An attempt at modelling memory effects in dynamics of spread of infectious disease

Dmitrii Rachinskii

Department of Applied Mathematics, University College Cork, Ireland

International Workshop on Hysteresis and Slow-Fast Systems
Lutherstadt Wittenberg, December 12-14, 2011

Joint work with:

Tom Kelly

Michael O'Callaghan

(University College Cork)

Andrei Korobeinikov

(University of Limerck)

Alexander Pimenov

(WIAS, Berlin)

SIR (Susceptible-Infectious-Recovered) compartmental model of Kermack and McKendrick (1927)

- Foot and Mouth Disease
- Severe Acute Respiratory Syndrome (SARS)
- West Nile virus
- ...

[1] Ferguson, N. M.; Donnelly, C. A.; Anderson, R. M. The Foot-and-Mouth Epidemic in Great Britain: Pattern of Spread and Impact of Interventions, **Science** 11 May 2001: 292, No. 5519, 1155–1160.

[2] Lipsitch, M. et all. Transmission Dynamics and Control of Severe Acute Respiratory Syndrome, **Science** 20 June 2003: 300, No. 5627, 1966–1970.

[3] Wonham, M.; Lewis, M.; Renclawowicz, J.; van den Driessche, P. Transmission assumptions generate conflicting predictions in host-vector disease models: a case study in West Nile virus, **Ecology Letters** 2006, 9, 706–725.

SIR model

$$S \longrightarrow I \longrightarrow R$$

- Horizontal transmission of the disease
- Law of mass action
- Recovered individuals retain immunity
- Population size remains constant: $S + I + R = N = const$ (characteristic time of epidemics much shorter than demographic processes)

$$\begin{aligned} \dot{S} &= -\beta SI \\ \dot{I} &= \beta SI - \gamma I \\ \dot{R} &= \gamma I \end{aligned}$$

β - incidence rate (infection rate constant)
γ - recovery rate

SIR model with birth and mortality (recruitment and departure)

$$\begin{aligned}
\dot{S} &= b - \beta SI - \mu S \\
\dot{I} &= \beta SI - \gamma I - (\mu + \sigma) I \\
\dot{R} &= \gamma I - \mu R
\end{aligned}$$

β - incidence rate (infection rate constant)
γ - recovery rate

μ - natural mortality rate
σ - rate of mortality inflicted by the disease
b - birth or recruitment rate

Basic reproduction number

$$R_0 = \frac{b\beta}{\mu(\gamma+\mu+\sigma)}$$

- the number of secondary cases produced by a single infected individual introduced into an entirely susceptible population

$R_0 \leq 1$ - infection dies out
Infection free equilibrium $I = 0$, $S = b/\mu$ is globally asymptotically stable.

$R_0 > 1$ - infection persists in the population
Endemic equilibrium is globally asymptotically stable.

People respond to the advent of an epidemic and adapt their behaviour to the danger

Change the ‘usual’ behaviour to a ‘safer’ behaviour:

- refraining from potentially dangerous contacts
- using vaccination in a case of influenza
- taking an extra portion of vitamin C in the case of a common cold
- other precautionary measures (adjusting life style...)

Each individual can choose between one of two possible modes of behaviour, **Risky** and **Safe**

$$S = S_R \cup S_S$$

Fractions of individuals who have adopted each behaviour:

$$P_R(t) + P_S(t) = 1$$

Effective incidence rate $\beta_S P_S(t) + \beta_R P_R(t)$ replaces β:

$$\begin{aligned} \dot{S} &= b - (\beta_S P_S + \beta_R P_R) SI - \mu S \\ \dot{I} &= (\beta_S P_S + \beta_R P_R) SI - (\gamma + \mu + \sigma) I \end{aligned}$$

Fraction of population P_S is affected by aggregated temporary stimuli such as

- awareness of the population about the current severity of the epidemic
- availability of effective and inexpensive medicines
- media and/or administrative pressure, etc.

Information that can influence a shift in the behaviour comes from a variety of sources, importantly, from

- Mass media, including any kind of messages spread by reporters, health officials, or volunteers

Closing the model

- How does P_S respond to aggregated stimuli (pressure) A to change behaviour to the Safe mode?
- How A depends on I, S?

NB: $A(t)$ can be the amount of money invested into promoting Safe bahaviour

Assumption that P_S is a function of A, A is a function of I, S leads to the SIR model with a nonlinear infection rate

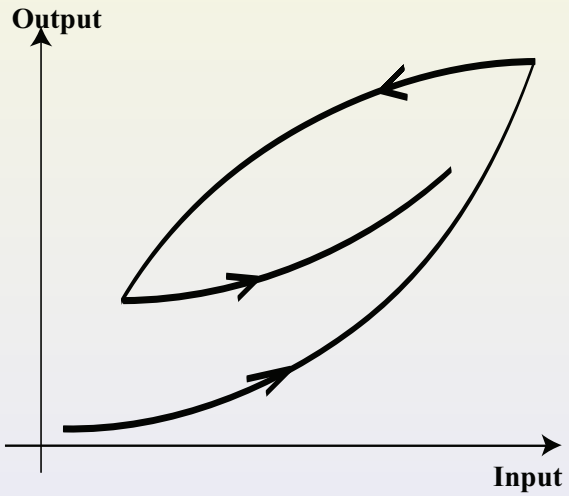
$$\begin{aligned} \dot{S} &= b - f(S,I) - \mu S \\ \dot{I} &= f(S,I) - (\gamma + \mu + \sigma) I \end{aligned}$$

Under the natural concavity assumption $\frac{\partial^2 f}{\partial I^2} \leq 0$, the basic reproduction number R_0 defines which equilibrium is globally stable in the same way as for the classical SIR model.

Memory

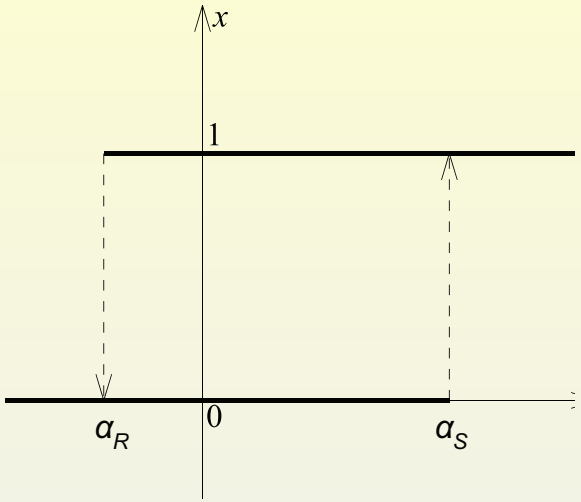
The relationship between P_S and A can have **memory**, that is $P_S(\tau)$ at a particular moment τ is determined by the history of the aggregate stimulus $A(t)$, $t \leq \tau$.

$A(t)$ - input, $P_S(t)$ - output



- On an individual level, we assume that the value of A at which an individual person switches to Safe mode is greater than the value at which the person switches back.

- A binary switch with two thresholds (relay) is adopted as a model for the behaviour of an individual.



  State $x = 1$, the **Safe** mode; state $x = 0$, the **Risky** mode.
- We treat the total population as an infinite ensemble. We assume that individuals behave independently.
- The pairs of thresholds (α_R, α_S) are distributed with some integrable density $\rho(\alpha_R, \alpha_S)$ (which is independent of time).
- The state $x(t, \alpha_R, \alpha_S)$ of the system describes the distribution of population between the Risky and Safe modes of behaviour at any moment $t \geq t_0$.

$x(t, \alpha_R, \alpha_S)$ is 0 whenever the individuals with the pair of thresholds (α_R, α_S) chooses the Risky behaviour, and 1 whenever they choose the Safe behaviour.

$A(t) \longrightarrow$ evolution of $x(t, \alpha_R, \alpha_S) \longrightarrow$ dynamics of $P_S(t)$

Aggregation:

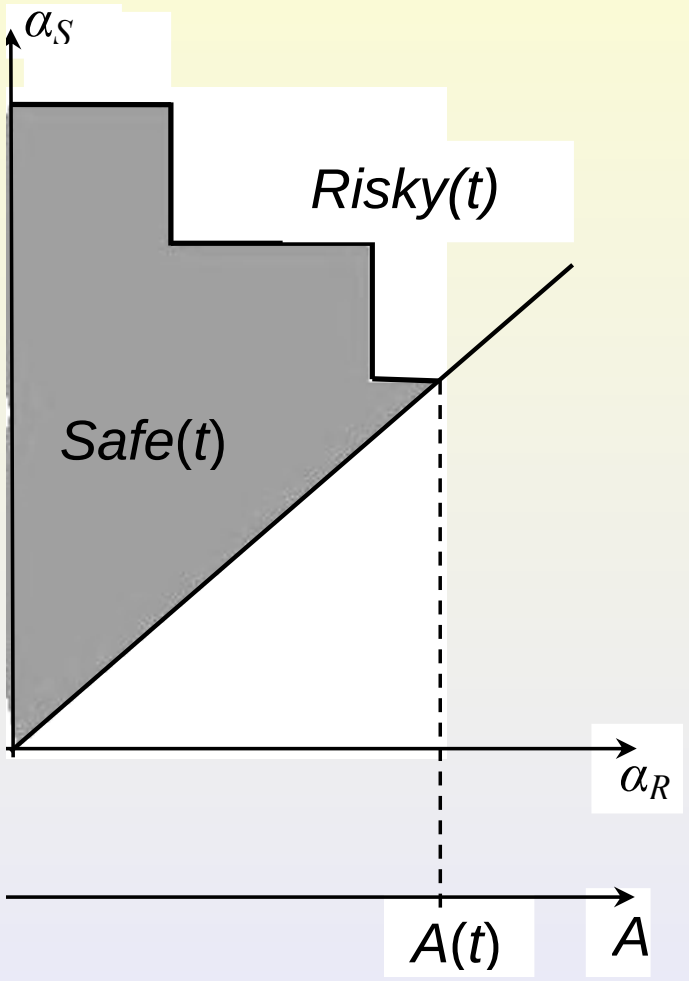
$$P_S(t) = \int\int_{\alpha_R < \alpha_S} x(t, \alpha_R, \alpha_S) \rho(\alpha_R, \alpha_S)\, d\alpha_R d\alpha_S$$

This is the **Preisach operator** $\mathscr{P}[x_0] : A(t) \to P_S(t)$:

$$P_S(t) = (\mathscr{P}[x_0]A)(t)$$

Here $x_0 = x_0(\alpha, \beta)$ is the initial state.

Preisach half-plane



The state $x(t, \alpha_R, \alpha_S)$ is the characteristic function of the grey domain, which changes in response to the variation of the input.

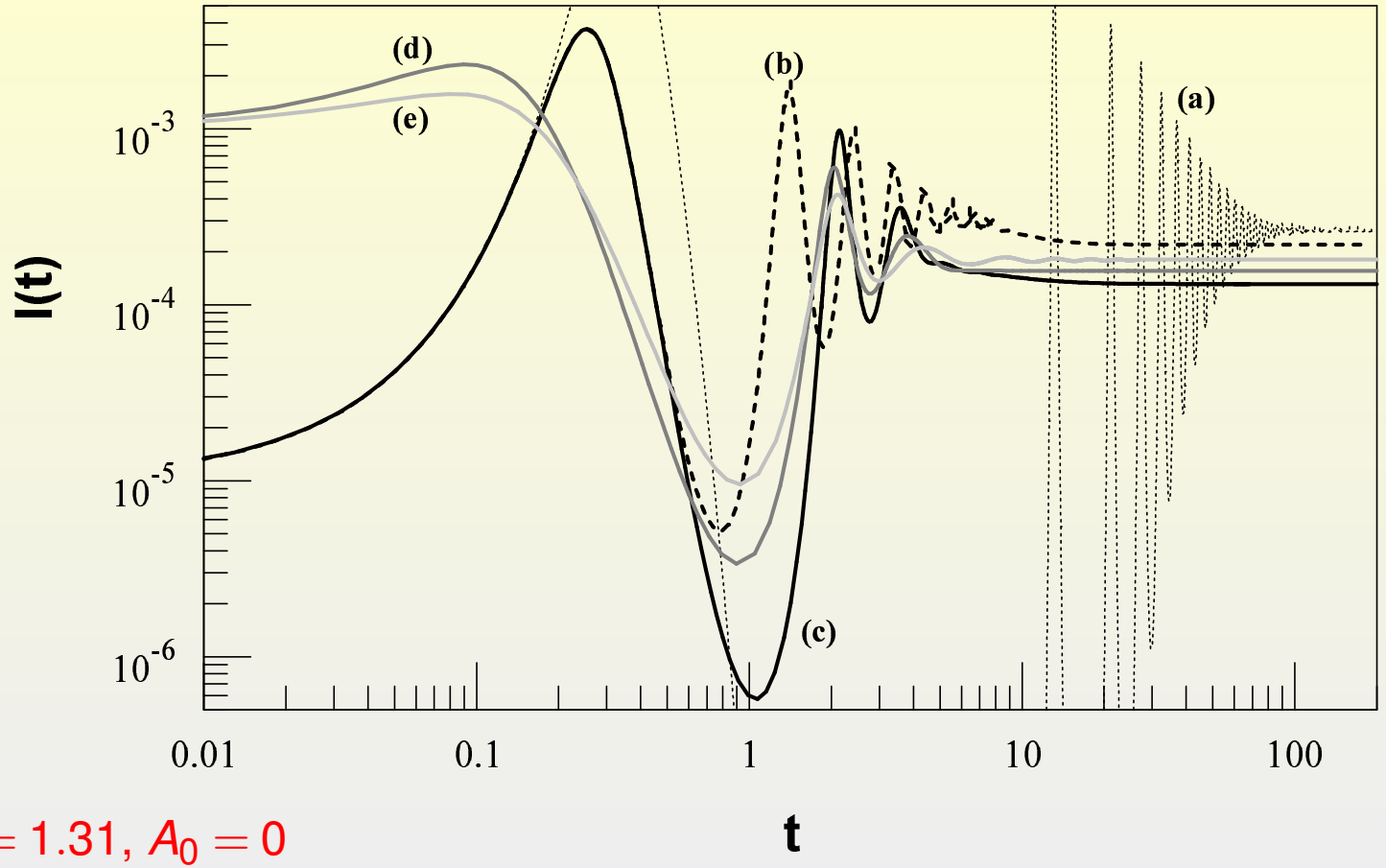
Closed SIR model with memory

$$
\begin{aligned}
\dot{S} &= b-(\beta_S P_S+\beta_R(1-P_S)SI-\mu S \\
\dot{I} &= (\beta_S P_S+\beta_R(1-P_S))SI-(\gamma+\mu+\sigma)I \\
P_S(t) &= (\mathscr{P}[x_0]A)(t) \\
A &= \kappa I+A_0
\end{aligned}
$$

Two susceptible groups with different incidence rates, $\beta_S < \beta_R$, where the sizes of both groups are given as fractions of the total susceptible population, $P_S(t)$ and $P_R(t) = 1 - P_S(t)$ respectively, and the fraction $P_S(t)$ is determined by the current value and history of dynamics of stimuli $A(t) = \kappa I(t) + A_0$.

Version with inductive delay

$$
\begin{aligned}
\dot{S} &= b - (\beta_S P_S + \beta_R (1 - P_S)) S I - \mu S \\
\dot{I} &= (\beta_S P_S + \beta_R (1 - P_S)) S I - (\gamma + \mu + \sigma) I \\
P_S(t) &= (\mathscr{P}[x_0] y)(t) \\
\dot{P}_S &= k(A(t) - y) \\
A &= \kappa I + A_0
\end{aligned}
$$



$R_0 = 1.31$, $A_0 = 0$

(a) Standard SIR, $P_S = 0$ - **dotted** $(I(0) = 10^{-5}, S(0) = 0.999)$
(b) Safe and Risky modes, no memory, $P_S = y^2$ - **dashed** $(I(0) = 10^{-5}, S(0) = 0.999)$
(c)-(e) SIR with memory - **solid**

(c) $I(0) = 10^{-5}, S(0) = 0.999$; (d) $I(0) = 10^{-3}, S(0) = 0.9$; (e) $I(0) = 10^{-3}, S(0) = 0.85$

$b = \mu = 0.1, \beta_R = 120, \beta_S = 1, \gamma = 365/4, \sigma = 0, \kappa = 1000, k = 1, y(0) = 0.01$

- Model with memory has multiple endemic equilibria
- The higher the initial peak of infection, the smaller the infected population at the endemic equilibrium

Now assume $A_0 > 0$

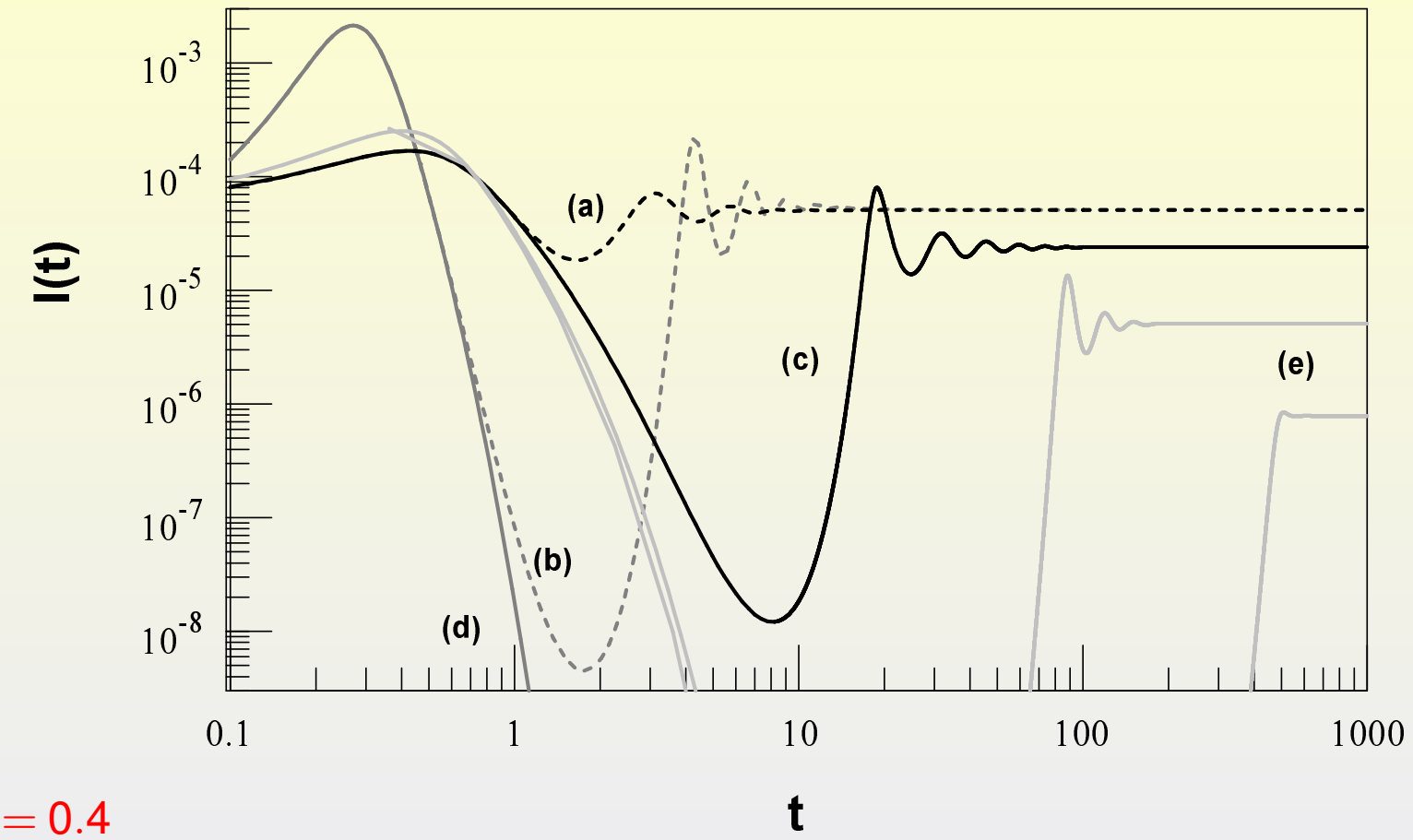
Effective basic reproduction number

$$\widetilde{R_0} = \frac{b(\beta_R(1-P_S)+\beta_S P_S)}{\mu(\gamma+\mu+\sigma)}$$

where P_S is evaluated at an infection free equilibrium state.

P_S is not necessary equal to zero, as emergence of a pathogen can be anticipated, and pro-active intervention measures can be taken prior to its actual appearance in a community.

$P_S > 0$ reflects the impact of an intervention policy or history.



$A_0 = 0.4$

(a)-(b) Safe and Risky modes, no memory, $P_S = y^2$ - **dashed**

(a) $I(0) = 10^{-5}, S(0) = 0.999$; (b) $I(0) = 5 \times 10^{-5}, S(0) = 0.953$

(c)-(e) SIR with memory - **solid**

(c) $I(0) = 10^{-5}, S(0) = 0.999$; (d) $I(0) = 5 \times 10^{-5}, S(0) = 0.953$

Equilibrium states

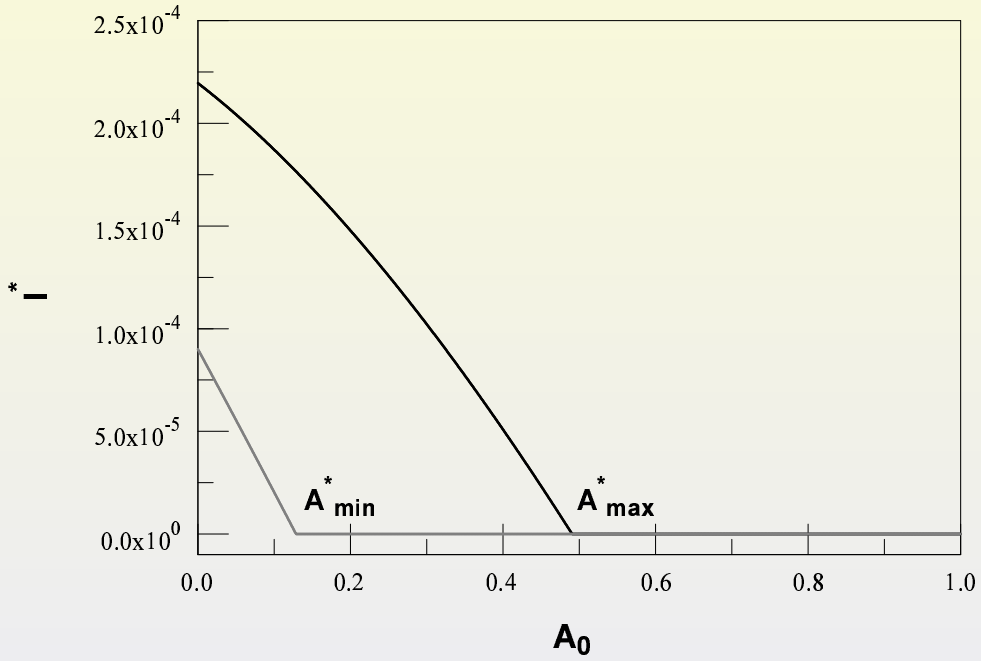
- Equilibrium state is described by a quadruplet (I, S, y, P_S), where P_S measures the fraction of population that adapted the Safe behaviour. Moreover, the state $x(\alpha, \beta)$ of the Preisach operator can be different for the same P_S.
- P_S is an additional degree of freedom which transforms an isolated equilibrium into a connected continuum of equilibria with different proportion of people in Safe mode.
- It is more convenient to use the variable

$$r = \frac{b(\beta_R(1 - P_S) + \beta_S P_S)}{\mu(\gamma + \mu + \sigma)}$$

  instead of P_S. An equilibrium is described by the quadruplet (I, S, y, r). At an infection free equilibrium, $r = \widetilde{R_0}$.
- There is some numerical indication that stability of an equilibrium is defined by the sign of $r - 1$ as in the case without memory.

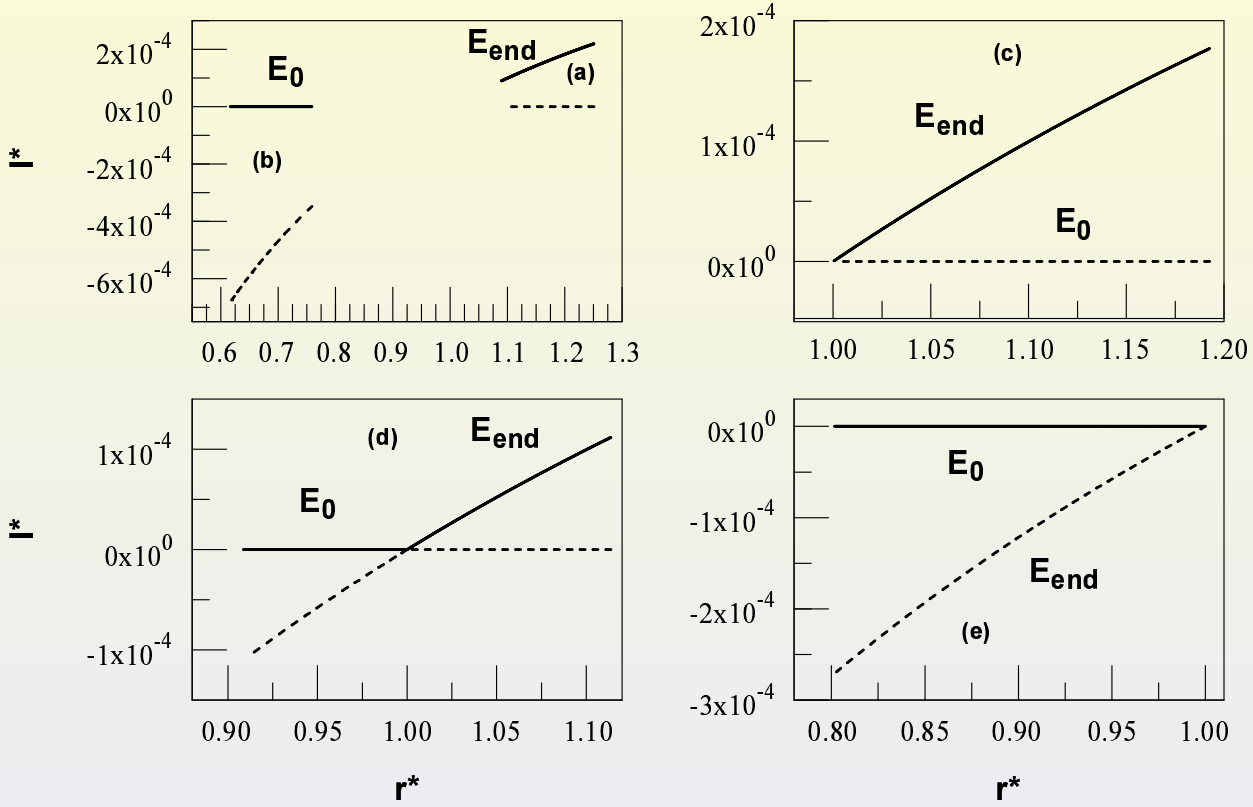
E_0 = curve of infection free equilibria ($I = 0$)

E_{end} = curve of endemic equilibria ($I > 0$)



Projection of the curve E_{end} of endemic equilibrium states on I-axis depending on parameter A_0 in $A = \kappa I + A_0$.

Projection of the curves E_0, E_{end} of equilibrium states on (r, I)-plane for different values of the parameter A_0



(a) $0 \le A_0 < A^*_{min}$; E_{end} is the global attractor, E_0 is unstable

(d) $A^*_{min} < A_0 < A^*_{max}$; the solid corner is the global attractor

(b) $A_0 > A^*_{max}$; E_0 is the global attractor

Conjectures based on numerical simulations for a limited number of initial conditions and a bit of analysis

- An infection free equilibrium is stable if $r < 1$ and unstable if $r > 1$
- Positive endemic equilibria are stable and have $r > 1$
- The connected set of all locally stable infection free and endemic equilibria is a global attractor for the domain $I > 0, S \geq 0$
- If $A^*_{min} < A_0 < A^*_{max}$, then a trajectory converges to an infection-free equilibrium if $y(\tau)$ at some point exceeds

$$Y = \frac{1}{2A_0}\left(A_0^2 + \frac{\beta_R}{\beta_R - \beta_S} - \frac{\mu(\gamma+\mu+\sigma)}{b(\beta_R - \beta_S)}\right)$$

and to an endemic equilibrium otherwise (the effect of permanent memory which is not erased even for $I = 0$). $\kappa I + A_0$ should also exceed Y to ensure $I \to 0$.

Further work, problems, questions

- Hopf bifurcation in SIRS model with saturable incidence rate
- Hopf bifurcation in a spatially inhomogeneous extension of the above SIRS model with diffusion; periodic spatially inhomogeneous solutions
- Predator-Prey model where prey has two modes of behaviour, or two patches: a Risky patch with plentiful food (higher carrying capacity and higher attack rate of the predator) and a Safe patch (lower carrying capacity and lower attack rate of the predator)
- Hysteresis (memory) in diffusion in spatially distributed population dynamics: How should it be modelled?
- Diffusion of population between compartments with different thresholds; Preisach operator the density $\rho(t; \alpha_R, \alpha_S)$
- Stability of systems with the Preisach operator
-