Hunting French Ducks in a Noisy Environment

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joint work with:
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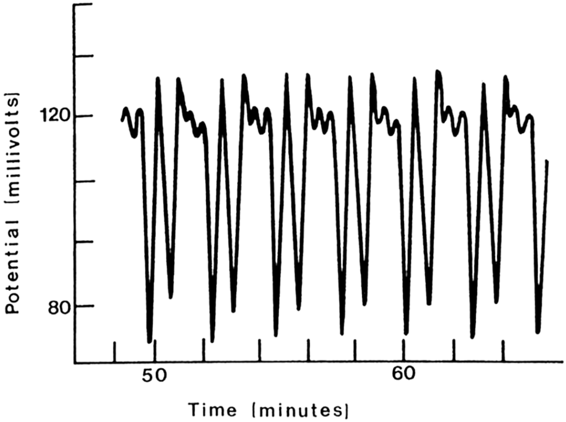
Outline

1. Motivation: Mixed-Mode Oscillations
2. Introduction to Multiple Time Scale Dynamics
3. Canards near a Folded Node
4. Stochastic Blow-Up and Linearization
5. Covariance Tubes
6. Conclusions

Mixed-Mode Oscillations (MMOs)

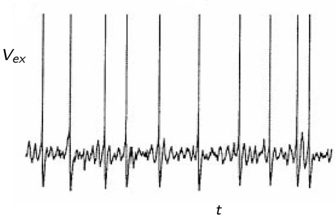
Belousov-Zhabotinsky reaction (Hudson, Hart and Marinko 1979):

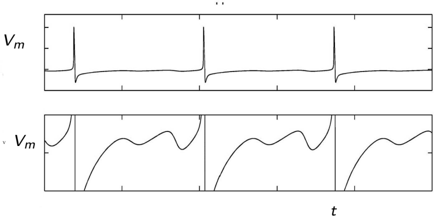


Notation: $\dots L_{j-1}^{s_{j-1}} L_j^{s_j} L_{j+1}^{s_{j+1}} \dots$ here: $L^s = 2^2$.

Mixed-Mode Oscillations (MMOs)

Another Example: Layer II Stellate Cells





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- (right) Model by Rotstein et al. (J. Comp. Neurosci. 2006)

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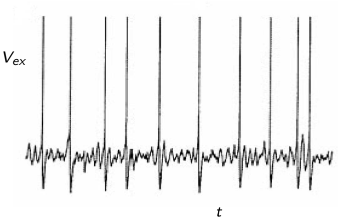

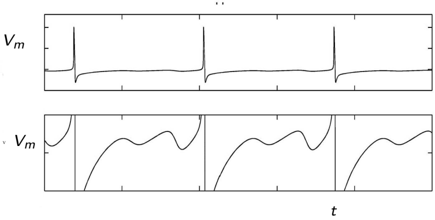

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- (right) Model by Rotstein et al. (J. Comp. Neurosci. 2006)

- Question 1: Mechanism for small-amplitude oscillations?
- Question 2: Influence of noise on SAOs?

Fast-Slow Systems

Fast variables $x \in \mathbb{R}^m$, slow variables $y \in \mathbb{R}^n$, time scale separation $0 < \epsilon \ll 1$.

$$\begin{cases} x' &= f(x,y) \\ y' &= \epsilon g(x,y) \end{cases} \quad \overset{\epsilon t = s}{\longleftrightarrow} \quad \begin{cases} \epsilon \dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{cases}$$

$$\downarrow \; \epsilon = 0 \qquad\qquad \downarrow \; \epsilon = 0$$

$$\begin{cases} x' &= f(x,y) \\ y' &= 0 \end{cases} \qquad\qquad \begin{cases} 0 &= f(x,y) \\ \dot{y} &= g(x,y) \end{cases}$$

fast subsystem — slow subsystem

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fast subsystem — slow subsystem

- $C := \{f = 0\}$ = critical manifold = equil. of fast subsystem.
- C is normally hyperbolic if $D_x f$ has no zero-real-part eigenvalues.
- **Fenichel's Theorem:** Normal hyperbolicity $\Rightarrow$ "nice" perturbation.

Folded Singularties in $\mathbb{R}^3$

Consider the following normal form:

$$\begin{aligned} \epsilon\dot{x} &= y - x^2, \\ \dot{y} &= -(\mu+1)x - z, \\ \dot{z} &= \frac{\mu}{2}, \end{aligned}$$

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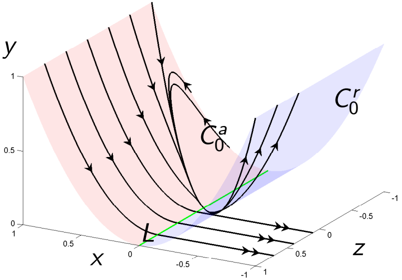
The critical manifold decomposes as:

$$C_0 = \{(x,y,z) \in \mathbb{R}^3 : y = x^2\} = C_0^a \cup L \cup C_0^r$$



Let's calculate the slow flow

$$0 = y - x^2, \qquad \Rightarrow \quad \dot{y} = 2x\dot{x}.$$

Therefore the slow subsystem is

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Set $s \to 2x\ s$; the **desingularized slow subsystem** is

$$\begin{aligned} \dot{x} &= -(\mu+1)x - z, \\ \dot{z} &= \mu x. \end{aligned}$$

Equilibrium $(x, z) = (0, 0)$ for desingularized slow flow.
Eigenvalues are

$$(\lambda_s, \lambda_w) := (-1, -\mu).$$

The origin $(0, 0)$ is a **folded node** for $\mu \in (0, 1)$.

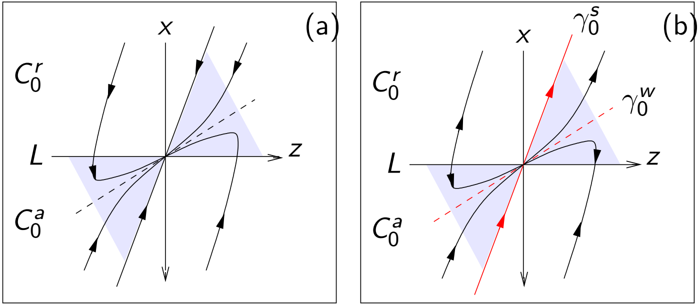

Figure: Strong singular canard γ_0^s; weak singular canard γ_0^w.

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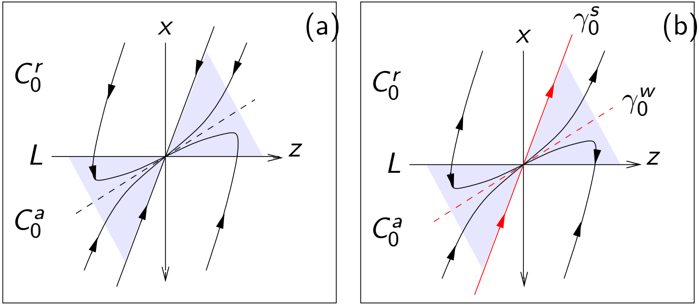

Figure: Strong singular canard γ_0^s; weak singular canard γ_0^w.

Definition: A **maximal canard** is an orbit in $C_\epsilon^a \cap C_\epsilon^r$.

Theorem (Benoît 1990; Szmolyan/Wechselberger/Krupa 2000)

For $\epsilon > 0$ *sufficiently small the singular strong canards* $\gamma_0^{s,w}$ *perturb to maximal canards* $\gamma_\epsilon^{s,w}$. *Suppose* $k \in \mathbb{N}$ *and*

$$2k+1 < \mu^{-1} < 2k+3 \qquad \text{and} \qquad \mu^{-1} \neq 2(k+1).$$

there are k *other maximal canards that rotate around* γ_ϵ^{w}.

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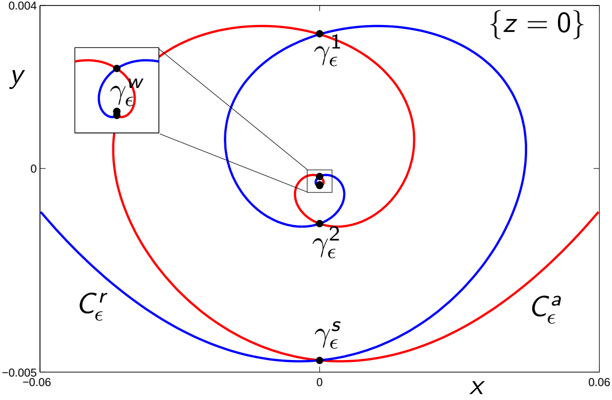

Stochastic Folded Nodes

Consider the normal form

$$
\begin{aligned}
dx_s &= \frac{1}{\epsilon}(y_s - x_s^2)ds + \frac{\sigma}{\sqrt{\epsilon}}dW_s^{(1)}, \\
dy_s &= [-(\mu+1)x_s - z_s]\, ds + \sigma' dW_s^{(2)}, \\
dz_s &= \frac{\mu}{2}ds.
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Main Idea: Control sample paths near deterministic solution.

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Main Idea: Control sample paths near deterministic solution.

Strategy:

1. Geometric desingularization (Blow-Up).
2. Linearization around deterministic solution.
3. Covariance evolution provides tubular neighbourhoods.
4. Stay inside tubes for $-1 < z < \sqrt{\mu}$.
5. Need to control nonlinearity and diffusion.

Blow-up (desingularize) the normal form

$$(x, y, z, s) = \left(\sqrt{\epsilon}\bar{x}, \epsilon\bar{y}, \sqrt{\epsilon}\bar{z}, \sqrt{\epsilon}\bar{s}\right)$$

then (dropping overbars for convenience)

$$\begin{aligned} dx_s &= (y_s - x_s^2)ds + \tfrac{\sigma}{\epsilon^{3/4}} dW_s^{(1)}, \\ dy_s &= \left[-(\mu+1)x_s - z_s\right] ds + \tfrac{\sigma'}{\epsilon^{3/4}} dW_s^{(2)}, \\ dz_s &= \tfrac{\mu}{2} ds. \end{aligned}$$

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We also re-scale the noise level parameters and set

$$(\epsilon^{3/4}\sigma, \epsilon^{3/4}\sigma') =: (\bar{\sigma}, \bar{\sigma}')$$

Observe: Can use s or z as "time" variable.

The Stochastic Variational Equation

Focusing on $(x_z, y_z) = (x_z^{\text{det}} + \xi_z, y_z^{\text{det}} + \eta_z)$ we get

$$
\begin{aligned}
d\xi_z &= \tfrac{2}{\mu}(\eta_z - \xi_z^2 - 2x_z^{\text{det}}\xi_z)dz + \tfrac{\sqrt{2}\sigma}{\sqrt{\mu}} dW_z^{(1)}, \\
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Proposition

Linearize the variational equation; set $V(z) := \sigma^{-2} Cov(z)$ then

$$\begin{aligned} \dot{v}_{11} &= -8x^{\mathrm{det}}(z)v_{11} + 4v_{12} + 2, \\ \dot{v}_{22} &= -4(\mu+1)v_{12} + 2(\sigma'/\sigma)^2, \\ \dot{v}_{12} &= -2(\mu+1)v_{11} + 2v_{22} - 4x^{\mathrm{det}}(z)v_{12}. \end{aligned}$$

Note $v_{12} = Cov(\xi_z, \eta_z) = Cov(\eta_z, \xi_z) = v_{21}$.

Neighbourhoods of Deterministic Solutions

Let $(x(z), y(z)) =: w(z)$ be a deterministic solution. Define a **tube-shaped**-neighbourhood

$$\begin{aligned}\mathcal{B}(r) &= \{(x,y,z) : z_0 \leq z \leq \sqrt{\mu}, \\ &\quad [(x,y) - w(z)] \cdot V(z)^{-1}[(x,y) - w(z)] < r^2\}.\end{aligned}$$

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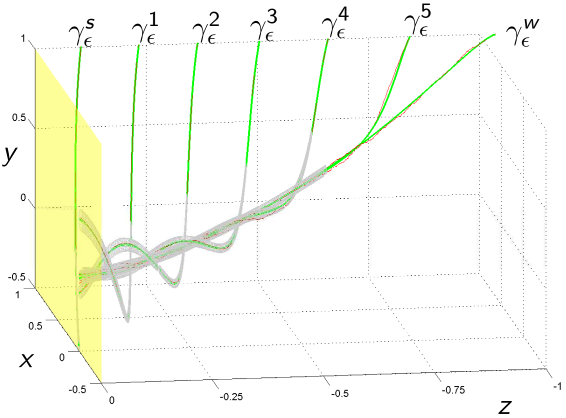
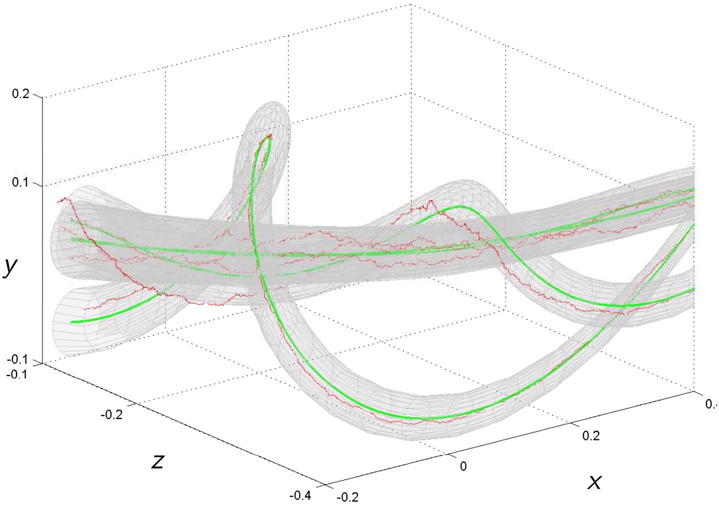
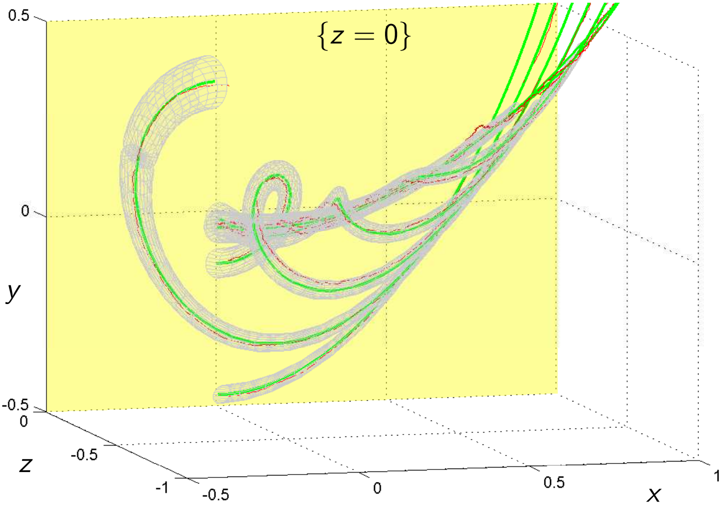


Zoom



Front View



Covariance Estimates

Theorem (Covariance Tubes)

On the section $\{z = 0\} = \{\bar{z} = 0\}$ we have (as $\mu \to 0$):

$$v_1 = \mathcal{O}(1/\sqrt{\mu}), \quad v_2 = \mathcal{O}(1/\sqrt{\mu}), \quad v_3 = \mathcal{O}(1), \quad (v_1 - v_2) = \mathcal{O}(1)$$

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Sketch of Proof.

1. Change coordinates in the variational equation.
2. Use a symmetry to reduce it to a planar system.
3. A complex eigenvalue pair crosses the imaginary axis at $z = 0$.
4. View the planar system as a fast subsystem with slow time z.
5. Apply the delayed Hopf bifurcation theory.

□

The Nonlinear Variational SDE

It turns out that in suitable coordinates we have to deal with

$$d\zeta_z = \frac{1}{\mu}\big[A(z)\zeta_z + b(\zeta_z, z)\big]\,dz + \frac{\sigma}{\sqrt{\mu}}F(z)dW_z,$$

where $\zeta_z = (\xi_z, \eta_z)$ and $A(z)$ is now given by

$$A(z) = \begin{pmatrix} -2x(z) & \omega_2(z) \\ -\omega_2(z) & -2x(z) \end{pmatrix}.$$

Staying inside $\mathcal{B}(r)$ ...

Theorem (Staying inside Covariance Tubes)

There exists a function $K(z, z_0)$ such that for $\kappa = 1 - \mathcal{O}(\cdot)$

$$\mathbb{P}\left\{\tau_{\mathcal{B}(r)} < z\right\} \leq K(z, z_0) \exp\left\{-\kappa \frac{r^2}{2\sigma^2}\right\}$$

holds for all z such that $z_0 \leq z \leq \sqrt{\mu}$.

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Sketch of Proof.

1. Consider a short time interval $[z_1, z_2]$.
2. Consider the fundemental solution $U(z, u)$ for $\mu\dot{\zeta} = A(z)\zeta$.
3. Set $\Upsilon_u := U(z, u)\zeta_u$ and observe $\Upsilon_u = \Upsilon_u^0 + \Upsilon_u^1$

$$\begin{aligned} \Upsilon_u^0 &= \frac{\sigma}{\sqrt{\mu}} \int_{z_0}^{u} U(z, v) F(v) dW_v, \\ \Upsilon_u^1 &= \frac{1}{\mu} \int_{z_0}^{u} U(z, v) b(\zeta_v, v) dv. \end{aligned}$$

Sketch of Proof (continued).

4. Υ_u^0 is a Gaussian martingale.
5. Doob's submartingale inequality, let $M_u := \|Q(z_1, z_2)\Upsilon_u^0\|$

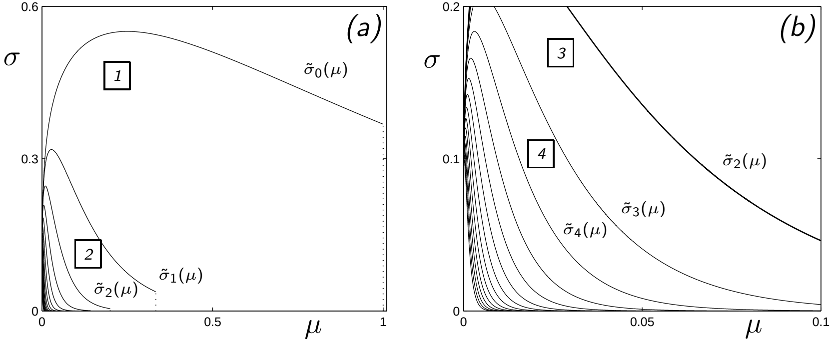
$$\mathbb{P}\left\{\sup_{z_1 \leq u \leq z_2} e^{M_u^2} \geq e^{r^2}\right\} \leq \frac{1}{e^{r^2}}\mathbb{E}\left[e^{M_{z_2}}\right] \leq (\cdots)\mathcal{O}\left(e^{-\frac{r^2}{2\sigma^2}}\right)$$

   where $Q(z_1, z_2)$ is defined via the covariance matrix V.
6. From last step we bound $\mathbb{P}\left\{\sup_{z_1 \leq u \leq z_2} M_u \geq r\right\}$.
7. Estimate $\|Q(z_1, z_2)\Upsilon_u^1\|$ directly and show that it is small.
8. We find that escape during a short time is highly unlikely.
9. Piece previous result together for a "nice" partition of $[z_0, z]$.

$\square$

Theorem (Noise, Canards and SAOs)

Depending on noise intensity $\tilde{\sigma}$ and bifurcation parameter μ the "noisy interactions" of canards are:



$$\tilde{\sigma}_k(\mu) = \mu^{1/4} e^{-(2k+1)^2 \mu}$$

Conclusions

Overview I:

- Fast-slow systems can have intricate singularities.
- The SAOs of MMOs are often caused by these mechanisms.
- Deterministic scenario is often unrealistic (biology!).

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- Deterministic scenario is often unrealistic (biology!).

Overview II:

- Metastable sample paths for SDEs are natural extension.
- Variational equations around solutions play a key role.
- Use Doob's inequality to control sample paths.
- (Early jumps after passage through folded node region.)
- Intricate dependencies between σ, μ and ϵ.

Main References:

(1) N. Berglund, B. Gentz, **CK**, *Hunting French ducks in a noisy environment*, arXiv:1011.3193.

(2) M. Desroches, B. Krauskopf, J. Guckenheimer, **CK**, H. Osinga & M. Wechselberger, *Mixed Mode Oscillations with Multiple Time Scales*, SIAM Review, in press.

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(4) J. Guckenheimer & **CK**, *"Computing slow manifolds of saddle type"*, SIADS, 2009.

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Prediction of Critical Transitions / Tipping Points:

(6) **CK**, *A mathematical framework for critical transitions: bifurcations, fast-slow systems and stochastic dynamics*, Physica D, 2011.
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Webpage:

- http://www.asc.tuwien.ac.at/∼ckuehn/

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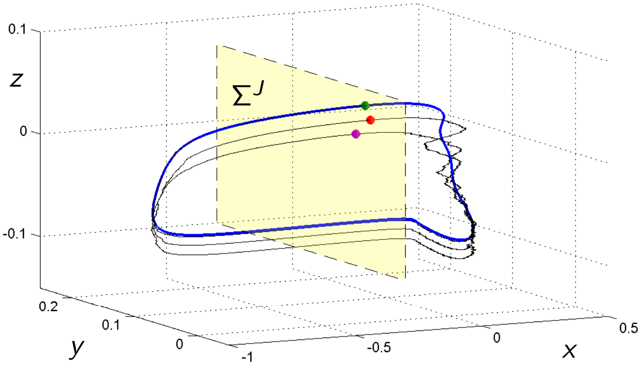
Thank you for your attention.

"Appendix" - Early Jumps

For $z > \sqrt{\mu}$ beyond the folded node, SDE paths jump early.

$$\begin{aligned} dx &= \tfrac{1}{\epsilon}(y - x^2 - x^3)ds + \tfrac{\sigma}{\sqrt{\epsilon}} dW_s^{(1)} , \\ dy &= [-(\mu+1)x - z]\, ds + \sigma' dW_s^{(2)} , \\ dz &= \left[\tfrac{\mu}{2} + ax + bx^2\right] ds . \end{aligned}$$



Theorem (Escape of Sample Paths)

In blow-up coordinates, consider $z > \sqrt{\mu}$ *and let* $\mathcal{D}$ *be a tube around* γ^w *that grows like* $\mathcal{O}(\sqrt{z})$*. Then the probability that a sample path stays in* $\mathcal{D}$ *becomes small as soon as*

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where $\nu > 0$.

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where $\nu > 0$.

Sketch of Proof.

1a. **Diffusion-dominated escape** from small set near γ^w.

1b. Subdivide again, need Markov property to re-start.

2a. **Drift-dominated escape** from $\mathcal{D}$.

2b. Change to polar coordinates.

2c. Use averaging to consider radius SDE.

2d. Show that drift dominates diffusion.

□