Weierstrass Institute for
Applied Analysis and Stochastics

Hysteresis operators for vector-valued inputs and their representation by functions on strings

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Content

Motivation

- Brokate–Sprekels 1996:
  - Representation of hysteresis operators with scalar, piecewise monotone inputs and scalar outputs by functions acting on alternating strings with elements of $\mathbb{R}$
  - Investigation of operators by considering their representations
- Current talk:
  Introduction of appropriate functions space for input functions and of appropriate set for strings allowing a similar representation result for hysteresis operators with vectorial inputs

Some Notations and Definitions I

- Let $T > 0$ denote some final time.
- Let $(X, \|\cdot\|_X)$ be some normed vector space.
- Let Y be some nonempty set, and let $\mathrm{Map}\,([0,T],Y) := \{v : [0,T] \to Y\}$.
- Let $\mathrm{C}\,([0,T];X)$ denote the set of all continuous functions $u : [0,T] \to X$.
- $\alpha : [0,T] \to [0,T]$ is an *admissible time transformation* $:\Longleftrightarrow$ α is continuous and increasing (not necessary strictly increasing), $\alpha(0) = 0$ and $\alpha(T) = T$.
- Let $\mathrm{C}_{\mathrm{pm}}([0,T]) \subset \mathrm{C}\,([0,T];\mathbb{R})$ be the set of all piecewise monotone and continuous functions from $[0,T]$ to $\mathbb{R}$.

- Let $\mathcal{H} : D(\mathcal{H})(\subseteq \mathrm{Map}\,([0,T],X)) \to \mathrm{Map}\,([0,T],Y)$ with $D(\mathcal{H}) \neq \emptyset$ be given.
  - $\mathcal{H}$ a *hysteresis operator* $:\Longleftrightarrow$ $\mathcal{H}$ is rate-independent and causal.
  - $\mathcal{H}$ is *rate-independent* $:\Longleftrightarrow \forall\, v \in D(\mathcal{H}), \forall$ admissible time transformation $\alpha : [0,T] \to [0,T]$ with $v \circ \alpha \in D(\mathcal{H}), \forall\, t \in [0,T]$:

$$\mathcal{H}[v \circ \alpha](t) = \mathcal{H}[v](\alpha(t)).$$

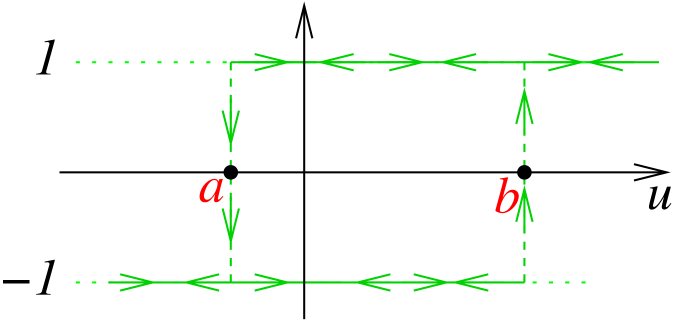
  - $\mathcal{H}$ is *causal* $:\Longleftrightarrow \forall\, v_1, v_2 \in D(\mathcal{H}), \forall\, t \in [0,T]$:
    If $v_1(\tau) = v_2(\tau) \quad \forall\, \tau \in [0,t]$ then $\mathcal{H}[v_1](t) = \mathcal{H}[v_2](t)$.
- Let $D_0 \subset \mathrm{Map}\,([0,T],X)$ be nonempty. Let Z be a nonempty set of *initial states*. Let $\mathcal{G} : Z \times D_0 \to \mathrm{Map}\,([0,T],Y)$ be given.
  $\mathcal{G}$ is a *hysteresis operator* $:\Longleftrightarrow$ for all $z_0 \in Z$ it holds that $\mathcal{G}[z_0, \cdot]$ is a hysteresis operator

For $a < b$ let the *relay operator* $\mathcal{R}_{a,b} : \{-1,1\} \times \mathrm{C}([0,T];\mathbb{R}) \to \mathrm{Map}([0,T],\{-1,1\})$ be the operator mapping $\eta_0 \in \{-1,1\}$ and $u \in \mathrm{C}([0,T];\mathbb{R})$ to

$$\mathcal{R}_{a,b}[\eta_0,u](t) := \begin{cases} \zeta_{a,b}(u(t)), & \text{if} \quad u(t) \notin]a,b[, \\ \eta_0, & \text{if} \quad u(s) \in]a,b[\quad \forall\, s \in [0,t], \\ \zeta_{a,b}\left(u\left(\max\left\{s \in [0,t] : u(s) \notin]a,b[\,\right\}\right)\right), & \text{otherwise,} \end{cases}$$

with $\zeta_{a,b} \ :]-\infty,a] \cup [b,\infty[\to \{-1,1\}$ defined by

$$\zeta_{a,b}(w) = \begin{cases} 1, & \text{if} \quad w \geq b, \\ -1, & \text{if} \quad w \leq a. \end{cases} \tag{1}$$



- The relay operator is a hysteresis operator.

- Set of alternating strings:

$$S_A = \{(v_0, v_1, \ldots, v_n) \in \mathbb{R}^{n+1} \,|\, n \geq 1, \\ (v_i - v_{i-1})(v_{i+1} - v_i) < 0, \quad \forall\, 1 \leq i < n\}.$$

- For any function $u : [0, T] \to \mathbb{R}$ being piecewise monotone the *standard monotonicity partition of* $[0, T]$ *for* u := unique defined decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$ such that for $1 \leq i \leq n$ holds: t_i is the maximal number in $]t_{i-1}, T]$ such that u is monotone on $[t_{i-1}, t_i]$.
  It holds

$$(u(t_0), u(t_1), \ldots, u(t_{i-1}), u(t)) \in S_A, \quad \forall t \in]t_{i-1}, t_i], \quad 1 \leq i \leq n.$$

Brokate-Sprekels 1996:

Theorem

- *Let a function* $G : S_A \to \mathbb{R}$ *be given.*
  For $u \in \mathrm{C_{pm}}([0,T])$ *let* $\mathcal{H}_G^{\mathrm{gen}}[u] : [0,T] \to \mathbb{R}$ *be defined by considering the standard monotonicity partition* $0 = t_0 < t_1 < \cdots < t_n = T$ *of* $[0,T]$ *for* u *and defining*

  $$\mathcal{H}_G^{\mathrm{gen}}[u](t) := G\left(u(t_0), u(t)\right), \quad \forall\, t \in [t_0, t_1], \tag{2a}$$

  $$\mathcal{H}_G^{\mathrm{gen}}[u](t) := G\left(u(t_0), \ldots, u(t_{i-1}), u(t)\right), \qquad \forall\, t \in]t_{i-1}, t_i],\ 2 \le i \le n. \tag{2b}$$

  The mapping $u \mapsto \mathcal{H}_G^{\mathrm{gen}}[u]$ *is a hysteresis operator on* $\mathrm{C_{pm}}([0,T])$*.*
- *Every hysteresis operator on* $\mathrm{C_{pm}}([0,T])$ *can be uniquely generated by this method.*

Consequences of the representation results

- The representation results above yields that for evaluating a hysteresis operator for continuous piecewise monotone inputs one needs not to memorize the exact evolution, but it is sufficient to keep track of the values of u in the past local extrema.
- Many properties of this hysteresis operators can be conveniently formulated and investigated by considering the functional generated by the operator.

Functional generating the scalar relay

For $\eta_0 \in \{-1, 1\}\mathbb{R}$ it holds: The restriction of the $\mathcal{R}_{a,b}[\eta_0, \cdot]$ to $\mathrm{C}_{\mathrm{pm}}([0, T])$ is equal to $\mathcal{H}^{\mathrm{gen}}_{G_{\mathcal{R},a,b,\eta_0}}$ with $G_{\mathcal{R},a,b,\eta_0} : S_A \to \mathbb{R}$ defined by

$$G_{\mathcal{R},a,b,\eta_0}((v_0, v_1, \ldots, v_n)) \\ = \begin{cases} \zeta_{a,b}(v_n), & \text{if} \quad v_n \notin\ ]a, b[, \\ \eta_0, & \text{if} \quad v_i \in]a, b[, \quad \forall\, i \in \{0, \ldots, n\}, \\ \zeta_{a,b}\left(v_{i_{\max}}\right), \text{ with } i_{\max} := \max\big\{i \in \{0, \ldots, n-1\} \mid v_i \notin]a, b[\big\} & \text{otherwise} \end{cases} \\ \forall\, (v_0, v_1, \ldots, v_n) \in S_A$$

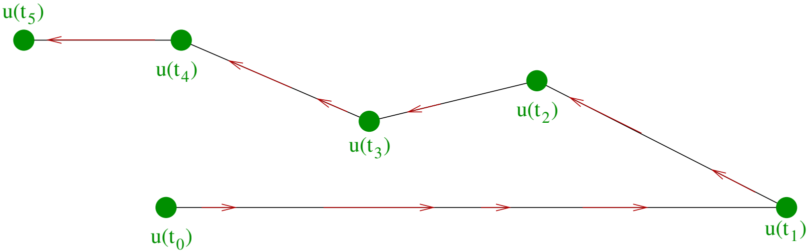
with $\zeta_{a,b}$ as in (1).

Monotaffine function = composition of a monotone with an affine function

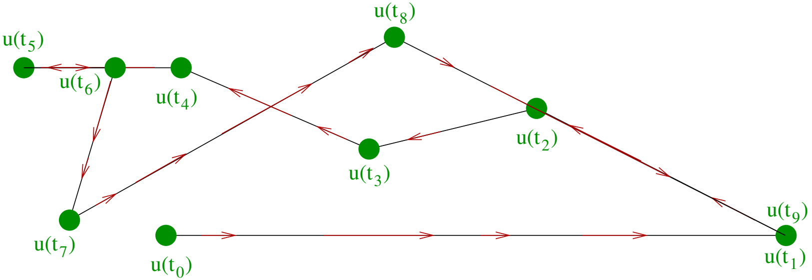
Kl. 2011,2012:

- Let some function $u : [0,T] \to X$ be given Let some $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ be given.
  u is *monotaffine on* $[t_1, t_2]$ $:\Longleftrightarrow$
  $\exists\, \beta : [t_1, t_2] \to [0,1]$ monotone increasing (not necessary strictly increasing) such that
  $$u(t) = (1 - \beta(t))u(t_1) + \beta(t)u(t_2), \quad \forall\, t \in [t_1, t_2].$$
- $u : [0,T] \to X$ is denoted as *piecewise monotaffine* $:\Longleftrightarrow$
  there exists a decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ such that for $1 \le i \le n$: u is monotaffine on $[t_{i-1}, t_i]$.
- Let $\mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X)$ be the set of all piecewise monotaffine functions in $C([0,T];X)$.
- Let $u \in \mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X)$ be given. The *standard monotaffinicity partition of* $[0,T]$ *for* u is the uniquely defined decomposition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0,T]$ such that for $1 \le i \le n$ holds: t_i is the maximal number in $]t_{i-1}, T]$ such that u is monotaffine on $[t_{i-1}, t_i]$.

Example for a monotaffine function I



Example for a monotaffine function II



- For $u \in \mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X)$ and every admissible time transformation $\alpha : [0,T] \to [0,T]$ is holds that $u \circ \alpha \in \mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X)$.

Kl. 2011:

- A *string of elements of X* is any $(v_0, \dots, v_n) \in X^{n+1}$ with $n \in \mathbb{N}$.
  It is a *convexity triple free string of elements of X* $:\Longleftrightarrow \quad v_i \notin \mathrm{conv}(v_{i-1}, v_{i+1})$ for all $1 \leq i < n$.
- For $v, w \in X$: $\mathrm{conv}(v, w) := \{(1-\lambda)v + \lambda w \,|\, \lambda \in [0,1]\}$.
- Let $S_F(X) := \{V \in X^{n+1} \,|\, n \in \mathbb{N}$ and V is a convexity triple free string of elements of $X\}$.

Theorem

It holds

$$S_A = S_F(\mathbb{R}).$$

Theorem

Let $u \in \mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X)$ be given. Let $0 = t_0 < t_1 < \dots < t_n = T$ of $[0,T]$ be the standard monotaffinicity partition of $[0,T]$ for u. Then it holds.

$$(u(t_0), u(t_1), \dots, u(t_{i-1}), u(t)) \in S_F(X), \quad \forall t \in]t_{i-1}, t_i], \quad 1 \leq i \leq n.$$

Kl.-2011, 2012

Theorem

1. *Every function* $G : S_F(X) \to Y$ *generates a hysteresis operator* $\mathcal{H}_G^{\text{gen}} : \mathrm{C}_{\text{p.w.m.a.}}([0,T];X) \to$ *Map*$([0,T];Y)$ *by considering for* $u \in \mathrm{C}_{\text{p.w.m.a.}}([0,T];X)$ *the corresponding standard monotaffinicity decomposition* $0 = t_0 < t_1 < \cdots < t_n = T$ *of* $[0,T]$ *and defining* $\mathcal{G}_G^{\text{gen}}[u] : [0,T] \to Y$ *by*

$$\mathcal{H}_G^{\text{gen}}[u](t) = G\left(u(t_0), u(t)\right), \forall\, t \in [t_0, t_1], \tag{3a}$$

$$\mathcal{H}_G^{\text{gen}}[u](t) = G\left(u(t_0), \ldots, u(t_{i-1}), u(t)\right), \forall\, t \in]t_{i-1}, t_i],\ 2 \le i \le n. \tag{3b}$$

2. *For* ***every hysteresis operator*** $\mathcal{B} : \mathrm{C}_{\text{p.w.m.a.}}([0,T];X) \to$ ***Map***$([0,T];Y)$ *there* ***exists a unique function*** $G : S_F(X) \to Y$ *such that* $\mathcal{B} = \mathcal{H}_G^{\text{gen}}$.

Generating function for vectorial hysteresis operators

- For a hysteresis operators $\mathcal{H} : D(\mathcal{H})(\subset \operatorname{Map}([0,T],X)) \to \operatorname{Map}([0,T],Y)$ with $\mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X) \subseteq D(\mathcal{H})$ we denote by *the string function generated by $\mathcal{H}$* the function $G : S_F(X) \to Y$ such that we have

$$\mathcal{H}\big|_{\mathrm{C}_{\mathrm{p.w.m.a.}}([0,T];X)} = \mathcal{G}_G^{\mathrm{gen}}.$$

  This function can be determined by evaluating the operator for piecewise affine functions.

- If one is evaluating a hysteresis operator acting on continuous monotaffine functions, then it is sufficient to keep track of the positions of the changes of direction of the input function.

Reformulation of scalar relay operator

For $v \in \mathbb{R}$, $r > 0$, $\eta_0 \in \{-1, 1\}$, and $u \in \mathrm{C}([0,T];\mathbb{R})$ it holds

$$\mathcal{R}_{v-r,v+r}[\eta_0, u](t) = \begin{cases} \tilde{\zeta}_v(u(t)), & \text{if} \quad |u(t) - v| \geq r, \\ \eta_0, & \text{if} \quad |u(s) - v| < r \quad \forall\, s \in [0,t], \\ \tilde{\zeta}_v\left(u\left(\max\left\{s \in [0,t] : |u(s) - v| \geq r\right\}\right)\right), & \text{otherwise,} \end{cases}$$

with

$$\tilde{\zeta}_v(w) := \frac{w - v}{|w - v|}.$$

Following Löschner-Brokate 2008, Löschner-Greenberg 2008:

For $\xi \in \mathbb{R}^N$ and $r > 0$ let the vectorial relay operator be defined by

$$\mathcal{R}_{\xi,r} : \mathbb{R}^N \times \mathrm{C}\left([0,T];\mathbb{R}^N\right) \to \mathrm{Map}\left([0,T],\mathbb{R}^N\right),$$

$$\mathcal{R}_{\xi,r}[\eta^0,u](t) := \begin{cases} \zeta_\xi(u(t)), & \text{if } \|u(t)-\xi\|_{\mathbb{R}^N} \geq r, \\ \eta^0, & \text{if } \|u(s)-\xi\|_{\mathbb{R}^N} < r, \quad \forall s \in [0,t], \\ \zeta_\xi\left(u\left(\max\{s \in [0,t] \mid \|u(s)-\xi\|_{\mathbb{R}^N} \geq r\}\right)\right), & \text{otherwise,} \end{cases}$$

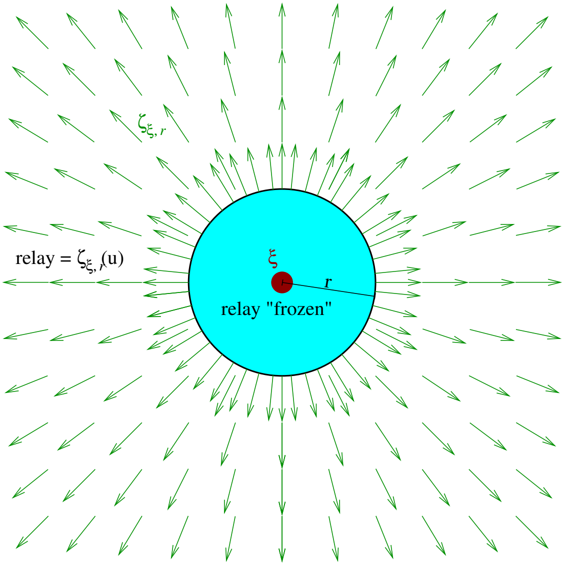
with

$$\zeta_\xi : \mathbb{R}^N \setminus \{\xi\} \to \partial B_1(0_X), \quad \zeta_\xi(v) := \frac{v-\xi}{\|v-\xi\|_{\mathbb{R}^N}}. \tag{4}$$

The vectorial relay operator is a hysteresis operator.

For $\xi, \eta_0 \in \mathbb{R}^N$, $r > 0$ and $u \in \mathrm{C}\left([0,T];\mathbb{R}^N\right)$ it holds:

- $\mathcal{R}_{\xi,r}[\eta_0, u] = \zeta_\xi(u(t))$ if $u(t)$ is not in the blue circle,
- If $u(t)$ enters the blue circle $\mathcal{R}_{\xi,r}[\eta_0, u]$ becomes constant, i.e. $\mathcal{R}_{\xi,r}[\eta_0, u]$ “freezes”, (notation following Löschner-Greenberg 2008)
- figure on the right-hand side follows inspiration from Löschner-Greenberg 2008



String function generated by vectorial relay

For $\eta_0 \in \mathbb{R}^N$ it holds: The string function generated by $\mathcal{R}_{(\xi,r)}[\eta_0,\cdot]$ is $G_{\mathcal{R},(\xi,r),\eta_0} : S_F(\mathbb{R}^N) \to \mathbb{R}^N$ defined by

$$G_{\mathcal{R},(\xi,r),\eta_0}\left((v_0,v_1,\cdots,v_n)\right)$$

$$:= \begin{cases} \zeta_\xi(v_n) \quad \text{if} \quad \|v_n-\xi\|_{\mathbb{R}^N} \geq r, \\ \eta^0, \quad \text{if} \quad \|v_i-\xi\|_{\mathbb{R}^N} < r \quad \forall\, i \in \{0,1,\dots,n\}, \\ \zeta_\xi\left((1-\chi_{\max})v_k + \chi v_{k+1}\right) \quad \text{with} \\ \qquad \chi_{\max} := \max\big\{\chi \in [0,1] \,\big|\, \|(1-\chi)v_k+\chi v_{k+1}-\xi\| \geq r\big\}, \\ \qquad k := \max\big\{i \in \{0,\dots,n-1\} \,\big|\, \|v_i-\xi\| \geq r,\big\}, \\ \qquad \text{otherwise} \end{cases}$$

with ζ_ξ as in (4).

Let a Lebesgue-integrable *Preisach density function* $\omega : \mathbb{R}^N \times [0, \infty[\to \mathbb{R}$ and some *measurable initial state* $\eta^0 : \mathbb{R}^N \times [0, \infty[\to \partial B_1(0)$ with $B_1(0)$ being the ball in $\mathbb{R}^N$ with radius 1 around 0 be given

Considering, as for the scalar Preisach operator, a weighted superposition of relays, one obtains the *vectorial Preisach operator* introduced in Brokate- Löschner-Greenberg 2008, Löschner-Greenberg 2008, (omitting the dependence on the initial state)

$$\mathcal{PR}^{vec} : \mathrm{C}\left([0,T];\mathbb{R}^N\right) \to \mathrm{Map}\left([0,T],\mathbb{R}^N\right),$$

$$\mathcal{PR}^{vec}[u](t) := \int_0^\infty \int_{\mathbb{R}^N} \omega(\xi,r)\mathcal{R}_{(\xi,r)}[\eta^0(\xi,r),u](t)\,\mathrm{d}\xi\,\mathrm{d}r.$$

The string function generated by $\mathcal{PR}^{vec}$ is

$$G_{\mathcal{PR},vec} : S_F(X) \to \mathbb{R},$$

$$G_{\mathcal{PR},vec}(V) = \int_0^\infty \int_{\mathbb{R}^N} \omega(\xi,r) G_{\mathcal{R},(\xi,r),\eta_0(\xi,r)}(V)\,\mathrm{d}\xi\,\mathrm{d}r.$$

Many vectorial relay considered in the literature can be rewritten as an operator of the following form:

For a nonempty, open subset O of X and a function $\zeta : X \setminus O \to Y$ we deal with the following generalized relay operator:

$$\mathcal{R}_{O,\zeta} : Y \times \mathrm{C}\left([0,T];X\right) \to \mathrm{Map}\left([0,T],Y\right),$$

$$\mathcal{R}_{O,\zeta}[\eta^0,u](t) := \begin{cases} \zeta(u(t)), & \text{if} \quad u(t) \notin O, \\ \eta^0, & \text{if} \quad u([0,t]) \subset O, \\ \zeta\left(u\left(\left\{s \in [0,t] \,\middle|\, u(s) \notin O\right\}\right)\right), & \text{otherwise.} \end{cases}$$

- The generalized relay operator is a hysteresis operator.
- Example: Vectorial Relay as in A. Visintin's talk:
  $\mathcal{R}_{a,b,\theta}[u] = \mathcal{R}_{a,b}[u \cdot e_\theta]e_\theta$ corresponds to

$$O = \{v \in \mathbb{R}^2 \,|\, v \cdot e_\theta \in]a,b[\},$$
$$\zeta(v) = r_{a,b}(v \cdot e_\theta)e_\theta.$$

Function on strings generated by the vectorial relay

For $\eta_0 \in Y$ holds: The string function generated by $\mathcal{R}_{O,\zeta}[\eta_0,\cdot]$ is $G_{\mathcal{R},O,\zeta,\eta} : S_F(X) \to Y$ defined by

$$G_{\mathcal{R},O,\zeta,\eta_0}(v_0, v_1, \ldots, v_n)$$

$$= \begin{cases} \zeta(v_n), & \text{if} \quad v_n \notin O, \\ \eta^0, & \text{if} \quad \operatorname{conv}(v_i, v_{i+1}) \subset O \quad \forall i \in \{0,\ldots,n-1\}, \\ \zeta\left((1-\chi_{\max})v_k + \chi_{\max} v_{k+1}\right) & \text{with} \\ \qquad \chi_{\max} := \max\big\{\chi \in [0,1] \,\big|\, (1-\chi)v_k + \chi v_{k+1} \notin O\big\}, \\ \qquad k := \max\big\{i \in \{0,\ldots,n-1\} \,\big|\, \operatorname{conv}(v_i, v_{i+1}) \not\subset O\big\}, \\ \quad \text{otherwise.} \end{cases}$$

Concluding remarks

- Using the monotaffine functions and convexity triple free strings one can extend the representation result in Brokate-Sprekels 1996 to the vectorial case.
- If one is considering piecewise affine or piecewise monotaffine and continuous inputs, one needs not the memorizes the whole evolution of the input to be able to evaluate the value of the hysteresis operator, one needs only to keep track of the positions of the directions changes of the input.
- The string representation of hysteresis operators can be used to formulated and investigated properties of these operators. For example:
  - Which elements of a string can be removed without changing the value returned by the string function, c.f. Madelung deletion in Brokate-Sprekels 1996?
  - The level-set functions generalizing those considered in Brokate-Sprekels simplify the investigation of the hysteresis operator.