Singularities arising from lattice polytopes

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1 Introduction

(1.1) Assume that an affine variety $Y \subseteq \mathbb{C}^w$ is defined by certain binomials $\underline{z}^a - \underline{z}^b$ ($a, b \in \mathbb{N}^w$); for example take $Y := [z^n - xy] \subseteq \mathbb{C}^3$. Then, the ring of regular functions on Y equals the semigroup algebra $\mathbb{C}[S]$ with S obtained from $\mathbb{N}^w$ via identifying a and b. If, moreover, the semigroup S is easy to handle (for instance, if S is the set of lattice points in a rational, polyhedral, convex cone in some $\mathbb{R}^k$), then one might hope that important features of the algebraic variety Y can be encoded with combinatorial data.
This is, more or less, the main idea of the concept of (affine) toric varieties. Even more work has been done: If a bunch of polyhedral cones comes in a so-called fan, then the affine toric varieties associated to these cones glue together; if the fan arises from the inner normals of a lattice polytope, then we obtain a projective variety. Those ideas were formed in the last 20 years, and many textbooks are available. For a detailed treatment we refer to [Da], [Fu], [Ke], or [Od]. For a short introduction into the subject without proofs see §2.

(1.2) As just mentioned, lattice polyhedra are related to projective toric varieties. Hence it is no surprise that (affine) cones over projective varieties, in the algebro-geometric sense, arise from cones over lattice polytopes in the sense of convex geometry.
Doing toric geometry, one has to deal with cones and their duals as well (cf. (2.1)). This implies that lattice polytopes have a second chance to induce a certain class of affine toric varieties; it was first observed by Ishida in [Ish], that it consists exactly of the affine toric Gorenstein varieties. A more detailed explanation of both methods to construct singularities from lattice polytopes is given in (2.6) and (2.7).

(1.3) The main purpose of the present paper is to give a survey about the deformation theory of toric singularities (or equivalently, of affine toric varieties) known so far. In the very beginning, deformation theory appeared as the investigation of how complex structures may vary on a fixed compact, smooth manifold (cf. [Kod]). In a similar manner, we may regard deformations of germs of analytic spaces. If $Y = (Y, 0)$ is such a germ (often called "singularity", since smooth germs are not the interesting ones), we define the following functor:

$$\mathrm{Def}_Y((T,0)) := \{\text{isomorphism classes of flat } g: Z \to T \text{ , together with } g^{-1}(0) \stackrel{\sim}{\to} Y\}\,.$$

Good references for facts about deformation theory of germs are Artin's Lecture notes [Art], the large introduction in Palamodov's paper [Pa], or Stevens' Habilitationsschrift [St 2]. In many cases, for instance for isolated singularities, there exists the so-called mini-versal deformation. By definition, it yields every possible deformation via specialization of parameters, i.e. via base change. The mini-versal deformation is, up to non-canonical isomorphism, uniquely determined and may

be considered a source of numerical invariants of the original singularity. If Y is a complete intersection, then every perturbation of the defining equations yields a (flat) deformation of Y; in particular, its mini-versal deformation is a family over a smooth base space with well-known dimension.
However, as soon as we leave this class of singularities, the structure of the versal family or even the base space will be more complicated. The first example showing that the situation is not always as boring as in the complete intersection case was given by Pinkham, cf. [Pi]. He studied the cone over the rational normal curve of degree four; here the versal base space consists of two irreducible components of dimension one and three, respectively. In the following we will see similar examples in higher dimensions; even non-reduced points will occur as versal base spaces.

(1.4) What are the major issues we are concerned with? In the sequel we will discuss the following three:

(1) Study the vector spaces T^1_Y of infinitesimal deformations and T^2_Y containing the obstructions for lifting deformations onto larger base spaces. There are different possibilities to define them (cf. (3.1)-(3.3)); but in any case, they are as multigraded as the semigroup algebra $\mathbb{C}[S]$ is. Hence, the problem is to spot the multidegrees R contributing to these vector spaces and to determine the dimensions of $T^p_Y(-R)$. (The minus sign just makes some upcoming formulas easier.)

(2) Let us return to the trivial example of the A_{n-1}-singularity mentioned in the very beginning. The algebra $A := \mathbb{C}[x,y,z]/(z^n - xy)$ is $\mathbb{Z}^2$-graded via deg $x := [n,-1]$, deg $y := [0,1]$, and deg $z := [1,0]$. The infinitesimal deformations equal $T^1_Y = \mathbb{C}[z]/(z^{n-1})$, and the one-parameter deformation assigned to the element $z^{n-k} \in T^1_Y$ equals $(z^n - xy) + t^{(k)} z^{n-k}$. Substituting $T := z^k + t^{(k)}$, the total space of this deformation is defined by a binomial again, by $Tz^{n-k} - xy$.
The general goal is to look for so-called genuine deformations, i.e. for those which are no longer infinitesimal, but defined over parameter spaces which are reduced or even smooth. To be able to use the language of polyhedral cones, we would like to remain somehow in the category of toric varieties. It turns out to be a good idea to look for deformations having toric total spaces as just seen for A_{n-1}.

(3) The best possible result is the description of the whole versal deformation. This might be done by listing equations, by providing information about its irreducible components, or by different methods.

After a short introduction into the subject of toric varieties we will discuss the progress in each of these questions in a separate section. Emphasis will be put on both cones over projective varieties and toric Gorenstein singularities, i.e. on those toric varieties induced by lattice polytopes.
Our survey does not contain any proofs; the claims are either standard, or easy exercises, or we refer to the original papers.

2 Convex geometry and toric varieties

(2.1) We are going to introduce briefly the notions we need from convex geometry. It should be considered a good opportunity to fix notation, on the one hand, and to bring readers from algebraic geometry in the mood for cones and polytopes, on the other hand. References for the details are the new book [Zi] or the appendix in [Od].

Convex cones: Throughout the paper we use the word *cone* for rational, convex, polyhedral cones. If N, M are two mutually dual, free Abelian groups of finite rank, then a cone $\sigma \subseteq N_{I\!R} := N \otimes_{Z\!\!Z} I\!R$ can be given either by its fundamental generators

$$\sigma = \langle a^1, \ldots, a^m \rangle := \sum_{i=1}^{m} I\!R_{\geq 0} \cdot a^i \qquad (a^1, \ldots, a^m \in N)$$

or by finitely many inequalities

$$\sigma = \{a \in N_{I\!R} \,|\, \langle a, r^j \rangle \geq 0\,;\, j = 1, \ldots, K\} \qquad (r^1, \ldots, r^K \in M)\,.$$

The elements $a^i \in N$ and $r^j \in M$ can be normalized by asking for primitive ones, i.e. not being proper multiples. (I hope the reader will not be confused by abuse of notation: We use the symbol $\langle \ldots \rangle$ for both the pairing between the mutually dual lattices N, M as well as for indicating the generators of cones; "$<$" also denotes the face relation.)
The concept of duality interchanges both representations: The cone dual to σ is defined as

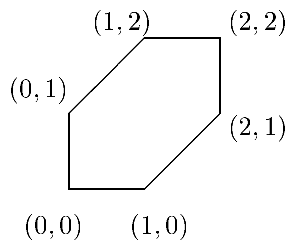
$$\sigma^\vee := \{r \in M_{I\!R} \,|\, \langle a, r \rangle \geq 0 \text{ for all } a \in \sigma\}\,.$$

It has $r^1, \ldots, r^K \in M$ as fundamental generators, or it can be given by the inequalities provided by $a^1, \ldots, a^m \in N$.

(2.2) *Polytopes and polyhedra:* Let $(L_{I\!R}, L)$ be a finite-dimensional real vector (or maybe affine) space with a lattice. Rational polyhedra in $(L_{I\!R}, L)$ are given as intersections of finitely many rational half spaces. If additionally compact, they will be called polytopes. A polyhedron is said to be a lattice polyhedron if its vertices are contained in L.

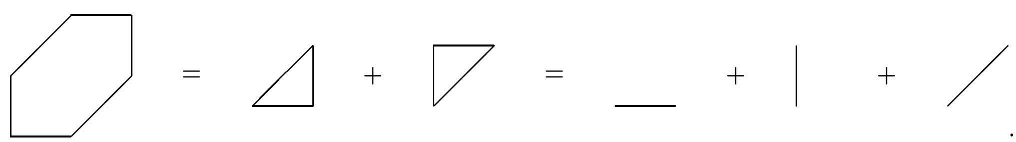
Definition: *For two polyhedra $Q', Q'' \subseteq L_{I\!R}$ we define their Minkowski sum as the polyhedron $Q' + Q'' := \{p' + p'' \,|\, p' \in Q', p'' \in Q''\}$. Obviously, this notion also makes sense for translation classes of polyhedra, hence for affine instead of vector spaces $L_{I\!R}$.*

Example: The plane hexagon $Q := \text{Conv}\,\{(0,0), (1,0), (2,1), (2,2), (1,2), (0,1)\} \subseteq I\!R^2$



hexagon Q

splits into



Every polyhedron $Q \subseteq L_{I\!R}$ is decomposable into the Minkowski sum $Q = Q^{\rm c} + Q^\infty$ of a (compact) polytope $Q^{\rm c} \in L_{I\!R}$ and the so-called cone of unbounded directions Q^∞; the latter one is contained in the *vector* space associated to $L_{I\!R}$ which, however, will be identified with $L_{I\!R}$. The cone Q^∞ is uniquely determined by Q, the compact summand is not. However, we can take for $Q^{\rm c}$ the minimal one – given as the convex hull of the vertices of Q itself. If Q was already compact, then $Q^{\rm c} = Q$ and $Q^\infty = 0$.

Example: Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone, and fix some primitive element $R \in M$. Then $L_{\mathbb{R}} := [R = 1] := \{a \in N_{\mathbb{R}} \mid \langle a, R\rangle = 1\} \subseteq N_{\mathbb{R}}$ is an affine space with lattice $L := [R = 1] \cap N$. We define the crosscut of σ in degree R as the polyhedron $Q := \sigma \cap [R = 1] \subseteq L_{\mathbb{R}}$. It has the cone of unbounded directions $Q^\infty = \sigma \cap [R = 0] \subseteq N_{\mathbb{R}}$. The compact part Q^c of Q is obtained by describing its vertices: Obviously, they correspond exactly to those fundamental generators a^i of σ meeting $\langle a^i, R\rangle \geq 1$ – the actual vertices equal $\bar{a}^i = a^i / \langle a^i, R\rangle$.
Fundamental generators contained in $R^\perp$ can still be "seen" as edges in Q^∞, but those with $\langle \bullet, R\rangle < 0$ are "invisible" in Q. In particular, we can recover the cone σ from Q if and only if $R \in \sigma^\vee$.

(2.3) One of the most frequently used notions will be that of *Minkowski summands* of a given polyhedron $Q \subseteq L_{\mathbb{R}}$. Of course, a Minkowski summand Q' of Q should be at least a summand in the usual sense, i.e. there has to be a Q'' such that $Q = Q' + Q''$. However, since $Q = Q' + Q^\infty$ is true for every $Q^c \subseteq Q' \subseteq Q$, this might not be enough; we would like to avoid additional face structure of Q' (not "coming" from Q). We take the following definition from [Sm]:

Definition: *A polyhedron Q' is called a Minkowski summand of Q if there is a Q'' such that $Q = Q' + Q''$ and if $(Q')^\infty = Q^\infty$.*

It is not difficult to see that the faces of $Q' + Q''$ equal the Minkowski sums of the corresponding faces (defined by the same hyperplane in $L_{\mathbb{R}}$) of Q' and Q''. In particular, up to dilatation, the set of edges of $Q' + Q''$ equals the union of the corresponding sets for Q' and Q'', respectively. That means, a Minkowski summand has not only the same cone of unbounded directions, but, up to dilatation with a factor ≥ 0, also the same compact edges as the original polyhedron.
This is the point for describing the "moduli space" of all Minkowski summands of Q, as it was done in [Al 3]. After choosing orientations, denote the compact edges of Q by $d^1, \ldots, d^N \in L_{\mathbb{R}}$:

Definition: *For every two-dimensional compact face ("two-face") $\varepsilon < Q$ we define its sign vector $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_N) \in \{0, \pm 1\}^N$ by*

$$\varepsilon_i := \begin{cases} \pm 1 & \text{if } d^i \text{ is an edge of } \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

such that the oriented edges $\varepsilon_i \cdot d^i$ fit to a cycle along the boundary of ε. This determines $\underline{\varepsilon}$ up to sign, and any choice will do. In particular, $\sum_i \varepsilon_i d^i = 0$. Then, let $V(Q)$ be the vector space

$$V(Q) := \{(t_1, \ldots, t_N) \in \mathbb{R}^N \mid \sum_i t_i\, \varepsilon_i\, d^i = 0 \text{ for every compact two-face } \varepsilon < Q\}.$$

It is obvious that the points of the cone $C(Q) := V(Q) \cap \mathbb{R}^N_{\geq 0}$ parametrize the set of Minkowski summands of positive multiples of Q via measuring the dilatation factors of the compact edges. In particular, $\underline{1} \in C(Q)$ corresponds to Q itself.

(2.4) *Affine toric varieties:* Let N, M be two mutually dual, free Abelian groups of finite rank. From now on, the lattice structure of cones and polyhedra becomes important. In particular, isomorphisms between those objects are always assumed to be induced from isomorphisms of the lattices. We are going to describe how to get affine algebraic varieties from convex cones:

Definition: *If $\sigma \subseteq N_{\mathbb{R}}$ is a cone with apex, then we define by $Y_\sigma := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ the associated, affine toric variety. ($\mathbb{C}[\sigma^\vee \cap M]$ denotes the semigroup ring - obtained by regarding elements $r \in \sigma^\vee \cap M$ as exponents of some "abstract symbol" x.)*
Let $\sigma_1 \subseteq N^1_{\mathbb{R}}$, $\sigma_2 \subseteq N^2_{\mathbb{R}}$ be two cones. Then, a $\mathbb{Z}$-linear map $f : N^1 \to N^2$ such that $f(\sigma_1) \subseteq \sigma_2$ induces an algebraic morphism $f : Y_1 \to Y_2$ in an obvious way. Those maps will be regarded as the morphisms in the category of affine, toric varieties.

The semigroup $S := \sigma^\vee \cap M$ is generated by the finite set E of its irreducible elements. E is often called the Hilbert basis of that semigroup. Assigning to each element $r \in E$ a variable z_r, our affine toric variety Y_σ can be embedded into $\mathbb{C}^E$. It is defined by the binomial equations obtained from "raising" linear dependencies between the r's into the exponents of the z_r's. Just to give an example, the relation $r + 2s = 3t + u$ turns into $z_r\, z_s^2 = z_t^3\, z_u$.

Examples:

(1) The cone $\sigma := \mathbb{R}^k_{\geq 0} \subseteq \mathbb{R}^k$ with $N := \mathbb{Z}^k$ yields $\sigma^\vee \cap M = \mathbb{N}^k$, hence $Y_\sigma = \mathbb{C}^k$.

(2) Let $E \subseteq \sigma^\vee \cap M$ be the Hilbert basis for an arbitrary cone $\sigma \subseteq N_\mathbb{R}$. Then, assigning to each $a \in N$ the E-tuple $(\langle a, r\rangle)_{r\in E} \in \mathbb{Z}^E$, defines a $\mathbb{Z}$-linear map $N \to \mathbb{Z}^E$ sending σ into $\mathbb{R}^E_{\geq 0}$. At the level of toric varieties, this yields exactly the embedding $Y_\sigma \hookrightarrow \mathbb{C}^E$ described above.

(3) Let $n, q \in \mathbb{Z}$ be relatively prime numbers. With $(\alpha, \beta) \mapsto (-q\,\alpha + \beta,\, n\,\alpha)$, we obtain a $\mathbb{Z}$-linear map $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ sending $\mathbb{R}^2_{\geq 0}$ onto $\sigma := \langle (1,0),\, (-q,n)\rangle \subseteq \mathbb{R}^2$. (By the way, any two-dimensional cone can be written that way.) At the toric level this means we have a morphism $\pi : \mathbb{C}^2 \to Y_\sigma$.
The dual cone equals $\sigma^\vee = \langle [0,1],\, [n,q]\rangle$, and we obtain for $f^*(\sigma^\vee \cap \mathbb{Z}^2)$ the semigroup

$$f^*(\sigma^\vee \cap \mathbb{Z}^2) = \mathbb{N}^2 \cap \{(r_1, r_2) \in \mathbb{Z}^2 \mid r_1 + q\, r_2 \equiv 0 \bmod n\}\,.$$

Hence, the affine coordinate ring $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2]$ of Y_σ equals the subring of $\mathbb{C}[z_1, z_2]$ consisting of polynomials invariant under $\begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix}$ with ξ being a primitive n^{th} root of unity. In particular, Y_σ is a cyclic quotient singularity, and π is the quotient map.
For $q = -1$ we obtain the A_{n-1}–singularity mentioned in the very beginning; $n = 4$, $q = 1$ yields Pinkham's example for singularities with reducible versal base space.

We have seen that almost all two-dimensional cones yield singular toric varieties. This reflects the general situation - smooth, affine toric varieties are boring: If σ is a top-dimensional cone, then *Y_σ is smooth if and only if σ is a simplex generated by a $\mathbb{Z}$-basis of N*, i.e. the determinant of its fundamental generators has to be ± 1. Then, Y_σ is isomorphic to the affine space.
Toric varieties got their name because they always contain the torus $T = \operatorname{Spec} \mathbb{C}[M]$. This algebraic group acts on them and causes a (finite) stratification into T-orbits. The unique closed orbit in an affine toric variety (if σ is top-dimensional, then it is a point) is the most singular one. In higher dimensions, most cones are no longer simplicial. That means, the singularities get worse than quotient singularities.

(2.5) *General toric varieties:* As already mentioned, morphisms between affine toric varieties arise from $\mathbb{Z}$-linear maps $f : N^1 \to N^2$ such that $f(\sigma_1) \subseteq \sigma_2$. A very important special case is that of $f : N \to N$ being the identical map and σ_1 being a face of σ_2. If $r \in \sigma_2^\vee \cap M$ is actually cutting out this face (i.e. $\sigma_1 = \sigma_2 \cap r^\perp$), then $\mathbb{C}[\sigma_2^\vee \cap M]$ equals the localization of $\mathbb{C}[\sigma_1^\vee \cap M]$ by the element x^r. In particular, the induced map $Y_{\sigma_1} \to Y_{\sigma_2}$ is an open embedding identifying the first variety with the open subset $[x^r \neq 0] \subseteq Y_{\sigma_1}$. Moreover, every open embedding in our category arises that way.

Definition: *If Σ is a fan in $N_\mathbb{R}$ (i.e. a finite collection of cones such that $\sigma, \tau \in \Sigma$ always implies $\tau \cap \sigma \leq \tau, \sigma$, and such that Σ contains with every cone all of its faces), then the toric variety Y_Σ is obtained by gluing together the affine pieces Y_σ ($\sigma \in \Sigma$) along common faces of Σ-cones.*
A map between toric varieties $Y_{\Sigma^1}, Y_{\Sigma^2}$ is given by a $\mathbb{Z}$-linear $f : N^1 \to N^2$ such that for each $\sigma_1 \in \Sigma^1$ there is some $\sigma_2 \in \Sigma^2$ meeting $f(\sigma_1) \subseteq \sigma_2$.

It is well known that complete toric varieties arise from fans with $|\Sigma| := \bigcup_{\sigma\in\Sigma} \sigma = N_\mathbb{R}$. If, moreover, Σ is the inner normal fan of some lattice polytope P in M, then $Y_P := Y_\Sigma$ is projective.

The polytope itself reflects the projective embedding; the lattice points of P correspond in an easy way to a natural basis of the global sections of an ample line bundle $\mathcal{L}_P$.
More general, there is a one-to-one correspondence between certain lattice polytopes, on the one hand, and globally generated invertible sheaves on Y_Σ, on the other; Minkowski addition translates into the tensor product. Hence, it seems quite natural that, for projective toric varieties, the vector space $V(P)$ of (2.3) appears as $V(P) = \operatorname{Pic} Y_P \otimes_{\mathbb{Z}} \mathbb{R}$.

Examples:

(1) The k-dimensional fan spanned by the canonical basis vectors $e^1, \ldots, e^k$ of $\mathbb{Z}^k$ and $-e$, with $e := e^1 + \ldots + e^k$, defines the projective space $\mathbb{P}^k$.

(2) The subdivision of the smooth cone $\sigma = \mathbb{R}^k_{\geq 0} = \langle e^1, \ldots, e^k \rangle$ into the union of $\sigma_i := \langle e^1, \ldots, \hat{e}^i, \ldots, e^k, e \rangle$ $(i = 1, \ldots, k)$ describes the blowing up of the origin in the affine k-space.

(3) If $\sigma = \langle a^1, \ldots, a^m \rangle$ is an arbitrary cone (with apex), then the normalized blowing up of Y_σ in the closed orbit is given by the subdivision of σ into the union of $\sigma_r := \{a \in \sigma \mid \langle a, r \rangle \leq \langle a, E \rangle\}$ with r running through the Hilbert basis E of $\sigma^\vee \cap M$.

In general, every fan can be subdivided into a "smooth" fan. That means, every toric variety admits a toric desingularization.

(2.6) Let $P \subseteq (L_{\mathbb{R}}, L)$ be a lattice polytope. Embedding P via $P \subseteq L_{\mathbb{R}} \cong L_{\mathbb{R}} \times \{1\} \hookrightarrow L_{\mathbb{R}} \times \mathbb{R} =: M_{\mathbb{R}}$ into the next dimension, we obtain rational, polyhedral cones $\sigma^\vee := \operatorname{cone}(P) := \mathbb{R}_{\geq 0} \cdot P \subseteq M_{\mathbb{R}}$ and $\sigma := \sigma^{\vee\vee} \subseteq N_{\mathbb{R}}$. Since

$$\mathbb{C}[S] := \mathbb{C}[\sigma^\vee \cap M] = \oplus_{d \geq 0} \mathbb{C}[d\,P \cap M] = \oplus_{d \geq 0} H^0(Y_P, \mathcal{L}_P^{\otimes d}),$$

the toric variety Y_σ equals the *affine cone over the projective variety* $(Y_P, \mathcal{L}_P)$. There is a distinguished point $a^* := (\underline{0}, 1) \in \sigma \subseteq N_{\mathbb{R}}$; the equation $[a^* = 1]$ recovers P from $\sigma^\vee$. Moreover, a^* may be used to make the ring $\mathbb{C}[\sigma^\vee \cap M]$ $\mathbb{Z}$-graded ($\deg x^r := \langle a^*, r \rangle$), and we obtain $Y_P = \operatorname{Proj} \mathbb{C}[\sigma^\vee \cap M]$ while $Y_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$.
We may elucidate the relation between Y_P and its affine cone Y_σ also from another point of view: The open subset $Y_\sigma \setminus \{0\} \subseteq Y_\sigma$ is given by the fan $\partial\sigma$ contained in $N_{\mathbb{R}}$, and the projection $\pi : Y_\sigma \setminus \{0\} \to Y_P$ is provided by the $\mathbb{Z}$-linear map $\pi : N \twoheadrightarrow N/\mathbb{Z} \cdot a^* = L_{\mathbb{R}}{}^*$. It sends proper faces of σ isomorphically onto Σ-cones meaning that $Y_\sigma \setminus \{0\}$ is a $\mathbb{C}^*$-bundle over Y_P.

(2.7) Finally, we would like to explain Ishida's relation (cf. [Ish], Theorem 7.7.) between lattice polytopes and Gorenstein singularities: The differential form $\omega_0 := \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_k}{x_k}$ on the torus $(\mathbb{C}^*)^k \cong \operatorname{Spec}\mathbb{C}[M] = Y_0 \subseteq Y_\sigma$ does, up to sign, *not* depend on the special choice of coordinates. Multiples $x^r \cdot \omega_0$ are holomorphic on Y_σ if and only if r belongs to the interior of $\sigma^\vee$. Since Y_σ is normal, this means that we can describe the canonical module as $\omega_Y = \big(\oplus_{r \in (\operatorname{int}\sigma^\vee) \cap M} \mathbb{C} \cdot x^r\big) \cdot \omega_0$. In particular, Y_σ is Gorenstein, i.e. ω_Y is invertible if and only if there is an $R^* \in M$ such that $(\operatorname{int}\sigma^\vee) \cap M = R^* + (\sigma^\vee \cap M)$. Replacing ω_Y by its g-th tensor power, we have obtained the following criterion:

Let $Y_\sigma = \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$ be an affine toric variety given by a cone $\sigma = \langle a^1, \ldots, a^m \rangle$. Then, Y is $\mathbb{Q}$-Gorenstein if and only if there is a primitive element $R^* \in M$ and a natural number $g \in \mathbb{N}$ such that

$$\langle a^i,\, R^* \rangle = g \quad \textit{for each } i = 1, \ldots, m.$$

Y is Gorenstein if and only if, additionally, $g = 1$.

In particular, toric Gorenstein singularities are obtained by putting a lattice polytope $Q \subseteq (L_{I\!R}, L)$ into the affine hyperplane $L_{I\!R} \times \{1\} \subseteq L_{I\!R} \times I\!R =: N_{I\!R}$ and defining $\sigma := \mathrm{cone}(Q) = I\!R_{\geq 0} \cdot Q$. The canonical degree R^* equals $[\underline{0}, 1]$ in this setting. As an example, lattice intervals of length n provide the two-dimensional A_{n-1}-singularities; see (2.4)(3).

3 Infinitesimal deformations

(3.1) If $Y \subseteq \mathbb{C}^w$ is defined by an ideal $I = (f_1, \ldots, f_s) \subseteq \mathbb{C}[z_1, \ldots, z_w]$, then we denote by $A := \mathbb{C}[\underline{z}]/I$ the algebra of regular functions, by $\mathcal{R} \subseteq \mathbb{C}[\underline{z}]^s$ the $\mathbb{C}[\underline{z}]$-module of linear relations between $f_1, \ldots, f_s$, and by $\mathcal{R}_0 \subseteq \mathcal{R}$ the so-called Koszul relations generated by all $f_j e^k - f_k e^j \in \mathbb{C}[\underline{z}]^s$. In particular, we have the exact sequences

$$0 \to \mathcal{R} \to \mathbb{C}[\underline{z}]^s \to I \to 0 \quad \text{and} \quad 0 \to I \to \mathbb{C}[\underline{z}] \to A \to 0\,.$$

Then, the easiest way to define the vector spaces T^1_Y and T^2_Y is

$$T^1_Y := {\mathrm{Hom}_A(I/I^2, A)} \Big/ {\mathrm{Hom}_A(A^w, A)} \quad \text{via } d: I/I^2 \to A^w \quad \text{with } d(f_j) := \Big(\frac{\partial f_j}{\partial z_1}, \ldots \frac{\partial f_j}{\partial z_w}\Big)$$

and

$$T^2_Y := {\mathrm{Hom}_{\mathbb{C}[z]}(\mathcal{R}/\mathcal{R}_0, A)} \Big/ {\mathrm{Hom}_{\mathbb{C}[z]}(\mathbb{C}[\underline{z}]^s, A)}\;.$$

These definitions seem to depend on the embedding of Y, but they do not. In the above straightforward formulas, the relation of the vector spaces T^p_Y to deformation theory becomes apparent. For instance, if $\xi \in \mathrm{Hom}_A(I/I^2, A)$, then the infinitesimal deformation represented by ξ may be obtained by replacing f_j with the perturbed equation $f_j + \varepsilon\, \xi(f_j)$.

(3.2) A more fancy way to obtain T^1_Y and T^2_Y is to consider them as the first and second André-Quillen cohomology groups, respectively. This cohomology theory is obtained from the cotangent complex which is defined for any $\mathbb{C}$-algebras; it is closely related to Hochschild and Harrison cohomology. A nice introduction into this subject may be found in [Lo], §3.5 - §4.5.
To calculate the vector spaces T^p_Y for affine toric varieties Y_σ, the Harrison cohomology approach was most successful. It yields the following results (cf. [AlSl]):

The T^p_Y admit an M-(multi) grading as it does any other natural module over $A = \mathbb{C}[\sigma^\vee \cap M]$; let us fix an element $R \in M$. For any face $\tau \leq \sigma$ we define

$$K^R_\tau := \Big[\sigma^\vee \cap (R - \mathrm{int}\,\tau^\vee) \cap M\Big] \setminus \{0\}\,.$$

Definition: Let $K \subseteq M$ be an arbitrary subset of the lattice M. A function $f: K \to \mathbb{C}$ is called *quasilinear* if $f(r) + f(s) = f(r+s)$ for any r and s with $r, s, r+s \in K$. The vector space of quasilinear functions is denoted by $\overline{\mathrm{Hom}}\,(K, \mathbb{C})$.

The sets K^R_τ admit the following properties:

(i) $K^R_0 = (\sigma^\vee \cap M) \setminus \{0\}$, and $K^R_i := K^R_{a^i} = \{r \in K_0 \mid \langle a^i, r\rangle < \langle a^i, R\rangle\}$ with $i = 1, \ldots, m$ are "thick strips" along the facets of $\sigma^\vee$.

(ii) For $\tau \neq 0$ the equality $K^R_\tau = \bigcap_{a^i \in \tau} K^R_i$ holds. Moreover, if σ is a top-dimensional cone, $K^R_\sigma = K_0 \cap \big(R - \mathrm{int}\,\sigma^\vee\big)$ is a (diamond shaped) finite set.

(iii) The dependence of the sets K_τ^R on τ is a contravariant functor. This gives rise to the complex $\overline{\operatorname{Hom}}(K_\bullet^R, \mathbb{C})$ with $\overline{\operatorname{Hom}}(K_p^R, \mathbb{C}) := \oplus_{\tau\le\sigma, \dim\tau=p} \overline{\operatorname{Hom}}(K_\tau^R, \mathbb{C})$ $(0 \le p \le \dim\sigma)$ and the usual differentials.

(iv) If $\tau \le \sigma$ is a smooth face, then the injections $\operatorname{Hom}(\operatorname{span}_{\mathbb{C}} K_\tau^R, \mathbb{C}) \hookrightarrow \overline{\operatorname{Hom}}(K_\tau^R, \mathbb{C})$ are also isomorphisms. Moreover, $\operatorname{span}_{\mathbb{C}} K_\tau^R = \bigcap_{a^i\in\tau} \operatorname{span}_{\mathbb{C}} K_i^R$, and the latter vector spaces equal $\operatorname{span}_{\mathbb{C}} K_i^R = M_{\mathbb{C}}$, $(a^i)^\perp$, or 0 if $\langle a^i, R\rangle \ge 2$, $= 1$, or ≤ 0, respectively.

Now, we may express the André-Quillen cohomology groups T_Y^p in terms of the sets K_τ^R:

Theorem: ([AlSl])
1) Let σ be an arbitrary rational, polyhedral cone with apex in 0. Then, for every $R \in M$,

$$T_Y^p(-R) = H^p\big(\overline{\operatorname{Hom}}(K_\bullet^R, \mathbb{C})\big) \qquad \textit{for } p = 0, 1, 2.$$

Moreover, for $p = 0, 1$, this vector space equals $H^p\big((\operatorname{span}_{\mathbb{C}} K_\bullet^R)^*\big)$, too.
2) If Y_σ is Gorenstein in codimension two, then $T_Y^2(-R) = H^2\big((\operatorname{span}_{\mathbb{C}} K_\bullet^R)^*\big)$.
3) Let Y_σ be an isolated singularity. Then, the André-Quillen cohomology in degree $-R$ equals

$$T_Y^p(-R) = \begin{cases} H^p\big(\overline{\operatorname{Hom}}(K_\bullet^R, \mathbb{C})\big) = H^p\big((\operatorname{span}_{\mathbb{C}} K_\bullet^R)^*\big) & \textit{for } 0 \le p \le \dim\sigma - 1 \\ H^{\dim\sigma}\big(\overline{\operatorname{Hom}}(K_\bullet^R, \mathbb{C})\big) & \textit{for } p = \dim\sigma \\ HA^{p-\dim\sigma+1}(K_\sigma^R; \mathbb{C}) & \textit{for } p \ge \dim\sigma + 1\,. \end{cases}$$

Here, the vector spaces $HA^\bullet(K; \mathbb{C})$ are defined as the cohomology of the complex

$$C^q(K; \mathbb{C}) := \Big\{\varphi : \{(r_1, \dots, r_q) \in K^q \mid \textstyle\sum_v r_v \in K\} \to \mathbb{C} \;\Big|\; \varphi \text{ is shuffle invariant}\Big\}$$

with differential $\delta^q : C^{q-1}(K; \mathbb{C}) \to C^q(K; \mathbb{C})$

$$(\delta^q\varphi)(r_1, \dots, r_q) := \varphi(r_2, \dots, r_q) + \sum_{v=1}^{q-1} (-1)^v \varphi(r_1, \dots, r_v + r_{v+1}, \dots, r_q) + (-1)^q \varphi(r_1, \dots, r_{q-1})\,.$$

Remark: The general framework for the above theorem is the existence of a spectral sequence $E_1^{p,q} = \oplus_{\dim\tau=p} HA^q(K_\tau^R; \mathbb{C}) \Longrightarrow T_Y^{p+q-1}(-R)$ together with the vanishing result $HA^q(K_\tau^R; \mathbb{C}) = 0$ for smooth faces $\tau \le \sigma$ and $q \ge 2$.

(3.3) Under some additional assumptions, the results for T_Y^1 and T_Y^2 have already been obtained in [Al 2] with different methods. The paper uses the formula $T_Y^p = \operatorname{Ext}^p(\Omega_{A|\mathbb{C}}, A)$, which is true for normal rings A and $p = 1, 2$. Moreover, [Al 2] contains the relation between the above combinatorial formulas for T_Y^p and those mentioned in (3.1). For T_Y^1 it looks like the following:

Recall from (2.4) that $E \subseteq \sigma^\vee \cap M$ denotes the Hilbert basis. It provides a natural map $\pi : \mathbb{Z}^E \twoheadrightarrow M$ which is dual to $N \to \mathbb{Z}^E$ mentioned in Example (2.4)(2). Then, the ideal $I \subseteq \mathbb{C}[\underline{z}]$ defining Y_σ is generated, as a $\mathbb{C}$-vector space, by the binomials $\underline{z}^a - \underline{z}^b$ with $a, b \in \mathbb{N}^E$ and $\pi(a) = \pi(b)$.
With $E_\tau^R := E \cap K_\tau^R$ we denote by $L_{\mathbb{C}}(E_\tau^R)$ the $\mathbb{C}$-vector space consisting of the linear relations among E_τ^R-elements, i.e. $L_{\mathbb{C}}(E_\tau^R) = (\ker \pi_{\mathbb{C}}) \cap \mathbb{C}^{E_\tau^R}$. In particular, there is an exact sequence

$$0 \to L_{\mathbb{C}}(E_\tau^R) \to \mathbb{C}^{E_\tau^R} \to \operatorname{span}_{\mathbb{C}} E_\tau^R \to 0\,.$$

Theorem: ([Al 2]) *Let $R \in M$. The above sequence for the corresponding complexes provides* $T_Y^1(-R) = H^1\big((\operatorname{span}_{\mathbb{C}} K_\bullet^R)^*\big) = H^1\big((\operatorname{span}_{\mathbb{C}} E_\bullet^R)^*\big) \cong \Big[L_{\mathbb{C}}\big(\bigcup_{i=1}^m E_i^R\big) / \sum_{i=1}^m L_{\mathbb{C}}(E_i^R)\Big]^*$.

If, moreover, $\varphi : L_{\mathbb{C}}(E) \to \mathbb{C}$ induces some element of $T^1_Y(-R)$, then the A-linear map $I/I^2 \to A$ sending $\underline{z}^a - \underline{z}^b$ onto $\varphi(a-b) \cdot x^{\pi(a)-R}$ yields the same element via the T^1_Y-formula in (3.1).

(3.4) What do the above formulas mean for the simplest example, the two-dimensional cyclic quotient singularities? We use the notation of Example (2.4)(3), i.e. $\sigma = \langle (1,0); (-q,n) \rangle \subseteq \mathbb{R}^2$. We develop $\frac{n}{n-q}$ into a (negative) continued fraction

$$\frac{n}{n-q} = a_2 - \cfrac{1}{a_3 - \frac{1}{a_4 - \dots}} \qquad \text{(with } a_v \geq 2\text{)}.$$

Then $E \subseteq \sigma^\vee \cap \mathbb{Z}^2$ is given as the set $E = \{r^1, \dots, r^w\} \subseteq \langle [0,1], [n,q] \rangle \cap \mathbb{Z}^2$ with

- $r^1 = [0,1]$, $r^2 = [1,1]$, $r^w = [n,q]$;
- $r^{v-1} + r^{v+1} = a_v \cdot r^v$ $(v = 2, \dots, w-1)$ (cf. [Ri] or [Od]).

Pinkham has already obtained that, if $w \geq 4$, $\dim T^1_Y = \sum_{v=2}^{w-1} a_v - 2$. The previous theorems should give the same result. Denoting $a^1 = (1,0)$ and $a^2 = (-q,n)$, they mean

$$T^1_Y(-R) = \left((\operatorname{span}_{\mathbb{C}} E^R_1) \cap (\operatorname{span}_{\mathbb{C}} E^R_2) \Big/ \operatorname{span}_{\mathbb{C}}(E^R_1 \cap E^R_2) \right)^* .$$

There are four different cases for the multidegree $R \in M \cong \mathbb{Z}^2$; we assume $w \geq 3$, i.e. Y is not smooth:

(i) $R = r^2$ (or analogously $R = r^{w-1}$): We obtain $E^R_1 = \{r^1\}$ and $E^R_2 = \{r^3, \dots, r^w\}$, i.e. $\operatorname{span}_{\mathbb{C}} E^R_1 = (a^1)^\perp$, $\operatorname{span}_{\mathbb{C}} E^R_2 = \mathbb{C}^2$ (or $(a^2)^\perp$, if $w = 3$), and $\operatorname{span}_{\mathbb{C}} E^R_{12} = 0$. Hence $T^1_Y(-R) \cong \mathbb{C}$ (or 0, if $w = 3$).

(ii) $R = r^v$ $(3 \leq v \leq w-2)$: We obtain $E^R_1 = \{r^1, \dots, r^{v-1}\}$ and $E^R_2 = \{r^{v+1}, \dots, r^w\}$, hence $\operatorname{span}_{\mathbb{C}} E^R_1 = \operatorname{span}_{\mathbb{C}} E^R_2 = \mathbb{C}^2$, $\operatorname{span}_{\mathbb{C}} E^R_{12} = 0$, and the theorem yields $T^1_Y(-R) \cong \mathbb{C}^2$.

(iii) $R = p \cdot r^v$ $(2 \leq v \leq w-1,\ 2 \leq p < a_v$ for $w \geq 4$; or $v = 2 = w-1,\ 2 \leq p \leq a_2$ for $w = 3)$: We obtain $E^R_1 = \{r^1, \dots, r^v\}$ and $E^R_2 = \{r^v, \dots, r^w\}$, i.e. $\operatorname{span}_{\mathbb{C}} E^R_1 = \operatorname{span}_{\mathbb{C}} E^R_2 = \mathbb{C}^2$ and $\operatorname{span}_{\mathbb{C}} E^R_{12} = \mathbb{C} \cdot R$. The theorem yields $T^1_Y(-R) \cong \mathbb{C}$.

(iv) For the remaining $R \in M$, either $E^R_1 \subseteq E^R_2$ or $E^R_2 \subseteq E^R_1$ or $\#(E^R_1 \cap E^R_2) \geq 2$. In all these cases the theorem yields $T^1_Y(-R) = 0$.

Summing up, we get Pinkham's formula back. The vector space T^2_Y may be obtained in a similar way.

(3.5) We would like to have an interpretation for the vector spaces $T^p_Y(-R)$ in terms of convex geometry. In fact, we do not know any for T^2_Y, but for T^1_Y this was done in [Al 4]:
To any $R \in M$ we assign the crosscut $Q(R) := \sigma \cap [R = 1]$, cf. Example (2.2), and two $\mathbb{R}$-vector spaces $V(R)$, $W(R)$ defined as follows: $V(R) := V(Q(R))$ is the space of "generalized" (allowing negative dilatations) Minkowski summands of $Q(R)$ introduced in (2.3), and

$$W(R) := \mathbb{R}^{\#\{Q(R)\text{-vertices not belonging to } N\}}.$$

The latter provides coordinates s_i for each vertex $\bar{a}^i \in Q(R) \setminus N$, i.e. for each fundamental generator $a^i \in \sigma$ with $\langle a^i, R \rangle \geq 2$. To each compact edge $d^{ij} = \overline{\bar{a}^i \bar{a}^j}$ we associate a set of equations G_{ij} dealing with elements of $V(R) \oplus W(R)$. These sets are of one of the following three types:

(0) $G_{ij} = \emptyset$,

(1) $G_{ij} = \{s_i - s_j = 0\}$ provided both coordinates exist in $W(R)$, set $G_{ij} = \emptyset$ otherwise, or

(2) $G_{ij} = \{t_{ij} - s_i = 0,\; t_{ij} - s_j = 0\}$, dropping equations that do not make sense.

Restricting $V(R) \oplus W(R)$ to the (at most) three coordinates t_{ij}, s_i, s_j, the actual choice of G_{ij} is made such that these equations yield a subspace of dimension $1 + \dim T^1_{\langle a^i, a^j\rangle}(-R)$; since $\langle a^i, a^j\rangle$ is a two-dimensional cone, we do know $\dim T^1_{\langle a^i, a^j\rangle}(-R)$ from the previous section. Anyway, if Y is smooth in codimension two, then G_{ij} is always of type (2).

Theorem: ([Al 4]) *The infinitesimal deformations of Y_σ in degree $-R$ equal*

$$T^1_Y(-R) = \{(\underline{t},\, \underline{s}) \in V_{\mathbb{C}}(R) \oplus W_{\mathbb{C}}(R) \,|\, (\underline{t},\, \underline{s}) \text{ fulfills all the equations } G_{ij}\} \;/\; \mathbb{C} \cdot (\underline{1},\, \underline{1})\,.$$

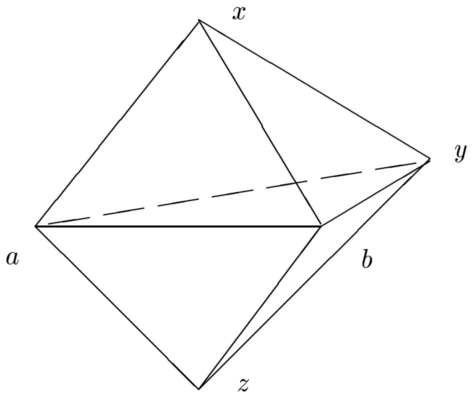
Corollary: *If Y is smooth in codimension two, then $T^1_Y(-R)$ is contained in $V_{\mathbb{C}}(R)/\,\mathbb{C} \cdot (\underline{1})$. It is built from those $\underline{t}$ such that $t_{ij} = t_{jk}$ whenever d^{ij}, d^{jk} are compact edges with a common non-lattice vertex $\bar{a}^j$ of $Q(R)$.*
Thus, $T^1_Y(-R)$ equals the set of equivalence classes of those Minkowski summands of $\mathbb{R}_{\geq 0} \cdot Q(R)$ that preserve up to homothety the stars of non-lattice vertices of $Q(R)$.

(3.6) In the last three sections we would like to mention some applications and examples. First, assume that Y_σ is smooth in codimension two. For the cone $\sigma \subseteq N_{\mathbb{R}}$ this assumption means that any two-dimensional face is generated by two elements of N forming a part of a $\mathbb{Z}$-basis of N.

Proposition: *There are only finitely many $R \in \sigma^\vee \cap M$ such that $T^1_Y(-R) \neq 0$. If, moreover, the three-dimensional faces of σ are simplicial (i.e. built from only three fundamental generators), then $T^1_Y(-R) = 0$ for each $R \in \operatorname{int}\sigma^\vee \cap M$.*

Degrees from $-(\operatorname{int}\sigma^\vee \cap M)$ somehow play the role of negative degrees when dealing with singularities with $(\mathbb{C})^*$-action. In contrast to the Gorenstein case below, there might be infinitesimal deformations in non-negative degrees, even for isolated singularities. In particular, it is not always true that cones which are smooth in codimension three provide rigid singularities:

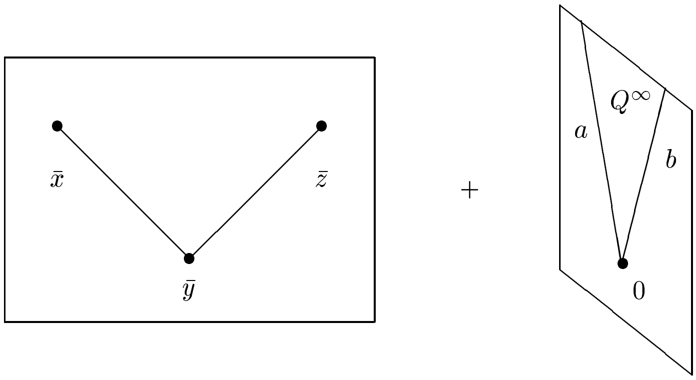
Example: Let $\sigma = \langle a, b, x, y, z\rangle$ be the (four-dimensional) cone over some double tetrahedron.



$$\begin{aligned} a &= (1,0,0,0)\\ b &= (0,1,0,0)\\ y &= (0,0,1,0)\\ x &= (1,1,1,-1)\\ z &= (1,1,1,1) \end{aligned}$$

The two partial cones generated by a, b, y, x and a, b, y, z, respectively, are smooth. So are the proper faces of σ, i.e. Y_σ is an isolated singularity. Let $R = [0,0,1,0]$; then, the compact part of $Q(R)$ consists just of two edges containing the lattice-vertices $\bar{x}, \bar{y}, \bar{z}$.

$Q(R) =$



In particular, $\dim T^1_Y(-R) = 1$.

(3.7) If Y_σ is the affine cone over a projective toric variety Y_P (cf. (2.6)), then we may use the distinguished element $a^* \in N$ to transform M-multidegrees into ordinary integers. In particular, elements $R \in P \cap M \subseteq M$ yield $T^1_Y(-R) \subseteq T^1_Y(-1)$. For cones over smooth, projective surfaces, the reverse implication is also true:

Proposition: *Let P be a plane lattice polygon defining a smooth, projective surface. Except for when P is isomorphic to the unit square (yielding $Y = [xy - zw = 0] \subseteq \mathbb{C}^3$), we have*

$$(T^1_Y)_- := \oplus_{k\geq 1} T^1_Y(-k) = T^1_Y(-1) = \oplus_{R\in P} T^1_Y(-R)\,.$$

Moreover, it is possible to list the series of polygons P provided $T^1_Y(-1) \neq 0$; they are at most hexagons such that the inner lattice points are contained in a line. These polygons are closely related to the so-called reflexive polygons occurring in [Bat].

(3.8) Finally, we consider $\mathbb{Q}$-Gorenstein singularities Y_σ provided by some lattice polytope $Q \subseteq M_{\mathbb{R}}$ sitting in height g (cf. (2.7)). Since $R^* \in M$ is a distinguished element, we may expect the homogeneous part $T^1_Y(-R^*)$ to be special.

Theorem: ([Al 1]) *Let $\sigma = \operatorname{cone}(Q)$ be a $\mathbb{Q}$-Gorenstein cone which is at least three-dimensional. Assume that T^1_Y is finite-dimensional (for instance, if Y_σ has an isolated singularity). Then,*

(1) Y_σ is rigid unless it is Gorenstein.

(2) Let Y_σ be Gorenstein, i.e. $g = 1$; then $T^1_Y = T^1_Y(-R^*) = V_{\mathbb{C}}(Q)/\mathbb{C}\cdot \underline{1}$.
If, moreover, Y_σ is smooth in codimension three, then Y_σ is rigid, too.

If the restriction $\dim T^1_Y < \infty$ is dropped, the situation will be more complicated. Examples are the three-dimensional, non-isolated Gorenstein singularities; they are given by plane lattice polygons Q. Each Q-edge $\overline{a^i a^{i+1}}$ (with $a^{m+1} := a^1$) of length n stands for a one-dimensional A_{n-1}-singularity inside Y_σ. Besides deformations with degree in $-\mathbb{N}\cdot R^*$, edges of length n provide $(n-1)$ infinite series of degrees $R \in M$ with $\dim T^1_Y(-R) = 1$. Since

$$T^2_A(-R)^* = \bigcap_{i=1}^m \left(\operatorname{span}_{\mathbb{C}} K^R_{i,i+1}\right) \Big/ \operatorname{span}_{\mathbb{C}}\left(\bigcap_{i=1}^m K^R_{i,i+1}\right),$$

these deformations are, for each edge separately, unobstructed. They lift the versal deformation of A_{n-1}. Details may be found in [Al 4].

4 q-parameter families

(4.1) In the previous chapter we have seen that infinitesimal deformations of Y_σ in degree $-R$ are closely related to Minkowski summands of the crosscut $Q(R) = \sigma \cap [R = 1]$. Now, we would like to describe how certain Minkowski decompositions of $Q(R)$ give rise to genuine deformations living over smooth parameter spaces. We begin with negative degrees; let $R \in \sigma^\vee \cap M$. Assume we are given a Minkowski decomposition

$$Q(R) \;=\; Q_0 + Q_1 + \ldots + Q_q$$

meeting the following conditions:

(i) $Q_0 \subseteq [R = 1]$ and $Q_1, \ldots, Q_q \in [R = 0]$ are polyhedra with $Q(R)^\infty$ as their common cone of unbounded directions.

(ii) Each supporting hyperplane t of $Q(R)$ defines faces $F(Q_0, t), \ldots, F(Q_q, t)$ of the indicated polyhedra; obviously, their Minkowski sum equals $F\big(Q(R), t\big)$. With at most one exception (depending on t), these faces should contain lattice vertices, i.e. vertices belonging to N.

Remark: If $\sigma = \mathrm{cone}(Q)$ defines a Gorenstein singularity as in (2.7), then for $R = R^*$ the above conditions become easier: Since $Q(R^*) = Q$ is a lattice polytope, all summands $Q_0, \ldots, Q_q$ have to be lattice polytopes, too.

These data provide a q-parameter deformation of Y_σ in degree $-R$ by the following *construction*: Defining $\tilde N := N \oplus \mathbb{Z}^q$ (and $\tilde M := M \oplus \mathbb{Z}^q$), we embed the Minkowski summands as $(Q_0,\, 0)$, $(Q_1,\, e^1), \ldots, (Q_q,\, e^q)$ into the vector space $\tilde N_{\mathbb{R}}$; $\{e^1, \ldots, e^q\}$ denotes the canonical basis of $\mathbb{Z}^q$. Together with $(Q(R)^\infty,\, 0)$, these polyhedra generate a cone $\tilde\sigma \subseteq \tilde N$ containing σ via $N \hookrightarrow \tilde N$, $a \mapsto (a; \langle a, R\rangle, \ldots, \langle a, R\rangle)$. Actually, σ equals $\tilde\sigma \cap N_{\mathbb{R}}$, and we obtain an inclusion $Y_\sigma \hookrightarrow X_{\tilde\sigma}$ between the associated toric varieties.
On the other hand, $[R, 0] : \tilde N \to \mathbb{Z}$ and $\mathrm{pr}_{\mathbb{Z}^q} : \tilde N \to \mathbb{Z}^q$ induce regular functions $g : X_{\tilde\sigma} \to \mathbb{C}$ and $(g^1, \ldots, g^q) : X_{\tilde\sigma} \to \mathbb{C}^q$, respectively. If restricted on Y_σ, they coincide.

Theorem: ([Al 1]) *The resulting map $(g^1 - g, \ldots, g^q - g) : X_{\tilde\sigma} \to \mathbb{C}^q$ is flat and has $Y_\sigma \hookrightarrow X_{\tilde\sigma}$ as special fiber.*

(4.2) Having both a description of T^1_Y and a recipe to construct certain deformations, one should ask for the Kodaira-Spencer map. Here it comes: Let $R \in \sigma^\vee \cap M$ and $Q(R) = Q_0 + \ldots + Q_q$ be a decomposition satisfying (i) and (ii) mentioned above. Denote by $(\bar a^i)_v$ the vertex of Q_v induced from $\bar a^i \in Q(R)$, i.e. $\bar a^i = (\bar a^i)_0 + \ldots + (\bar a^i)_q$.

Theorem: ([Al 4]) *The Kodaira-Spencer map of the corresponding toric deformation $X_{\tilde\sigma} \to \mathbb{C}^q$ is*

$$\varrho : \mathbb{C}^q \;=\; T_{\mathbb{C}^q, 0} \longrightarrow T^1_Y(-R) \subseteq V_{\mathbb{C}}(R) \oplus W_{\mathbb{C}}(R) / \mathbb{C} \cdot (\underline{1}, \underline{1})$$

sending e^v onto the pair $[Q_v,\, \underline{s}^v] \in V(R) \oplus W(R)$ $(v = 1, \ldots, q)$ with

$$s^v_i := \begin{cases} 0 & \text{if the vertex } (\bar a^i)_v \text{ of } Q_v \text{ belongs to the lattice } N \\ 1 & \text{if } (\bar a^i)_v \text{ is not a lattice point.} \end{cases}$$

Remark: Setting $e^0 := -(e^1 + \ldots + e^q)$, we obtain $\varrho(e^0) = [Q_0,\, \underline{s}^0]$ with $\underline{s}^0$ defined similar to $\underline{s}^v$ in the previous theorem.

(4.3) If $R \notin \sigma^\vee \cap M$, then the previous construction fails. Moreover, in [Al 1] it was shown that deformations such that

- the total space together with the embedding of the special fiber still belong to the toric category and
- the Kodaira-Spencer map points injectively into some homogeneous piece $T^1_Y(-R) \subseteq T^1_Y$

are of the above type, including $R \in \sigma^\vee \cap M$. Nevertheless, in [Al 4] it has been shown that Minkowski decompositions of $Q(R)$ fulfilling (i) and (ii) yield q-parameter deformations in the general case, too. They are obtained from the previous construction for the adapted cone $\sigma' := \sigma \cap [R \geq 0]$ via blowing down. This causes the total space to loose its toric structure.

(4.4) Examples:

(4.4.1) *Cones over Del Pezzo surfaces:* Del Pezzo surfaces arising from $I\!\!P^2$ by blowing up not more than three points are toric. The affine cones over them are Gorenstein. Hence they fit in both patterns (2.6) and (2.7), and the polytopes $P \subseteq M_{I\!\!R}$ and $Q \subseteq N_{I\!\!R}$ are mutually dual.
Let Q be the hexagon from (2.2); it provides the cone over the Del Pezzo surface of degree 6. Identifying the hyperplanes $[R = 0]$ and $[R = 1]$, the two different Minkowski decompositions shown in (2.2) induce families with one and two parameters, respectively. The total spaces are the cones over $I\!\!P^1 \times I\!\!P^1 \times I\!\!P^1$ and $I\!\!P^2 \times I\!\!P^2$.

(4.4.2) *Pinkham's example:* The two-dimensional cone $\sigma := \langle (1,0), (-1,4) \rangle$ yields the cone over the rational normal curve of degree four; see Example (2.4)(3). T^1_Y consists of two one-dimensional parts in the degrees $-[1,1]$ and $-[1,3]$, respectively, and of a two-dimensional part in degree $-[1,2]$. Considering the latter, we obtain the real interval $[-1/2, 1/2] \subseteq I\!\!R^1$ as crosscut $Q(R)$. According to our rules (i) and (ii), it allows two different Minkowski decompositions

$$[-1/2, 1/2] = [-1/2, 0] + [0, 1/2] = \{1/2\} + [-1, 0]$$

inside $I\!\!R^1 \cong [R = 0] \cong [R = 1]$. The associated one-parameter deformations cause two lines in the versal base space of Y_σ; they belong to the two different components of dimension one and three, respectively.

(4.4.3) The A_{n-1}*-singularity* from (1.4)(2) is given by the cone $\sigma = \langle (1,0), (1,n) \rangle$. The vector space T^1_Y is spread in the degrees $-[k,0]$ with $2 \leq k \leq n$, and for each k the previous method yields the perturbation $(z^n - xy) + t^{(k)} z^{n-k}$.
On the other hand, it is even possible to get all these families with one strike: Despite T^1_Y vanishing in this degree, we consider $R^* = [1,0]$. The cross cut $Q(R^*)$ is an integer interval of length n. It splits into a Minkowski sum of n unit intervals causing an $(n-1)$-parameter family via the previous construction. However, since $T^1_Y(-R^*) = 0$, the Kodaira-Spencer map vanishes. The reason is that we have not obtained the versal deformation of A_{n-1}, but an $n!$-fold cover of it.

5 Versal deformations

(5.1) The *two-dimensional toric singularities*, i.e. the two-dimensional cyclic quotient singularities from Example (2.4)(3), have been investigated in [Ri], [KoSh], [Arn], [Ch], and [St 1]. Using Kollár/ Shepherd-Barron's fundamental result about the one-to-one correspondence between so-called P-resolutions and components of the versal base space, Christophersen and Stevens have been able to give a detailed description of these components:

For a given two-dimensional cone σ let $a_2, \ldots, a_{w-1}$ be the positive integers introduced in (3.4). Since the A_{n-1}-singularities behave a little bit differently, assume $w \geq 4$ for simplicity. We define

$K(\sigma)$ as the finite set

$$K(\sigma) := \Big\{ (k_2, \ldots, k_{w-1}) \in \mathbb{Z}^{w-2} \,\Big|\, 0 \le k_v \le a_v, \text{ and the continued fraction } k_2 - \frac{1}{k_3 - \ldots} \text{ is well defined and yields } 0 \Big\}.$$

Theorem: ([Ch], [St 1]) *The set $K(\sigma)$ parametrizes the irreducible components of the reduced, versal base space of Y_σ. Moreover, the component assigned to $\underline{k}$ is smooth and has dimension $\sum_{v=2}^{w-1}(a_v - 2k_v + 3) - 4$. Via the Kodaira-Spencer map it sits in the degrees $p \cdot r^v$ with $2 \le v \le w-1$ and $1 \le p \le a_v - k_v$ and, additionally, with the second dimension in some of the r^v with $3 \le v \le w-2$.*
The element $(1, 2, 2, \ldots, 2, 1) \in K(\sigma)$ represents the Artin component.

In [Al 5] there is a straightforward description of how to find, to a given chain representing 0, the P-resolution as an explicit subdivision of the cone σ. The paper makes use of the fact that the singularities allowed on P-resolutions, the so-called T-singularities (cf. [KoSh]), may be easily described in combinatorial terms. They are the two-dimensional toric singularities arising from cones over intervals with integer length, i.e. over shifted lattice intervals.

(5.2) In higher dimensions, we have no correspondence relating deformation theory to certain partial resolutions which might be easier to describe. However, in [Al 3] the approach via Minkowski decompositions was used to obtain the versal deformation, even with its non-reduced structure, for isolated, three-dimensional, toric Gorenstein singularities. The concept works similarly in more general cases, i.e. for arbitrary dimensions and without the Gorenstein assumption, as long as the following two conditions are satisfied:

- Y_σ is smooth in codimension two (excluding the two-dimensional cyclic quotients) and
- T^1_Y is concentrated in one single multidegree $-R \in M$.

It is Theorem (3.8) telling us that the isolated, three-dimensional Gorenstein singularities fulfill these assumptions (with $R = R^*$). For the sake of simplicity, we will use the rest of the paper to focus on this special case.

(5.3) Let $Q \subseteq \mathbb{R}^2$ be a lattice polygon such that the edges do not contain any interior lattice points. In particular, they are given by clockwise oriented primitive vectors $d^1, \ldots, d^m \in \mathbb{Z}^2$. The vector space $V(Q) \subseteq \mathbb{R}^m$ is given by the single equation $\sum_i t_i\, d^i = 0$; hence it is $(m-2)$-dimensional. Recalling from (2.3) that elements $\underline{t} \in C(Q) = V(Q) \cap \mathbb{R}^m_{\ge 0}$ represent Minkowski summands $Q_{\underline{t}}$ of $\mathbb{R}_{\ge 0} \cdot Q$, we may define the tautological cone as

$$\tilde{C}(Q) := \big\{ (a, \underline{t}) \,\big|\, \underline{t} \in C(Q);\ a \in Q_{\underline{t}} \big\} \subseteq \mathbb{R}^2 \times V(Q) \subseteq \mathbb{R}^2 \times \mathbb{R}^m .$$

$\tilde{C}(Q)$ is, as $C(Q)$, a rational, polyhedral cone. By definition, there is a natural projection $\pi : \tilde{C}(Q) \twoheadrightarrow C(Q)$ yielding $\sigma := \operatorname{cone}(Q)$ as the pull back of $\mathbb{R}_{\ge 0} \cdot \underline{1} \subseteq C(Q)$. We obtain the following fiber product diagram of rational polyhedral cones:

$$\begin{array}{ccccc} \tilde{C}(Q) & \xrightarrow{\ \pi\ } & C(Q) & \to & \mathbb{R}^m_{\ge 0} \\ \uparrow & & \uparrow & \nearrow_{\Delta} & \\ \sigma & \longrightarrow & \mathbb{R}_{\ge 0} & & \end{array}$$

with Δ denoting the diagonal embedding. Applying the functor of building toric varieties from cones yields affine varieties $X_{\tilde{C}(Q)}$ and $S_{C(Q)}$ fitting into the fiber product diagram

$$\begin{array}{ccccccc} X_{\tilde{C}(Q)} & \xrightarrow{\pi} & S_{C(Q)} & \longrightarrow & \mathbb{C}^m & \xrightarrow{\ell} & \mathbb{C}^m/\mathbb{C}\cdot\underline{1} \\ \uparrow & & \uparrow & \nearrow_{\Delta} & & & \uparrow \\ Y_\sigma & \longrightarrow & \mathbb{C} & & \longrightarrow & & \{0\} \end{array}$$

with closed embeddings in the vertical directions. The map $S_{C(Q)} \to \mathbb{C}^m$ is finite, and, for the sake of simplicity, we will pretend in the following that it is a closed embedding, too. Denote by $\bar{\mathcal{M}} \subseteq \mathbb{C}^m/\mathbb{C}\cdot\underline{1}$ the largest closed subscheme such that $\mathcal{M} := \ell^{-1}(\bar{\mathcal{M}}) \subseteq \mathbb{C}^m$ is also contained in $S_{C(Q)}$. Theorem (3.8) says that both $\bar{\mathcal{M}}$ and T^1_Y sit in the same space $\mathbb{C}^m/\mathbb{C}\cdot\underline{1}$.

Theorem: ([Al 3]) *Restricting the map $X_{\tilde{C}(Q)} \to \mathbb{C}^m/\mathbb{C}\cdot\underline{1}$ onto $\bar{\mathcal{M}} \subseteq \mathbb{C}^m/\mathbb{C}\cdot\underline{1}$, the outer frame of the above commutative diagram yields the versal deformation of Y_σ.*

(5.4) What does $\bar{\mathcal{M}}$ look like? To answer this question, we define for each integer $k \geq 1$ the (vector-valued) polynomial

$$g_k(t_1,\ldots,t_m) := \sum_{i=1}^m t_i^k\, d^i$$

generating an ideal

$$\mathcal{J} := (g_k(\underline{t}) \mid k \geq 1) \subseteq \mathbb{C}[t_1,\ldots,t_m]\,.$$

Since $\sum_{i=1}^m d^i = 0$, this ideal may alternatively be generated by $g_k(t_1 - t_1, \ldots, t_m - t_1)$ with $k \geq 1$, i.e. $\mathcal{J}$ originally comes from the subring $\mathbb{C}[t_i - t_j \mid 1 \leq i,j \leq m]$.

Theorem: ([Al 3])

(1) $\mathcal{M}$ and $\bar{\mathcal{M}}$ are the affine schemes associated to the ideals $\mathcal{J} \subseteq \mathbb{C}[\underline{t}]$ and $\mathcal{J} \cap \mathbb{C}[t_i - t_j] \subseteq \mathbb{C}[t_i - t_j]$, respectively.

(2) If Q is contained in two different strips defined by pairs of parallel lines of lattice-distance $\leq k_0$ each, then the polynomials g_k with $k > k_0$ are not necessary for generating $\mathcal{J}$ or $\mathcal{J} \cap \mathbb{C}[t_i - t_j]$.

Since $g_1(\underline{t})$ are the linear forms defining $V(Q) \subseteq \mathbb{R}^m$, we see that $\mathcal{M}$ is not only contained in $\mathbb{C}^m$, but also in $V_{\mathbb{C}}(Q)$.

Example: Let Q be the hexagon from (2.2). Starting with $d^1 := \overline{(0,0)(1,0)}$, the anticlockwise oriented edges equal

$$d^1 = (1,0);\;\; d^2 = (1,1);\;\; d^3 = (0,1);\;\; d^4 = (-1,0);\;\; d^5 = (-1,-1);\;\; d^6 = (0,-1).$$

Obviously, Q is contained in at least three different strips of thickness two. Hence, $\mathcal{J}$ is generated in degree ≤ 2. We obtain

$$\mathcal{J} = \big(t_1 + t_2 - t_4 - t_5,\;\; t_2 + t_3 - t_5 - t_6,\;\; t_1^2 + t_2^2 - t_4^2 - t_5^2,\;\; t_2^2 + t_3^2 - t_5^2 - t_6^2\big).$$

(5.5) Finally, we would like to mention the structure of the underlying reduced spaces of $\mathcal{M}$ and $\bar{\mathcal{M}}$. Let $Q = R_0 + \ldots + R_q$ be a decomposition of Q into a Minkowski sum of $q+1$ lattice polytopes. Then, denoting with $[R_v] \in C(Q)$ the point representing the summand R_v, the m-tuples $[R_0], \ldots, [R_q] \in \mathbb{R}^m$ have only 0 and 1 as entries and sum up to $(1,\ldots,1)$. In particular, the $(q+1)$-plane $\mathbb{C}\cdot[R_0] + \ldots + \mathbb{C}\cdot[R_q] \subseteq \mathbb{C}^m$ (or its q-dimensional image via ℓ) is contained in $\mathcal{M}$ (in $\bar{\mathcal{M}}$, respectively). It is not difficult to see that there are no other points, i.e.

$\mathcal{M}_{\text{red}}$ and $\bar{\mathcal{M}}_{\text{red}}$ equal the union of those flats corresponding to maximal Minkowski decompositions of Q into lattice polytopes.

Example: Considering the hexagon again, the linear equations of $\mathcal{M}$ allow the substitution $t := t_1$, $s_1 := t_1 - t_3$, $s_2 := t_4 - t_2$, and $s_3 := t_1 - t_4$. The two quadratic equations transform into $s_1\, s_3 = s_2\, s_3 = 0$. Since $\mathcal{M} = \ell^{-1}(\bar{\mathcal{M}})$, the equations do not contain t; it is still a variable for $\mathcal{M}$, but not for $\bar{\mathcal{M}}$.
We see that $\bar{\mathcal{M}}$ is the union of a line and a two-plane. The corresponding one- and two-parameter families are exactly those obtained in (4.4.1) corresponding to the Minkowski decompositions drawn in (2.2).

(5.6) In the same manner as we did with the hexagon, we may investigate the lattice rectangle $Q = \text{conv}\big\{(0,0);\, (2,1);\, (2,2);\, (1,2)\big\}$ corresponding to the cone over the Del Pezzo surface of degree eight. Here the versal base space equals the fat point Spec $\mathbb{C}[\varepsilon]/\varepsilon^2$.

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