Slow-fast asymptotics for delay differential equations

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1. Delay equation?
2. $2\tau$ – periodic square-wave oscillations
3. $\tau$ – periodic square-wave oscillations
1. Delay Equation?

\[
\frac{dy(t)}{dt} = -ky(t) \rightarrow y = A \exp(-kt)
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\frac{dy(t)}{dt} = -ky(t) \rightarrow y = A \exp(-kt)
\]

\[
\frac{dy(t)}{dt} = -ky(t - \tau) \rightarrow y = A \sin(\omega t)
\]

>> if \( \tau = \frac{\pi}{2k} \)
2. $2\tau -$ periodic square-waves

$$\varepsilon x' = -x + f(x(t-1))$$

$$\varepsilon = \tau^{-1}$$

$$f = -ax + x^3$$

$$a = 1.2, \varepsilon = 0.02$$
2. $2\tau$ – periodic square-waves

\[ \varepsilon x' = -x + f(x(t-1)) \]
\[ \varepsilon = \tau^{-1} \]
\[ f = -ax + x^3 \]
\[ a = 1.2, \varepsilon = 0.02 \]

\[ \varepsilon \rightarrow 0 \]
\[ -x + f(x(t-1)) = 0 \]
\[ x_{n+1} = f(x_n) \]

Period 2 fixed points = square-wave plateaus

Experiments using an optoelectronic oscillator

Larger et al. JOSA B 18, 1063 (2001)

\[ \varepsilon x' = -x + f(x(t-1)) \]

\[ f = \beta \left[ 1 + \frac{1}{2} \cos(x(t-1)) \right] \]
Experiments using an optoelectronic oscillator
Larger et al. JOSA B 18, 1063 (2001)

\[ \varepsilon x' = -x + f(x(t-1)) \]

\[ f(x) = \beta \left[ 1 + \frac{1}{2} \cos(x) \right] \]

\[ -x + f(x(t-1)) = 0 \]

\[ x_{n+1} = f(x_n) \]

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3. $\tau$ – periodic square – waves

Experiments using an optoelectronic oscillator

Feedback gain
Model equations

\[\mu x' = -x - \theta^{-1} \int_{0}^{t} x(t) dt' + \beta \left[ \cos^2(x(t - \tau) + \Phi) - \cos^2(\Phi) \right] \]

\[s \equiv t / \tau, \quad y \equiv \tau^{-1} \int_{0}^{t} x(t') dt'\]

\[y' = x\]

\[\varepsilon x' = -x - \delta y + \beta \left[ \cos^2(x(s - 1) + \Phi) - \cos^2(\Phi) \right] \]

\[\varepsilon \equiv \mu \tau^{-1} = 10^{-3}, \quad \delta \equiv \tau \theta^{-1} = 8 \times 10^{-3}\]
Numerical Simulations

\[ y' = x \]

\[ \varepsilon x' = -x - \delta y + \beta \left[ \cos^2 (x(s - 1) + \Phi) - \cos^2 (\Phi) \right] \]

\[ \varepsilon = 10^{-3} \quad \delta = 8.43 \times 10^{-3} \]

\[ \Phi = -\pi / 4 + 0.1 \]

\[ \beta = 1.2 \]

\[ x = -1 \quad (-1 \leq s < -1/3) \]

\[ x = 1 \quad (-1/3 \leq s < 0) \]

\[ y(0) = 0 \]
Asymptotics

\[ y' = x \]
\[ \varepsilon x' = -x - \delta y + \beta \left[ \cos^2(x(s - 1) + \Phi) - \cos^2(\Phi) \right] \]

Part 1. Seek a \( T \)-periodic solution:

\[ x(s - T) = x(s) \]

where \( T = 1 + \varepsilon \alpha(\varepsilon) \)
Asymptotics

\[ y' = x \]
\[ \varepsilon x' = -x - \delta y + \beta \left[ \cos^2(x(s - 1) + \Phi) - \cos^2(\Phi) \right] \]

Part 1. Seek a \( T \)-periodic solution

\[ x(s - T) = x(s) \]

where \( T = 1 + \varepsilon \alpha(\varepsilon) \)

leading \( \varepsilon = 0 \)

\[ y' = x \]
\[ 0 = -x - \delta y + \beta \left[ \cos^2(x + \Phi) - \cos^2(\Phi) \right] \]

\( \varepsilon = 0 \) → ignore fast transitions layers
\[ y' = x \]
\[ 0 = -x - \delta y + \beta \left[ \cos^2 (x + \Phi) - \cos^2 (\Phi) \right] \]

Part 2. Try

\[ y = \delta^{-1} y_0 (s) + y_1 (s) + \ldots \]
\[ x = x_{0,j} (s) + \delta x_{1,j} (s) + \ldots \]
\[ j = 1 \ (0 < s < s_0), \ j = 2 \ (s_0 < s < 1) \]
\[ y' = x \]
\[ 0 = -x - \delta y + \beta \left[ \cos^2(x + \Phi) - \cos^2(\Phi) \right] \]

Part 2. Try

\[ y = \delta^{-1} y_0(s) + y_1(s) + \ldots \]
\[ x = x_{0,j}(s) + \delta x_{1,j}(s) + \ldots \]
\[ j = 1 \quad (0 < s < s_0), \quad j = 2 \quad (s_0 < s < 1) \]

leading \( \delta = 0 \)

\[ y_0' = 0 \]
\[ 0 = -x_{0,j} - y_0 + \beta \left[ \cos^2(x_{0,j} + \Phi) - \cos^2(\Phi) \right] \]

\[ y_0' = 0 \rightarrow y_0 = \text{cst} ? \]
O(\delta)

\[ y_1' = x_{0j} \]

\[ 0 = -x_{1j} - y_1 - 2\beta \sin(2x_{0j} + 2\Phi)x_{1j} \]

Solve for \( j = 1, 2 \)

Periodicity conditions

continuity condition at \( s = s_0 \)
Bifurcation diagrams

Numerical solution       Asymptotic analysis

\[ \beta = 1.2 \]
Bifurcation point near $\beta = 1$

$\beta = 1.01$

- Not branching from Hopf bifurcations
- Branching from a SN of limit-cycles?
Conclusions

1. Stable $\tau$ – periodic asymmetric square-waves not possible for first order scalar DDEs
   \[ \varepsilon x' = -x + f(x(t-1)) \]

2. Possible for second order scalar DDEs
   \[ y' = x \]
   \[ \varepsilon x' = -x - \delta y + f(s - 1) \]

Note:
1. Other periodic solutions coexist with the asymmetric square waves
2. No connection with the Hopf bifurcations from the zero solution
3. Isolated branch of periodic solutions
4. We have ignored the fast transition