left with considering the situation of Step 3. There we used a simplicial fan subdivision of σ^\vee with maximal cones λ_i . This defines birational morphisms $\varphi_i: \tilde{X} \rightarrow \tilde{X}_i$ and commutative diagrams

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow{\varphi_i} & \tilde{X}_i \\
 \pi \searrow & & \swarrow \pi_i \\
 & Y &
 \end{array}$$

Choose i such that $u = \text{deg}(f)$ lies in λ_i , and write $f = \varphi_i^*(f_i)$. Then the diagram directly gives $\pi(\tilde{X}_f) \subset Y_f$. For the converse inclusion, note first that φ_i induces dominant morphisms of the fibers $\tilde{X}_y \rightarrow \tilde{X}_{i,y}$. Now, let $y \in Y$ such that f vanishes along $\pi^{-1}(y)$. Then, by fiber-wise dominance of φ_i , the function f_i vanishes along $\pi_i^{-1}(y)$. Hence, the previous cases result in $y \notin Y_f$.

If Y_f is affine, then so is $\pi^{-1}(Y_f)$ and hence, by $\tilde{X}_f \subset \pi^{-1}(Y_f)$, also $\tilde{X}_f = \pi^{-1}(Y_f)_f$. In particular, we have an isomorphism $\tilde{X}_f \rightarrow X_f$ in this case. The last statement of (iii) can be proven as follows: for any $v \in \sigma^\vee \cap M$, we have

$$\begin{aligned}
 \Gamma(Y_f, \mathcal{A}_v) &= \{g \in \mathbb{K}(Y); \operatorname{div}(g) + \mathfrak{D}(v) \geq 0 \text{ on } Y_f\} \\
 &= \{g \in \mathbb{K}(Y); \exists k \geq 0 : \operatorname{div}(g) + \mathfrak{D}(v) + k \cdot (\operatorname{div}(f) + \mathfrak{D}(u)) \geq 0\} \\
 &= \{g \in \mathbb{K}(Y); \exists k \geq 0 : gf^k \in A_{v+ku}\} \\
 &= (A_f)_v.
 \end{aligned}$$

Step 6. The varieties \tilde{X} and X are normal.

It suffices to show that \tilde{X} is normal. This is a local problem; hence, we may assume in this step that Y is affine and, moreover, that for all $u \in \sigma^\vee \cap M$, the homogeneous pieces $A_u := \Gamma(Y, \mathcal{A}_u)$ of A are non-trivial. We may use any $g_u \in A_u$ to obtain an embedding

$$\iota_u : A_u \hookrightarrow Q(A)^T = Q(A_0) = \mathbb{K}(Y), \quad f \mapsto f/g_u,$$

where $Q(A)$ stands for the fraction field of A . Note that the equality $Q(A)^T = Q(A_0)$ holds because the quotient space Y is of dimension $\dim(\tilde{X}) - \dim(T)$. The image of the above embedding can be described as follows:

$$\iota_u(A_u) = \{h \in Q(A_0); \operatorname{div}(h) \geq -\mathfrak{D}(u) - \operatorname{div}(g_u)\}.$$

Now, we consider the integral closure \bar{A} of A in $Q(A)$. Recall that \bar{A} is also M -graded, see e.g. [3, Prop. V.8.21]. Thus, in order to show $A = \bar{A}$, we only have to verify that a homogeneous $f \in Q(A)$, say of degree $u \in M$, belongs to A provided it satisfies a homogeneous equation of integral dependence with certain $h_i \in A$:

$$f^n = h_1 f^{n-1} + \dots + h_{n-1} f + h_n.$$

This equation implies, in particular, that the degree $u \in M$ belongs to the weight cone σ^\vee . Hence, we may choose an element $g_u \in A_u$. Suppose, for the moment, that $\mathfrak{D}(u)$ is an integral Cartier divisor with its sheaf being locally generated, without loss of generality, by g_u . Then the above equation expressing the integral dependence of f takes place over

$$B := \bigoplus_{n \in \mathbb{N}} A_{nu} = \bigoplus_{n \in \mathbb{N}} \Gamma(Y, \mathcal{O}(n\mathfrak{D}(u))) = A_0[g_u],$$

and we are done because of $f \in g_u Q(A_0) \subset Q(B)$ and the integral closedness of B . In the general case, we choose an $m > 0$ such that $m\mathfrak{D}(u)$ is an integral Cartier divisor. The previous argument yields $f^m \in A_{mu}$. Hence, by the description of A_{mu} in terms of the injection $\iota_{mu} : A_{mu} \rightarrow Q(A_0)$, this means that

$$\operatorname{div}(f^m/g_u^m) \geq -m \mathfrak{D}(u) - m \operatorname{div}(g_u).$$

Dividing this inequality by m shows $f/g_u \in \iota_u(A_u)$. This in turn means $f \in A_u$, which concludes the proof. □

5. Ingredients from GIT

In this section, we recall crucial ingredients from Geometric Invariant Theory for the proof of Theorem 3.4 and also for the applications presented later. The central statement is a description of the GIT-equivalence classes arising from linearizations of the trivial bundle over an affine variety with torus action in terms of a quasifan. For torus actions on \mathbb{K}^n , this is well known; the describing quasifan then is even a fan, and is called a Gelfand-Kapranov-Zelevinsky decomposition, compare [21]. For details on the general case as presented here, we refer to [2].

Let us remark that there are analogous, and even further going results in the projective case. Brion and Procesi [4] observed that, for a torus action on a projective variety, the collection of all GIT-quotients arising from the different linearizations of a given ample bundle comes along with a piecewise linear structure. This has been generalized in [27], [7] and [24] to arbitrary reductive groups and the collection of all GIT-quotients arising from linearized ample bundles; see also [15] for some work in the toric setup.

Let us fix the setup. By M , we denote a lattice, and A is an integral, affine, M -graded \mathbb{K} -algebra:

$$A = \bigoplus_{u \in M} A_u.$$

Let $X := \text{Spec}(A)$ denote the affine variety associated to A . Then the M -grading of A defines an action of the algebraic torus $T := \text{Spec}(\mathbb{K}[M])$ on X .

For convenience, we briefly recall the basic concepts from [19] in a down-to-earth manner. A T -linearization of a line bundle $L \rightarrow X$ is a fiberwise linear T -action on the total space L such that the bundle projection $L \rightarrow X$ becomes T -equivariant. Any T -linearization of the trivial bundle over X is of the form

$$t \cdot (x, z) = (t \cdot x, \chi^u(t)z), \tag{5.1}$$

where $\chi^u : T \rightarrow \mathbb{K}^*$ denotes the character corresponding to $u \in M$. Note that the n -fold tensor product of the above linearization corresponds to the character χ^{nu} . Any T -linearization of a line bundle defines a representation of T on the space of its sections via

$$(t \cdot s)(x) := t \cdot (s(t^{-1} \cdot x)).$$

The set of *semistable points* associated to a T -linearized line bundle $L \rightarrow X$ is defined as the union of all affine sets of the form X_f , where f is a T -invariant section of some positive tensor power $L^{\otimes n}$. The invariant sections for the linearization 5.1 are precisely the elements $f \in A_{nu}$, where $n \in \mathbb{Z}_{>0}$, and the corresponding set of semistable points is

$$X^{ss}(u) := \bigcup_{f \in A_{nu}, n \in \mathbb{Z}_{>0}} X_f.$$

Two linearized bundles are called *GIT-equivalent* if they define the same sets of semistable points. The description of the GIT-equivalence classes arising from the linearizations of the trivial bundle presented in [2] works in terms of orbit cones. Let us recall the definition of these and other orbit data.

Definition 5.1. Consider a point $x \in X$.

- (i) The orbit monoid associated to $x \in X$ is the submonoid $S(x) \subset M$ consisting of all $u \in M$ that admit an $f \in A_u$ with $f(x) \neq 0$
- (ii) The orbit cone associated to $x \in X$ is the convex cone $\omega(x) \subset M_{\mathbb{Q}}$ generated by the orbit monoid.
- (iii) The orbit lattice associated to $x \in X$ is the sublattice $M(x) \subset M$ generated by the orbit monoid.

The orbit cones are polyhedral, and each of them is contained in the weight cone $\omega \subset M_{\mathbb{Q}}$, which in turn is generated by the $u \in M$ with $A_u \neq \{0\}$. The geometric meaning of the above orbit data is made clear by the following:

Proposition 5.2. Consider a point $x \in X$.

- (i) The orbit lattice $M(x)$ consists of all $u \in M$ which admit a u -homogeneous function $f \in \mathbb{K}(X)$ that is defined and invertible near x .
- (ii) The isotropy group $T_x \subset T$ of the point $x \in X$ is the diagonalizable group given by $T_x = \text{Spec}(\mathbb{K}[M/M(x)])$.
- (iii) The orbit closure $\overline{T \cdot x}$ is isomorphic to $\text{Spec}(\mathbb{K}[S(x)])$; it comes along with an equivariant open embedding of the torus $T/T_x = \text{Spec}(\mathbb{K}[M(x)])$.
- (iv) The normalization of the orbit closure $\overline{T \cdot x}$ is the toric variety corresponding to the cone $\omega(x)^\vee$ in $\text{Hom}(M(x), \mathbb{Z})$.

In terms of orbit cones, there is a simple description of the sets $X^{ss}(u)$ of semistable points. Namely, we have

$$X^{ss}(u) = \{x \in X; u \in \omega(x)\}.$$

Definition 5.3. The GIT-cone associated to an element $u \in \omega \cap M$ is the intersection of all orbit cones containing u :

$$\lambda(u) := \bigcap_{x \in X; u \in \omega(x)} \omega(x).$$

The GIT-cones turn out to be polyhedral cones as well. Their importance is that they correspond to the GIT-equivalence classes. The main results of [2, Section 2] can be summarized as follows:

Theorem 5.4. Let A be an integral affine algebra graded by a lattice M . Then, for the action of $T := \text{Spec}(\mathbb{K}[M])$ on $X := \text{Spec}(A)$, the following statements hold.

- (i) The GIT-cones $\lambda(u)$, where $u \in M$, form a quasifan Λ in $M_{\mathbb{Q}}$.
- (ii) The support of the quasifan Λ is the weight cone $\omega \subset M_{\mathbb{Q}}$.
- (iii) For any two elements $u_1, u_2 \in \omega \cap M$, we have

$$X^{ss}(u_1) \subset X^{ss}(u_2) \iff \lambda(u_1) \supset \lambda(u_2).$$

The set of semistable points of a T -linearized line bundle over X is an open T -invariant subset of X , and it admits a good quotient by the action of T . For the linearization 5.1, the quotient space $Y_u := X^{ss}(u) // T$ is given by

$$Y_u = \text{Proj}(A_{(u)}), \quad \text{where } A_{(u)} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{nu} \subset A.$$

Note that each quotient space Y_u is projective over the affine variety $Y_0 = \text{Spec}(A_0)$. For our purposes, the following observation concerning the dimension of quotient spaces will be needed.

Lemma 5.5. *Suppose that the action of T on X is effective. Then, for every $u \in \text{relint}(\omega) \cap M$, we have*

$$\dim(X^{ss}(u) // T) = \dim(X) - \dim(T).$$

Proof. Note that the orbit cone of a generic orbit $T \cdot x_0 \subset X$ equals the weight cone ω . Thus, for $u \in \text{relint}(\omega)$, the orbit $T \cdot x_0$ is a closed subset of $X^{ss}(u)$. Thus, $T \cdot x_0$ is a generic fiber of $X^{ss}(u) \rightarrow X^{ss}(u) // T$. Since the T -action is effective, we have $\dim(T \cdot x_0) = \dim(T)$. The assertion follows. \square

6. Proof of Theorem 3.4

In this section, we prove Theorem 3.4. The setup is the following: M is a lattice, and A is a M -graded affine \mathbb{K} -algebra. We consider $X := \text{Spec}(A)$ and the action of $T := \text{Spec}(\mathbb{K}[M])$ on X defined by the grading; we assume that X is normal and that the T -action is effective. By $\omega \subset M_{\mathbb{Q}}$, we denote the weight cone of the T -action, and Λ is the quasifan in $M_{\mathbb{Q}}$ consisting of the GIT-cones as discussed in Theorem 5.4.

Then, for every $\lambda \in \Lambda$, the map $u \mapsto X^{ss}(u)$ is constant on the relative interior $\text{relint}(\lambda)$. We denote by $W_\lambda \subset X$ the set of semistable points defined by any of those $u \in \text{relint}(\lambda)$ and, moreover, by $q_\lambda: W_\lambda \rightarrow Y_\lambda$, the corresponding good T -quotient. In particular, we have $W_0 = X = \text{Spec}(A)$ and $Y_0 = \text{Spec}(A_0)$. The spaces Y_λ are normal; the morphisms q_λ are affine, and each of their fibers contains exactly one closed T -orbit, and hence it is connected.

The quotient maps $q_\lambda: W_\lambda \rightarrow Y_\lambda$, where $\lambda \in \Lambda$, can be put together to an inverse system with $q_0: W_0 \rightarrow Y_0$ sitting at the end. Let us consider its inverse limit. If $\gamma \preceq \lambda$, then we have an open embedding $W_\lambda \subset W_\gamma$ inside X . Set

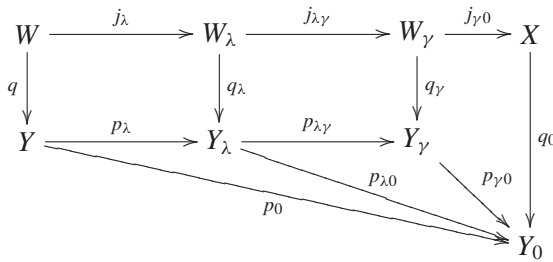
$$W := \varprojlim W_\lambda = \bigcap_{\lambda \in \Lambda} W_\lambda.$$

The inverse limit Y' of the induced maps $p_{\lambda\gamma} : Y_\lambda \rightarrow Y_\gamma$ between the quotient spaces is a nested fiber product over Y_0 . The inverse limit of all quotient maps is the canonical map $q' : W \rightarrow Y'$.

In general, Y' might be reducible, but there is a canonical irreducible component: the closure of the image $q'(W)$. We obtain an irreducible, normal variety Y by taking the normalization of this canonical component:

$$Y := \text{normalization} \left(\overline{q'(W)} \right).$$

By the universal property of the normalization, we have an induced morphism $q : W \rightarrow Y$. In summary, we obtain for each pair $\gamma \preceq \lambda$ in Λ a commutative diagram



Lemma 6.1. *The morphisms $p_\lambda : Y \rightarrow Y_\lambda$ and $p_{\lambda\gamma} : Y_\lambda \rightarrow Y_\gamma$ are projective surjections with connected fibers. Moreover, if $\dim(Y_\lambda) = \dim(X) - \dim(T)$, for example if λ intersects $\text{relint}(\omega)$, then the morphism $p_\lambda : Y \rightarrow Y_\lambda$ is birational.*

Proof. First, recall that each quotient space Y_λ is projective over Y_0 .

It follows that Y is projective over Y_0 , and thus, that each of the maps p_λ is projective, too. Since every Y_λ is dominated by W , all morphisms $p_\lambda : Y \rightarrow Y_\lambda$ are dominant. Together with properness, this implies surjectivity of each p_λ . The same reasoning leads to these properties for the $p_{\lambda\gamma}$.

Let us show the connectedness of the fibers. In general, the fibers of a projective morphism between normal varieties are connected if and only if its generic fiber is connected; use, e.g., Stein factorization. Thus, it suffices to check that the generic fiber of $p_\lambda : Y \rightarrow Y_\lambda$ is irreducible.

The image $q(W) \subset Y$ is constructible. Hence, we can choose a closed proper subset $C \subset Y$ such that $q(W)$ and C cover Y . Let $y \in Y_\lambda$ be a generic point. Then, the fiber $p_\lambda^{-1}(y) \subset Y$ splits into

$$p_\lambda^{-1}(y) = \overline{q(q^{-1}(p_\lambda^{-1}(y)))} \cup (p_\lambda^{-1}(y) \cap C) = \overline{q(q_\lambda^{-1}(y) \cap W)} \cup (p_\lambda^{-1}(y) \cap C).$$

Since we know that the generic fiber of $q_\lambda : W_\lambda \rightarrow Y_\lambda$ is irreducible [1, Prop. 4], this also holds for the first part of the previous expression. Thus, it suffices to show that this part actually fills the whole fiber $p_\lambda^{-1}(y)$.

Assume to the contrary that there is an irreducible component $C_0 \subset C$ dominating Y_λ and containing some irreducible component $F_0 \subset p_\lambda^{-1}(y)$. By general properties of morphisms, see for example [11, II.3.22] or [14, I.4.1 and I.4.3], this would imply

$$\dim(F_0) \geq \dim(Y) - \dim(Y_\lambda) > \dim(C_0) - \dim(Y_\lambda).$$

On the other hand, consider the restriction $\pi_\lambda : C_0 \rightarrow Y_\lambda$ of p_λ . Since $y \in Y_\lambda$ is generic, the dimension of $\pi_\lambda^{-1}(y)$ equals $\dim(C_0) - \dim(Y_\lambda)$. This contradicts the previous estimation.

Finally, we need to prove the claim about the maps p_λ being birational. If we are given two cones $\gamma \preceq \lambda$ both satisfying the assumption $\dim(Y_\gamma) = \dim(Y_\lambda) = \dim(X) - \dim(T)$, then the connecting map $p_{\lambda\gamma} : Y_\lambda \rightarrow Y_\gamma$ is birational, because it induces the identity map on

$$\mathbb{K}(Y_\gamma) = \mathbb{K}(Y_\lambda) = \mathbb{K}(X)^T.$$

But Y can also be obtained from the complete inverse system provided by all cones $\lambda \in \Lambda$ which intersect $\text{relint}(\omega)$. Thus, Y can be built from a system of birational maps, and the common open subset (where all the $p_{\lambda\gamma}$ are isomorphisms) survives in Y . □

We will now investigate certain coherent sheaves on the quotient spaces Y_λ . As mentioned earlier, we have $Y_\lambda = \text{Proj}(A_{(u)})$ with the ring

$$A_{(u)} = \bigoplus_{n \in \mathbb{N}} A_{nu},$$

where $u \in \text{relint}(\lambda) \cap M$ may be any element. This allows us to associate to u a sheaf on Y_λ , namely

$$\mathcal{A}_{\lambda,u} := \mathcal{O}_{Y_\lambda}(1) = (q_\lambda)_*(\mathcal{O}_{W_\lambda})_u,$$

where in the last expression, the subscript “ u ” indicates that we mean the sheaf of semiinvariants with respect to the character $\chi^u : T \rightarrow M$.

Remark 6.2. In the terminology of [11], our $\mathcal{A}_{\lambda,u}$ is nothing but the sheaf on the Proj associated to the graded module $A_{(u)}(1)$.

We call an element $u \in M$ *saturated* if the ring $A_{(u)}$ is generated in degree one. It is well known [3, Prop. II.1.3] that every $u \in M$ admits a positive multiple $nu \in M$ such that all positive multiples of nu are saturated. Moreover, as before, denote by $Q(A)$ the field of fractions of A .

Lemma 6.3. *Let $\lambda \in \Lambda$ and $u \in \text{relint}(\lambda)$. For $f \in A_{nu}$, let $Y_{\lambda,f} := q_\lambda(X_f)$ be the corresponding affine chart of Y_λ .*

- (i) On $Y_{\lambda, f}$, the sheaf $\mathcal{A}_{\lambda, u}$ is the coherent \mathcal{O}_{Y_λ} -module associated to the $(A_f)_0$ -module $(A_f)_u$.
- (ii) If u is saturated, then $\mathcal{A}_{\lambda, u}$ is an ample invertible sheaf on Y_λ , and on the charts $Y_{\lambda, f}$, where $f \in A_u$, we have

$$\mathcal{A}_{\lambda, u} = f \cdot (A_f)_0 = f \cdot \mathcal{O}_{Y_\lambda}.$$

- (iii) If $g \in Q(A)$ and $n \in \mathbb{Z}_{>0}$, then $g^n \in \mathcal{A}_{\lambda, nu}$ implies $g \in \mathcal{A}_{\lambda, u}$.
- (iv) The global sections of $\mathcal{A}_{\lambda, u}$ are $\Gamma(Y_\lambda, \mathcal{A}_{\lambda, u}) = A_u$.

Proof. The first two assertions are obvious. To prove the third one, let $g \in Q(A)$ such that $g^n \in (A_f)_{nu}$. We may assume that f appears in the denominator with a power divisible by n . Then, there is some $k \geq 0$ such that $(gf^k)^n \in A$. The normality of A implies $gf^k \in A$; thus, $g \in (A_f)_u$.

We turn to the last statement. Let $u \in \text{relint}(\lambda) \cap M$. Since $\Gamma(Y_\lambda, \mathcal{A}_{\lambda, u})$ equals $\Gamma(W_\lambda, (\mathcal{O}_{W_\lambda})_u)$, we need to prove that any u -homogeneous function g on W_λ extends to X . By normality of X , it suffices to show that g has non-negative order along any prime divisor contained in $X \setminus W_\lambda$. For the latter, we may also take any positive power g^n . Thus, we may assume that u is saturated.

Consider a prime divisor $D \subset X \setminus W_\lambda$. Choose $f \in A_u$ such that the order $\nu_D(f)$ of f along D is minimal. Regarding g as an element of $(A_f)_u$, we find a $k \geq 0$ and an $h \in A_{(k+1)u}$ such that $g = h/f^k$. Since the elements of $A_{(k+1)u}$ are polynomials in elements of A_u , the minimality of $\nu_D(f)$ provides $\nu_D(h) \geq (k + 1)\nu_D(f)$; hence $\nu_D(g) \geq \nu_D(f) \geq 0$. □

The sheaves $\mathcal{A}_{\lambda, u}$ live on different spaces. By pulling them back, we obtain for every $u \in \omega \cap M$ a well defined coherent sheaf on Y :

$$\mathcal{A}_u := p_\lambda^*(\mathcal{A}_{\lambda, u}), \text{ where } \lambda \in \Lambda \text{ is the cone with } u \in \text{relint}(\lambda).$$

Lemma 6.4. *Let $u, u' \in \omega \cap M$.*

- (i) We have $\mathbb{K}(Y) = Q(A)_0$, and the natural transformation $p_\lambda^*q_{\lambda*} \rightarrow q_*j_\lambda^*$ sends \mathcal{A}_u into $Q(A)_u$.
- (ii) Let u be saturated. Then \mathcal{A}_u is a globally generated, invertible sheaf. On the (not necessarily affine) sets $Y_f := p_\lambda^{-1}(Y_{\lambda, f})$ with $f \in A_u$, we have

$$\mathcal{A}_u = f \cdot \mathcal{O}_Y \subset f \cdot \mathbb{K}(Y) = Q(A)_u.$$

Moreover, for the global sections of \mathcal{A}_u , we obtain $\Gamma(Y, \mathcal{A}_u) = A_u$.

- (iii) If u, u' and $u + u'$ are saturated, then we have $\mathcal{A}_u \mathcal{A}_{u'} \subset \mathcal{A}_{u+u'}$. If, moreover, u and u' lie in a common cone of Λ , then equality holds.

Proof. Using an arbitrary cone $\lambda \in \Lambda$ which intersects $\text{relint}(\omega)$ and some homogeneous $f \in A$ with $\text{deg}(f) \in \text{relint}(\lambda)$, we obtain

$$\mathbb{K}(Y) = \mathbb{K}(Y_\lambda) = Q((A_f)_0) \subset Q(A)_0.$$

Conversely, starting with an element $a/b \in Q(A)_0$, then $\deg a = \deg b$ is sitting in the interior of some cone $\gamma \in \Lambda$; thus, we have

$$a/b \in Q((A_b)_0) = \mathbb{K}(Y_\gamma) \subset \mathbb{K}(Y).$$

The second assertion is a direct consequence of Lemma 6.3. For the last part, we use that the adjunction map $\mathcal{A}_{\lambda,u} \rightarrow p_{\lambda*}p_{\lambda}^*\mathcal{A}_{\lambda,u}$ locally looks like $\mathcal{O}_{Y_\lambda} \rightarrow p_{\lambda*}p_{\lambda}^*\mathcal{O}_{Y_\lambda}$. Hence, because of Lemma 6.1, it is an isomorphism.

Eventually, to prove the third assertion, we have to deal with the product $\mathcal{A}_u\mathcal{A}_{u'}$. Due to saturatedness, we obtain

$$\mathcal{A}_u\mathcal{A}_{u'} = (A_u \cdot \mathcal{O}_Y)(A_{u'} \cdot \mathcal{O}_Y) = A_uA_{u'} \cdot \mathcal{O}_Y \subset A_{u+u'} \cdot \mathcal{O}_Y = \mathcal{A}_{u+u'}.$$

Now let $u \in \text{relint}(\gamma)$, $u' \in \text{relint}(\gamma)'$ and, moreover, $\gamma, \gamma' \subset \lambda \in \Lambda$ (with λ being minimal). Then we have $W_\lambda \subset W_\gamma \cap W_{\gamma'}$. Conversely, again by saturatedness, we obtain

$$W_\gamma \cap W_{\gamma'} = \left(\bigcup_{f \in A_u} X_f \right) \cap \left(\bigcup_{f' \in A_{u'}} X_{f'} \right) = \bigcup_{\substack{f \in A_u \\ f' \in A_{u'}}} X_{ff'} \subset \bigcup_{g \in A_{u+u'}} X_g = W_\lambda.$$

In particular, the sets $X_{ff'}$ cover W_λ , hence, the $Y_{\lambda,ff'}$ cover Y_λ , hence, so do the $Y_{ff'}$ with Y . On the other hand, the inclusion $X_{ff'} \subset X_f$ induces a morphism $Y_{\lambda,ff'} \rightarrow Y_{\gamma,f}$ which also applies to f' . Thus, $Y_{ff'} \subset Y_f \cap Y_{f'}$, and it follows that $\mathcal{A}_u = f \cdot \mathcal{O}_Y$, $\mathcal{A}_{u'} = f' \cdot \mathcal{O}_Y$ and $\mathcal{A}_{u+u'} = ff' \cdot \mathcal{O}_Y$ on $Y_{ff'}$. \square

Proof of Theorem 3.4. Let Y be the semiprojective variety defined at the beginning of this section. We will construct the desired pp-divisor on Y as a convex piecewise linear map $\omega \rightarrow \text{CaDiv}_{\mathbb{Q}}(Y)$ in the sense of Definition 2.9. The previous two Lemmas will be used implicitly.

Our construction requires a (non-canonical) choice of a homomorphism $s : M \rightarrow Q(A)^*$ such that for every $u \in M$ the function $s(u)$ is homogeneous of degree u . Since T acts effectively on X , such “sections” $s : M \rightarrow Q(A)^*$ always exist.

Now, if $u \in \omega \cap M$ is any saturated element, then there is a unique Cartier divisor $\mathfrak{D}(u) \in \text{CaDiv}(Y)$ such that

$$\mathcal{O}_Y(\mathfrak{D}(u)) = \frac{1}{s(u)} \cdot \mathcal{A}_u \subset \mathbb{K}(Y).$$

The local equation for $\mathfrak{D}(u)$ on Y_f with $f \in A_u$ is $s(u)/f$. For general $u \in \omega$, we can choose a saturated multiple nu and define

$$\mathfrak{D}(u) := \frac{1}{n} \cdot \mathfrak{D}(nu) \in \text{CaDiv}_{\mathbb{Q}}(Y).$$

Obviously, this definition does not depend on n , and one directly checks the properties of Definition 2.9 for the map $u \mapsto \mathcal{D}(u)$. Moreover, we can recover the M -graded ring A via

$$A_u = s(u) \cdot \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u))).$$

For saturated u , this is clear by the construction of \mathcal{D} . For general degrees $u \in \omega \cap M$, we have to argue as usual: If $g \in \mathbb{K}(Y)$, then

$$g \in \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(u))) \Leftrightarrow g^n \in \Gamma(Y, \mathcal{O}_Y(\mathcal{D}(nu))) \Leftrightarrow (gs(u))^n \in A_{nu} \Leftrightarrow gs(u) \in A_u.$$

Note that for the last step one uses normality of the ring A . □

7. Fibers of the quotient map

The construction of the affine T -variety X associated to a pp-divisor \mathcal{D} on a normal semiprojective variety Y involves, as an intermediate step, the construction of a certain T -variety \tilde{X} over Y . The aim of this section is to describe the geometry of the fibers of the canonical map $\pi : \tilde{X} \rightarrow Y$ in terms of the defining pp-divisor.

Fix a normal semiprojective variety Y ; let N be a lattice, and $\sigma \subset N_{\mathbb{Q}}$ a pointed cone. As outlined in Section 3, any pp-divisor $\mathcal{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ gives rise to a sheaf of graded \mathcal{O}_Y -algebras

$$\mathcal{A} = \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}(\mathcal{D}(u)),$$

where M is the dual lattice of N . The variety $\tilde{X} := \text{Spec}_Y(\mathcal{A})$ comes with an action of the torus $T := \text{Spec}(\mathbb{K}[M])$, and the canonical map $\pi : \tilde{X} \rightarrow Y$ is a good quotient for the action of T on \tilde{X} .

A first step in the study of the fibers of $\pi : \tilde{X} \rightarrow Y$ is an investigation of certain bouquets of toric varieties associated to σ -polyhedra $\Delta \subset N_{\mathbb{Q}}$. For the definition of these objects, recall that one associates a normal quasifan $\Lambda(\Delta)$ to Δ ; the faces of Δ correspond to the cones of $\Lambda(\Delta)$ via

$$F \mapsto \lambda(F) = \{u \in M_{\mathbb{Q}}; \langle u, v - v' \rangle \geq 0 \text{ for all } v \in \Delta, v' \in F\}.$$

Having obtained a quasifan Λ from the σ -polyhedron Δ , we associate to this quasifan a graded algebra $\mathbb{K}[\Lambda]$ by a frequently used procedure, and then define the toric bouquet associated to Δ as the spectrum of the graded algebra $\mathbb{K}[\Lambda]$.

Definition 7.1. *Let M be a lattice, and let Λ be a quasifan with convex support $\omega \subseteq M_{\mathbb{Q}}$. The fan ring associated to Λ is the affine \mathbb{K} -algebra defined by*

$$\mathbb{K}[\Lambda] := \bigoplus_{u \in \omega \cap M} \mathbb{K}\chi^u, \quad \chi^u \chi^{u'} := \begin{cases} \chi^{u+u'} & \text{if } u, u' \in \lambda \text{ for some } \lambda \in \Lambda, \\ 0 & \text{else.} \end{cases}$$

Note that the fan ring $\mathbb{K}[\Lambda]$ may as well be viewed as something like a semi-group algebra, if we define $u + u' := \infty$, whenever u and u' do not belong to a common cone.

Definition 7.2. Let $\Delta \subset N_{\mathbb{Q}}$ be a σ -polyhedron. The toric bouquet associated to Δ is $X(\Delta) := \text{Spec}(\mathbb{K}[\Lambda])$, where $\Lambda = \Lambda(\Delta)$ is the normal quasifan of Δ .

We collect some basic geometric properties of these toric bouquets; in particular, we note that they have equidimensional toric varieties as their irreducible components, whence the name. The proofs of the statements only use standard toric geometry and therefore are left to the reader.

Proposition 7.3. Let $\Delta \subset N_{\mathbb{Q}}$ be a σ -polyhedron, let $\Lambda = \Lambda(\Delta)$ be the normal quasifan of Δ , and let $X(\Delta) = \text{Spec}(\mathbb{K}[\Lambda])$ be the corresponding toric bouquet.

- (i) The M -grading of $\mathbb{K}[\Lambda]$ defines an effective algebraic action of the torus $T := \text{Spec}(\mathbb{K}[M])$ on $X(\Delta)$.
- (ii) The T -orbits of $X(\Delta)$ are in dimension reversing one-to-one correspondence with the faces of Δ via $F \mapsto T \cdot x_F$, where $x_F \in X(\Delta)$ is defined by

$$\chi^u(x_F) = \begin{cases} 1 & \text{if } u \in \lambda(F), \\ 0 & \text{else.} \end{cases}$$

- (iii) For a face $F \preceq \Delta$, let $I(F) \subset \mathbb{K}[\Lambda]$ denote the ideal generated by the χ^u 's with $u \notin \lambda(F)$. Then the closure of the orbit through x_F is given by

$$\overline{T \cdot x_F} = V(I(F)).$$

This orbit closure is a toric variety, and, denoting by $\text{lin}(F) \subset N_{\mathbb{Q}}$ the vector space generated by all $v - v'$ with $v, v' \in F$, its defining cone is

$$\mathbb{Q}_{\geq 0} \cdot (\Delta - F) / \text{lin}(F) \subset (N / (\text{lin}(F) \cap N))_{\mathbb{Q}}.$$

- (iv) The irreducible components of $X(\Delta)$ are precisely the orbit closures corresponding to the vertices $v \in \Delta$. As toric varieties, they correspond to the cones

$$\lambda(v)^{\vee} = \mathbb{Q}_{\geq 0} \cdot (\Delta - v).$$

For our description of the fibers of the canonical map $\pi : \tilde{X} \rightarrow Y$, we have to generalize the toric bouquets discussed so far. We have to consider certain subalgebras of the fan ring arising from a σ -polyhedron.

Definition 7.4. Let Λ be the normal quasifan in $M_{\mathbb{Q}}$ of a σ -polyhedron $\Delta \subset N_{\mathbb{Q}}$. Let $\omega = \sigma^{\vee}$ and $S \subset \omega \cap M$ be a subset such that for each cone $\lambda \in \Lambda$ we have

$$S_{\lambda} := \lambda \cap S = \lambda \cap M_{\lambda}$$

with a lattice $M_\lambda \subset M \cap \text{lin}(\lambda)$ of full rank in $\text{lin}(\lambda)$. Then we obtain a finitely generated subalgebra:

$$\mathbb{K}[\Lambda, S] := \bigoplus_{u \in S} \mathbb{K}\chi^u \subset \mathbb{K}[\Lambda].$$

Example 7.5. Let $N := \mathbb{Z}$, and $\sigma := \{0\}$. Then any interval $\Delta \subset \mathbb{Q}$ is a σ -polyhedron, and the corresponding normal fan Λ consists of

$$\lambda_- := (-\infty, 0], \quad \lambda_0 := \{0\}, \quad \lambda_+ := [0, \infty).$$

We have $\omega \cap M = \mathbb{Z}$, and fixing a subset $S \subset \mathbb{Z}$ means to establish a “conewise varying lattice structure”. For example, we may obtain a picture



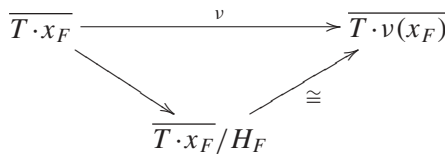
where the subset $S \subset \mathbb{Z}$ is indicated by the fat points; it is obtained by means of choice of sublattices $M_\lambda \subset \mathbb{Z}$ compatible with the fan structure:

$$S \cap \lambda_- := (-\infty, 0] \cap 2\mathbb{Z}, \quad S \cap \lambda_0 := \{0\}, \quad S \cap \lambda_+ := [0, \infty) \cap 3\mathbb{Z}.$$

Geometrically, the passage from $\mathbb{K}[\Lambda]$ to a subalgebra $\mathbb{K}[\Lambda, S]$ as in Definition 7.4 corresponds to taking componentwise quotients of a toric bouquet by a possibly varying finite group. More precisely, we obtain:

Proposition 7.6. *Let Δ, Λ, S , etc. be as in Definition 7.4. Then the inclusion $\mathbb{K}[\Lambda, S] \subset \mathbb{K}[\Lambda]$ gives rise to a finite equivariant morphism $v: X(\Delta) \rightarrow X(\Delta, S)$ of affine T -varieties. Moreover, the following statements hold:*

- (i) $\overline{T \cdot x_F} = v^{-1}(\overline{T \cdot v(x_F)})$ for every orbit closure $\overline{T \cdot x_F} \subset X(\Delta)$, and on $\overline{T \cdot x_F}$, the map v is the quotient by a finite group H_F :



Setting $M_{F,S} := (\text{lin}(\lambda(F)) \cap M) / M_{\lambda(F)}$, the finite group H_F is obtained as follows

$$H_F = \text{Spec}(\mathbb{K}[M_{F,S}]) = T_{v(x_F)} / T_{x_F}.$$

- (ii) The orbit closure $\overline{T \cdot v(x_F)}$ is a toric variety with big torus $T / T_{v(x_F)}$. Its corresponding cone is the image of $\mathbb{Q}_{\geq 0} \cdot (\Delta - F)$ under $N \rightarrow \text{Hom}(M_{\lambda(F)}, \mathbb{Z})$.

Proof. For every face $F \preceq \Delta$, we have the ideal $I(F, S) := \nu^*(I(F))$ in $\mathbb{K}[\Lambda, S]$. The corresponding homomorphism of factor algebras is the inclusion of a certain Veronese subalgebra:

$$\bigoplus_{u \in \lambda(F) \cap M_{\lambda(F)}} \mathbb{K}\chi^u \subset \bigoplus_{u \in \lambda(F) \cap M} \mathbb{K}\chi^u.$$

In particular, the left hand side is the ring of invariants of the group H_F . Moreover, by Proposition 7.3 (iii), this inclusion of algebras describes the map ν over $\nu(\overline{T \cdot x_F})$. The assertions follow directly. \square

We now associate a couple of fiber data to any $y \in Y$. We say that a divisor $D \in \text{Div}_{\mathbb{Q}}(Y)$ is *principal at* $y \in Y$ if near $y \in Y$ it is the divisor of a function $f \in \mathbb{K}(Y)$.

Definition 7.7. Let $\mathfrak{D} = \sum \Delta_i \otimes D_i$ be a representation such that all D_i are prime, and let $y \in Y$.

(i) We define the fiber polyhedron of $y \in Y$ to be

$$\Delta_y := \sum_{y \in D_i} \Delta_i \in \text{Pol}_{\sigma}(N_{\mathbb{Q}}).$$

(ii) The normal quasifan of Δ_y is denoted by Λ_y .

(iii) We define the fiber monoid complex of $y \in Y$ as

$$S_y := \{u \in \omega \cap M; \mathfrak{D}(u) \text{ is principal at } y\}.$$

(iv) For $\lambda \in \Lambda_y$, we denote by $M_{y,\lambda} \subset M$ the sublattice generated by $S_y \cap \lambda$.

Example 7.8. Let $Y := \mathbb{K}^1$; take $N := \mathbb{Z}$, and let $\sigma \subset \mathbb{Q}$ be the zero cone. Consider the pp-divisor

$$\mathfrak{D} = \left[\frac{1}{3}, \frac{1}{2} \right] \otimes \{0\}.$$

Then, for $y = 0$, the fiber polyhedron is $[1/3, 1/2]$, and the fiber monoid complex is as the picture in Example 7.5. For any other point $y \in Y$, the fiber polyhedron is $\sigma = \{0\}$, and the fiber monoid complex is just \mathbb{Z} .

Lemma 7.9. Let $y \in Y$. Then every $\lambda \in \Lambda_y$ satisfies $S_y \cap \lambda = M_{y,\lambda} \cap \lambda$. In particular, $S_y \cap \lambda$ is a finitely generated semigroup, and we obtain a finitely generated algebra

$$\mathbb{K}[\Lambda_y, S_y] := \bigoplus_{u \in S_y} \mathbb{K}\chi^u \subset \mathbb{K}[\Lambda_y].$$

Now we are ready to begin with the study of the fibers of the canonical map $\pi : \tilde{X} \rightarrow Y$. Below is the first statement.

Proposition 7.10. *Let $\mathcal{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$. Then, for every $y \in Y$, the reduced fiber $\pi^{-1}(y)$ of the associated map $\pi: \tilde{X} \rightarrow Y$ is T -equivariantly isomorphic to $X(\Delta_y, S_y) = \text{Spec}(\mathbb{K}[\Lambda_y, S_y])$ where Λ_y is the normal quasifan of Δ_y .*

Proof. Fix $y \in Y$, and consider the group $G \subset \text{CaDiv}_{\mathbb{Q}}(Y)$ generated by those $\mathcal{D}(u)$ with $u \in \omega \cap M$ that are principal at y . By Lemma 7.9, G is finitely generated. Thus, we may choose a basis $E_1, \dots, E_r \in G$. After possibly shrinking Y , we may assume that Y is affine, and that $E_i = \text{div}(g_i)$ holds with $g_i \in \mathbb{K}(Y)$. For $D \in G$, set

$$g_D := g_1^{a_1} \dots g_r^{a_r}, \quad \text{where } D = a_1 E_1 + \dots + a_r E_r.$$

These functions satisfy $g_{D'+D} = g_{D'} g_D$ for all $D', D \in G$. Consequently, denoting by Λ_y the normal quasifan of Δ_y , we may define a graded epimorphism

$$\Phi: \Gamma(Y, \mathcal{A}) \rightarrow \mathbb{K}[\Lambda_y, S_y], \quad \Gamma(Y, \mathcal{A})_u \ni h \mapsto \begin{cases} (g_{\mathcal{D}(u)} h)(y) \chi^u & \text{if } u \in S_y, \\ 0 & \text{else.} \end{cases}$$

To see multiplicativity, let $h_i \in \Gamma(Y, \mathcal{A})_{u_i}$. Then, $g_{\mathcal{D}(u_1+u_2)} h_1 h_2$ vanishes at y if the u_i do not belong to the same $\lambda \in \Lambda$, or if one of the $\mathcal{D}(u_i)$ is not principal at y .

To conclude the proof, we have to show that the kernel of the above epimorphism Φ equals the radical of the ideal of the fiber $\pi^{-1}(y)$. The fiber ideal is given by

$$I_y := \langle hf; h \in \Gamma(Y, \mathcal{O}), h(y) = 0, f \in \Gamma(Y, \mathcal{A})_u, u \in \omega \cap M \rangle \subset \Gamma(Y, \mathcal{A}).$$

Obviously, $I_y \subset \ker(\Phi)$. Conversely, for any $f \in \ker(\Phi)$, say homogeneous of degree u , we have $f^n \in I_y$ as soon as $\mathcal{D}(nu)$ is principal at y . □

Combining the above result with Propositions 7.3 and 7.6 and the notions of Definition 5.1 gives the following information on the geometry of the fibers of $\pi: \tilde{X} \rightarrow Y$:

Corollary 7.11. *For $y \in Y$, consider the affine T -variety $\pi^{-1}(y)$.*

- (i) *For any $\tilde{x} \in \pi^{-1}(y)$, we have $\omega(\tilde{x}) \in \Lambda_y$, and this sets up a one-to-one correspondence between the T -orbits of $\pi^{-1}(y)$ and the cones of Λ_y (corresponding to the faces of Δ_y). The orbit lattice of \tilde{x} is $M(\tilde{x}) = M_{y, \omega(\tilde{x})}$.*
- (ii) *The irreducible components of $\pi^{-1}(y)$ are the orbit closures $T \cdot \tilde{x}$ with $\omega(\tilde{x})$ maximal in Λ_y . They are normal toric varieties with big torus $T/T_{\tilde{x}}$, where $T_{\tilde{x}} = \text{Spec}(\mathbb{K}[M/M_{y, \omega(\tilde{x})}])$.*

In the proof of Proposition 7.10, we had to compare the fiber ideal I_y and its radical $\ker(\Phi)$. Looking a little bit closer at these data gives the following:

Proposition 7.12. *A fiber $\pi^{-1}(y)$ is reduced if and only if all $\mathcal{D}(u)$, where $u \in \omega \cap M$ are principal at $y \in Y$.*

Example 7.13. As in Example 7.8, take $Y := \mathbb{K}^1$, let $N := \mathbb{Z}$, and $\sigma := \{0\}$. Again consider the pp-divisor

$$\mathcal{D} = \left[\frac{1}{3}, \frac{1}{2} \right] \otimes \{0\}.$$

Then the \mathbb{K}^* -variety \tilde{X} associated to \mathcal{D} is the affine space \mathbb{K}^2 together with the \mathbb{K}^* -action given by

$$t \cdot (z, w) = (t^3 z, t^{-2} w).$$

The canonical map $\pi : \tilde{X} \rightarrow Y$ may, as a good quotient for this \mathbb{K}^* -action, concretely be written as

$$\pi : \mathbb{K}^2 \mapsto \mathbb{K}, \quad (z, w) \mapsto z^2 w^3.$$

The fiber $\pi^{-1}(0)$ is the union of three orbits: the origin and the orbits through $(1, 0)$ and $(0, 1)$.

Combinatorially, this is reflected as follows. We have $\Delta_0 = [1/3, 1/2]$ as the fiber polyhedron in $0 \in \mathbb{K}$; the associated normal fan Λ_0 consists of three cones as in Example 7.5, and the fiber monoid complex S_0 has a varying lattice structure:



This varying lattice structure indicates that \mathbb{K}^* acts on one coordinate axis of \mathbb{K}^2 with generic isotropy group of order two, and on the other with generic isotropy group of order three.

8. Functoriality properties

Our first results, Theorems 3.1 and 3.4, establish correspondences between pp-divisors on semiprojective varieties, on the one hand, and affine varieties with effective torus action on the other. In this section, we present the functoriality properties of these assignments; the proofs of the results are given in Section 9.

Going from polyhedral divisors to varieties is functorial in an almost evident manner, but the reverse direction is more delicate. Nevertheless, in an appropriate setup, we obtain an equivalence of categories, and our results allow us to decide when two pp-divisors define isomorphic T -varieties, see Corollaries 8.14 and 8.17.

First, we have to fix the respective notions of morphisms. Concerning varieties with torus action, we will work with the following concept.

Definition 8.1. *Let X and X' be varieties endowed with effective actions of tori T and T' . By an equivariant morphism from X to X' , from now on we mean a morphism $\varphi : X \rightarrow X'$ admitting an accompanying homomorphism $\tilde{\varphi} : T \rightarrow T'$ such that $\varphi(t \cdot x) = \tilde{\varphi}(t) \cdot \varphi(x)$ holds for all $(t, x) \in T \times X$.*

So, a morphism of two T -varieties is equivariant in the usual sense, if and only if it has the identity as an accompanying homomorphism. Note that in case of a dominant morphism $\varphi : X \rightarrow X'$, the accompanying homomorphism is uniquely determined.

We turn to pp-divisors. To define the notion of a map between two pp-divisors, we first have to introduce the concept of a “polyhedral principal divisor”.

Definition 8.2. *Let Y be a normal variety, N a lattice and $\sigma \subset N_{\mathbb{Q}}$ a pointed cone.*

- (i) A plurifunction with respect to the lattice N is an element of $\mathbb{K}(Y, N)^* := N \otimes_{\mathbb{Z}} \mathbb{K}(Y)^*$.
- (ii) For $u \in M = \text{Hom}(N, \mathbb{Z})$, the evaluation of a plurifunction $f = \sum v_i \otimes f_i$ with respect to N is

$$f(u) := \prod f_i^{(u, v_i)} \in \mathbb{K}(Y)^*.$$

- (iii) The “polyhedral principal” divisor with respect to $\sigma \subset N_{\mathbb{Q}}$ of a plurifunction $f = \sum v_i \otimes f_i$ with respect to N is

$$\text{div}(f) := \sum (v_i + \sigma) \otimes \text{div}(f_i) \in \text{CaDiv}(Y, \sigma).$$

Note that evaluating and taking the divisor with respect to $\sigma \subset N_{\mathbb{Q}}$ of a plurifunction commute for $u \in \sigma^{\vee} \cap M$. We are ready to define the notion of a map of pp-divisors.

Definition 8.3. *Let Y, Y' be normal semiprojective varieties, N, N' lattices, $\sigma \subset N_{\mathbb{Q}}$ and $\sigma' \subset N'_{\mathbb{Q}}$ pointed cones, and consider pp-divisors*

$$\mathfrak{D} = \sum \Delta_i \otimes D_i \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma), \quad \mathfrak{D}' = \sum \Delta'_i \otimes D'_i \in \text{PPDiv}_{\mathbb{Q}}(Y', \sigma').$$

- (i) For morphisms $\psi : Y \rightarrow Y'$ such that none of the supports $\text{Supp}(D'_i)$ contains $\psi(Y)$, we define the (not necessarily proper) polyhedral pull back as

$$\psi^*(\mathfrak{D}') := \sum \Delta'_i \otimes \psi^*(D'_i) \in \text{CaDiv}_{\mathbb{Q}}(Y, \sigma').$$

- (ii) For linear maps $F : N \rightarrow N'$ with $F(\sigma) \subset \sigma'$, we define the (not necessarily proper) polyhedral push forward as

$$F_*(\mathfrak{D}) := \sum (F(\Delta_i) + \sigma') \otimes D_i \in \text{Div}_{\mathbb{Q}}(Y, \sigma').$$

- (iii) A map $\mathfrak{D} \rightarrow \mathfrak{D}'$ is a triple (ψ, F, f) with a dominant morphism $\psi : Y \rightarrow Y'$, a linear map $F : N \rightarrow N'$ as in (ii), and a plurifunction $f \in \mathbb{K}(Y, N')^*$ such that

$$\psi^*(\mathfrak{D}') \leq F_*(\mathfrak{D}) + \text{div}(f).$$

Note that the relation “ \leq ” among pp-divisors is equivalent to requiring the opposite inclusion for the respective polyhedral coefficients in the representation of the pp-divisors as linear combinations of prime divisors.

Example 8.4. With the notation of the previous definition, we obtain natural adjunction maps:

- (i) For any generically finite morphism $\psi : Y \rightarrow Y'$, the pullback $\psi^*(\mathcal{D}')$ is, if defined at all, a pp-divisor on Y , and the triple $(\psi, \text{id}_N, 1)$ defines a map $\psi^*(\mathcal{D}') \rightarrow \mathcal{D}'$.
- (ii) For any lattice homomorphism $F : N \rightarrow N'$, the triple $(\text{id}_Y, F, 1)$ defines a map $\mathcal{D} \rightarrow F_*(\mathcal{D})$, provided that $F_*(\mathcal{D})$ is a pp-divisor.

In order to obtain a category of pp-divisors, we still have to introduce composition. For this, note that along the lines of 8.3 (i) and (ii), we can also define pullback and pushforward of plurifunctions.

Definition 8.5. *The identity map of a pp-divisor is the triple $(\text{id}, \text{id}, 1)$. The composition of two maps of pp-divisors (ψ, F, \mathfrak{f}) and $(\psi', F', \mathfrak{f}')$ is defined as the map of pp-divisors $(\psi' \circ \psi, F' \circ F, F'_*(\mathfrak{f}) \cdot \psi^*(\mathfrak{f}'))$.*

Let us now demonstrate how to obtain functoriality. The construction of Theorem 3.1 associates to a given pp-divisor $\mathcal{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ on a normal semi-projective variety Y the affine variety

$$\mathfrak{X}(\mathcal{D}) := X := \text{Spec}(\Gamma(Y, \mathcal{A})), \quad \text{where } \mathcal{A} = \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}(\mathcal{D}(u)).$$

Proposition 8.6. *The assignment $\mathcal{D} \mapsto \mathfrak{X}(\mathcal{D})$ is a faithful covariant functor from the pp-divisors on normal semiprojective varieties to the normal affine varieties with torus action.*

Proof. Let \mathcal{D} and \mathcal{D}' be pp-divisors on Y and Y' respectively. Write $X := \mathfrak{X}(\mathcal{D})$ and $X' := \mathfrak{X}(\mathcal{D}')$, and denote the acting tori by T and T' . By definition, any map $\mathcal{D} \rightarrow \mathcal{D}'$, given by (ψ, F, \mathfrak{f}) , induces homomorphisms of $\Gamma(Y', \mathcal{O})$ -modules:

$$\Gamma(Y', \mathcal{O}(\mathcal{D}'(u))) \rightarrow \Gamma(Y, \mathcal{O}(\mathcal{D}(F^*u))), \quad h \mapsto \mathfrak{f}(u)\psi^*(h). \tag{8.1}$$

These maps fit together to a graded homomorphism $\Gamma(Y', \mathcal{A}') \rightarrow \Gamma(Y, \mathcal{A})$. This in turn gives an equivariant morphism $\varphi : X \rightarrow X'$ with the map $\tilde{\varphi} : T \rightarrow T'$ defined by $F : N \rightarrow N'$ as an accompanying homomorphism.

Obviously, the identity map of pp-divisors defines the identity on the level of equivariant morphisms. Compatibility with composition follows from the definition of the equivariant morphism associated to a map of pp-divisors via 8.1 and the fact that we always have

$$F'_*(\mathfrak{f}) \cdot \psi^*(\mathfrak{f}')(u) = \mathfrak{f}((F')^*(u))\psi^*(\mathfrak{f}'(u)).$$

In order to see that the functor is faithful, i.e. injective on morphisms, consider two maps (ψ_i, F_i, f_i) of pp-divisors \mathcal{D} on Y and \mathcal{D}' on Y' that define the same equivariant morphism $\varphi: X \rightarrow X'$.

To obtain $F_1 = F_2$, it suffices to check that $\varphi(X)$ contains points with free T' -orbit: by equivariance, this will fix the accompanying homomorphism $\tilde{\varphi}: T \rightarrow T'$, which in turn determines the lattice homomorphisms $F_i: N \rightarrow N'$.

According to the properties of the maps ψ_i , there is an open set $V \subset Y'$ with $\psi_i(Y) \cap V \neq \emptyset$ such that any $u \in \omega \cap M$ admits a section $f_u \in \mathcal{A}_u(V)$ without zeroes in V . Consider

$$U := \bigcap X \setminus \text{div}(\varphi^*(f_u)).$$

Using Theorem 3.1 (iii), one sees that U is nonempty. Moreover, for each $x \in U$ and each $u \in M$, the image $\varphi(x)$ admits a homogeneous rational function $f \in \mathbb{K}(X')_u$ defined near $\varphi(x)$ with $f(\varphi(x)) \neq 0$. In other words, $\varphi(x)$ has a free T' -orbit.

In order to see $\psi_1 = \psi_2$, it suffices to show that $\psi_1^*, \psi_2^*: \mathbb{K}(Y') \rightarrow \mathbb{K}(Y)$ coincide. Given a function $f \in \mathbb{K}(Y')$, we may write it as $f = g/h$ with $g, h \in \Gamma(Y', \mathcal{A}_u)$ for some $u \in M'$, because we have $\mathbb{K}(Y') = \mathbb{K}(X')^T$, and on X' any invariant rational function is the quotient of two semiinvariants. Thus, we obtain

$$\psi_i^*(f) = \frac{f_i(u)\psi_i^*(g)}{f_i(u)\psi_i^*(h)} = \frac{\varphi^*(g)}{\varphi^*(h)}.$$

Finally, the equality $f_1 = f_2$ follows from their appearance in the comorphism of $\varphi: X \rightarrow X'$: since the weight monoid of the T' -action on X' generates the lattice M' , there are enough homogeneous sections h to fix f_i by using the defining formula 8.1 of φ^* and $\psi_1 = \psi_2$. □

To proceed, we introduce a notion of minimality for a pp-divisor $\mathcal{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$. Since all evaluations $\mathcal{D}(u)$ are semiample, they come along with a natural contraction map, being birational whenever $u \in \text{relint}(\sigma^\vee)$:

$$\vartheta_u: Y \rightarrow Y_u := \text{Proj} \left(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Gamma(Y, \mathcal{O}(\mathcal{D}(nu))) \right).$$

Denoting by X the normal, affine T -variety associated to the pp-divisor \mathcal{D} , we can recover the semiprojective varieties Y_u as GIT-quotient spaces associated to linearizations of the trivial bundle; namely, we have

$$Y_u = X^{ss}(u) // T.$$

From this we see, similarly as in Section 6, that the spaces Y_u fit into an inverse system with projective morphisms $\vartheta_{uw}: Y_u \rightarrow Y_w$, whenever $u \in \lambda$ and $w \in \gamma$ for two cones $\gamma \preceq \lambda$ of the GIT-quasifan of the T -action on X .

Clearly, we have $\vartheta_w = \vartheta_{uw} \circ \vartheta_u$, whenever composition is possible. Thus, the morphisms $\vartheta_u : Y \rightarrow Y_u$ lift to a (projective, birational) morphism to the inverse limit of the system of the GIT-quotient spaces:

$$\vartheta : Y \rightarrow \varprojlim Y_u.$$

Recall from Section 6 that $\varprojlim Y_u$ comes with a canonical component, dominated by the intersection W over all $X^{ss}(u)$, where $u \in \sigma^\vee \cap M$. By construction, ϑ maps Y onto this component.

Definition 8.7. We say that a pp-divisor $\mathcal{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ is minimal if the morphism $\vartheta : Y \rightarrow \varprojlim Y_u$ is the normalization of the canonical component of $\varprojlim Y_u$.

Note that the pp-divisors constructed in the proof of Theorem 3.4 are minimal. Moreover, on a curve Y , every pp-divisor is minimal.

The following result makes precise, up to what extent we can describe equivariant morphisms in terms of maps of pp-divisors.

Theorem 8.8. Let $\mathcal{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ and $\mathcal{D}' \in \text{PPDiv}_{\mathbb{Q}}(Y', \sigma')$ be pp-divisors and let $\varphi : \mathfrak{X}(\mathcal{D}) \rightarrow \mathfrak{X}(\mathcal{D}')$ be a dominant, equivariant morphism. Then, there exist a projective birational morphism $\kappa : \tilde{Y} \rightarrow Y$, a map (ψ, F, \mathfrak{f}) from $\kappa^*\mathcal{D}$ to \mathcal{D}' , and a commutative diagram

$$\begin{array}{ccc}
 & \mathfrak{X}(\kappa^*\mathcal{D}) & \\
 \mathfrak{X}(\kappa, \text{id}, 1) \swarrow & & \searrow \mathfrak{X}(\psi, F, \mathfrak{f}) \\
 \mathfrak{X}(\mathcal{D}) & \xrightarrow{\varphi} & \mathfrak{X}(\mathcal{D}')
 \end{array}$$

\cong (between $\mathfrak{X}(\kappa, \text{id}, 1)$ and φ)

If φ is an isomorphism and \mathcal{D}' is minimal, then one may take κ as the identity and obtains $\varphi = \mathfrak{X}(\psi, F, \mathfrak{f})$ where $F : N \rightarrow N'$ is an isomorphism of lattices sending σ to σ' , and $\psi : Y \rightarrow Y'$ is birational and projective; if also \mathcal{D} is minimal, then ψ is an isomorphism.

The theorem shows that minimal pp-divisors may serve as a tool for the study of equivariant automorphism groups, where “equivariant” is understood in the usual sense, i.e., with $\tilde{\varphi} = \text{id}$ in the language of Definition 8.1.

Corollary 8.9. Let $\mathcal{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ be a minimal pp-divisor. Then the automorphisms $\varphi : \mathfrak{X}(\mathcal{D}) \rightarrow \mathfrak{X}(\mathcal{D})$ satisfying $\varphi(t \cdot x) = t \cdot \varphi(x)$ correspond to pairs (ψ, \mathfrak{f}) , where $\psi : Y \rightarrow Y$ is an automorphism and $\mathfrak{f} \in \mathbb{K}(N; Y)^*$ satisfies $\psi^*(\mathcal{D}) = \mathcal{D} + \text{div}(\mathfrak{f})$.

Example 8.10. Given a lattice N and a cone $\sigma \subset N_{\mathbb{Q}}$, we obtain the associated affine toric variety as $X_{\sigma} = \mathfrak{X}(\mathcal{D})$ for the trivial pp-divisor $\mathcal{D} = 0$ living

on $Y = \{y\}$, see Example 3.2. For the big torus $T \subset X_\sigma$, we have canonical identifications

$$T \cong N \otimes_{\mathbb{Z}} \mathbb{K}^* \cong \mathbb{K}(Y, N)^*.$$

Hence, the translation $X_\sigma \rightarrow X_\sigma$ by a torus element $t \in T$ is the equivariant morphism $\mathfrak{X}(\mathfrak{D}) \rightarrow \mathfrak{X}(\mathfrak{D})$ associated to the map $(\text{id}_Y, \text{id}_N, f)$ of \mathfrak{D} , where $f \in \mathbb{K}(N; Y)^*$ is the plurifunction corresponding to $t \in T$.

Example 8.11. Let $\mathfrak{D} = \sum_{y \in Y} \Delta_y \otimes \{y\}$ be a pp-divisor on an elliptic curve Y . In order to obtain the equivariant automorphisms of $\mathfrak{X}(\mathfrak{D})$, we have to figure the automorphisms $\psi : Y \rightarrow Y$ satisfying

$$\sum_{y \in Y} (\Delta_{\psi(y)} - \Delta_y) \otimes \{y\} = \psi^*(\mathfrak{D}) - \mathfrak{D} = \text{div}(f)$$

with a plurifunction $f \in \mathbb{K}(Y, N)^*$. Note that by completeness of Y , the plurifunction $f \in \mathbb{K}(Y, N)^*$ is determined by its divisor up to a ‘‘constant’’ from $N \otimes_{\mathbb{Z}} \mathbb{K}^* \cong T$; so, $\text{div}(f)$ determines the automorphism up to translation by a torus element.

The left thand side difference is a polyhedral principal divisor if and only if there are elements $v_y \in N$ such that $\Delta_{\psi(y)} = \Delta_y + v_y$ as polyhedra, $\sum v_y = 0$ in N , and, using the group law on the elliptic curve, $\sum v_y \otimes y = 0$ in $N \otimes_{\mathbb{Z}} Y$. In particular, unless $\Delta_y \in N + \sigma$ for all $y \in Y$, the automorphism ψ must be of finite order.

As another immediate consequence, we can answer the question, when two given pp-divisors define equivariantly isomorphic varieties.

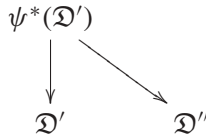
Corollary 8.12. *Two pp-divisors $\mathfrak{D}_i \in \text{PPDiv}_{\mathbb{Q}}(Y_i, \sigma_i)$ define equivariantly isomorphic varieties $\mathfrak{X}(\mathfrak{D}_i)$ if and only if there are projective birational morphisms $\psi_i : Y_i \rightarrow Y$ and a pp-divisor $\mathfrak{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ with $\mathfrak{D}_i \cong \psi_i^* \mathfrak{D}$.*

In order to turn the functor \mathfrak{X} into an equivalence of categories, we restrict ourselves to those maps of polyhedral divisors that define dominant equivariant morphisms. Let us call these for the moment *dominating*.

Remark 8.13. If (ψ, F, f) is a map of pp-divisors $\mathfrak{D} \rightarrow \mathfrak{D}'$ such that F has finite cokernel, then (ψ, F, f) is dominating.

However, the main obstruction for \mathfrak{X} to yield an equivalence is the fact that, for a projective, birational map $\psi : Y \rightarrow Y'$, the morphism $\psi^*(\mathfrak{D}') \rightarrow \mathfrak{D}'$ is not an isomorphism, but $\mathfrak{X}(\psi^*(\mathfrak{D}')) \rightarrow \mathfrak{X}(\mathfrak{D}')$ is. Hence, similar to the construction process of derived categories, we have to localize by those maps: We extend the morphisms of our category of pp-divisors by formally introducing an inverse of

$\psi^*(\mathcal{D}') \rightarrow \mathcal{D}'$. The correct way to do this is to define a new morphism $\mathcal{D}' \rightarrow \mathcal{D}''$ as a diagram



of traditional ones with some projective, birational ψ . Now, as an immediate consequence of Theorem 8.8, we obtain the following statement.

Corollary 8.14. *The functor \mathfrak{X} induces an equivalence from the localized category of pp-divisors with dominating maps to the category of normal affine varieties with effective torus action and dominant equivariant morphisms.*

Finally, we want to describe the isomorphism classes of normal, affine T -varieties. Fixing T and the weight cone ω of its action, there are only two types of isomorphisms in our localized category of pp-divisors. First, the adding of divisors of plurifunctions: As we do with principal divisors in the traditional setting, we handle this by introducing the Picard group.

Definition 8.15. *Dividing by the group of polyhedral principal divisors, we may define the polyhedral Picard group and the rational polyhedral Picard group as*

$$\text{Pic}(Y, \sigma) := \text{CaDiv}(Y, \sigma) / \mathbb{K}(Y, N)^*, \quad \text{Pic}_{\mathbb{Q}}(Y, \sigma) := \text{CaDiv}_{\mathbb{Q}}(Y, \sigma) / \mathbb{K}(Y, N)^*.$$

Note that, by abuse of notation, we actually divide by the image of $\mathbb{K}(Y, N)^*$ – there is always a kernel. Moreover, the rational polyhedral Picard group is *not* the rational vector space associated to the (integral) polyhedral Picard group.

Example 8.16 (Cf. 2.6). Let $N = \mathbb{Z}$ and $\sigma = \mathbb{Q}_{\geq 0}$. Then we have $\text{Pol}_{\sigma}(N) = \mathbb{Z}$, and the polyhedral Picard group is the usual one, i.e. $\text{Pic}(Y, \sigma) = \text{Pic}(Y)$. Moreover, the following sequence is exact:

$$0 \rightarrow \text{Pic}(Y) \rightarrow \text{Pic}_{\mathbb{Q}}(Y, \sigma) \rightarrow \text{CaDiv}(Y) \otimes_{\mathbb{Z}} \mathbb{Q} / \mathbb{Z} \rightarrow 0.$$

The second type of isomorphisms in the localized category of pp-divisors consists of the new isomorphisms coming from (birational) modifications of Y . This yields

Corollary 8.17. *The isomorphism classes of normal affine varieties with effective T -action and fixed weight cone $\omega = \sigma^{\vee}$ are in 1-1 correspondence with the pp-classes in $\varinjlim \text{Pic}_{\mathbb{Q}}(\bullet, \sigma)$ where the limit is taken over modifications of the variety carrying the respective pp-divisors.*

Remark 8.18. The normal affine T -varieties X with $\dim(T) = \dim(X) - 1$ are precisely those arising from pp-divisors on smooth curves. Since there are no non-trivial modifications in the curve case, no localization is needed in this case.

9. Proof of Theorem 8.8

We begin with two auxiliary statements; the first one is an elementary general observation on semiample divisors.

Lemma 9.1. *Let D and D' be semiample \mathbb{Q} -Cartier divisors on a normal variety Y . If $\Gamma(Y, \mathcal{O}(nD)) \subset \Gamma(Y, \mathcal{O}(nD'))$ holds for infinitely many $n > 0$, then we have $D \leq D'$.*

Proof. Write $D = \sum \alpha_i D_i$ and $D' = \sum \alpha'_i D'_i$ with prime divisors D_i on Y . Then, for any $y \in Y$, we obtain new divisors by removing all prime components from D and D' that do not contain y :

$$D_y := \sum_{y \in D_i} \alpha_i D_i, \quad D'_y := \sum_{y \in D'_i} \alpha'_i D'_i.$$

By semiampleness of D , there are an $n > 0$ and a global section $f \in \Gamma(Y, \mathcal{O}(nD))$ with $\text{div}(f)_y + nD_y = 0$. Since f is a global section of $\mathcal{O}(D')$ as well, we have $\text{div}(f)_y + nD'_y \geq 0$. This gives $D_y \leq D'_y$ for all $y \in Y$, which implies $D \leq D'$. □

Let us call the minimal pp-divisors as produced in the proof of Theorem 3.4 for the moment *GIT-constructed*. So, given any X (or a pp-divisor $\mathfrak{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$ defining X), the associated GIT-constructed pp-divisors are minimal, and live on the normalization \bar{Y} of the canonical component of the limit over the GIT-quotients in question.

Lemma 9.2. *Let $\mathfrak{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$. Then, for every associated GIT-constructed $\bar{\mathfrak{D}} \in \text{PPDiv}_{\mathbb{Q}}(\bar{Y}, \sigma)$, we have $\mathfrak{D} = \vartheta^* \bar{\mathfrak{D}} + \text{div}(f)$ with a plurifunction $f \in \mathbb{K}(Y, N)^*$, where $\vartheta : Y \rightarrow \bar{Y}$ is the canonical morphism.*

Proof. As usual, denote by \mathcal{A} the \mathcal{O}_Y -algebra associated to \mathfrak{D} , let $\tilde{X} := \text{Spec}_Y(\mathcal{A})$, set $A := \Gamma(Y, \mathcal{A})$, and $X := \text{Spec}(A)$. Then, denoting by $r : \tilde{X} \rightarrow X$ the contraction map, we have commutative diagrams

$$\begin{array}{ccc}
 r^{-1}(X^{ss}(u)) & \xrightarrow{r} & X^{ss}(u) \\
 \downarrow & & \downarrow \\
 Y & \xrightarrow{\vartheta_u} & Y_u \\
 & \searrow \vartheta & \uparrow p_u \\
 & & \bar{Y}
 \end{array}$$

Now we are ready to compare \mathfrak{D} and the pullback $\vartheta^*(\bar{\mathfrak{D}})$. For this, recall that the divisors $\bar{\mathfrak{D}}(u)$ have been defined in the proof of Theorem 3.4 via

$$\mathcal{O}(\bar{\mathfrak{D}}(u)) := \frac{1}{s(u)} \cdot \bar{\mathcal{A}}_u \subset \mathbb{K}(\bar{Y})$$

with $s : M \rightarrow \mathbb{K}(X)$ being a section of the degree “map”, and $\bar{\mathcal{A}}_u$ being certain sheaves on \bar{Y} with global sections A_u . Using the above diagram, we see

$$\Gamma(Y, \mathcal{O}(\vartheta^*(\bar{\mathcal{D}}(u)))) = \frac{1}{s(u)} \cdot A_u \subset \mathbb{K}(Y).$$

On the other hand, our present X comes from the pp-divisor \mathcal{D} . Thus, there is a canonical multiplicative map, forgetting the grading:

$$\bigcup_{u \in \omega \cap M} \Gamma(Y, \mathcal{D}(u)) \rightarrow \mathbb{K}(Y), \quad f_u \mapsto f_u.$$

This map extends to the multiplicative system of all homogeneous rational functions on X , and hence we may view $s(u)$ as an element of $\mathbb{K}(Y)$. This gives

$$\begin{aligned} \Gamma(Y, \mathcal{O}(\vartheta^*\bar{\mathcal{D}}(u))) &= \frac{1}{s(u)} \cdot A_u \\ &= \frac{1}{s(u)} \cdot \Gamma(Y, \mathcal{O}(\mathcal{D}(u))) = \Gamma(Y, \mathcal{O}(\mathcal{D}(u) - \text{div}(s(u)))). \end{aligned}$$

By Lemma 9.1, this implies that $\vartheta^*(\bar{\mathcal{D}}(u))$ equals $\mathcal{D}(u) - \text{div}(s(u))$ for every $u \in \omega \cap M$. It follows that $u \mapsto s(u)$ defines the desired plurifunction. \square

Proof of Theorem 8.8. Writing $X := \mathfrak{X}(\mathcal{D})$ and $X' := \mathfrak{X}(\mathcal{D}')$, we are given a dominant, equivariant morphism $\varphi : X \rightarrow X'$. The ring $A := \Gamma(X, \mathcal{O})$ is graded by the character lattice M of the torus $T = \text{Spec}(\mathbb{C}[M])$. We will use the analogous notation A', M' etc. for the X' -world.

Let us first consider the case that the pp-divisors \mathcal{D} and \mathcal{D}' are GIT-constructed. Let $F^* : M' \rightarrow M$ denote the lattice homomorphism arising from the accompanying homomorphism $\tilde{\varphi} : T \rightarrow T'$. By dominance of $\varphi : X \rightarrow X'$, every element $u \in M'$ gives rise to a nonempty set

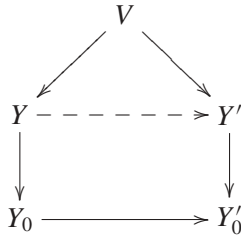
$$\varphi^{-1}((X')^{ss}(u)) = \bigcup_{f \in A_{\mathbb{N}u}} X_{\varphi^*(f)} \subset X^{ss}(F^*(u)).$$

By the construction of GIT-quotients, the set in the middle, and thus that on the left hand side, is a full inverse image under the quotient map $X^{ss}(F^*(u)) \rightarrow Y_{F^*(u)}$. Hence, we obtain commutative diagrams (of dominant morphisms):

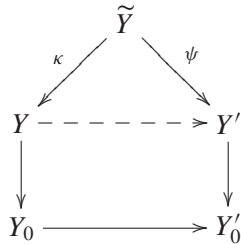
$$\begin{array}{ccccc} X^{ss}(F^*(u)) & \supset & \varphi^{-1}((X')^{ss}(u)) & \xrightarrow{\varphi} & (X')^{ss}(u) \\ \parallel T \downarrow & & \parallel T \downarrow & & \downarrow \parallel T' \\ Y_{F^*(u)} & \supset & \varphi^{-1}((X')^{ss}(u)) \parallel T & \xrightarrow{\psi_u} & Y'_u \end{array}$$

Now we consider the normalizations Y and Y' of the canonical components of the respective limits of the GIT-quotients of X and X' . The above ψ_u fit together

to a dominant rational map $Y \dashrightarrow Y'$, defined over some open $V \subset Y$, and we have a commutative diagram

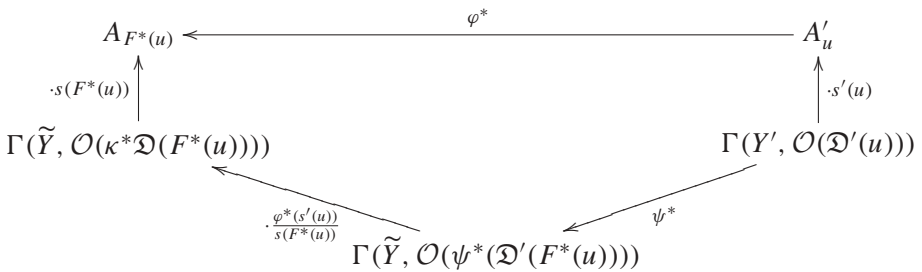


Let \tilde{Y} denote the normalization of the closure of the graph of $V \rightarrow Y'$ in $Y \times Y'$. Then, with the projections $\kappa: \tilde{Y} \rightarrow Y$ and $\psi: \tilde{Y} \rightarrow Y'$, we obtain a new commutative diagram



Since \tilde{Y} is projective over the graph of $Y_0 \rightarrow Y'_0$ and hence over Y_0 , the birational map $\kappa: \tilde{Y} \rightarrow Y$ is also projective. Moreover, under the identification $\mathbb{K}(X)^T = \mathbb{K}(Y)$ the pullback homomorphisms φ^* and ψ^* coincide.

Now, let $s: M \rightarrow \mathbb{K}(X)^*$ and $s': M' \rightarrow \mathbb{K}(X')^*$, be the sections defining the minimal pp-divisors \mathcal{D} and \mathcal{D}' respectively, compare Section 6. Then we obtain a commutative diagram:



The assignment $u \mapsto \varphi^*(s'(u))/s(F^*(u))$ defines a plurifunction $f \in \mathbb{K}(Y, N')$. Using Lemma 9.1, one directly verifies that the triple (ψ, F, f) describes a map of pp-divisors $\kappa^*\mathcal{D} \rightarrow \mathcal{D}'$ with the properties claimed in the first part of the assertion. To see the part concerning the case of an isomorphism $\varphi: X \rightarrow X'$, note

that then no resolution of indeterminacies $\tilde{Y} \rightarrow Y$ is needed: we can take κ to be the identity, and $\psi: Y \rightarrow Y'$ is the induced isomorphism.

So, the assertion is proved in the case of GIT-constructed pp-divisors \mathcal{D} and \mathcal{D}' . In the slightly more general case of minimal \mathcal{D} and \mathcal{D}' , the assertion follows immediately from Lemma 9.2 and the fact, that by definition of minimality, the morphisms $\vartheta: Y \rightarrow \bar{Y}$ and $\vartheta': Y' \rightarrow \bar{Y}'$ onto the normalized canonical components are isomorphisms.

Now, we turn to the case that \mathcal{D}' is minimal but \mathcal{D} is not. The part of the assertion concerning the case of an isomorphism $\varphi: X \rightarrow X'$ is easily settled by using Lemma 9.2 and the statement verified so far.

To see the first part of the assertion, let $\mathcal{D}_1 \in \text{PPDiv}_{\mathbb{Q}}(Y_1, \sigma)$ be any GIT-constructed pp-divisor for X . Consider the canonical birational projective morphism $\vartheta: Y \rightarrow Y_1$. By Lemma 9.2, the pullback $\vartheta^*\mathcal{D}_1$ and \mathcal{D} differ only by the divisor of a plurifunction f . Moreover, by the preceding considerations, we have a projective, birational morphism $\kappa_1: \tilde{Y}_1 \rightarrow Y_1$, and a commutative diagram

$$\begin{array}{ccc}
 & \mathfrak{X}(\kappa_1^*\mathcal{D}_1) & \\
 \mathfrak{X}(\kappa_1, \text{id}, 1) \swarrow \cong & & \searrow \mathfrak{X}(\psi_1, F_1, f_1) \\
 \mathfrak{X}(\mathcal{D}_1) & & \mathfrak{X}(\mathcal{D}') \\
 \cong \uparrow & & \uparrow = \\
 X & \xrightarrow{\varphi} & X'
 \end{array}$$

Consider the fiber product $Y \times_{Y_1} \tilde{Y}_1$. Since all maps are birational, this space contains a nonempty open subset projecting isomorphically onto open subsets of Y and \tilde{Y}_1 . Let \tilde{Y} be the normalization of the closure of this subset, and consider the canonical projective, birational morphisms $\kappa: \tilde{Y} \rightarrow Y$ and $\vartheta_1: \tilde{Y} \rightarrow \tilde{Y}_1$. Then $\kappa^*(\mathcal{D})$ and $\vartheta_1^*\kappa_1^*\mathcal{D}_1$ differ only by the divisor of the plurifunction κ^*f . This allows us to define the desired map $\kappa^*\mathcal{D} \rightarrow \mathcal{D}'$.

Finally, we turn to the general case. Let $\mathcal{D}'_1 \in \text{PPDiv}_{\mathbb{Q}}(Y'_1, \sigma')$ be a GIT-constructed pp-divisor for X' . Then, by what we proved so far, there is a projective birational map $\kappa_1: \tilde{Y}_1 \rightarrow Y_1$ and a commutative diagram.

$$\begin{array}{ccc}
 & \mathfrak{X}(\kappa_1^*\mathcal{D}) & \\
 \mathfrak{X}(\kappa_1, \text{id}, 1) \swarrow \cong & & \searrow \mathfrak{X}(\psi_1, F_1, f_1) \\
 \mathfrak{X}(\mathcal{D}) & & \mathfrak{X}(\mathcal{D}'_1) \\
 = \uparrow & & \uparrow \cong \\
 X & \xrightarrow{\varphi} & X'
 \end{array}$$

Let $\vartheta' : Y' \rightarrow Y_1$ be the canonical projective map such that $(\vartheta')^*\mathcal{D}'_1$ and \mathcal{D}' differ only by the divisor of a plurifunction. Consider the fiber product of \tilde{Y}_1 and Y' over Y'_1 and, similarly as before, the normalization \tilde{Y} of the canonical component. Then we have canonical birational projective morphisms $\kappa_2 : \tilde{Y} \rightarrow \tilde{Y}_1$ and $\psi : \tilde{Y} \rightarrow Y'$. Set $\kappa := \kappa_1 \circ \kappa_2$. Then $\kappa_1^*\mathcal{D} \rightarrow \mathcal{D}'_1$ lifts to a map $\kappa^*\mathcal{D} \rightarrow (\vartheta')^*\mathcal{D}'_1$, which allows to define the desired map $\kappa^*\mathcal{D} \rightarrow \mathcal{D}'$. \square

10. The orbit decomposition

In this section, we use the language of polyhedral divisors to study the orbit decomposition of a normal affine variety with torus action. We determine the orbit cones of Definition 5.1, and we describe the collection of orbits in terms of a defining pp-divisor. As an application, we show how to compute the GIT-fan of an affine variety with torus action directly from its defining pp-divisor.

Let us fix the setup. As usual, Y is a semiprojective variety, N is a lattice with dual lattice M , and $\sigma \subset N_{\mathbb{Q}}$ is a pointed cone. Let $\mathcal{D} \in \text{PPDiv}_{\mathbb{Q}}(Y, \sigma)$, and denote the associated sheaf of graded algebras by

$$\mathcal{A} := \bigoplus_{u \in \sigma^{\vee} \cap M} \mathcal{O}(\mathcal{D}(u)).$$

Then we have the variety $\tilde{X} := \text{Spec}_Y(\mathcal{A})$, the torus $T := \text{Spec}(\mathbb{K}[M])$, and the canonical map $\pi : \tilde{X} \rightarrow Y$. Moreover, there is a T -equivariant contraction $r : \tilde{X} \rightarrow X$ onto the affine T -variety $X = \text{Spec}(A)$, where $A := \Gamma(Y, \mathcal{A})$.

Our task is to describe the T -orbits of X in terms of \mathcal{D} . In Definition 7.7, we associated to any point $y \in Y$ a fiber polyhedron $\Delta_y \subset M_{\mathbb{Q}}$ with normal quasifan Λ_y , and a fiber monoid complex S_y . By Corollary 7.11, there is a bijection

$$\begin{aligned} \{(y, F); y \in Y, F \preceq \Delta_y\} &\rightarrow \{T\text{-orbits in } \tilde{X}\} \\ (y, F) &\mapsto B_{\tilde{X}}(y, F), \end{aligned}$$

where $B_{\tilde{X}}(y, F) \subset \pi^{-1}(y)$ is the unique T -orbit having $\lambda(F) \in \Lambda_y$ as its orbit cone. Besides these orbit data, our description of the collection of T -orbits in X involves the canonical maps

$$\vartheta_u : Y \rightarrow Y_u, \quad \text{where } Y_u = \text{Proj} \left(\bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Gamma(Y, \mathcal{O}(\mathcal{D}(nu))) \right),$$

being induced from the semiample divisors $\mathcal{D}(u)$. In a neighborhood of $y \in Y$, the variety Y_u as well as the map ϑ_u does not depend on u , but only on the Λ_y -cone containing u in its relative interior.

Theorem 10.1. *The T -equivariant contraction map $r : \tilde{X} \rightarrow X$ induces a surjection*

$$\begin{aligned} \{(y, F); y \in Y, F \preceq \Delta_y\} &\rightarrow \{T\text{-orbits in } X\} \\ (y, F) &\mapsto B_X(y, F) := r(B_{\tilde{X}}(y, F)). \end{aligned}$$

One has $B_X(y_1, F_1) = B_X(y_2, F_2)$ if and only if the following two conditions are satisfied:

$$\lambda(F_1) = \lambda(F_2) \subset M_{\mathbb{Q}}, \quad \vartheta_u(y_1) = \vartheta_u(y_2) \text{ for some } u \in \text{relint}(\lambda_{F_i}).$$

Moreover, for the geometry of the T -orbit $B_X(y, F) \subset X$ associated to a pair (y, F) , one obtains the following.

- (i) *For any $x \in B_X(y, F)$, its orbit cone is given by $\omega(x) = \lambda(F)$, and its orbit lattice is the sublattice $M(x) \subset M$ generated by $S_y \cap \lambda(F)$.*
- (ii) *The T -equivariant map of T -orbit closures $r : \overline{B_{\tilde{X}}(y, F)} \rightarrow \overline{B_X(y, F)}$ is the normalization.*

Note that, in contrast to the toric setting, nonnormal orbit closures show up quite frequently for actions of small tori. The simplest example is the action of \mathbb{K}^* on \mathbb{K}^2 by means of the weights 2 and 3 — there Neil’s parabola occurs as the generic orbit closure.

For the proof of Theorem 10.1, we first provide an auxiliary statement involving the GIT-quotient $q_u : X^{ss}(u) \rightarrow Y_u$ associated to a vector $u \in M$.

Lemma 10.2. *Let $x_1, x_2 \in X$. Then we have $T \cdot x_1 = T \cdot x_2$ if and only if $\omega(x_1) = \omega(x_2)$ and $q_u(x_1) = q_u(x_2)$ for some $u \in \text{relint}(\omega(x_i))$.*

Proof. Only the “if” part is nontrivial. So, suppose $\omega(x_1) = \omega(x_2)$ and $q_u(x_1) = q_u(x_2)$ for some $u \in \text{relint}(\omega(x_i))$. According to Proposition 5.2, any $f \in A_{nu}$, where $n > 0$, vanishes along $\overline{T \cdot x_i} \setminus T \cdot x_i$. Consequently, the T -orbits through x_1 and x_2 are closed in X_u . Since good quotients separate closed orbits, $q_u(x_1) = q_u(x_2)$ implies $T \cdot x_1 = T \cdot x_2$. □

Proof of Theorem 10.1. We first prove statements (i) and (ii) on the geometry of the orbits. Fix a pair (y, F) , and choose a point $\tilde{x} \in \pi^{-1}(y)$ with orbit cone $\omega(\tilde{x}) = \lambda(F)$. Then the associated orbit lattice $M(\tilde{x}) \subset M$ is generated by $S_y \cap \lambda(F)$. We will show (i) by checking that $x := r(\tilde{x})$ has orbit data $\omega(x) = \omega(\tilde{x})$ and $M(x) = M(\tilde{x})$. Since the T -orbit closure of \tilde{x} is normal, this also proves (ii).

In order to see $\omega(x) \subset \omega(\tilde{x})$, let $u \in \omega(x)$. Then there is an $f \in \Gamma(X, \mathcal{O}_{nu})$ with $n > 0$ such that $f(x) \neq 0$ holds. Thus, $r^* f(\tilde{x}) \neq 0$, which implies $u \in \omega(\tilde{x})$. For the reverse inclusion, note that we find for every $u \in \omega(\tilde{x})$ a $g \in \Gamma(Y, \mathcal{A}_{nu})$, where $n > 0$, such that $\pi(\tilde{x}) \notin Z(g)$. Then we have $g(\tilde{x}) \neq 0$. Moreover, $g = r^* f$ with $f \in \Gamma(X, \mathcal{O}_{nu})$ and $f(x) \neq 0$. This implies $u \in \omega(x)$.

Similar to the orbit cones, we see $M(x) \subset M(\tilde{x})$. To verify the reverse inclusion, let $u \in S(\tilde{x})$. Consider the contraction map $\vartheta_u: Y \rightarrow Y_u$. Then, $\mathfrak{D}(u) = \vartheta_u^*(E_u)$ with an ample divisor E_u on Y_u . In particular, we have $\mathfrak{D}(u) = \text{div}(h^{-1})$ on some neighbourhood $V = \vartheta_u^{-1}(V_u)$ of $y = \pi(\tilde{x})$ with $V_u \subset Y_u$ open and $h \in \mathbb{K}(V)$.

Recall that there is a good quotient $X_u \rightarrow Y_u$ for the set $X_u \subset X$ of semistable points associated to $u \in \omega \cap M$. Moreover, we may restrict $\pi: \tilde{X} \rightarrow Y$ to obtain a morphism $r^{-1}(X_u) \rightarrow Y$. Denoting by $W \subset r^{-1}(X_u)$ and $W_u \subset X_u$ the inverse images of $V \subset Y$ and $V_u \subset Y_u$ respectively, we arrive at a commutative cube:

$$\begin{array}{ccccc}
 & & r^{-1}(X_u) & \longrightarrow & X_u \\
 & \nearrow & \downarrow & & \nearrow \\
 W & \longrightarrow & W_u & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & \nearrow & Y & \longrightarrow & Y_u \\
 \downarrow & & \downarrow & & \downarrow \\
 V & \longrightarrow & V_u & &
 \end{array}$$

Now, consider $h \in \mathbb{K}(V)$ as a regular function on $W \subset \pi^{-1}(V)$. Since $u \in S(\tilde{x}) \subset \omega(\tilde{x})$ and $\omega(\tilde{x}) = \omega(x)$, we have $x \in X_u$. Hence $\tilde{x} \in r^{-1}(X_u)$, which gives us $\tilde{x} \in W$. Moreover, since $y \notin Z(h)$ holds, h is not trivial along $\pi^{-1}(y)$, and thus $u \in S(\tilde{x})$ yields $h(\tilde{x}) \neq 0$. Since $W \rightarrow W_u$ is proper and birational, the function $h \in \Gamma(W, \mathcal{O})_u$ is in fact a regular function on W_u . By construction, we have $x \in W_u$ and $h(x) \neq 0$. This implies $u \in M(x)$.

We come to the characterization of the equality $B_X(y_1, F_1) = B_X(y_2, F_2)$. Choose $x_i \in B_X(y_i, F_i)$. As we have just seen, $\omega(x_i) = \lambda(F_i)$ holds. Moreover, as remarked just before, we have a commutative diagram for every $u \in \omega \cap M$

$$\begin{array}{ccc}
 r^{-1}(X_u) & \xrightarrow{r} & X_u \\
 \pi \downarrow & & \downarrow q_u \\
 Y & \xrightarrow{\vartheta_u} & Y_u.
 \end{array}$$

Thus, the conditions $\lambda(F_1) = \lambda(F_2)$ and $\vartheta_u(y_1) = \vartheta_u(y_2)$ are equivalent to the conditions $\omega(x_1) = \omega(x_2)$ and $q_u(x_1) = q_u(x_2)$. According to Lemma 10.2, the latter conditions characterize $T \cdot x_1 = T \cdot x_2$. □

Putting together Theorem 10.1 with Corollary 7.11 and Propositions 7.10, 7.12 gives the following characterization for a pp-divisor to be an integral Cartier divisor.

Corollary 10.3. *The following statements are equivalent.*

- (i) *The polyhedral divisor \mathfrak{D} belongs to $\text{CaDiv}(Y, \sigma)$.*
- (ii) *The map $\pi: \tilde{X} \rightarrow Y$ has no multiple fibers.*
- (iii) *The torus T acts with connected isotropy groups on X .*

In a further application, we indicate how to read off the GIT-quasifan of X in the sense of Theorem 5.4 from its defining pp-divisor \mathfrak{D} .

Corollary 10.4. *Let $\mathfrak{D} = \Delta_1 \otimes D_1 + \dots + \Delta_r \otimes D_r$ with prime divisors D_i . Then the quasifan of GIT-cones associated to the T -linearizations of the trivial bundle on X is the normal quasifan of the Minkowski sum $\Delta_1 + \dots + \Delta_r$.*

Proof. According to Theorem 5.4, the GIT-quasifan Λ of X is the coarsest quasifan in $M_{\mathbb{Q}}$ refining all orbit cones $\omega(x)$, where $x \in X$. By Theorem 10.1, the orbit cones $\omega(x)$, where $x \in X$, are precisely the cones of the normal quasifans Λ_y of the fiber polyhedra Δ_y , where $y \in Y$. Thus, Λ is the coarsest common refinement of all Λ_y and hence equals the normal quasifan of the Minkowski sum of all the Δ_y where $y \in Y$. But the latter equals the normal quasifan of $\Delta_1 + \dots + \Delta_r$. \square

11. Calculating Examples

In this section, we indicate a recipe how to determine a pp-divisor for a given normal affine T -variety X . The strategy is first to treat the case of a toric variety X and then to settle the general case via equivariant embedding. The proof of the method is straightforward and will therefore be omitted.

Consider an affine toric variety X and the action of a subtorus $T \subset T_X$ of the big torus $T_X \subset X$. Let N_X be the lattice of one parameter subgroups of T_X , and let $\delta \subset (N_X)_{\mathbb{Q}}$ be the cone describing X . The inclusion $T \subset T_X$ corresponds to an inclusion $N = N_T \subset N_X$ of lattices, and we obtain a (non-canonical) split exact sequence

$$0 \longrightarrow N_T \xrightarrow{F} N_X \xrightarrow{P} N_Y \longrightarrow 0,$$

$\searrow \scriptstyle s$

where $N_Y := N_X/N_T$ and $s: N_X \rightarrow N_T$ satisfies $s \circ F = \text{id}$. Let Σ_Y be the coarsest fan in $(N_Y)_{\mathbb{Q}}$ refining all cones $P(\delta_0)$ where $\delta_0 \preceq \delta$. Then the toric variety Y corresponding to Σ_Y is the normalization of the closure of the image of T_X in the limit over all GIT-quotients of X , i.e. Y is as in Section 6. Note that, up to normalization, Y equals the Chow quotient of X by T as constructed in [15].

Let us indicate how to obtain a minimal pp-divisor for the T -variety X . Given a one-dimensional cone $\varrho \in \Sigma_Y$, let $v_{\varrho} \in \varrho$ denote the first lattice vector, and define a polyhedron

$$\Delta_{\varrho} := s(\delta \cap P^{-1}(v_{\varrho})) \subset N_{\mathbb{Q}} = (N_T)_{\mathbb{Q}}.$$

These polyhedra have $\sigma := \delta \cap (N_T)_{\mathbb{Q}}$ as their tail cone. Denoting by $R_Y \subset \Sigma_Y$ the set of one-dimensional cones, and by $D_{\varrho} \subset Y$ the prime divisor corresponding $\varrho \in R_Y$, we obtain a minimal pp-divisor for the T -variety X as

$$\mathfrak{D} = \sum_{\varrho \in R_Y} \Delta_{\varrho} \otimes D_{\varrho} \in \text{PPDiv}(Y, \sigma).$$

Example 11.1. Consider the affine toric variety $X = \mathbb{K}^4$ and the action of the two-dimensional torus $T = (\mathbb{K}^*)^2$ on X given, with respect to standard coordinates, by

$$t \cdot z = (t_1^4 z_1, t_1^3 z_2, t_2 z_3, t_1^{12} t_2^{-1} z_4).$$

The corresponding lattices are $N_T = \mathbb{Z}^2$, and $N_X = \mathbb{Z}^4$, and the quotient lattice is $N_Y = \mathbb{Z}^2$. The maps $F: N_T \rightarrow N_X$ and $P: N_X \rightarrow N_Y$ and a choice for $s: N_X \rightarrow N_T$ are given by the matrices

$$F = \begin{bmatrix} 4 & 0 \\ 3 & 0 \\ 0 & 1 \\ 12 & -1 \end{bmatrix}, \quad s = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 3 & 0 & -1 & -1 \\ 0 & 4 & -1 & -1 \end{bmatrix}.$$

From this we see that the fan Σ_Y is the standard fan of the projective plane, meaning that it has as its maximal cones

$$\text{cone}((1, 0), (0, 1)), \quad \text{cone}((0, 1), (-1, -1)), \quad \text{cone}((-1, -1), (1, 0)).$$

The cone $\sigma = s(\mathbb{Q}_{\geq 0}^4 \cap F(\mathbb{Q}^2))$ is generated by the vectors $(1, 0)$ and $(1, 12)$. The polyhedral coefficients of the pp-divisor $\sum \Delta_\varrho \otimes D_\varrho$ on $Y = \mathbb{P}^2$ are

$$\begin{aligned} \Delta_{\mathbb{Q}_{\geq 0}(1,0)} &= (1/3, 0) + \sigma &=: \Delta_0, \\ \Delta_{\mathbb{Q}_{\geq 0}(0,1)} &= (-1/4, 0) + \sigma &=: \Delta_1, \\ \Delta_{\mathbb{Q}_{\geq 0}(-1,-1)} &= (\{0\} \times [0, 1]) + \sigma &=: \Delta_\infty. \end{aligned}$$

Let us indicate how to handle the general case, i.e. a possibly non-toric normal affine variety X with an effective action of a torus T . First, choose a T -equivariant embedding into some \mathbb{K}^n , where T acts as a subtorus of $(\mathbb{K}^*)^n$, and X hits the big orbit of $(\mathbb{K}^*)^n$. Then apply the previous method to obtain a minimal pp-divisor

$$\mathcal{D}_{\text{toric}} = \sum \Delta_\varrho \otimes D_\varrho$$

for the T -variety \mathbb{K}^n living on some toric variety Y_{toric} . Then the desired variety Y lying over the GIT-quotients of X is the normalization of the closure of the image of $X \cap (\mathbb{K}^*)^n$ in Y_{toric} . Moreover, a pp-divisor for the T -variety X is obtained by pulling back $\mathcal{D}_{\text{toric}}$ to Y .

Example 11.2. Consider once more the affine threefold $X = V(z_1^3 + z_2^4 + z_3 z_4)$ in \mathbb{K}^4 discussed in the introduction of the paper. It is invariant under the action of $T = (\mathbb{K}^*)^2$ on \mathbb{K}^4 given by

$$t \cdot z = (t_1^4 z_1, t_1^3 z_2, t_2 z_3, t_1^{12} t_2^{-1} z_4).$$

In the preceding example, we computed a minimal pp-divisor $\mathcal{D}_{\text{toric}}$ living on $Y_{\text{toric}} = \mathbb{P}^2$ for the T -action on \mathbb{K}^4 . On the big tori $(\mathbb{K}^*)^4 \subset \mathbb{K}^4$ and $(\mathbb{K}^*)^2 \subset \mathbb{P}^2$, the projection is given by

$$(\mathbb{K}^*)^4 \rightarrow (\mathbb{K}^*)^2, \quad (t_1, t_2, t_3, t_4) \mapsto \left(\frac{t_1^3}{t_3 t_4}, \frac{t_2^4}{t_3 t_4} \right).$$

From this we see that the closure of the image of $X \cap (\mathbb{K}^*)^4$ in \mathbb{P}^2 is given with respect to its homogeneous coordinates $(w_0 : w_1 : w_2)$ by

$$Y = V(w_0 + w_1 + w_2).$$

Since $Y = \mathbb{P}^1$ is already normal, we can restrict $\mathcal{D}_{\text{toric}}$ to Y and thus obtain a minimal pp-divisor for the T -variety X by

$$\mathcal{D} = \Delta_0 \otimes \{0\} + \Delta_1 \otimes \{1\} + \Delta_\infty \otimes \{\infty\}.$$

Acknowledgements. We would like to especially thank J. A. Christophersen for valuable and stimulating discussions. Moreover, we are grateful to the referee, as well as to I.V. Arzhantsev and R. Vollmert for helpful remarks on earlier versions of the manuscript.

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