On the stability of differential-algebraic PDEs by time-delayed feedback control

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Outline

Consider the coupled system

$$\begin{aligned} u_t &= \Delta_x u + g(u,v), \;\; t > 0, \\ \varepsilon v_t &= f(u,v), \end{aligned}$$

where $0 \leqslant \varepsilon \ll 1$ is a singular perturbation parameter.

In the formal limit $\varepsilon \to 0$, differential-algebraic PDE (DA-PDE)

$$\begin{aligned} u_t &= \Delta_x u + g(u,v), \\ 0 &= f(u,v). \end{aligned}$$

Suppose that the system has an equilibrium (u_0, v_0) which solves

$$\begin{cases} g(u,v) &= 0, \\ f(u,v) &= 0. \end{cases}$$

Stabilization by time-delayed feedback control of Pyragas type

- Initiated by Pyragas (1992) [4] to stabilize unstable periodic orbits (UPOs) embedded in a chaotic attractor which is simulated by an ODE

$$\frac{dx}{dt} = Q(x,y), \;\; \frac{dy}{dt} = P(x,y) + F(t) \dashleftarrow F(t) = K(y(t-\tau) - y(t)).$$

where K is a feedback gain matrix.

- In parallel, stabilization of unstable steady states (USSs) became a field of increasing interest.
- The theory of USSs has been well studied [3], [7], etc., for the model simulated by ODEs.
- We wish to study models simulated by differential-algebraic equations (DAEs), possibly from discretized DA-PDEs.

Discretizing the spatial variable x to obtain a system

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U}(t) \\ \dot{V}(t) \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U(t) \\ V(t) \end{bmatrix} + \begin{bmatrix} g(U(t), V(t)) \\ f(U(t), V(t)) \end{bmatrix} - K \left(\begin{bmatrix} U(t) \\ V(t) \end{bmatrix} - \begin{bmatrix} U(t-\tau) \\ V(t-\tau) \end{bmatrix} \right), \quad t > 0,$$

where $U(t) \in \mathbb{C}^n$, $V(t) \in \mathbb{C}^m$.

Notice that the new system

$$\begin{aligned} \dot{U}(t) &= WU(t) + g(U, V), \quad t > 0, \\ 0 &= f(U, V), \end{aligned}$$

has the equilibrium $(U_0, V_0) := (u_0 \cdot e_n, v_0 \cdot e_m)$, where $e_n = [1 \quad 1 \dots 1]^T \in \mathbb{R}^n$.

W. l. o. g., we assume that $U_0 = 0, \; V_0 = 0$.

Linearizing at that equilibrium (U_0, V_0) we obtain

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U}(t) \\ \dot{V}(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U(t) \\ V(t) \end{bmatrix} - K \left(\begin{bmatrix} U(t) \\ V(t) \end{bmatrix} - \begin{bmatrix} U(t-\tau) \\ V(t-\tau) \end{bmatrix} \right), \quad (1)$$

where

$$A := W + \mathbf{J}_U g|_{(U_0,V_0)}, \qquad B := \mathbf{J}_V g|_{(U_0,V_0)},$$
$$C := \mathbf{J}_U f|_{(U_0,V_0)}, \qquad D := \mathbf{J}_V f|_{(U_0,V_0)}.$$

Equation (1) is of the form

$$E\dot{x}(t) = Ax(t) + Bx(t-\tau), \ \ t > 0,$$

which is delay differential-algebraic equation or delay DAE.

Problem statement: Design a feedback gain matrix K to stabilize the equilibrium $(0,0) \in \mathbb{C}^{n,m}$ of the delay DAE

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U}(t) \\ \dot{V}(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U(t) \\ V(t) \end{bmatrix} - K \left(\begin{bmatrix} U(t) \\ V(t) \end{bmatrix} - \begin{bmatrix} U(t-\tau) \\ V(t-\tau) \end{bmatrix} \right), \quad t > 0.$$

Definition

The desire unstable orbit is called

i) totally periodic if $u(t,x)$ and $v(t,x)$ are time-periodic of period τ. As a consequence, $U(t)$ & $V(t)$ are periodic of period τ.

ii) semi-periodic if only $u(t,x)$ is time-periodic of period τ. Therefore, only $U(t)$ is periodic of period τ.

If the desire orbit is *semi-periodic*, then K should be chosen as

$$K = \begin{bmatrix} K_{11} & 0 \\ K_{21} & 0 \end{bmatrix}.$$

Example

We consider the delay DAE

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix}, \quad t \geq 0.$$

The initial function $\phi = x|_{[-\tau,0]}$ need to satisfies

$$-\phi(0) = [k_3 \quad k_4]\phi(-\tau).$$

Definition

The initial function ϕ of the delay DAE

$$E\dot{x}(t) = Ax(t) + Bx(t-\tau) + f(t), \; t \geq 0,$$

is called consistent if with that ϕ, there exists a solution $x(t)$.

Consider the homogeneous delay DAE

$$E\dot{x}(t) = Ax(t) + Bx(t-\tau), \ t \geq 0,$$

where E, A, $B \in \mathbb{R}^{p,p}$; with an initial condition $x|_{[-\tau,0]} = \phi \in C([-\tau,0],\mathbb{C}^p)$.

$C([-\tau,0],\mathbb{C}^n)$ is equipped with the sup-norm $\|\cdot\|_C$.

Definition

Stability of delay DAEs

The trivial solution is called *stable (in Lyapunov sense)* if for any $\varepsilon > 0$ $\exists$ $\delta > 0$ such that for any **consistent initial condition** ϕ with $\|\phi\|_C \leqslant \delta$ then the solution $|x(t,\phi)| \leqslant \varepsilon$, $t \geq 0$.
In addition, if $\lim\limits_{t\to\infty} x(t,\phi) = 0$, then the trivial solution is called *asymptotically stable*.

Stabilization - DAEs vs. ODEs

ODE case:

$$\dot{x}(t) = Ax(t) - K(x(t) - x(t-\tau)), \leftarrow\text{--}\quad \text{control for the dynamic.}$$

DAE case:

$$\begin{aligned} \dot{U}(t) &= AU(t) + BV(t), \leftarrow\text{--}\quad \text{the dynamic?} \\ 0 &= CU(t) + DV(t), \leftarrow\text{--}\quad \text{the constraint?} \end{aligned}$$

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The answer is NO! Why?

Hidden constraints may exist inside the dynamic.

Example

Consider the DAE

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},$$

$$\rightarrow \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix},$$

$$\overset{remodel}{\longrightarrow} \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}.$$

Slow-fast system - DAEs vs. ODEs

ODE case:

$$\dot{x}(t) = f(x, y, \epsilon),$$
$$\varepsilon \dot{y}(t) = g(x, y, \epsilon),$$

with $0 < \varepsilon \ll 1$

Rewrite with some perturbation

$$\dot{x}(t) = f(x, y, \epsilon) + \delta f,$$
$$\dot{y}(t) = \frac{1}{\varepsilon} \left[g(x, y, \epsilon) + \delta g \right],$$

so x is called slow variable and y is fast variable.

DAE case:

$$\dot{x}(t) = f(x, y, 0),$$
$$0 = g(x, y, 0).$$

Slow-fast system - DAEs vs. ODEs

Example

Consider the DAE

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{y}(t) \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix},$$

$$\rightarrow \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{bmatrix},$$

$$\overset{remodel}{\rightarrow} \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{y}(t) \end{bmatrix} = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \begin{bmatrix} x_1(t) \\ x_2(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} f_1(t) \\ -\dot{f}_3(t) + f_2(t) \\ f_3(t) \end{bmatrix}.$$

From perturbation point of view, $\begin{bmatrix} x_2 \\ y \end{bmatrix}$ **is fast variable,** x_1 **is slow variable.**

So hidden fast components in the variable may exist. Why?

The reason is that in general $D = J_V f|_{U_0,V_0}$ is not invertible (eigenvalues with real part 0 appear).

If D is invertible, we have normally hyperbolic system, which is often considered in DAE theory as index 1 case. For index 1 case, the perturbation theory has been developed by Fenichel.

Determining exact fast, slow variables plays a key role in control theory of DAEs.

Even though our system is

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U}(t) \\ \dot{V}(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U(t) \\ V(t) \end{bmatrix} - K \left(\begin{bmatrix} U(t) \\ V(t) \end{bmatrix} - \begin{bmatrix} U(t-\tau) \\ V(t-\tau) \end{bmatrix} \right), \ t > 0,$$

we can consider a general time-delay feedback control system

$$E_1 \dot{x}(t) = A_1 x(t) - K_1 \left(x(t) - x(t-\tau) \right), \ t \geq 0. \tag{2}$$

Our strategy: transforming (2) into

$$E_2 \dot{y}(t) = A_2 y(t) - K_2 \left(y(t) - y(t-\tau) \right), \tag{3}$$

where $E_2 = PE_1Q$, $A_2 = PA_1Q$, $K_2 = K_1Q$, $x(t) = Qy(t)$, P and Q are invertible.

We study the stabilization of system (3) and then get back to (2).

Definition

- $\sigma(E, A) := \{s \in \mathbb{C} : det(sE - A) = 0\}$ is called the *spectrum* of matrix pair (E, A).
- The pair (E, A) is called regular, iff $det(\lambda E - A) \neq 0$ for some $\lambda \in \mathbb{C}$.

Theorem

[5] Kronecker-Weierstraß canonical form
Suppose that the pair (E, A) is regular. Then, there exist invertible matrices P, Q such that

$$(E, A) = \left(P \begin{bmatrix} \mathrm{I_d} & 0 \\ 0 & N \end{bmatrix} Q, P \begin{bmatrix} J & 0 \\ 0 & \mathrm{I_a} \end{bmatrix} Q \right),$$

N is nilpotent, J and N are in Jordan form.

The number $\nu = min\{i : N^i = 0, \; N^{i-1} \neq 0\}$ is called index of the system.

Suppose that the pair (E, A) is regular, using Kronecker-Weierstraß canonical form to transform the system

$$
\begin{aligned}
& E\dot{x}(t) = Ax(t), \\
\rightarrow \quad & P^{-1}EQ\dot{y}(t) = P^{-1}AQy(t), \\
\rightarrow \quad & \begin{bmatrix} \mathrm{I_d} & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & \mathrm{I_a} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix},
\end{aligned}
$$

where N is nilpotent of index ν, J and N are in Jordan form.
We rewrite system in details

$$
\begin{aligned}
\mathrm{I}_d \dot{y}_1 &= J y_1, \\
N \dot{y}_2 &= \mathrm{I}_a y_2.
\end{aligned}
$$

Lemma

An equation of the form $N\dot{y}_2(t) = \mathrm{I}_a y_2(t) + g(t)$ *has a unique solution*

$$
y_2(t) = -\sum_{i=0}^{\nu-1} N^i g^{(i)}(t).
$$

Control strategy

$$\mathrm{I}_d \dot{y}_1 = J y_1, \leftarrow\text{-- We apply time-delayed feedback control here}$$
$$N \dot{y}_2 = \mathrm{I}_a y_2. \dashrightarrow \text{Has a unique solution } y_2 = 0$$

Time-delayed feedback control system

$$\mathrm{I}_d \dot{y}_1(t) = J y_1(t) - \tilde{K}\left(y_1(t) - y_1(t-\tau)\right), \; t \geq 0,$$

has been deeply investigated, [6], [3], etc.

After obtaining $\tilde{K}$, we have a desire feedback of an original system

$$K = Q \begin{bmatrix} \tilde{K} & 0 \\ 0 & 0 \end{bmatrix}.$$

Disadvantage: Computing Kronecker-Weierstraß canonical form is very complicated, expensive and numerically unstable.

Theorem

[2] (qz-decomposition)
If E and A are in $\mathbb{C}^{n,n}$, then there exist unitary Q and Z such that

$$Q^H EZ = S,$$
$$Q^H AZ = T,$$

are upper triangular.
If for some k, t_{kk} and s_{kk} are both zero, then $\sigma(E, A) = \mathbb{C}$. Otherwise

$$\sigma(E, A) = \{\frac{t_{ii}}{s_{ii}} : s_{ii} \neq 0\}.$$

If $s_{ii} = 0$, and $t_{ii} \neq 0$ then we call $\frac{t_{ii}}{s_{ii}}$ an infinite eigenvalue of (E, A).

Algorithm

Input: Matrix pair (E, A) be regular.
Output: Gain matrix K.
Step 1: Using the qz-decomposition to decouple the system

$$\left[\begin{bmatrix} \tilde{E}_1 & \tilde{E}_2 \\ 0 & \tilde{E}_4 \end{bmatrix}, \begin{bmatrix} \tilde{A}_1 & \tilde{A}_2 \\ 0 & \tilde{A}_4 \end{bmatrix}, Q, Z\right] = qz(E, A),$$

where E_1, E_4, A_1, A_4 are upper triangular,

- spectrum of $(\tilde{E}_1, \tilde{A}_1)$ contains finite eigenvalues of (E, A),
- spectrum of $(\tilde{E}_4, \tilde{A}_4)$ contains infinite eigenvalues of (E, A).

Step 2: Computing the time-delay feedback control of the subsystem

$$\tilde{E}_1 \dot{y}_1(t) = \tilde{A}_1 y_1(t) - \tilde{K}\left(y_1(t) - y_1(t-\tau)\right),$$

Step 3: The desire gain matrix K is $K = Z' \begin{bmatrix} \tilde{K} & 0 \\ 0 & 0 \end{bmatrix}$.

In step 2 we need to calculate the feedback gain K for the subsystem

$$\tilde{E}_1 \dot{y}_1(t) = \tilde{A}_1 y_1(t) - \tilde{K}\left(y_1(t) - y_1(t-\tau)\right).$$

We notice that

- both $\tilde{E}_1$ and $\tilde{A}_1$ are upper triangular,
- the main diagonal of $\tilde{E}_1$ does not contain 0 element.

Therefore, $\tilde{K}$ can be chosen in the form

$$\tilde{K} = p\tilde{E}_1,$$

where p is a scalar, adjustable parameter.

Moreover, structure of $(\tilde{E}_1, \tilde{A}_1)$ suggests an extension on the theory of time-delayed feedback control of ODEs, for example, Floquet exponent, [6].

We could not use Kronecker-Weierstraß canonical form any more.

Extra assumption: $D = J_V f(U, V)|_{[U_0, V_0]}$ is invertible. ←-- normally hyperbolic system

We transform the system as follows

$$\begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U}(t) \\ \dot{V}(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U(t) \\ V(t) \end{bmatrix},$$

$$\to \begin{bmatrix} \mathrm{I} & -BD^{-1} \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U}(t) \\ \dot{V}(t) \end{bmatrix} = \begin{bmatrix} \mathrm{I} & -BD^{-1} \\ 0 & D^{-1} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} U(t) \\ V(t) \end{bmatrix},$$

$$\to \begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U}(t) \\ \dot{V}(t) \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & 0 \\ -D^{-1}C & \mathrm{I} \end{bmatrix} \begin{bmatrix} U(t) \\ V(t) \end{bmatrix}.$$

System in details

$\dot{U}(t) = \left(A - BD^{-1}C\right) U(t),$ ←-- We apply time-delayed feedback control here

$V(t) = -D^{-1}CU(t).$

Partitioning K and set

$$(E_0, A_0, B_0) = \left(\begin{bmatrix} \mathrm{I_n} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A - K_{11} & B - K_{12} \\ C - K_{21} & D - K_{22} \end{bmatrix}, \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \right)$$

We must choose K such that the matrix pair $(E_0, A_0 - K)$ is regular and of index at most one.

The necessary and sufficient condition is that $D - K_{22}$ is invertible.

The system reads in detail

$$\begin{aligned} \dot{U}(t) &= (A - K_{11})U(t) + (B - K_{12})V(t) + K_{11}U(t-\tau) + K_{12}V(t-\tau), \\ 0 &= (C - K_{21})U(t) + (D - K_{22})V(t) + K_{21}U(t-\tau) + K_{22}V(t-\tau). \end{aligned}$$

Since $D - K_{22}$ is invertible, then

$$\mathcal{D}(V(t)) = -(D - K_{22})^{-1}\Big((C - K_{21})U(t) + K_{21}U(t-\tau)\Big),$$

where $\mathcal{D}(V(t)) := V(t) + (D - K_{22})^{-1}K_{22}V(t-\tau)$.

The necessary condition for the stability is
[A1.] The difference operator

$$\mathcal{D}(V(t)) = V(t) + (D - K_{22})^{-1} K_{22} V(t - \tau),$$

is stable, i.e., the equation $\mathcal{D}(V(t)) = 0$ is asymptotically stable.

Sufficient condition for [A1.] is given by
[A2.] There exist some matrix operator norm $\|\cdot\|$ such that

$$\|(D - K_{22})^{-1} K_{22}\| < 1.$$

[A3.] The matrix $(D - K_{22})^{-1} K_{22}$ is Schur-Cohn stable, i.e., its spectrum is inside the unit circle in the complex plane.

Hypothesis

We assume that a block K_{22} in the feedback matrix K can be chosen such that one of the conditions **[A1]-[A3]** *is satisfied.*

Lemma

(E. Fridman [1])
Suppose that our time-delayed feedback system satisfies the Hypothesis. Moreover, assume that there exist positive numbers α, β, γ and a continuous functional $V : C([\tau, 0], \mathbb{C}^n) \to \mathbb{R}$ such that

$$\beta|\phi_1(0)|^2 \leqslant V(\phi) \leqslant \gamma\|\phi\|^2,$$
$$\dot{V}(\phi) \leqslant -\alpha|\phi(0)|^2,$$

and the function $\bar{V}(t) = V(x_t)$ is absolutely continuous for $x(t)$ satisfying the delay DAE, then the delay DAE is asymptotically stable.

V is called Lyapunov-Krasovskii functional along the orbit of the delay DAE.
The usually chosen functional is

$$V(x_t) = x^T(t)EPx(t) + \int_{t-\tau}^{t} x^T(s)Qx(s)ds,$$

where P and Q are symmetric positive definite.

Theorem

Assume that the Hypothesis holds. Then, the system is asymptotically stable if there exists two matrices $P, Q \in \mathbb{R}^{n,n}$ *such that the following LMIs hold*

$$\begin{cases} P > 0, \quad Q > 0, \\ \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} P = P \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \geq 0, \\ \begin{bmatrix} -Q & K^T P \\ P^T K & A_0^T P + P^T A_0 + Q \end{bmatrix} < 0, \end{cases}$$

where $A_0 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} - K.$

The last equation can be written in the Riccati form

$$A_0^T P + P^T A_0 + Q + P^T K Q^{-1} K^T P < 0.$$

By adjusting K, we aim to solve the system of LMIs and obtain a desire K if with that K the LMI system has at least one solution.

Example 1 [1]
Consider the system

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{U}(t)\\ \dot{V}(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -1\end{bmatrix}\begin{bmatrix}U(t)\\ V(t)\end{bmatrix},\ t>0.$$

The equilibrium $\begin{bmatrix}0\\ 0\end{bmatrix}$ is unstable since the system has a unique eigenvalue $\lambda = 2$.
We choose the feedback gain of the type

$$K=\begin{bmatrix}K_{11} & K_{12}\\ K_{21} & K_{22}\end{bmatrix}=\begin{bmatrix}\alpha & 1\\ -2 & \beta\end{bmatrix}$$

then we get

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{U}(t)\\ \dot{V}(t)\end{bmatrix}=\begin{bmatrix}-\alpha & 0\\ 0 & -1-\beta\end{bmatrix}\begin{bmatrix}U(t)\\ V(t)\end{bmatrix}+\begin{bmatrix}\alpha & 1\\ -2 & \beta\end{bmatrix}\begin{bmatrix}U(t-\tau)\\ V(t-\tau)\end{bmatrix},\ t>0.$$

The Hypothesis becomes

$$|(-1-\beta)^{-1}\beta|<1 \Leftrightarrow 0<\beta \text{ or } \frac{-1}{2}<\beta<0.$$

Using the LMI toolbox in Matlab to solve LMI system we obtain

$$P = \begin{bmatrix} 0.4142 & 0 \\ 0 & 0.4142 \end{bmatrix}, \quad Q = \begin{bmatrix} 13.0326 & -12.4530 \\ -12.4530 & 12.2895 \end{bmatrix},$$

with the gain matrix $K = \begin{bmatrix} 0.9800 & 1.0000 \\ -2.0000 & 0.9800 \end{bmatrix}$.

The delayed feedback system in the semi-periodic case is

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{U}(t) \\ \dot{V}(t) \end{bmatrix} = \begin{bmatrix} A - K_{11} & B \\ C - K_{21} & D \end{bmatrix} \begin{bmatrix} U(t) \\ V(t) \end{bmatrix} + \begin{bmatrix} K_{11} & 0 \\ K_{21} & 0 \end{bmatrix} \begin{bmatrix} U(t-\tau) \\ V(t-\tau) \end{bmatrix}, \quad t > 0,$$

If D is invertible, then

$$\dot{U}(t) = (A - K_{11} - BD^{-1}C + BD^{-1}K_{21})U(t) + (K_{11} - BD^{-1}K_{21})U(t-\tau),$$
$$V(t) = -D^{-1}\left((C - K_{21})U(t) + K_{21}U(t-\tau)\right)$$

Asymptotic stability of the 1^{st} equation dominates asymptotic stability of system and hence, no hypothesis is needed.

Theorem

The system is asymptotically stable if there exists two matrices $P, Q \in \mathbb{R}^{n,n}$ such that the following LMIs hold

$$P > 0, Q > 0,$$
$$\begin{bmatrix} -Q & B_1^T P \\ P^T B_1 & A_1 P + P^T A_1 + Q \end{bmatrix} < 0,$$

where $A_1 := A - BD^{-1}C - \left(K_{11} - BD^{-1}K_{21}\right),\ B_1 := K_{11} - BD^{-1}K_{21}.$

Remark

The desire feedback K is obtained by choosing $K_{21} = 0$, and K_{11} stabilize the system

$$\dot{U}_1(t) = \left(A - BD^{-1}C\right) U(t) - K_{11}(U(t) - U(t-\tau)),\ t \geq 0.$$

This result coincides with the result obtained by using eigenvalue method.

Observations on the singular pair case

Suppose that $\left(\begin{bmatrix} \mathrm{I} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$ is not regular, may not even square.

Consequently, we **do not** have

- Kronecker-Weierstraß form,
- *qz*-decomposition.

We consider a new concept for DAEs: **strangeness-index** (denoted by μ) introduced by Kunkel & Mehrmann [5].

Idea: Using differentiation (μ times) and equivalent transformations (left-right multiplications by P, Q) to transform the system into

$$\begin{bmatrix} \mathrm{I_d} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & \mathrm{I_a} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} U(t) \\ V(t) \end{bmatrix}.$$

And then, we fix 2^{st} equation (constraint) apply the time-delayed feedback control on 1^{st} equation (dynamic); or we can use x_3 like a control.

Conclusion

- Stabilization by time-delayed feedback control of Pyragas type has been studied by both eigenvalue method and Lyapunov functional method.
- Experiments show that eigenvalue method converges much faster than Lyapunov functional method.
- Using eigenvalue method is computationally cheap, which is suitable for discretized DA-PDEs.

Outlook

- The semi-periodic orbits case is essentially open.
- Stabilization of periodic orbits of DA-PDEs with hysteresis ... it is beyond my dream?

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Thank you for your attention!

Suggestions and comments are welcome!