# PARABOLIC OBSTACLE-TYPE PROBLEMS

*Daria Apushkinskaya*
**Saarland University**

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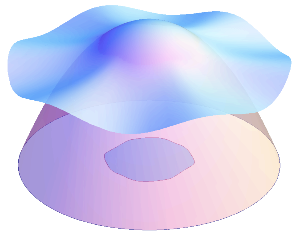
$$Hu = \Delta u - u_t = \chi_\Omega \text{ in } D$$

$$u = |\nabla u| = 0 \text{ in } \mathrm{D} - \Omega$$

$$\| u \|_{\infty, D} \leq M$$

Obstacle Problem:

$u \geq 0$



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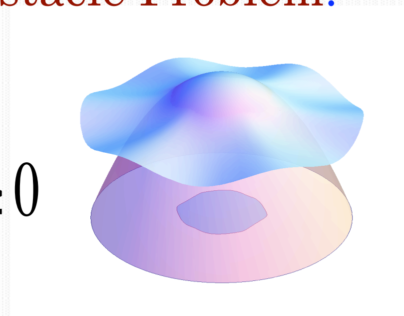
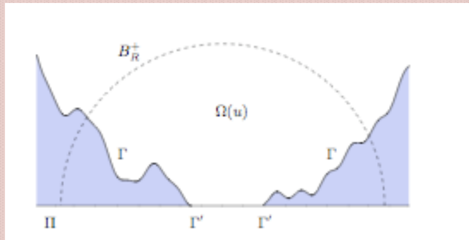


Contact Case

+

$u = 0$ on $\partial\mathrm{D}$



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## Obstacle Problem:

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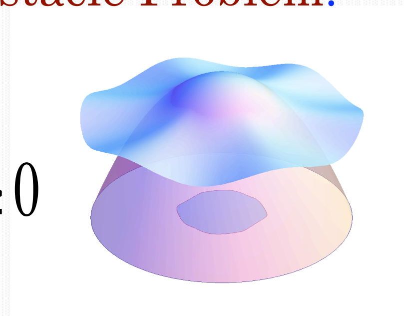
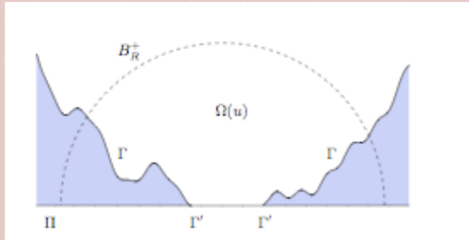


## Contact Case

+

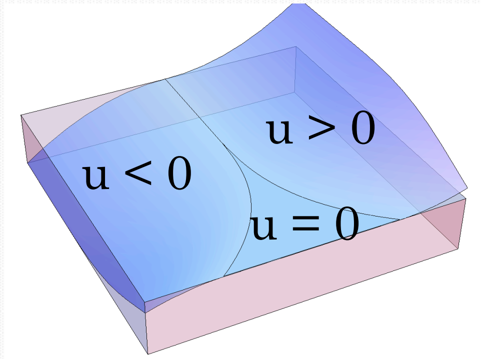
$$u = 0 \text{ on } \partial D$$




$$Hu = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}} \text{ in } D$$

$$\lambda_\pm > 0, \lambda_\pm \in \text{Lip}$$

## 2-phase Problem:



1. **How regular is** $u$ **?**

2. **How regular is** $\partial\Omega$ **?**

# 1-PHASE PROBLEM

## Optimal regularity of a solution

$$u \in C_x^{1,1} \cap C_t^{0,1}$$

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- **Remark.**

Classical PDE-Theory gives that $|Hu|$ is bounded

$$u \in C_x^{1,\alpha} \cap C_t^{0,\alpha}$$

We want $\alpha = 1$

# WHY $\alpha = 1$ IS DESIRABLE?

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$$u_r(x,t) = \frac{u(rx, r^2 t)}{r^2}$$

$$Hu_r = \chi_{\Omega_r}$$

If $\alpha = 1$ then $u_r$ is bounded

⬇

We can go for a limit as $r \to 0$

# WHY $\alpha = 1$ IS DESIRABLE?

$$u_r(x,t) = \frac{u(rx, r^2 t)}{r^2}$$

$$Hu_r = \chi_{\Omega_r}$$

$$u_r \to u_0$$

$$\Omega_r \to \Omega_0$$

If $\alpha = 1$ then $u_r$ is bounded

⬇

We can go for a limit as $r \to 0$

Moreover,

$$Hu_0 = \chi_{\Omega_0} \text{ in } \mathbb{R}^{n+1}_+$$

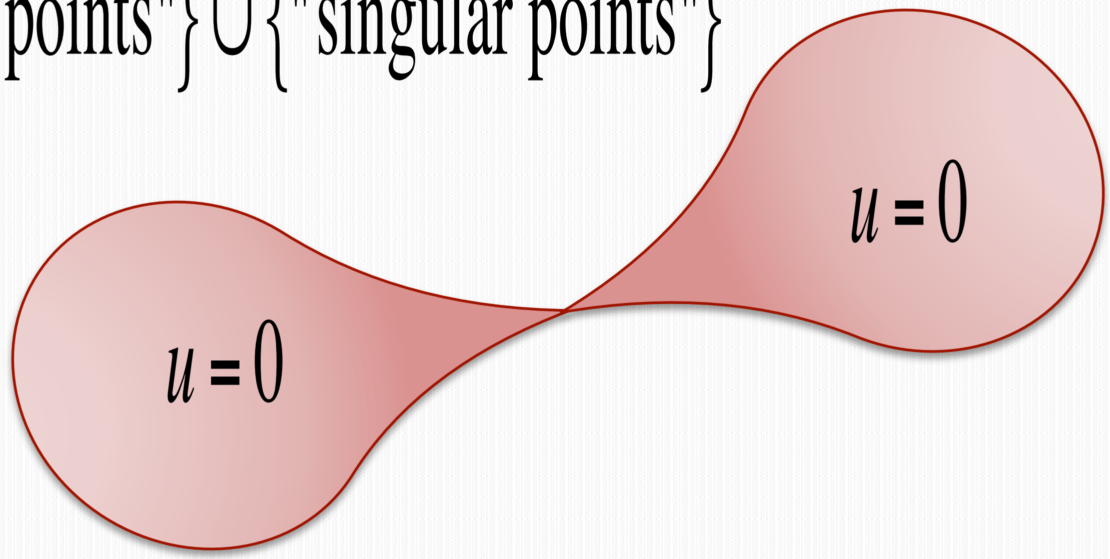
$$u_0 = |Du_0| = 0 \text{ in } \mathbb{R}^{n+1}_+ - \Omega_0$$

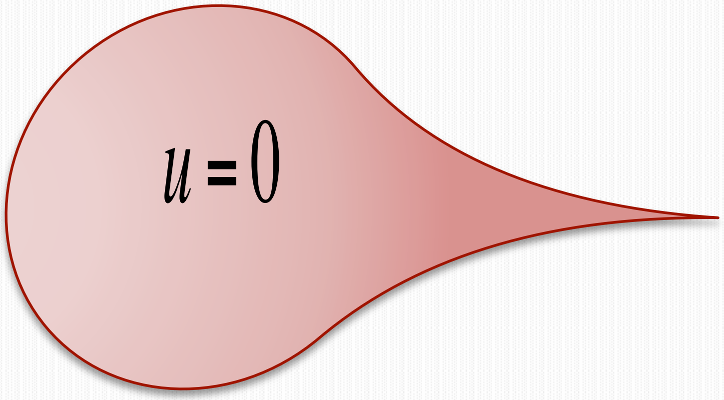
$$|u_0| \leq C\left(1 + |x|^2 + |t|\right)$$

# 1-PHASE PROBLEM

## Regularity of $\partial\Omega$

$$\partial\Omega = \{\text{"regular points"}\} \cup \{\text{"singular points"}\}$$



# 1-PHASE PROBLEM

Regularity of $\partial\Omega$

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Regularity of $\partial\Omega$

"thickness condition"

$$\begin{cases} Hu = \chi_\Omega \text{ in Q} \\ |\mathrm{u}| = |\nabla u| = 0 \; in \; Q - \Omega \end{cases}$$

**No contact points !!!**

$$\Rightarrow \quad \partial\Omega \cap R \in C^{1,\alpha}$$

# 1-PHASE PROBLEM

Regularity of $\partial\Omega$

"thickness condition"

$$\begin{cases} Hu = \chi_\Omega \text{ in } Q \\ |u| = |\nabla u| = 0 \text{ in } Q - \Omega \end{cases}$$

**No contact points !!!**

$$\partial\Omega \cap R \in C^{1,\alpha}$$

$$\begin{cases} Hu = \chi_\Omega \text{ in } Q^+ \\ u = |\nabla u| = 0 \text{ in } Q^+ - \Omega \\ u = 0 \text{ on } \{x_1 = 0\} \end{cases}$$

**Contact points may exists!!!**

$\partial\Omega \in \mathrm{Lip}$

$\partial\Omega \in C^{1,\alpha}$ at interior points only!!!

# 2-PHASE PROBLEM

## Optimal regularity of a solution

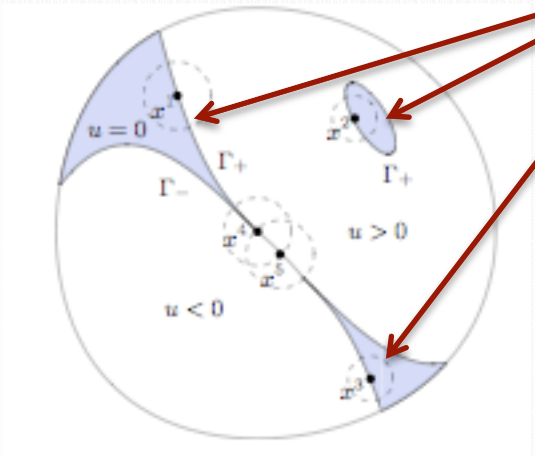
$\nabla u$ is Lipschitz continuous w.r.t. space variables

$\nabla u$ is Holder continuous with exponent 1/2 w.r.t. time variable

# 2-PHASE PROBLEM

Regularity of $\partial\Omega$

•**1-phase points**



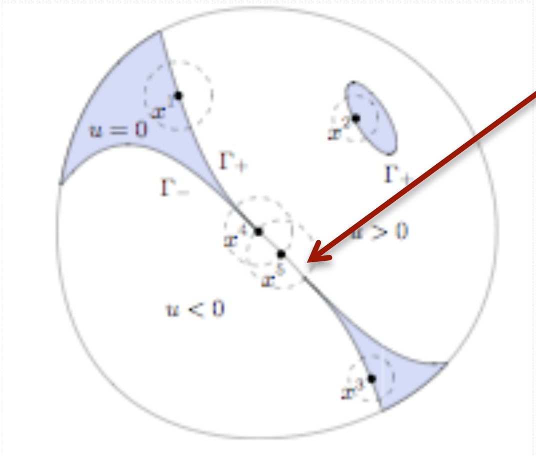

$$\partial\{u>0\}\cap\{|\nabla u|=0\}$$
$$\partial\{u<0\}\cap\{|\nabla u|=0\}$$

$$\partial\Omega\in C^{1,\alpha}$$

# 2-PHASE PROBLEM

Regularity of $\partial\Omega$

•**2-phase nonbranch points**




$$\{u=0\}\cap\{|\nabla u|\neq 0\}$$

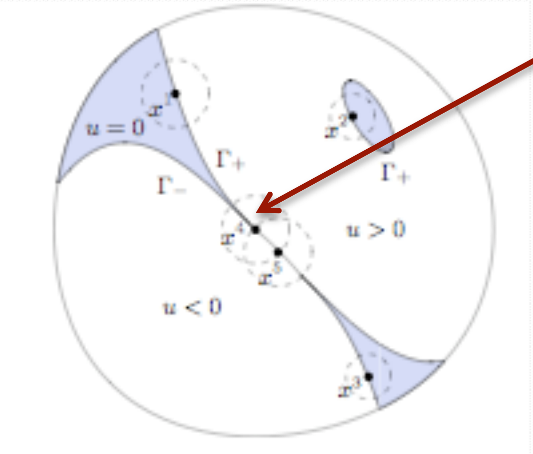
**Implicit function Theorem**

$$\partial\Omega\in C^1$$

# 2-PHASE PROBLEM

Regularity of $\partial\Omega$

•**2-phase branch points**




near

$$\partial\{u>0\}\cap\partial\{u<0\}\cap\{|\nabla u|=0\}$$

$\partial\Omega$ is a union of two Lip-graphs,
which are continuously differentiable w.r.t space variables

| FBP | Regularity of $u$ | Regularity of the free boundary |
|---|---|---|
| $\begin{cases} Hu = \chi_\Omega \text{ in } Q \\ \lvert u\rvert = \lvert\nabla u\rvert = 0 \ in \ \Omega^c \end{cases}$ <br> **No contact points** | $u \in C_x^{1,1} \cap C_t^{0,1}$ <br> **Optimal !** | $\partial\Omega = \{\text{"regular"}\} \cup \{\text{"singular"}\}$ <br> "thickness condition" <br> $\Downarrow$ <br> $\partial\Omega \cap R \in C^{1,\alpha}$ |
| $\begin{cases} Hu = \chi_\Omega \text{ in } Q^+ \\ u = \lvert\nabla u\rvert = 0 \text{ in } \Omega^c \\ u = 0 \text{ on } \{x_1 = 0\} \end{cases}$ <br> **Contact points may exist** | $u \in C_x^{1,1} \cap C_t^{0,1}$ <br> **Optimal !** | $\partial\Omega \in \text{Lip}$ <br> $\partial\Omega \in C^{1,\alpha}$ at interior points only!!! |
| $Hu = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}}$ | $\nabla u \in C_x^{0,1} \cap C_t^{0,1/2}$ | $\partial\{u>0\} \cap \{\nabla u = 0\} \in C^{1,\alpha}$ <br> $\{u=0\} \cap \{\nabla u \neq 0\} \in C^1$ <br> $\partial\{u>0\} \cap \partial\{u<0\} \cap \{\nabla u = 0\} \in \text{Lip}$ |

# References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3. H.Shahgholian, N.Uraltseva, G.Weiss, *Adv. Math. (2009)*

# THANK YOU !!!