Torsion of differentials on toric varieties

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Abstract

We introduce an invariant for semigroups with cancellation property. When the semigroup equals the set of lattice points in a rational, polyhedral cone, then this invariant describes the torsion of the differential sheaf on the associated toric variety.
Finally, as an example, we present the case of two-dimensional cones (corresponding to two-dimensional cyclic quotient singularities).

1 An invariant for semigroups

(1.1) Let S be a commutative semigroup with 0 and cancellation property (i.e. $a+s=b+s$ implies $a=b$ for $a,b,s\in S$). In particular, S can be embedded into a group, and the notion $-a$ for $a\in S$ makes sense. Assume that (inside this group) $S\cap(-S)=\{0\}$; then via

$$a\geq b \quad :\Longleftrightarrow \quad a-b\in S\,,$$

S turns also into a partially ordered set.

For each $\ell\in S$ we will define a certain abelian group T_ℓ. Their direct sum $T:=\oplus_{\ell\in S}T_\ell$ or the T_ℓ's itself should be considered an invariant of the original semigroup. If S is finitely generated, then so will the T_ℓ's. In case S equals the set of lattice points of some rational, polyhedral cone (i.e. generated by a finite set of rays through rational points) in $\mathbb{R}^n$, we will show that $T\otimes\mathbb{C}$ coincides with the torsion of the differential sheaf Ω^1_Y of the affine toric variety $Y=\operatorname{Spec}\mathbb{C}[S]$ ($\mathbb{C}[S]$ denotes the semigroup algebra of S over $\mathbb{C}$).

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(1.2) For any subset $P \subseteq S$ we denote by $L(P)$ the abelian group of integral relations between elements of P, i.e.

$$L(P) := \ker\left(\mathbb{Z}^{\oplus P} \to \operatorname{span}_{\mathbb{Z}} P \subseteq S - S\right) .$$

For each $q \in L(P)$ we define its support as

$$\operatorname{supp} q := \{s \in P \mid q_s \neq 0\} ;$$

it is a finite subset of P. Then, every $q \in L(P)$ is uniquely representable as a difference $q = q^+ - q^-$ with $q^+, q^- \in \mathbb{N}^{\oplus P}$ and $\operatorname{supp} q^+ \cap \operatorname{supp} q^- = \emptyset$. In particular,

$$\sum_{s \in S} q_s^+ \cdot s = \sum_{s \in S} q_s^- \cdot s \in \operatorname{span}_{\mathbb{N}} P \subseteq S ,$$

and we denote this element by $\bar{q}$ (the "*bar-value*" of q).

Definition: For every $\ell \in S$ let $S_\ell := \{s \in S \mid s \leq \ell\}$. Then, we define

$$T_\ell := L(S_\ell) \big/ \langle q \in L(S_\ell) \mid \bar{q} \leq \ell \rangle = \langle q \in L(S) \mid \operatorname{supp} q \leq \ell \rangle \big/ \langle q \in L(S) \mid \bar{q} \leq \ell \rangle .$$

($\langle q \in L(S_\ell) \mid \bar{q} \leq \ell \rangle$ means the subgroup generated by those q's, and "$\operatorname{supp} q \leq \ell$" is just an abbreviation for "$s \leq \ell$ for all $s \in \operatorname{supp} q$".)

(1.3) **Proposition:** *Let $E \subseteq S$ be a subset that generates the semigroup S. Denoting $E_\ell := E \cap S_\ell = \{s \in E \mid s \leq \ell\}$, we obtain for $\ell \in S$*

$$T_\ell = L(E_\ell) \big/ \langle q \in L(E_\ell) \mid \bar{q} \leq \ell \rangle = \langle q \in L(E) \mid \mathit{supp}\, q \leq \ell \rangle \big/ \langle q \in L(E) \mid \bar{q} \leq \ell \rangle .$$

Proof: We regard the canonical map

$$\varphi : L(E_\ell) \big/ \langle q \in L(E_\ell) \mid \bar{q} \leq \ell \rangle \longrightarrow L(S_\ell) \big/ \langle q \in L(S_\ell) \mid \bar{q} \leq \ell \rangle .$$

φ is surjective: Suppose a relation $q \in L(S_\ell)$ involves $s^1, \ldots, s^N \in S_\ell$ (i.e. $\operatorname{supp} q = \{s^1, \ldots, s^N\}$). Each s^v can be represented as a sum of elements from E providing a special relation $q(s^v)$ with $\overline{q(s^v)} = s^v \leq \ell$. Then, if q equals the relation $[\sum_v q_v \cdot s^v = 0]$, we obtain $q - \sum_v q_v \cdot q(s^v) \in L(E_\ell)$.
(By the way, $\tilde{q} := q - \sum_v q_v \cdot q(s^v)$ can be regarded as similar to q, but each occurrence of s^v is replaced by an appropriate sum of elements from E. In particular, the bar-value of $\tilde{q}$ is less or equal than that of q.)

φ is injective: Let $q \in L(E_\ell)$ such that $\varphi(q) = 0$, i.e. it splits into a sum $q = \sum_i q^i$ with $q^i \in L(S_\ell)$, $\bar{q}^i \leq \ell$. Treating each of the summands as we did q previously, we obtain

$$q^i = \tilde{q}^i + \sum_v q_v^i \cdot q(s^v) \quad \text{with } \tilde{q}^i \in L(E_\ell),\ \overline{\tilde{q}^i} \leq \ell, \text{ and } s^v \in S_\ell.$$

Hence, up to relations from $\langle q \in L(E_\ell) \,|\, \bar{q} \leq \ell \rangle$, we may assume that

$$q = \sum_{s^v \in S_\ell \setminus E_\ell} g_v \cdot q(s^v) \in L(S_\ell) \subseteq \mathbb{Z}^{S_\ell}$$

for some integer coefficients g_v. However, since q is actually contained in $L(E_\ell) \subseteq \mathbb{Z}^{E_\ell}$, evaluating this equation at each of the $s^v \in S_\ell \setminus E_\ell$ yields always $g_v = 0$. □

Corollary: If S is a finitely generated semigroup, then the T_ℓ are finitely generated abelian groups.

(1.4) We conclude this general section with a trivial, but important lemma.

Lemma: *Let $\ell \in S$, and define $S' \subseteq S$ as the semigroup generated by S_ℓ. The relation "$\leq_{S'}$" associated to S' is not the restriction of that of S; in general, "$\leq_{S'}$" is stricter than "$\leq_S$". However,*

(i) *$(S')_\ell = S_\ell$, and*

(ii) *the abelian groups T_ℓ coincide for S and S'.*

Proof: For (i) assume that $s \in S$ fulfills $s \leq_S \ell$. By definition, there is an $t \in S$ such that $s + t = \ell$. In particular, $s, t \in S_\ell \subseteq S'$, and we obtain $s \leq_{S'} \ell$.
The same trick works for (ii): If $q \in L(S_\ell)$ is a relation with $\bar{q} \leq_S \ell$, then there exists a $t \in S$ such that $\bar{q} + t = \ell$. In particular, $t \in S_\ell \subseteq S'$, hence $\bar{q} \leq_{S'} \ell$. □

Hence, to compute T_ℓ, we may always change the semigroup and assume that $E = E_\ell$.

2 Rational, polyhedral cones

(2.1) For this section we assume that $S \subseteq \mathbb{Z}^n$ equals the set of lattice points of some rational, polyhedral cone in $\mathbb{R}^n$. In particular, it defines an affine toric variety $Y := \operatorname{Spec} \mathbb{C}[S]$. (For general facts about toric varieties see [Da] or [Od]).

Definition: Let $Y = \operatorname{Spec} B$ be an arbitrary affine scheme over the complex numbers, i.e. let B be an arbitrary (commutative) $\mathbb{C}$-algebra. Then, the B-module of Kähler differentials Ω^1_Y together with the canonical differential $d : B \to \Omega^1_Y$ are defined by the following universal property:

(i) d is additive, kills the constants, and it fulfills the rule

$$d(fg) = f\, d(g) + g\, d(f) \quad \text{for } f, g \in B.$$

(ii) For each B-module Q all maps $d' : B \to Q$ meeting the properties similar to (i) factorize through d by a uniquely determined B-linear map $\Omega^1_Y \to Q$.

Indeed, the Kähler differentials Ω^1_Y do exist, and they are uniquely determined by the previous definition. For a proof of these facts and basic properties (for instance the two fundamental exact sequences) see Chapter 10 of [Ma].

Theorem: *In the toric situation the $\mathbb{C}[S]$-module of Kähler differentials Ω^1_Y and its torsion submodule $tors(\Omega^1_Y)$ are $\mathbb{Z}^n$-graded. For $\ell \in S$ we have*

$$tors(\Omega^1_Y)(\ell) = T_\ell \otimes_{\mathbb{Z}} \mathbb{C} \,,$$

and $tors(\Omega^1_Y)$ vanishes in the remaining degrees. Moreover, if $s \in S$, then the canonical map $T_\ell \to T_{\ell+s}$ describes the multiplication with $x^s \in \mathbb{C}[S]$.

(2.2) **Proof:** *Step 1:* Let $E = \{s^0, \ldots, s^w\}$ be a generating set of the semigroup S; in particular, we have a surjective map $\pi : \mathbb{N}^{w+1} \twoheadrightarrow S,\ a \mapsto \sum_i a_i\, s^i$. Let $m := \{(a,b) \mid a,b \in \mathbb{N}^{w+1};\ \pi(a) = \pi(b) \text{ in } S\}$.
Then, Y can be regarded as the closed subset of $\mathbb{C}^{w+1}$ defined by the ideal $I = (\underline{z}^a - \underline{z}^b \mid (a,b) \in m) \subseteq \mathbb{C}[z_0, \ldots, z_w]$. We obtain the standard exact sequence (cf. Theorem 58 in [Ma])

$$I/I^2 \stackrel{d}{\longrightarrow} \Omega^1_{\mathbb{C}^{w+1}} \otimes_{\mathbb{C}[z_0,\ldots,z_w]} \mathbb{C}[S] \longrightarrow \Omega^1_Y \longrightarrow 0\,,$$

inducing

$$\mathbb{C}[S]^{\oplus m} \stackrel{\alpha}{\longrightarrow} \mathbb{C}[S]^{w+1} \longrightarrow \Omega^1_Y \longrightarrow 0$$

via composing d with the canonical surjection $\mathbb{C}[S]^{\oplus m} \twoheadrightarrow I/I^2$.

The maps in the latter exact sequence could be described as follows: Denote by $\{e^{ab}\}$ and $\{e^i\}$ the standard bases of $\mathbb{C}[S]^{\oplus m}$ and $\mathbb{C}[S]^{w+1}$, respectively; and for an $s \in S$ denote by $x^s \in \mathbb{C}[S]$ the corresponding element in the semigroup algebra, e.g. $x^{s^i} \equiv z_i \pmod{I}$. Then, the image of e^{ab} in I/I^2 is the equation $\underline{z}^a - \underline{z}^b$, and this maps onto $d(\underline{z}^a - \underline{z}^b) = \sum_{i=0}^w (a_i - b_i)\, x^{\pi(a)-s^i}\, d\, z_i \in \Omega^1_{\mathbb{C}^{w+1}} \otimes \mathbb{C}[S]$. In particular,

$$\alpha(e^{ab}) = \sum_{i=0}^{w} (a_i - b_i) \cdot x^{\pi(a)-s^i} \cdot e^i\,.$$

(At the first glance, α might not always map into $\mathbb{C}[S]^{w+1}$. However, if $\pi(a) - s^i \notin S$, then this would imply $a_i = b_i = 0$.)
Finally, $e^i \in \mathbb{C}[S]^{w+1}$ maps onto $d\, x^{s^i} \in \Omega^1_Y$ by the second map.

Step 2: The $\mathbb{C}$-linear map $\mathbb{C}[S] \to \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{C}[\mathbb{Z}^n]\,,\ x^s \mapsto s \otimes x^s$ is a $\mathbb{C}$-derivation: It kills the constants, and for $s,t \in S$ we have $x^s(t \otimes x^t) + x^t(s \otimes x^s) = (s+t) \otimes x^{s+t}$. Hence, by definition of the Kähler differentials, we obtain a $\mathbb{C}[S]$-linear map

$$\Omega^1_Y \longrightarrow \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{C}[\mathbb{Z}^n]\,, \quad d\, x^s \mapsto s \otimes x^s\,.$$

On the other hand, $\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{C}[\mathbb{Z}^n]$ can be identified with the module of Kähler differentials on the torus $(\mathbb{C}^*)^n = \operatorname{Spec} \mathbb{C}[\mathbb{Z}^n]$, and the previous map corresponds to the restriction of differentials from Y onto the open subset $(\mathbb{C}^*)^n \subseteq Y$. Since $\Omega^1_{(\mathbb{C}^*)^n}$ is just a localisation of Ω^1_Y, we obtain that this map has exactly $\operatorname{tors}(\Omega^1_Y)$ as its kernel.

Putting this fact together with the information from the first step, we obtain the sequence of $\mathbb{C}[S]$-modules

$$(*) \qquad \begin{array}{ccccc} \mathbb{C}[S]^{\oplus m} & \stackrel{\alpha}{\longrightarrow} & \mathbb{C}[S]^{w+1} & \stackrel{\beta}{\longrightarrow} & \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{C}[\mathbb{Z}^n] \\ & & e^i & \mapsto & s^i \otimes x^{s^i} \end{array}$$

and, moreover, $\operatorname{tors}(\Omega^1_Y) = \ker \beta / \operatorname{im} \alpha$.

Step 3: Defining $\deg(e^{ab}) := \pi(a) = \pi(b) \in S$ for $\mathbb{C}[S]^{\oplus m}$, $\deg(e^i) := s^i \in S$ for $\mathbb{C}[S]^{w+1}$, and $\deg(s \otimes x^t) := t \in \mathbb{Z}^n$ for $\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{C}[\mathbb{Z}^n]$, the sequence $(*)$ turns out to be $\mathbb{Z}^n$-graded. In particular, to calculate $\ker \beta / \operatorname{im} \alpha$, we can deal with each degree $\ell \in S$ separately (degrees from $\mathbb{Z}^n \setminus S$ do not appear).

Claim: $T_\ell \otimes_{\mathbb{Z}} \mathbb{C} \to (\ker \beta / \operatorname{im} \alpha)_\ell\,,\ q \mapsto \sum_{s^i \in E} q_i \cdot x^{\ell - s^i} \cdot e^i$ is an isomorphism of $\mathbb{C}$-vector spaces.

Proof: For $q \in L(E_\ell)$, the sum $\sum_{s^i \in E} q_i \cdot x^{\ell - s^i} \cdot e^i$ defines an element of $(\mathbb{C}[S]^{w+1})_\ell$. Applying β, we obtain

$$\begin{array}{rcl} \beta(\sum_{s^i \in E} q_i \cdot x^{\ell - s^i} \cdot e^i) & = & \sum_{s^i \in E} q_i \cdot x^{\ell - s^i} \cdot (s^i \otimes x^{s^i}) \\ & = & (\sum_{s^i \in E} q_i\, s^i) \otimes x^\ell \\ & = & 0\,. \end{array}$$

Obviously, each element of $(\ker \beta)_\ell$ can be obtained that way: If $\sum_{s^i \in E} q_i(x) \cdot e^i \in \mathbb{C}[S]^{w+1}$ is of degree ℓ, we obtain $q_i(x) = q_i\, x^{\ell - s^i}$ ($q_i \in \mathbb{C}$ with $q_i = 0$ unless $\ell \geq s_i$). Moreover, we have just seen that β kills this element if and only if $\sum_{s^i \in E} q_i\, s^i = 0$, i.e. it comes from a relation $q \in L(E_\ell)$.

On the other hand, $q \in L(E_\ell)$ with $\bar{q} \leq \ell$ can be written as $q = a - b$ ($a, b \in \mathbb{N}^{w+1}$; $\pi(a) = \pi(b) \leq \ell$). Then,

$$\sum_{s^i \in E} (a_i - b_i) \cdot x^{\ell - s^i} \cdot e^i = x^{\ell - \pi(a)} \cdot \alpha(e^{ab})\,,$$

and those elements generate $(\operatorname{im} \alpha)_\ell$. □

(2.3) **Remark:** We never used the fact that $Y = \operatorname{Spec} \mathbb{C}[S]$ is a normal variety. Hence, Theorem (2.1) remains true, if $S \subseteq \mathbb{Z}^n$ is just supposed to be a finitely generated semigroup with $S \cap (-S) = \{0\}$. In particular, over $\mathbb{R}$, S generates a rational, polyhedral cone $\sigma \subseteq \mathbb{R}^n$, but there may exist finitely many points of $\sigma \cap \mathbb{Z}^n$ not belonging to S.

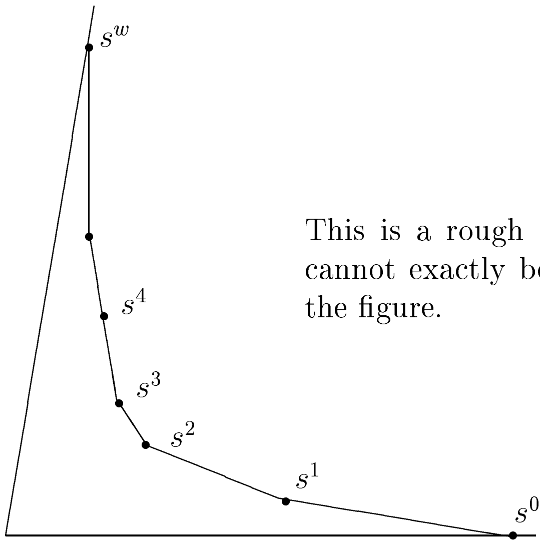
Example: If $S \subseteq I\!N$, then Y is a curve singularity. For instance, $S := I\!N \setminus \{1\}$ is generated by the integers 2 and 3, and $Y \subseteq \mathbb{C}^2$ is the cusp given by the equation $y^2 - x^3 = 0$.
Let us compute the torsion of Ω^1_Y in this case: With $E = \{s^0 := 2, s^1 := 3\}$ we have the only relation $q := [3s^0 - 2s^1 = 0]$. Since $q \in S_5$, on the one hand, but $\bar{q} = 6$, on the other hand, we obtain $T = T_5 = \mathbb{Z} \cdot q$, i.e. tors (Ω^1_Y) is a one-dimensional vector space concentrated in the singular point.

3 Two-dimensional, cyclic quotient singularities

(3.1) Two-dimensional, cyclic quotient singularities coincide with the two-dimensional, affine toric varieties; they were first investigated by Riemenschneider, [Ri]. We want to compute our invariant for this special case.

Let S be the set of lattice points in the two-dimensional cone spanned by some primitive vectors $s^0, s^w \in \mathbb{Z}^2$. Then, following §1.6 in [Od], $E = \{s^0, \ldots, s^w\}$ is built from all lattice points of the compact part of the boundary of conv $(S \setminus \{0\})$; in particular, $\partial(\text{conv } (S \setminus \{0\}))$ consists of w primitive edges (containing no interior lattice point). Every pair of adjacent elements s^{i-1}, s^i $(i = 1, \ldots, w)$ provides a $\mathbb{Z}$-basis for the lattice $\mathbb{Z}^2$, and there are relations

$$s^{i-1} + s^{i+1} = a_i \cdot s^i \quad (a_i \in I\!N,\ a_i \geq 2;\ i = 1, \ldots, w-1)\,.$$



This is a rough draft of S; in fact it cannot exactly be shaped as shown in the figure.

Remark: (cf. Lemma 1.20 in [Od]) The coefficients a_i can be obtained from the continued fraction

$$\frac{n}{n-q} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \ldots}}$$

with $n := \det(s^0, s^w)$ and $q \in \{1, \dots, n-1\}$ such that $n|(qs^0 + s^w)$.

(3.2) Since $a_i \geq 2$ $(i = 1, \dots, w-1)$, the sets $E_\ell \subseteq E$ are "connected" (i.e. they are shaped as $E_\ell = \{s^p, s^{p+1}, \dots, s^{p+k}\}$ for some p, k) for every $\ell \in S$. Hence, to compute T_ℓ, we may assume that $E_\ell = \{s^0, \dots, s^w\}$ (cf. Lemma (1.4)).

Lemma: *Let $\ell \in S$ such that $E_\ell = \{s^0, \dots, s^w\}$. Then,*

(i) $\ell \geq s^0 + s^w$, and

(ii) $T_\ell = 0$ unless $\ell = s^0 + s^w$.

Proof: (i) The vectors s^0 and s^w form a $\mathbb{R}$-basis of $\mathbb{R}^2$, and, for an $s \in \mathbb{Z}^2$, the condition "$s \in S$" is equivalent to having non-negative coordinates only. Hence, the inequalities $\ell \geq s^0, s^w$ imply that both coordinates of ℓ are not smaller than 1, i.e. $\ell - s^0 - s^w$ has non-negative coordinates.

(ii) Let $\ell > s^0 + s^w$ be given; we will show that $T_\ell = 0$. We distinguish two cases:

Case 1: $\ell - (s^0 + s^w) \geq s^0$ (or similarly $\ell - (s^0 + s^w) \geq s^w$).
Then, since $s^i \leq s^0 + s^w$, we know that $s^0 + s^i \leq 2s^0 + s^w \leq \ell$ for $i = 2, \dots, w$. Summing up the equations $s^{j-1} + s^{j+1} = a_j\, s^j$ $(1 \leq j \leq i-1)$ provides relations q^i

$$(q^i) \qquad s^0 + s^i = (s^1 + s^{i-1}) + \sum_{j=1}^{i-1} (a_j - 2)\, s^j \qquad (i = 2, \dots, w)$$

having exactly $s^0 + s^i \leq \ell$ as their bar-values. In particular, they yield zero in T_ℓ. On the other hand, if we are given an arbitrary relation $q \in L(E)$, then we can use $q^w, \dots, q^2$ to eliminate $s^w, \dots, s^2$ step by step from the support of q (without changing its value in T_ℓ). Since s^0 and s^1 are linearly independent, q has to be trivial then.

Case 2: Not Case 1; in particular, $\ell - (s^0 + s^w) \geq s^i$ for some $i = 1, \dots, w-1$.
Then, we have $\ell - (s^0 + s^w) = \sum_{j=1}^{w-1} g_j\, s^j$ with non-negative integers g_j and, moreover, $g_i \geq 1$. (There are no summands involving s^0 or s^w, because this would fit in the first case.)
Again, we start with an arbitrary $q \in L(E)$. First, we use the relation $s^0 + s^w = (s^1 + s^{w-1}) + \sum_{j=1}^{w-1} (a_j - 2)\, s^j$ (bar-value $s^0 + s^w \leq \ell$) to eliminate s^0 (if $i = 1$) or s^w (if $i \geq 2$) from $\operatorname{supp} q$.
If $w = 2$, then we are already done. Otherwise, we use the similar relations expressing $s^i + s^w$ (if $i = 1$) or $s^0 + s^i$ (if $i \geq 2$) by the generators in between to eliminate s^w or s^0, respectively. The result is a relation q with $\operatorname{supp} q \subseteq \{s^1, \dots, s^{w-1}\}$.
Finally, we know that

$$\ell = (s^0 + s^w) + \sum_{j=1}^{w-1} g_j\, s^j = (s^1 + s^{w-1}) + \sum_{j=1}^{w-1} (g_j + a_j - 2)\, s^j\,.$$

Hence, the fact $T_\ell = 0$ for the cone spanned by s^1 and s^{w-1} (induction by w) tells us that $L(s^1, \ldots, s^{w-1})$ is spanned by relations with bar-value not greater than ℓ. In particular, our q can be reduced to zero. $\square$

(3.3) **Lemma:** *Assume $\ell = s^0 + s^w$ (including $E_\ell = \{s^0, \ldots, s^w\}$). Then,*

$$T_\ell \cong \begin{cases} 0 & \text{for } w \leq 2 \\ \mathbb{Z}^{[(w-1)/2]} & \text{for } a_1 = \ldots = a_{w-1} = 2 \\ \mathbb{Z} & \text{otherwise}. \end{cases}$$

(The second case ($a_1 = \ldots = a_{w-1} = 2$) means that the points $s^0, \ldots, s^w$ are sitting on an affine line; the corresponding cyclic quotient singularity equals the cone over the rational normal curve of degree w.)

Proof: The case $w \leq 2$ is obvious. Assuming $a_1 = \ldots = a_{w-1} = 2$, the entire collection of relations with bar-value not greater than ℓ is given by the $[w/2]$ equations

$$s^0 + s^w = s^1 + s^{w-1} = \ldots = s^{[w/2]} + s^{w-[w/2]},$$

and we obtain $(w-1) - [w/2] = [(w-1)/2]$ for the rank of T_ℓ.
Finally, if $w \geq 3$ and not $a_1 = \ldots = a_{w-1} = 2$, then

$$\ell = s^0 + s^w = s^1 + s^{w-1} + \sum_{i=1}^{w-1} (a_i - 2)\, s^i >_{\langle s^1, s^{w-1} \rangle} s^1 + s^{w-1}$$

shows (cf. Lemma (3.2)) that $L(s^1, \ldots, s^{w-1}) \subseteq \langle q \in L(E) \mid \bar{q} \leq \ell \rangle$. On the other hand, the only relations involving s^0 or s^w and having a bar-value $\leq \ell$ are those representing $s^0 + s^w$ as linear combination of $s^1, \ldots, s^{w-1}$.
(Indeed, if for instance $s^0 + \sum_{i=0}^{w} g_i\, s^i = \sum_{i=1}^{w} h_i\, s^i$ is such a relation ($g_i, h_i \geq 0; g_i\, h_i = 0$), then $s^0 + \sum_{i=0}^{w} g_i\, s^i \leq \ell = s^0 + s^w$ implies $\sum_i g_i\, s^i \leq s^w$. Hence $g_w = 0$ or 1, and $g_i = 0$ for $i \neq w$. Moreover, since s^0 is not representable by other generators, this implies that $s^0 + s^w$ forms the left hand side of the relation - and the right hand side has to be built from $s^1, \ldots, s^{w-1}$ then.) $\square$

(3.4) We are gathering our results and obtain the following description of the invariants T_ℓ for a two-dimensional cyclic quotient singularity:

Theorem: *Let S and $a_1, \ldots, a_{w-1}$ as in (3.1); define $a_0 := a_w := 3$.*

(1) Let $s^p, s^{p+k} \in E$ $(0 \leq p < p+k \leq w)$ be elements such that

(i) $k \geq 3$, and

(ii) at least one of the numbers $a_p, \ldots, a_{p+k}$ is greater than two.

Then, $\ell := s^p + s^{p+k}$ *uniquely determines* p *and* k*, and we have*

$$T_\ell \cong \begin{cases} \mathbb{Z}^{[(k-1)/2]} & \text{for } a_{p+1} = \ldots = a_{p+k-1} = 2 \\ \mathbb{Z} & \text{otherwise}. \end{cases}$$

The abelian group T_ℓ *vanishes in the remaining degrees.*

(2) *For* $T = \oplus_\ell T_\ell$ *we obtain* $T \cong \mathbb{Z}^{(w-1)(w-2)/2}$*. In particular,*

$$dim\,(tors\,(\Omega^1_Y)) = \binom{w-1}{2}.$$

($w+1$ equals the embedding dimension of the cyclic quotient singularity Y*).*

Proof: For (1) assume that we are given some $\ell \in S$. Then, E_ℓ is shaped as $E_\ell = \{s^p, \ldots, s^{p+k}\}$, and by Lemma (3.2) we know that $T_\ell = 0$ unless $\ell = s^p + s^{p+k}$. On the other hand, if $\ell = s^p + s^{p+k}$, then Lemma (3.3) tells us about T_ℓ. The only thing being left is asking the other way around: What is the condition for an $\ell := s^p + s^{p+k}$ to yield exactly $E_\ell = \{s^p, \ldots, s^{p+k}\}$?
Obviously, E_ℓ does always contain $\{s^p, \ldots, s^{p+k}\}$, and we show that it is exactly the condition $a_p = \ldots = a_{p+k} = 2$ saying that both sets are *not* equal:
Assume $E_\ell = \{s^{p-i}, \ldots, s^p, \ldots, s^{p+k}, \ldots, s^{p+k+j}\}$ (w.l.o.g. $i \geq j \geq 0$, $i \neq 0$), then we obtain a chain of inequalities

$$s^p + s^{p+k} = \ell \geq s^{p-i} + s^{p+k+j} \geq s^{p-i+1} + s^{p+k+j-1} \geq \ldots \geq s^{p-i+j} + s^{p+k}.$$

If $i > j$, this would imply that two different elements of E (s^p and s^{p-i+j}) would be comparable in S. Hence, $i = j$, and all signs in the previous chain turn into equalities implying $a_{p-i+1} = \ldots = a_p = \ldots = a_{p+k} = \ldots = a_{p+k+i-1} = 2$.
The reversed direction is easy; the equalities $a_p = \ldots = a_{p+k} = 2$ imply $\ell = s^{p-1} + s^{p+k+1}$, hence $s^{p-1}, s^{p+k+1} \in E_\ell$.

To prove the second part of the theorem, we have to count dimensions. Assume that the compact part of the boundary $\partial(\text{conv}\,(S \setminus \{0\}))$ consists of m edges, each containing $w_i - 1$ ($i = 1, \ldots, m$) interior lattice points. In particular, $\sum_{i=1}^m w_i = w$.

Then, we have

(a) $\binom{w+1}{2}$ possibilities of choosing two different points s^p and s^{p+k} from E;

(b) $2w - 1$ of those pairs with $1 \leq k \leq 2$;

(c) $\binom{w_i+1}{2}$ possibilities of choosing two different points s^p and s^{p+k} from the i-th edge;

(d) $2w_i - 1$ of those pairs with $1 \leq k \leq 2$.

Hence, we obtain

$$\binom{w+1}{2} - (2w-1) - \sum_{i=1}^{m} \binom{w_i+1}{2} + \sum_{i=1}^{m}(2w_i - 1) = \binom{w-1}{2} - \sum_{i=1}^{m} \binom{w_i-1}{2}$$

possibilities of choosing pairs $s^p, s^{p+k} \in E$ with $k \geq 3$ and such that at least one of the numbers $a_{p+1}, \ldots, a_{p+k-1}$ is greater than two. Those pairs yield $T_\ell = \mathbb{Z}$.

On the other hand, let $\{s^q, \ldots, s^{q+w_i}\}$ form the i-th edge. Then, its only pairs (s^p, s^{p+k}) meeting the assumption that at least one of the numbers $a_p, \ldots, a_{p+k}$ is greater than two are (s^q, s^{q+k}) and (s^{q+w_i-k}, s^{q+w_i}) with $1 \leq k \leq w_i$. Each of them providing a contribution of $\mathbb{Z}^{[(k-1)/2]}$ (which is automatically zero if $k = 1, 2$), we obtain for the entire i-th edge

$$2 \cdot \sum_{k=3}^{w_i - 1} \left[\frac{k-1}{2}\right] + 1 \cdot \left[\frac{w_i - 1}{2}\right] = \binom{w_i - 1}{2}$$

dimensions for T. □

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