Slow-fast asymptotics for delay differential equations

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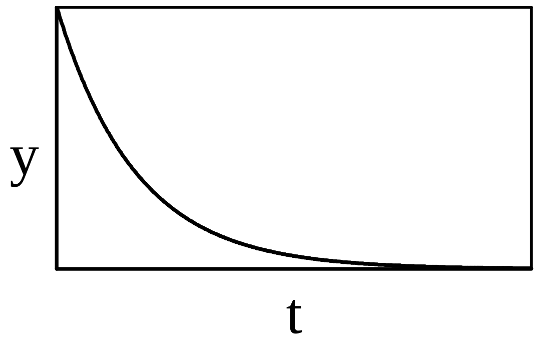
PLAN

1. Delay equation?
2. 2τ – periodic square-wave oscillations
3. τ – periodic square-wave oscillations

1. Delay Equation?

$$\frac{dy(t)}{dt} = -ky(t) \rightarrow \ y = A\exp(-kt)$$



1. Delay Equation?

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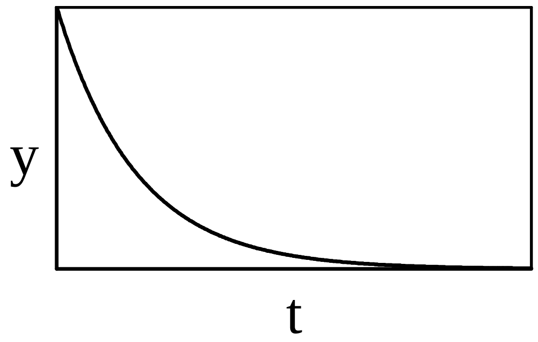
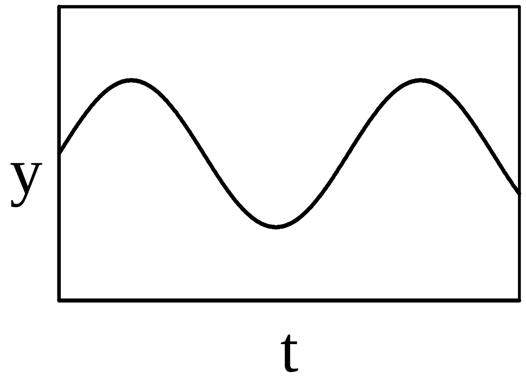


$$\frac{dy(t)}{dt} = -ky(t-\tau) \rightarrow \mathrm{y} = A\sin(\omega \mathrm{t})$$

$$\gg \;\; \text{if} \;\; \tau = \frac{\pi}{2\mathrm{k}}$$



2009 – 2011

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2. A. Balachandran, T. Kamár-Nagy, D.E. Gilsinn, Eds. **"Delay Differential Equations, Recent Advances and New Directions",** Springer (2009)
3. F.M. Atay, Ed. **"Complex Time-Delay Systems"**, Springer (2010)
4. Theme Issue **"Delay effects in brain dynamics"** compiled by G. Stepan, Phil. Trans. Roy. Soc A **367,** 1059 (2009)
5. Theme Issue **"Delayed complex systems"** compiled by W. Just, A. Pelster, M. Schanz and E. Schöll, Phil. Trans. Roy. Soc A **368,** 303 (2010)
6. Special Issue on **Time Delay Systems,** T. Kalmár-Nagy, N. Olgac, and G. Stépán, Eds., J. Vibration & Control**,** June/July 16 (2010)
7. Hal Smith, **An Introduction to Delay Differential Equations with Applications to the Life Sciences,** Springer (2010)
8. M. Lakshmanan and D.V. Senthilkumar, **Dynamics of Nonlinear Time-Delay Systems**, Springer Series in Synergetics (2011)
9. T. Insperger and G. Stépán, **Semi-discretization for time-delay systems – Engineering applications,** Springer (2011)
10. **Complex Systems, Fractionality, Time-delay and Synchronization,** A.C.J. Luo and J.-Q. Sun (Eds.) Springer (2012)

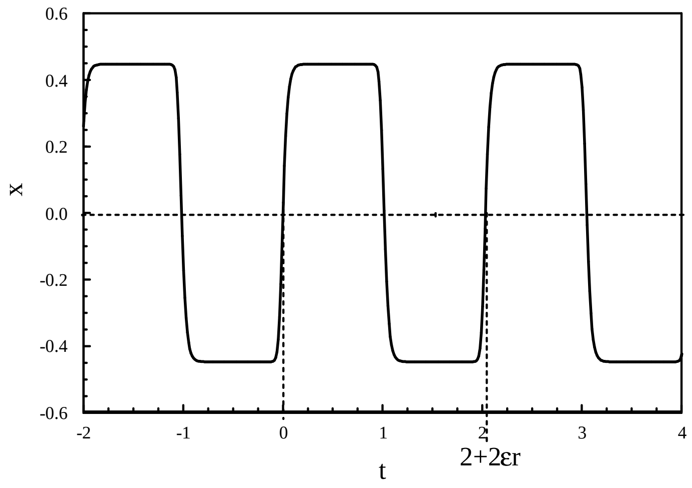
2. 2τ – periodic square-waves

$$\varepsilon x' = -x + f(x(t-1))$$

$$\varepsilon = \tau^{-1}$$

$$f = -ax + x^3$$

$$a = 1.2,\ \varepsilon = 0.02$$



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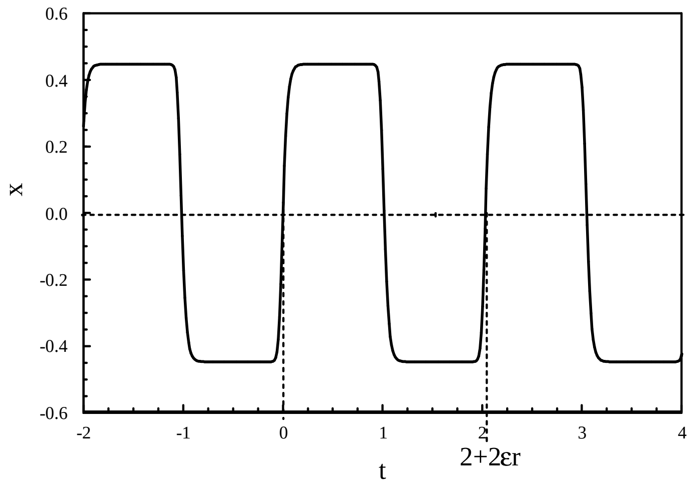


$$\varepsilon \to 0$$

$$-x + f(x(t-1)) = 0$$

$$x_{n+1} = f(x_n)$$

Period 2 fixed points = square-wave plateaus

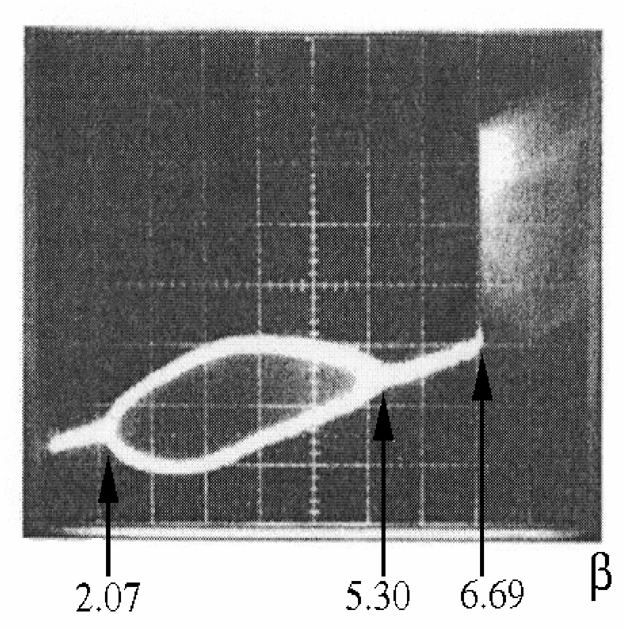
1983 - 1996: S.N. Chow, J. Mallet-Paret, R.D. Nussbaum, J.K. Hale and W. Huang

Experiments using an optoelectronic oscillator

Larger et al. JOSA B 18, 1063 (2001)

$$\varepsilon x' = -x + f(x(t-1))$$

$$f = \beta\left[1+\frac{1}{2}\cos(x(t-1))\right]$$



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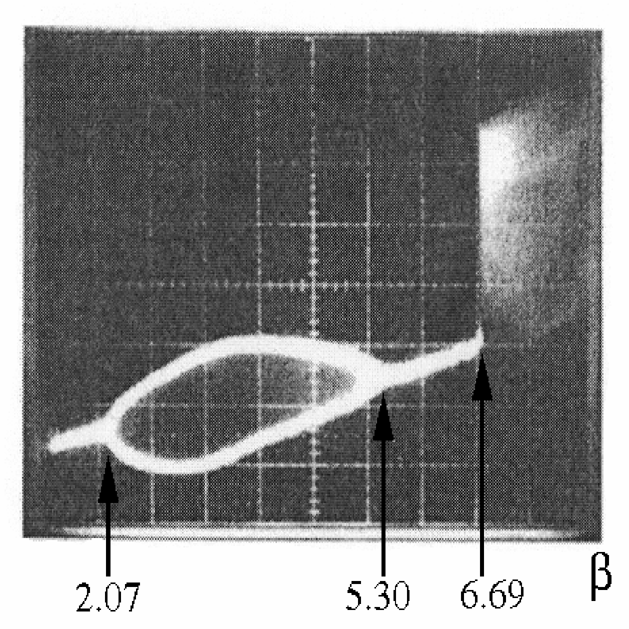
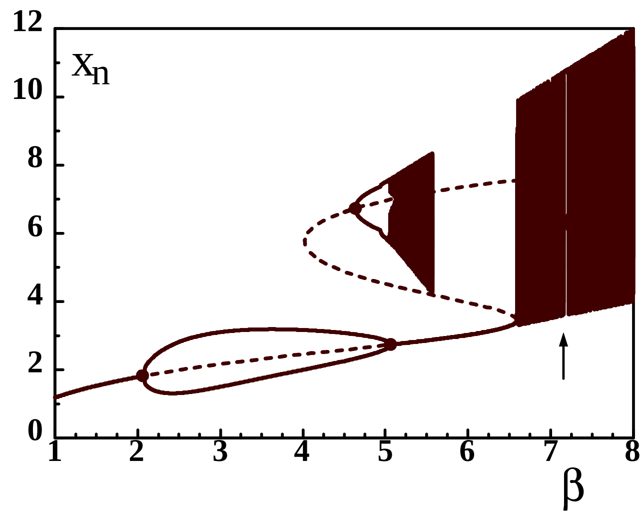


$$-x + f(x(t-1)) = 0$$

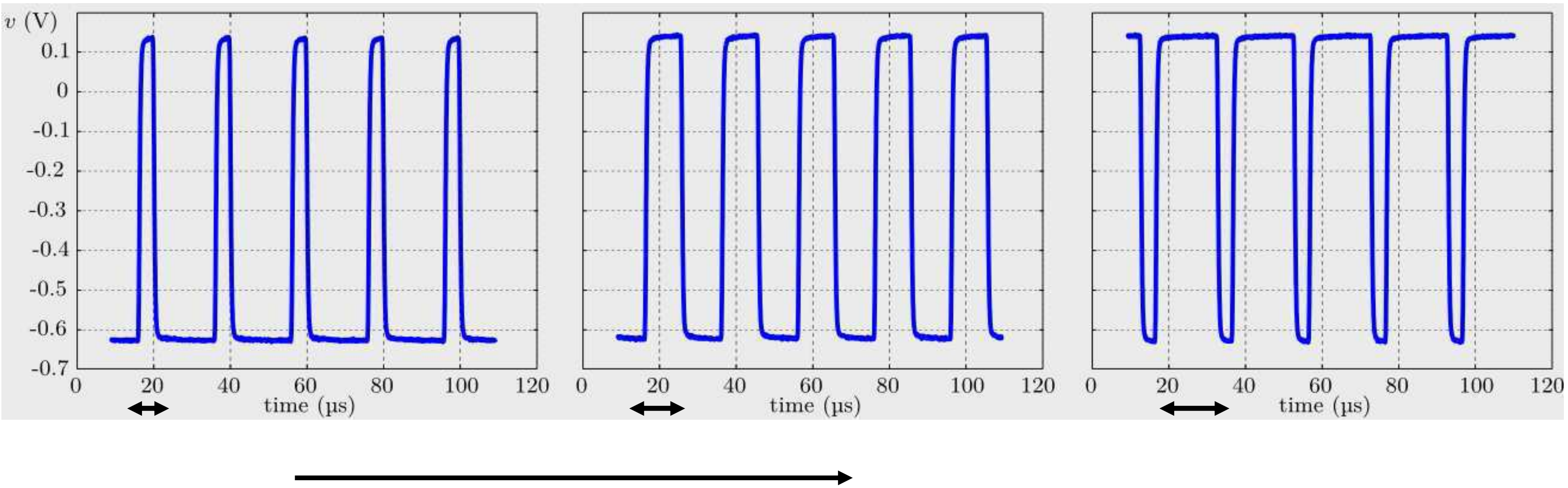
$$x_{n+1} = f(x_n)$$

	Hopf1	Hopf2	Chaos
map	2.08	5.04	6.59
experiment	2.07	5.30	6.69



3. τ – periodic square – waves

Experiments using an optoelectronic oscillator



Feedback gain

Model equations

Peil et al, Phys. Rev. E 79, 026208 (2009)

$$\mu x' = -x - \theta^{-1}\int_0^t x(t)dt' + \beta\left[\cos^2(x(t-\tau)+\Phi) - \cos^2(\Phi)\right]$$

$$s \equiv t/\tau, \quad y \equiv \tau^{-1}\int_0^t x(t')dt'$$

$$y' = x$$

$$\varepsilon x' = -x - \delta y + \beta\left[\cos^2(x(s-1)+\Phi) - \cos^2(\Phi)\right]$$

$$\varepsilon \equiv \mu\tau^{-1} = 10^{-3} \qquad \delta \equiv \tau\theta^{-1} = 8\times 10^{-3}$$

Numerical Simulations

$$y' = x$$

$$\varepsilon x' = -x - \delta y + \beta\left[\cos^2(x(s-1)+\Phi) - \cos^2(\Phi)\right]$$

$\varepsilon = 10^{-3}$ $\delta = 8.43 \times 10^{-3}$

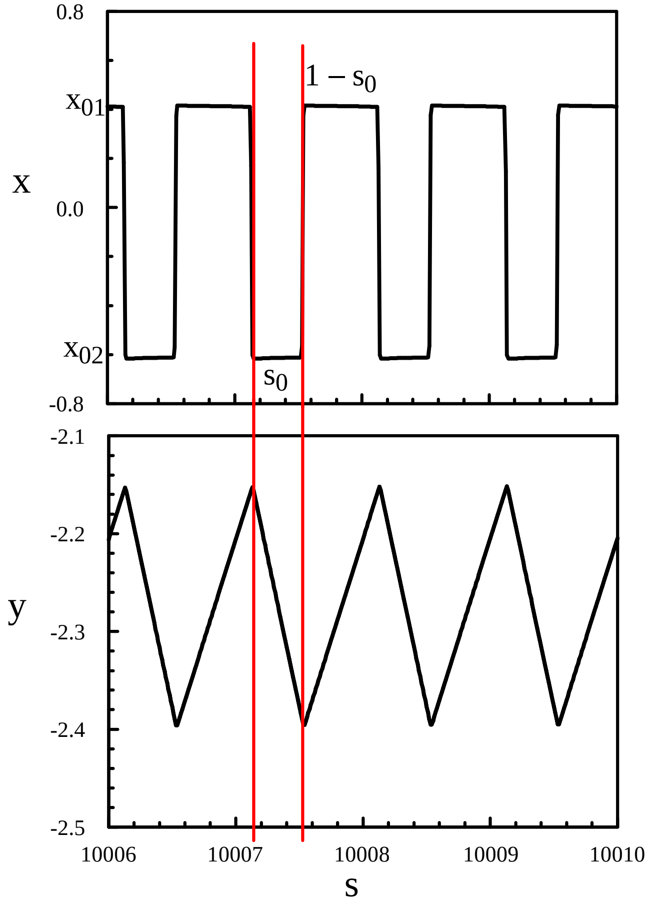
$\Phi = -\pi/4 + 0.1$

$\beta = 1.2$

$x = -1 \quad (-1 \le s < -1/3)$

$x = 1 \quad (-1/3 \le s < 0)$

$y(0) = 0$



Asymptotics

$$y' = x$$

$$\varepsilon x' = -x - \delta y + \beta\left[\cos^2(x(s-1)+\Phi) - \cos^2(\Phi)\right]$$

Part 1. Seek a T-periodic solution:

$$x(s-T) = x(s)$$

where $T = 1 + \varepsilon\alpha(\varepsilon)$

Asymptotics

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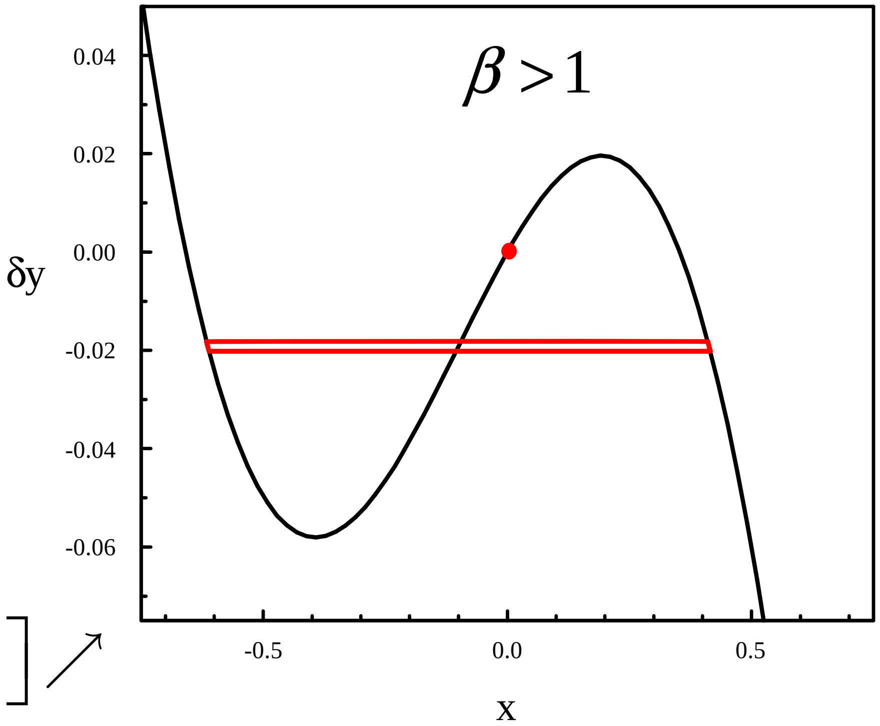
where $T = 1 + \varepsilon\alpha(\varepsilon)$

leading $\varepsilon = 0$



$$y' = x$$

$$0 = -x - \delta y + \beta\left[\cos^2(x+\Phi) - \cos^2(\Phi)\right] \nearrow$$

$\varepsilon = 0 \;\; \rightarrow$ ignore fast transitions layers

$$y' = x$$

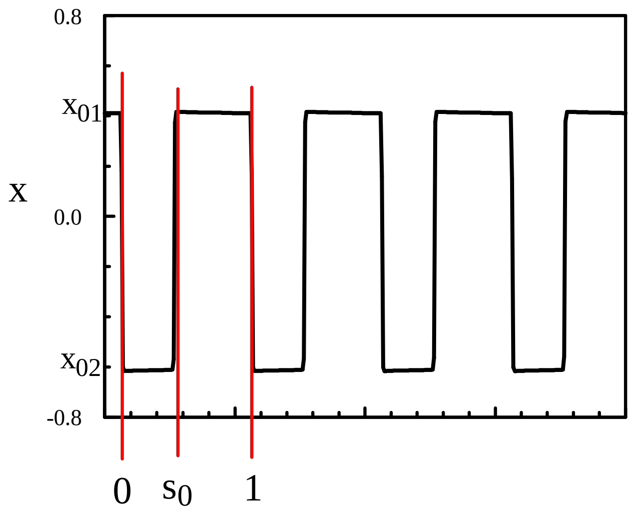
$$0 = -x - \delta y + \beta\left[\cos^2(x+\Phi) - \cos^2(\Phi)\right]$$

Part 2. Try

$$y = \delta^{-1} y_0(s) + y_1(s) + \dots$$

$$x = x_{0j}(s) + \delta x_{1j}(s) + \dots$$

$$j = 1 \;\; (0 < s < s_0),\; j = 2 \;\; (s_0 < s < 1)$$



$$y' = x$$

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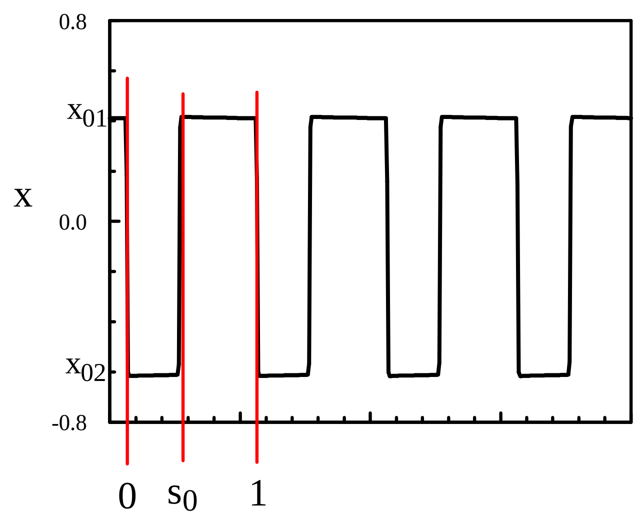
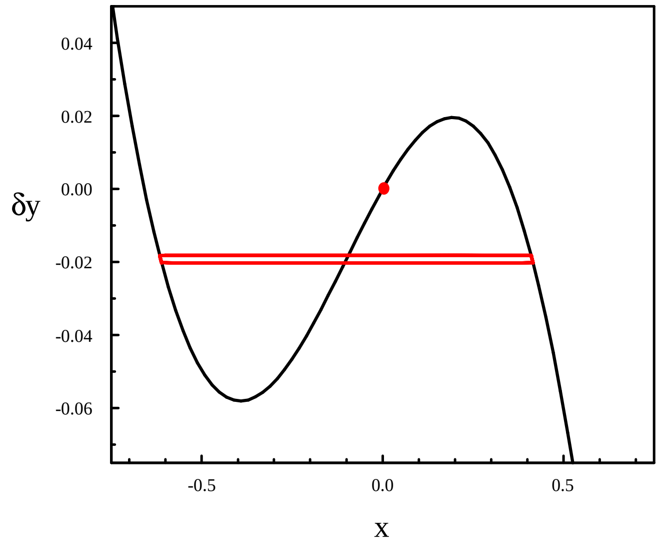


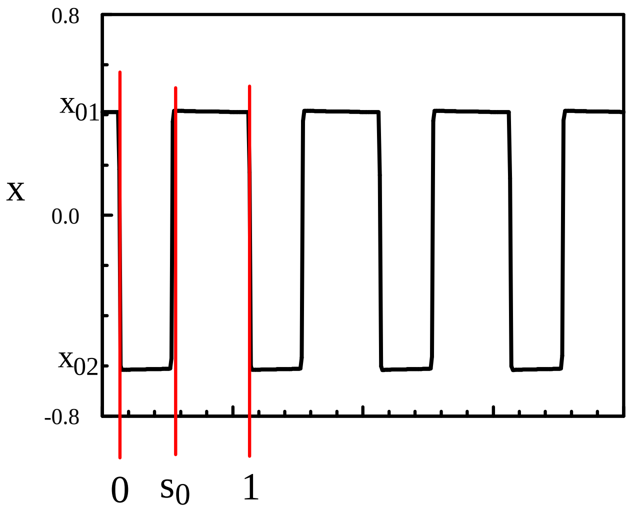
leading $\delta = 0$

$$y_0' = 0$$

$$0 = -x_{0j} - y_0 + \beta\left[\cos^2(x_{0j}+\Phi) - \cos^2(\Phi)\right]$$



$$y_0' = 0 \rightarrow y_0 = cst\,?$$



$O(\delta)$

$$y_1' = x_{0j}$$

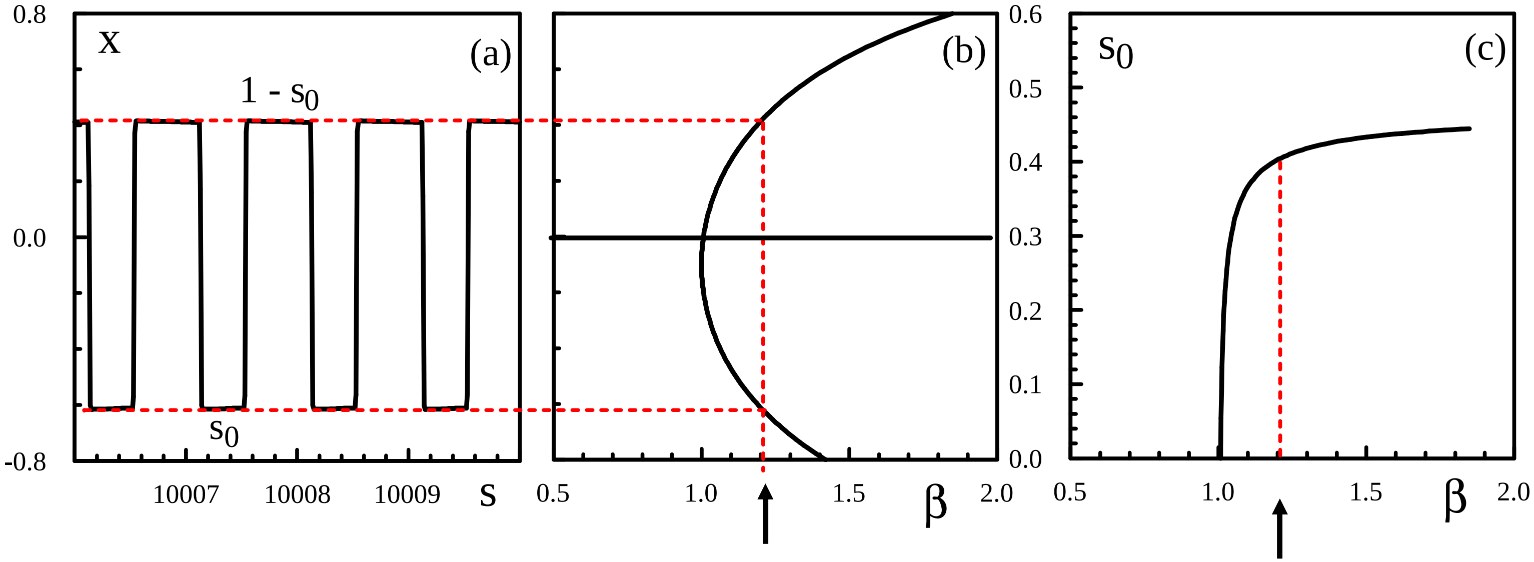
$$0 = -x_{1j} - y_1 - 2\beta \sin(2x_{0j} + 2\Phi) x_{1j}$$

Solve for $j = 1, 2$

Periodicity conditions

continuity condition at $s = s_0$

Bifurcation diagrams

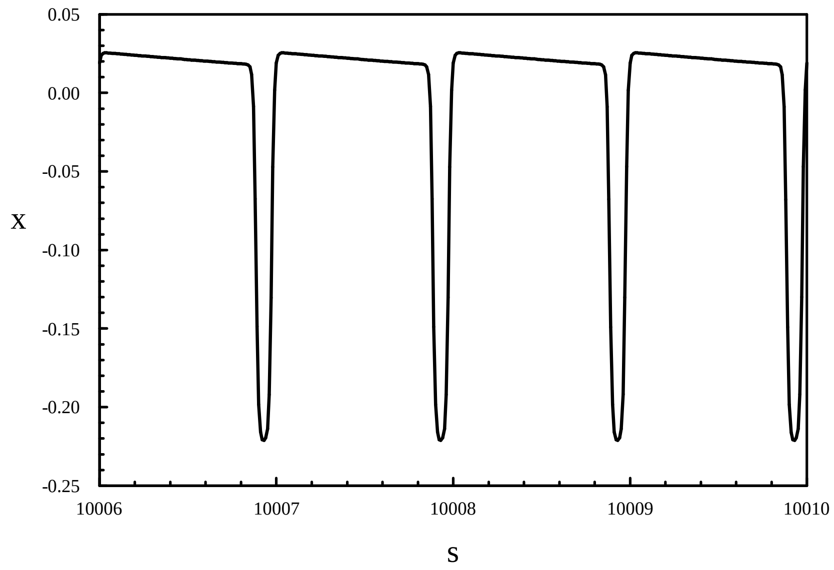


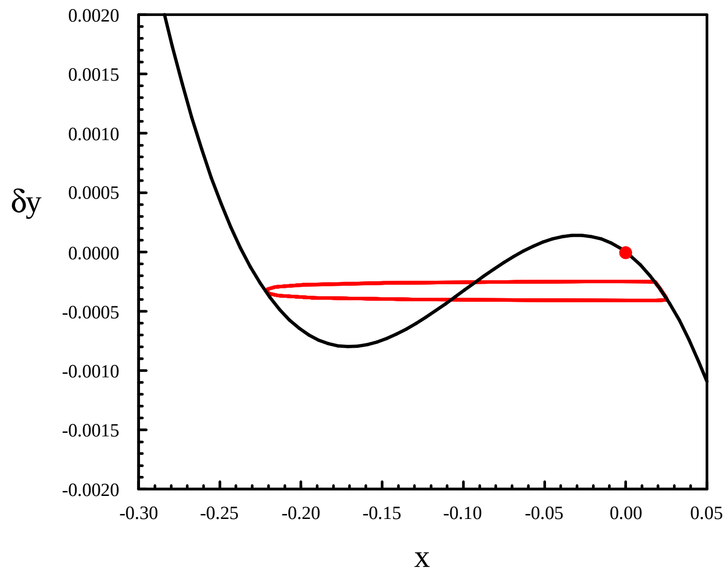
Numerical solution

Asymptotic analysis

$\beta = 1.2$

Bifurcation point near β = 1





$$\beta = 1.01$$

- Not branching from Hopf bifurcations

- Branching from a SN of limit-cycles?

Conclusions

1. Stable τ – periodic asymmetric square-waves not possible for first order scalar DDEs

$$\varepsilon x' = -x + f(x(t-1))$$

2. Possible for second order scalar DDEs

$$y' = x$$

$$\varepsilon x' = -x - \delta y + f(s-1)$$

Note:

1. Other periodic solutions coexist with the asymmetric square waves
2. No connection with the Hopf bifurcations from the zero solution
3. Isolated branch of periodic solutions
4. We have ignored the fast transition