Rate independent hysteresis as a limit case of slow-fast systems

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Limit behavior in a singularly perturbed problem

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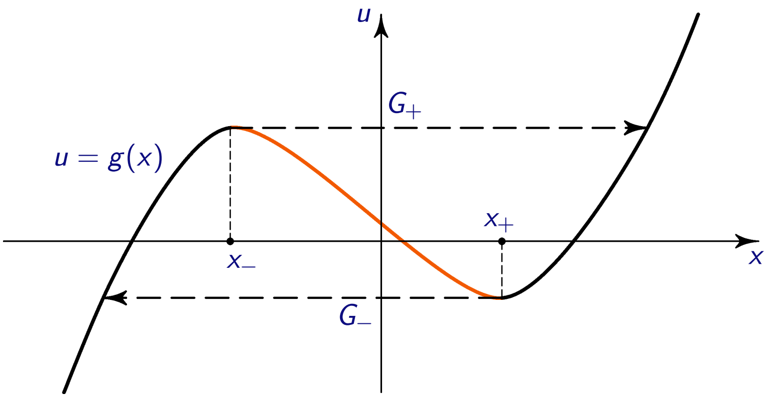

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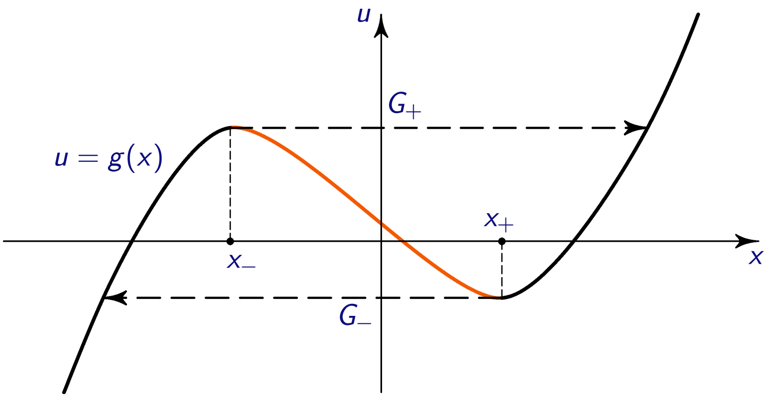

Which is the right convergence concept as $\alpha \to 0+$?

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where p and v are the pressure and velocity, respectively, ϱ is the mass density, and $K > 0$ is a material constant,

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The method of characteristics

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In dimensionless form, the problem reads

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Neither the space $C[0, T]$ of continuous functions, nor the space $BV(0, T)$ of functions of bounded variation are the right choice.

Functions of bounded variation

The space $BV(0, T)$ of functions of bounded variation on an interval $[0, T]$ contains all functions $u : [0, T] \to X$ which admit a constant $C > 0$ such that

$$\sum_{k=1}^{m} |u(t_k) - u(t_{k-1})| \leq C$$

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Let $\mathcal{D}_{0,T}$ be the set of all divisions of $[0, T]$. The number

$$\operatorname*{Var}_{[0,T]} u \ := \ \sup_{\mathcal{D}_{0,T}} \sum_{k=1}^{m} |u(t_k) - u(t_{k-1})|$$

is called the total variation of u on $[0, T]$.

Helly selection principle

Let $C > 0$ *be a constant, and let* $u_n \in BV(0, T)$ *for* $n \in \mathbb{N}$ *be a sequence such that*

$$|u_n(0)| + \operatorname*{Var}_{[0,T]} u_n \ \leq \ C \, .$$

Then there exists a subsequence $\{u_{n_k}\}$ *of* $\{u_n\}$ *and a function* $u \in BV(0, T)$ *such that* $u_{n_k}(t) \to u(t)$ *for* $k \to \infty$ *for all* $t \in [0, T]$.

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Extensions exist in the space of the so-called functions of bounded p-variation $BV_p(0,T) = \{f : [0,T] \to \mathbb{R}\,;\ \text{p-Var}_{[0,T]} f < \infty\}$ for $p > 1$, where

$$\operatorname*{p\text{-}Var}_{[0,T]} f \ := \ \sup_{\mathcal{D}_{0,T}} \sum_{k=1}^{m} |f(t_k) - f(t_{k-1})|^p\,.$$

(Chistyakov and Galkin (1998), J. E. Porter (2005)).

Functions of bounded φ-variation

More generally, for an arbitrary convex function $\varphi : [0, \infty[\to [0, \infty[$ such that $\varphi(0) = 0$, we define the φ-variation of $f : [0, T] \to \mathbb{R}$ as

$$\underset{[0,T]}{\varphi\text{-}\mathrm{Var}}\, f \;:=\; \sup_{\mathcal{D}_{0,T}} \sum_{k=1}^{m} \varphi(|f(t_k) - f(t_{k-1})|)\,,$$

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Clearly, the right and left limits $f(t+), f(t-)$ exist for all φ, all $f \in BV_\varphi(0, T)$, and all $t \in [0, T]$, with the convention $f(0-) = f(0), f(T+) = f(T)$.

Regulated functions

A function $f : [0, T] \to \mathbb{R}$ is said to be regulated, if the right and left limits $f(t+), f(t-)$ exist for all $t \in [0, T]$.
We denote the set of all regulated functions $f : [0, T] \to \mathbb{R}$ by the symbol $G(0, T)$. With the norm

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The Helly selection principle does not hold in $G(0, T)$!

Example: $f_n(t) = \sin nt$ for $t \in [0, \pi]$.

The Fraňková Theorem (1991)

Let $f_n \in G(0,T)$ *for* $n \in \mathbb{N}$ *be a sequence with the property*

$$\forall \varepsilon > 0 \;\; \exists L_\varepsilon > 0 \;\; \forall n \in \mathbb{N} \;\; \exists u_n \in BV(0,T):$$

$$\|u_n - f_n\|_\infty < \varepsilon\,, \;\; |u_n(0)| + \operatorname*{Var}_{[0,T]} u_n \;\leq\; L_\varepsilon\,.$$

Then there exists a subsequence $\{f_{n_k}\}$ *of* $\{f_n\}$ *and a function* $f \in G(0,T)$ *such that* $f_{n_k}(t) \to f(t)$ *for* $k \to \infty$ *for all* $t \in [0,T]$.

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Question: A criterion to check the condition of bounded ε-variation?

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Question: A criterion to check the condition of bounded ε-variation?

It is necessary to approximate regulated functions uniformly by BV-functions, keeping the total variations of the approximations uniformly bounded and depending only on ε.

Variational inequality

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A standard way to approximate a function f of unbounded variation by a function $u \in BV(0, T)$ consists in solving the differential inclusion

$$\dot{u}(t) \in \partial I_{[-\varepsilon,\varepsilon]}(f(t) - u(t)), \quad u(0) = f(0),$$

where $I_{[-\varepsilon,\varepsilon]}$ is the indicator function of $[-\varepsilon, \varepsilon]$, and ∂ is the symbol for the subdifferential.

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where $I_{[-\varepsilon,\varepsilon]}$ is the indicator function of $[-\varepsilon, \varepsilon]$, and ∂ is the symbol for the subdifferential. The mapping that with f associates u is called the play operator.

This inclusion is formally equivalent to the variational inequality

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which we have to reformulate in terms of the Kurzweil integral

$$\|f-u\|_\infty \le \varepsilon, \quad \int_0^t (f(\tau+) - u(\tau+) - z(\tau))\, \mathrm{d}u(\tau) \ge 0$$

$$\forall z \in G(0,T;[-\varepsilon,\varepsilon]) \ \ \forall t \in [0,T].$$

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Isolated singularities (i.e. points where $f(t)$ differs from both $f(t+)$ and $f(t-)$) have no influence on future evolution and can be removed. Hence, we can restrict ourselves e.g. to variational inequalities with left continuous inputs $f \in G_L(0, T)$

$$\boxed{\int_0^T (f(\tau+) - u(\tau+) - z(\tau))\,\mathrm{d}u(\tau) \geq 0 \quad \forall z \in G(0, T; [-\varepsilon, \varepsilon])}$$

which admits a unique left continuous solution $u \in BV_L(0, T)$.

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Theorem 1. *Let a set $F \subset G_L(0,T)$ and $\varepsilon > 0$ be given. Let there exist a convex increasing function φ and a constant $C_F > 0$ such that*

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The the set U_F of all solutions u of the Kurzweil variational inequality with inputs $f \in F$ has uniformly bounded variation.

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Theorem 2. *A set $F \subset G(0,T)$ has uniformly bounded φ-variation for some function φ if and only if it has uniformly bounded oscillation: For each $\varepsilon > 0$ there exists $N_{F,\varepsilon} > 0$ such that if n disjoint intervals $]a_k, b_k[$ satisfy the inequality $|f(b_k) - f(a_k)| > \varepsilon$ for some function $f \in F$, then $n \le N_{F,\varepsilon}$.*

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- f_n *have uniformly bounded oscillation;*
- *for each sequence* $g_n \in BV(0,T)$ *such that* $\mathrm{Var}_{[0,T]}\, g_n \le C$ *a* $g_n \to g$ *uniformly, we have*

$$\int_0^T f_n(t)\,\mathrm{d}g_n(t) \to \int_0^T f(t)\,\mathrm{d}g(t).$$

Bounded linear functionals on $G_L(0, T)$ a $G_R(0, T)$

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Theorem 4 (Hönig; Tvrdý; Brokate and Krejčí). *With each bounded linear functionals* P_L *on* $G_L(0,T)$ *and* P_R *on* $G_R(0,T)$ *we can associate uniquely determined functions* $g_L, g_R \in BV(0,T)$ *such that*

$$
\begin{aligned}
P_L(f) &= g_L(T)f(T) - \int_0^T g_L(t)\,\mathrm{d}f(t) \quad \forall f \in G_L(0,T)\,,\\
P_R(f) &= g_R(0)f(0) + \int_0^T g_R(t)\,\mathrm{d}f(t) \quad \forall f \in G_R(0,T)\,,
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with the properties

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Corollary. *The dual spaces to $G_L(0,T), G_R(0,T)$ are both isometrically isomorphic to $BV(0,T)$.*

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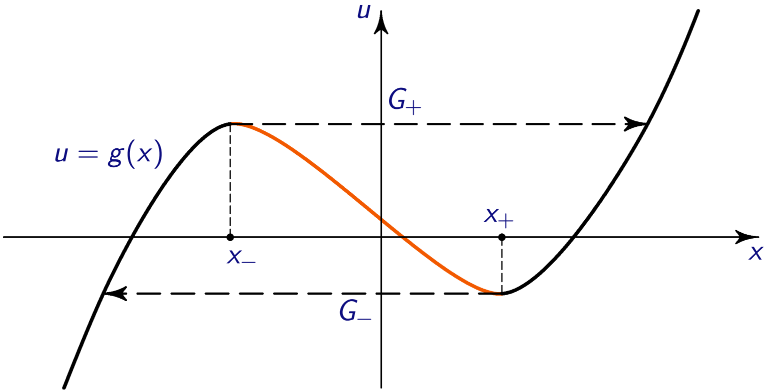
in $G_L(0,T)$ BO-converges, but does not converge weakly;
- Let $b_1 > a_1 > b_2 > a_2 > \cdots > 0$ be an infinite sequence of positive numbers. The sequence
$$f_n(t) = \sum_{k=n}^{2n} \chi_{]a_k,b_k]}(t)$$
converges weakly in $G_L(0,T)$, but does not have bounded oscillation.

Back to a singularly perturbed problem

$$\alpha\dot{x}(t) + g(x(t)) = u(t)\,, \quad x(0) = x_0$$



Uniformly bounded oscillation in singularly perturbed systems

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Theorem 5. *Let* $U \subset G_L[0,T]$ *be a bounded set with uniformly bounded oscillation, and let* $c > 0$ *be a constant. Then the set* $X \subset W^{1,\infty}[0,T]$ *of all solutions* x *to*

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In the proof, the play operator is used in a substantial way.

Convergence in singularly perturbed systems

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Theorem 6. *Let* $u \in G_L(0,T)$ *and* $x_0 \in \mathbb{R}$ *be given. Assume that one of the following two conditions holds:*

(i) $x_0 \notin]x_-, x_+[$.

(ii) $x_0 \in]x_-, x_+[$, $u(0+) \neq g(x_0)$.

Then there exists a function $x \in G_L(0,T)$ *such that* $x(t) \notin]x_-, x_+[$ *and* $g(x(t)) = u(t)$ *for every* $t \in]0,T]$, *and* x_α *BO-converge to* x .

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If $x_0 \in]x_-, x_+[$, $u(0+) = g(x_0)$, *then the subset* Y *of*

$$U = \{\hat{u} \in G_L(0,T) : \hat{u}(0+) = u(0+),\ \hat{u}(t) \in [G_-, G_+]\ \forall t \in [0,T]\}$$

containing all right hand sides for which $x_\alpha(t)$ *do not converge for any subsequence* $\alpha_j \to 0$, *is of the second Baire category.*

More general nonlinearities

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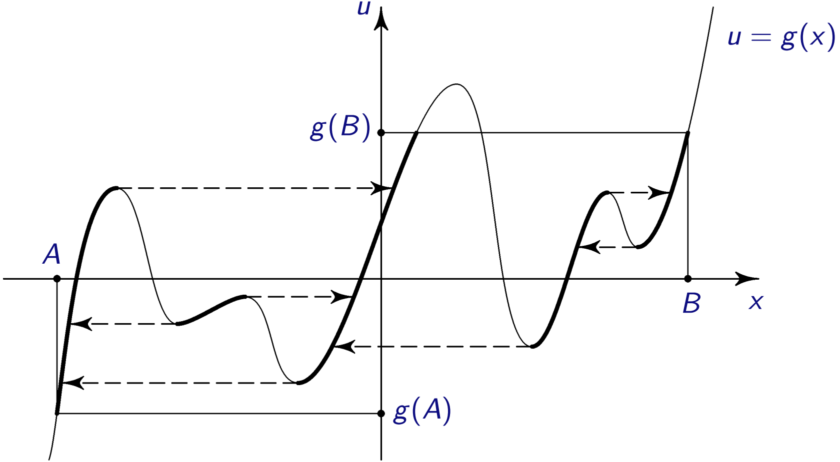
Similar statements hold for a large class of nonlinearities.



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- In bounded sets with uniformly bounded oscillation, the Fraňková extension of the Helly selection principle holds.
- Solutions of differential equations with a singular parameter in front of the derivative BO-converge to a rate independent hysteresis relation on a non-monotone graph.