Temperature-dependent Preisach shape memory model

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- Main result - Existence and Uniqueness Theorem

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- Main result - Existence and Uniqueness Theorem
- Proof of the main result

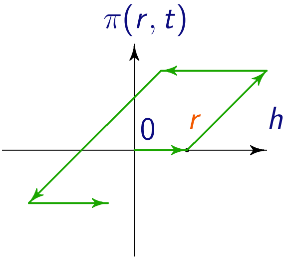
The play operator

Let $r > 0$ be a parameter, $h(t)$ a given input function on the time interval $I := [0, T]$ and $\pi_0(r) \in [h(0) - r, h(0) + r]$ an initial state. We consider a variational inequality

$$
\begin{aligned}
|h(t) - \pi(r,t)| &\le r \qquad \forall t \in I, \\
\pi_t(r,t)(h(t) - \pi(r,t) - x) &\ge 0 \\
&\quad \text{for a.e. } t \in I, \forall |x| \le r,
\end{aligned}
\tag{1}
$$

for the unknown $\pi(r,t)$. For an input $h \in W^{1,1}(I)$ this problem admits a unique solution $\pi(r,t) \in W^{1,1}(I)$. The play operator $\mathcal{P}_r[h] := \pi(r,t)$ with threshold r is defined as the solution operator for (1).

The play operator



$$
\begin{aligned}
|h(t) - \pi(r,t)| \le r & \qquad \forall t \in I, \\
\pi_t(r,t)(h(t) - \pi(r,t) - x) & \ge 0 \\
& \quad \text{for a.e. } t \in I, \forall |x| \le r,
\end{aligned}
$$

Shape memory models

Souza-Auricchio model

- very popular in the engineering community
- simplicity
- small number of model parameters which can easily be identified

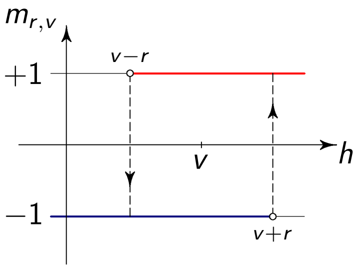
can be equivalently reformulated, in the 1D pure tension stress-controlled cases, as

$$\varepsilon = \frac{\sigma}{E} + \varepsilon_L Q\left(\frac{1}{E_h \varepsilon_L}\mathcal{P}_r[\sigma - f(\theta)]\right)$$

where σ is the stress, ε is the strain, θ is the absolute temperature, $E > 0$ is the elasticity modulus, $E_h > 0$ is the hardening modulus, $\varepsilon_L > 0$ is the reorientation strain, $r > 0$ is the yield stress, $Q : \mathbb{R} \to [0,1]$ is the projection of $\mathbb{R}$ onto $[0,1]$, and f is the piecewise affine function $f(\theta) = b(\theta - \theta_M)$, where b and θ_M are positive parameters.

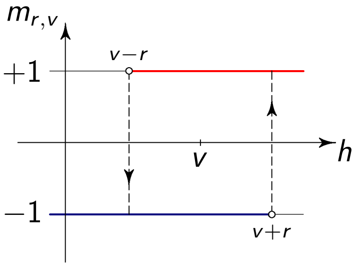
Preisach model: a two-parametric relay

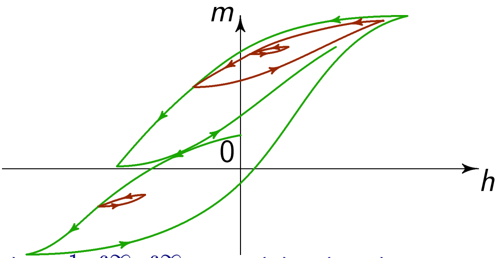
Let $h(t)$ be a given time dependent uniaxial magnetic field. Consider an elementary magnet ("relay") switching between two possible states: $+1$ and -1.



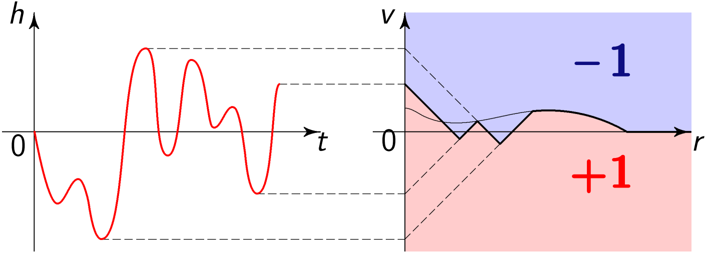
Switching occurs at value v ... interaction field
with delay $r > 0$ ... critical field of coercivity.

Preisach model





$$m(t) = \frac{1}{2}\int_0^\infty \int_{-\infty}^\infty m_{r,v}(t)\,\psi(r,v)\,dv\,dr$$



One-parametric description of the Preisach model

We can eliminate the relays by introducing for each fixed time t the memory function $\lambda_t(r)$ of the memory variable $r > 0$, describing the moving interface between the $+1$ and the -1 regions. The integral

$$m(t) = \frac{1}{2} \int_0^\infty \int_{-\infty}^\infty m_{r,v}(t)\, \psi(r,v)\, dv\, dr$$

defines the Preisach operator and can be written as

$$m(t) = \frac{1}{2} \left(\int_0^\infty \int_{-\infty}^0 - \int_0^\infty \int_0^\infty \right) \psi(r,v)\, dv\, dr$$

$$+ \int_0^\infty \int_0^{\lambda_t(r)} \psi(r,v)\, dv\, dr = C + \int_0^\infty g(r, \lambda_t(r))\, dr\,,$$

where we have set

$$g(r,v) = \int_0^v \psi(r,z)\, dz\,.$$

The play operator

For a fixed memory level $r > 0$, consider now the function

$$\pi(r, t) \; = \; \lambda_t(r)$$

as function of time t. Then the mapping $h \mapsto \pi(r, t)$ is the linear play characterized in each monotonicity interval $[t_0, t_1]$ of the input $h(t)$ by the formula

$$\pi(r, t) \; = \; \min\{h(t) + r, \max\{h(t) - r, \pi(r, t_0)\}\} \, .$$

Preisach energy balance

With the constitutive law $m(t) = \int_0^\infty g(r, \mathcal{P}_r[h])\, dr$,

we associate the potential energy

$$e(t) = \int_0^\infty G(r, \mathcal{P}_r[h])\, dr$$

with

$$G(r, v) = \int_0^v z\, \psi(r, z)\, dz = v\, g(r, v) - \int_0^v g(r, z)\, dz\,,$$

and the dissipation function

$$d(t) = \int_0^\infty r\, g(r, \mathcal{P}_r[h])\, dr\,.$$

For every given time evolution of $h(t)$ and at almost all times t we have the local energy balance

$$\boxed{h(t)\, \dot{m}(t) - \dot{e}(t) = |\dot{d}(t)| \geq 0\,.}$$

A rate dependent variational inequality

$$|h(t) - \pi(r,t)| \leq r, \forall t \in I,$$
$$(\mu_1(t)\pi_t(r,t) + \mu_2(t)(\pi(r,t) - h(t)))(h(t) - \pi(r,t) - x) \geq 0$$
$$\text{for a.e. } t \in I, \forall |x| \leq r,$$

where $h(t)$ is the input (strain), $\pi(r,t)$ is the output (memory state), and μ_1, μ_2 are given functions,
$\mu_2 = 0$, $\mu_1 > 0$ - rate independent plasticity - solution $\pi(r,t) = \mathcal{P}_r[h]$, where $\mathcal{P}_r[h]$ is the play operator,
$\mu_1 = 0$, $\mu_2 > 0$ - linear elasticity - gives a one-to-one correspondence between h and π, $\pi(r,t) = h(t)$,
$\mu_2 > 0$, $\mu_1 > 0$ - viscoelastoplasticity.

The main idea of the model

To derive a termodynamically consistent Preisach operator, we assume that the functions μ_1, μ_2 are temperature-dependent in such a way, that μ_1 vanishes (so the material is elastic) for high temperatures, and μ_2 vanishes (so the material is plastic) for low temperatures (and the material is viscoelastoplastic in between).

$$\begin{gathered} |h(t) - \pi(r,t)| \leq r, \quad \forall t \in I, \\ (\mu_1(\theta(t))\pi_t(r,t) + \mu_2(\theta(t))(\pi(r,t) - h(t)))(h(t) - \pi(r,t) - x) \geq 0 \\ \text{for a.e. } t \in I, \forall |x| \leq r. \end{gathered} \tag{2}$$

Assumptions on μ_1 and μ_2

More precisely, we assume that μ_1, μ_2 are continuous nonnegative functions on $(0, \infty)$, μ_1 is nonincreasing, μ_2 nondecreasing and that there exists $\theta_c > 0$ and constants $p > 1$, $C > 0$ such that $\mu_2(\theta_c) > 0$ and

$$\mu_1(\theta) \leq C((\theta_c - \theta)^+)^p.$$

Let us note that the behaviour of the model is regular as long as $\theta(t) < \theta_c$ or $\theta(t) > \theta_c$.

Temperature dependent Preisach model

To obtain the Preisach model, the individual contribution $\sigma_r(t)$ at level r is given by the formula

$$\sigma_r(t) := E(r)h(t) - g(r, \pi(r,t), \theta),$$

where $g : [0, \infty) \times R \times R^+ \longrightarrow R$ is a given function such that

$$\frac{\partial g(r, \pi, \theta)}{\partial \pi} \geq 0 \quad \text{a.e,} \quad g(r, 0, \theta) = 0,$$

and macroscopically σ is formally given by the integral

$$\sigma(t) = Eh(t) - \sigma_p = Eh(t) - \int_0^\infty g(r, \pi(r,t), \theta(t))dr.$$

Thermodynamical consistency of the model

Second principle of thermodynamics: There exist internal energy U and entropy S as functions of θ and $\pi(r,.)$ such that for every $h, \theta \in W^{1,1}_{\text{loc}}, \theta(t) > 0$ for all $t \in (0, T)$ we have

$$-\frac{\partial}{\partial t}U(\theta(t), \pi(r,t)) + \sigma(t)\frac{\partial h(t)}{\partial t} + \theta(t)\frac{\partial}{\partial t}S(\theta(t), \pi(r,t)) \geq 0$$

We consider

$$U = C_V\theta + \frac{E}{2}h^2(t) + \int_0^\infty U_r dr, \qquad U_r = G - \theta G_\theta - hg + h\theta g_\theta,$$

$$G(r,\pi,\theta) = \pi g(r,\pi,\theta) - \int_0^\pi g(r,l,\theta)dl$$

$$S = C_V log\left(\frac{\theta}{\theta_c}\right) + \int_0^\infty S_r dr, \qquad S_r(\theta,\pi) = -G_\theta + hg_\theta.$$

Wellpossedness of the variational inequality

Theorem

Let $h \in W^{1,1}(0,T)$ *and* $\theta \in C^{0,\frac{1}{p}}[0,T]$ *be given such that* $\theta(t) > 0$ *for every* $t \in [0,T]$*. Then for every initial condition* $\pi_{-1}(r) \in \{\pi \in W^{1,\infty}(0,\infty) : |\pi'| \leq 1 \text{ a.e. } \}$ *there exists a unique function* $\pi(r,.) \in C[0,T]$*, which satisfies for every* $r \geq 0$ *the variational inequality (2).*

If moreover $h_1, h_2 \in W^{1,1}(0,T)$, $\pi^1_{-1}(r), \pi^2_{-1}(r)$ are given and $\pi^{(1)}(r,.), \pi^{(2)}(r,.)$ are the corresponding solutions, then

$$\|\pi^{(1)}(r,.) - \pi^{(2)}(r,.)\|_{[0,T]} \leq$$

$$\leq \max\left\{|\pi^{(1)}(r,0) - \pi^{(2)}(r,0)|; \|h_1 - h_2\|_{[0,T]}\right\}.$$

Convergence result

Theorem

Let $h^{(n)} \in W^{1,1}(0,T)$ *and* $\theta^{(n)} \in C^{0,\frac{1}{p}}[0,T]$ *for* $n \in N$ *be given, such that* $\lim_{n\to\infty} \|h^{(n)} - h\|_{[0,T]} = 0$, $\lim_{n\to\infty} \|\theta^{(n)} - \theta\|_{[0,T]} = 0$, *there exists a constant independent of* n *such that*

$$|\theta^{(n)}(t) - \theta^{(n)}(s)| \leq C|t-s|^{1/p} \quad \forall t, s \in [0,T], \forall n \in N.$$

Let π^n_{-1} *be a sequence of admissible initial conditions,* $\pi^n_{-1} \to \pi_{-1}$ *uniformly in* $[0,\infty)$. *Let* $\pi^{(n)}$ *for* $n \in N$ *and* π *be the solutions to (2) corresponding to* $\theta^{(n)}$, $h^{(n)}$ *and* θ, h *respectively. If* $\theta(0) \neq \theta_c$, *then*

$$\lim_{n\to\infty} \|\pi^{(n)}(r,.) - \pi(r,.)\|_{[0,T]} = 0, \quad \forall r \geq 0.$$

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- Domain with low temperature $\theta < \theta_c$
  - time discrete model
  - time continuous model
- Local behaviour of the model in the neighborhood of a singularity
- Global well-posedness

Derivation of the model problem

Equation of motion - the field equation governing the space-time evolution:

$$\rho\, u_{tt} = \sigma_x + f_1$$

where $\rho > 0$ is a constant referential density, u is the displacement, f_1 is the volume force density and the stress is in the form

$$\sigma(t) = \sigma_v + Eh(t) - \sigma_p = \alpha h_t(t) + Eh(t) - \int_0^\infty g(r, \pi(r,t), \theta(t))dr.$$

The balance law of internal energy:

$$U_t = \sigma\, u_{xt} - q_x + f_2$$

where q is the heat flux, $q = -k\,\theta_x$ from the Fourier law, U is the total internal energy and f_2 is the heat source density.
Small deformation hypothesis $h = u_x$.

Balance equations

Consider the system

$$u_{tt} - \left(Eu_x - \sigma_p + \alpha u_{xt}\right)_x \;=\; f_1(\theta, x, t)\,,$$
$$\mathcal{U}_t - \theta_{xx} = \left(-\sigma_p + \alpha u_{xt}\right)\, u_{xt} + f_2(\theta, x, t)$$

with suitable initial and boundary conditions. Here, $\mathcal{U}$ is the internal energy functional

$$\mathcal{U} \;=\; C_V\theta + \int_0^\infty U_r\, dr\,, U_r = -hg + h\theta g_\theta + G - \theta G_\theta\,.$$

A necessary condition for the wellposedness of the problem is the positivity of the specific heat, that is,

$$\hat{C}_V \;=\; C_V - \theta \int_0^\infty \left(hg_{\theta\theta} - G_{\theta\theta}\right)\, dr \;>\; 0\,.$$

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- Space discretization

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- A-priori estimates
- Limit procedure using compact embeddings

Space discretization

Let $n > 1$ be a given integer. We get the following system of ODEs for unknown functions $u_1, \ldots, u_{n-1}$, $\theta_1, \ldots, \theta_n$,

$$
\begin{aligned}
\ddot{u}_k &= n(\sigma_{k+1} - \sigma_k) + f_k(\theta_k, t), \\
\frac{d}{dt}(C_V\theta_k &+ \mathcal{U}[\varepsilon_k, \theta_k]) = n^2\left(\theta_{k+1} - 2\theta_k + \theta_{k-1}\right) \\
&+ \dot{\varepsilon}_k(\mathcal{P}[\varepsilon_k, \theta_k] + \alpha\dot{\varepsilon}_k) + h_k(\theta_k, t), \\
\varepsilon_k &= n(u_k - u_{k-1}), \\
\sigma_k &= E\varepsilon_k - \mathcal{P}[\varepsilon_k, \theta_k] + \alpha\dot{\varepsilon}_k, \\
u_0 &= u_n = 0, \;\; \theta_0 = \theta_1, \;\; \theta_{n+1} = \theta_n, \\
f_k(\theta, t) &= n\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(\theta, x, t)\, dx, \, h_k(\theta, t) = n\int_{\frac{k-1}{n}}^{\frac{k}{n}} h(\theta, x, t)\, dx, \\
u_k(0) &= u^0\left(\frac{k}{n}\right), \dot{u}_k(0) = u^1\left(\frac{k}{n}\right), \theta_k(0) = \theta^0\left(\frac{k}{n}\right), \\
k &= 1, \ldots, n-1
\end{aligned}
$$

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