(Quasivariational) Sweeping processes on functions of bounded variation

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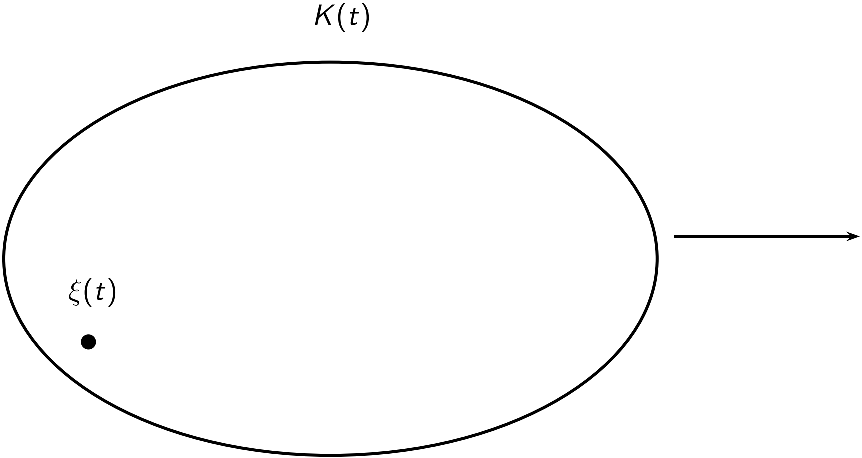
Agenda

The sweeping process

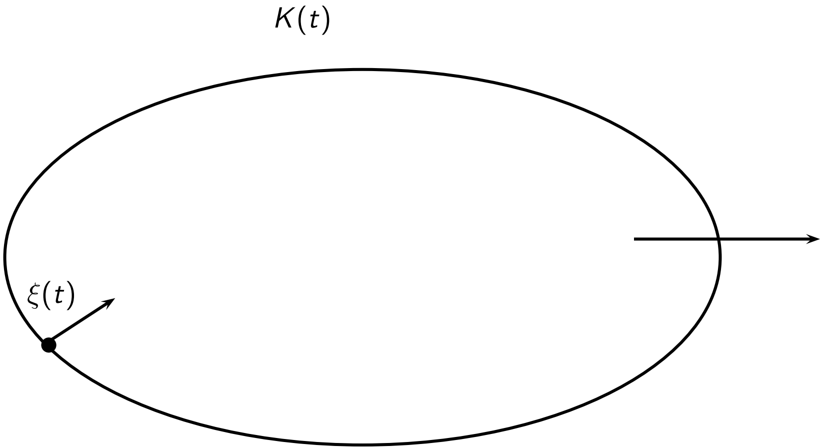
Definition (Moreau 1972/77)

Let $K(t) \subset X$ be a time dependent closed convex set and $\xi_0 \in K(0)$. A function $\xi \in W^{1,1}(0, T; X)$ is a solution to the sweeping process if $\xi(0) = \xi_0$ and

$$-\dot{\xi}(t) \in N_{K(t)}(\xi(t)) \text{ a.e. in } [0, T].$$



$K(t)$
$\xi(t)$



$K(t)$
$\xi(t)$

Existence and uniquenss of a solution

Theorem (Moreau 1972/77)

If there exists a monotone increasing function $w \in W^{1,1}(0,T)$ such that

$$\forall t, s \in [0,T], t \le s : d_H(K(t), K(s)) \le w(s) - w(t)$$

then there exists a unique solution $\xi \in W^{1,1}(0,T;X)$ to the sweeping process.

Approximation of solutionss by implicit time discretization

Implicit time discretisation

Let $(t_n)_{n=0}^N$ be defined by $t_n = \frac{n}{N} T$. Set $K_n = K(t_n)$. The time discretization of the sweeping process is given by

$$-\frac{1}{h}(\xi_n - \xi_{n-1}) \in N_{K_n}(\xi_n), \quad h = \frac{T}{N}$$

This is equivalent to

$$\xi_n \in K_n \;\wedge\; \langle \xi_{n-1} - \xi_n, \xi_n - y \rangle \geq 0 \;\; \forall y \in K_n$$

or

$$\xi_n = P_{K_n}(\xi_{n-1})$$

Extension to BV

Want to use BV inputs.
Requirements: Formulation has a meaning on BV and coincides with the sweeping process on $W^{1,1}$.
Still a lot of possibilities...

- Moreau, 1972/77
- Krejčí and Laurençot, 2002
- Mielke and Theil, 2002/04
- Recupero, 2009/10

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Definition (Sweeping process in the Kurzweil formulation)

Let $K(t) \subset X$ be closed and convex and $\xi_0 \in K(0)$. A function $\xi \in BV(0, T; X)$ satisfies the Kurzweil formulation if $\xi(0) = \xi_0$, $\xi(t) \in K(t)$ for all $t \in [0, T]$ and

$$\int_0^T \langle \xi(t+) - y(t), d\xi(t) \rangle \geq 0$$

for all $y \in G(0, T; X)$ such that $y(t) \in K(t+)$.

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Theorem (Krejčí and Liero, 2009)

There exists a unique solution to the sweeping process in BV

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Approximation by step functions $\rightarrow$ Projections

Agenda

Setting

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- Let $\xi \in BV(0, T; X)$ be the solution of the sweeping process.
- Question: What are the continuity properties of the map

$$(u, r, x_0) \mapsto \xi\,?$$

Answer: It depends on the shape of the convex set

In the case of absolutely continuous functions local Lipschitz continuity known if

- Z polyhedron (Krejčí and Vladmimirov, 2003)
- Z smooth (Brokate, Krejčí and Schnabel, 2004)

Theorem (Krejčí and R., 2010)

If $Z(r)$ has smooth boundary and depends smoothly on r, that is

$$d_H(Z(r), Z(s)) \leq c|r-s|\,,$$

for all r the outer normals are unique and

$$\forall x \in \partial Z(r), y \in \partial Z(s) : |n(r,x) - n(s,y)| \leq C\big(|r-s| + |x-y|\big)$$

then the sweeping process is locally Lipschitz continuous on BV, in the sense that for two solution ξ, η of the sweeping process with input (u, r, ξ_0) and (v, s, η_0) respectively it houlds

$$\begin{aligned}\mathrm{Var}(\xi - \eta) \leq \\ &C(u,r,v,s)\big(\mathrm{Var}(u-v) + \mathrm{Var}(r-s) + |\xi_0 - \eta_0|\big)\end{aligned}$$

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Quasivariational sweeping processes

Definition (Kunze and Monteiro Marques, 1998)

Let $K(t,\cdot) \subset X$ be a family of closed convex set and $\xi_0 \in K(0,\xi_0)$. A function $\xi \in W^{1,1}(0,T;X)$ is a solution to the quasivariational sweeping process if $\xi(0) = \xi_0$ and

$$-\dot{\xi}(t) \in \partial I_{K(t,\xi(t))}(\xi(t)) \text{ a.e. in } [0,T].$$

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- Existence results due to Kunze and Monteiro Marques (1998)
- First uniqueness result due to Brokate, Krejčí and Schnabel (2004) for smooth and bounded convex sets
- Generalization to smooth and unbounded convex sets due to Mielke and Rossi (2007)

Quasivariational sweeping processes on BV via the Kurzweil formulation

Theorem (Brokate, Krejčí and Schnabel, 2004)

Let $K(t,r)$ be a family of smooth and uniformly bounded convex sets, such that

$$\exists \mu < 1 : d_H(K(t,r), K(t,s)) \leq \mu |r - s| \,,$$
$$|n(r,x) - n(s,y)| \leq C\big(|r - s| + |x - y|\big)$$

and then there exists a unique solution to the quasivariational sweeping process on $W^{1,1}$.

Quasivariational sweeping processes on BV via the Kurzweil formulation

Theorem (R., 2010)

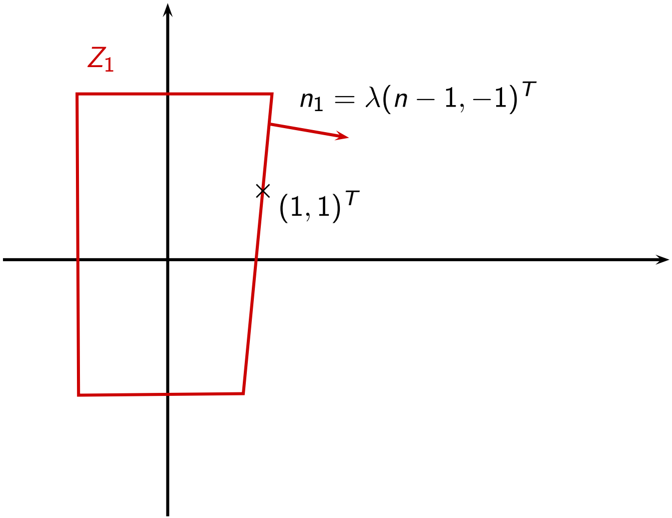
Let $K(t,r)$ be a family of smooth convex sets which smoothly depend on $r \in X$, such that

$$\exists \mu < 1 : d_H(K(t,r), K(t,s)) \leq \mu |r-s| \,,$$
$$|n(r,x) - n(s,y)| \leq C\big(|r-s| + |x-y|\big)$$

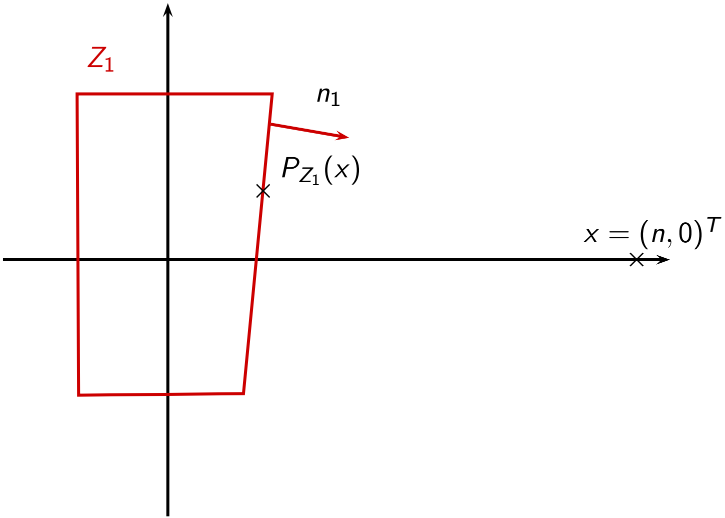
and if the size of the discontinuities remains small enough, i.e.

$$\forall t \in [0,T] : \quad d_H\big(K(t,r), K(t+,r)\big) + |u(t+) - u(t)| \leq c$$

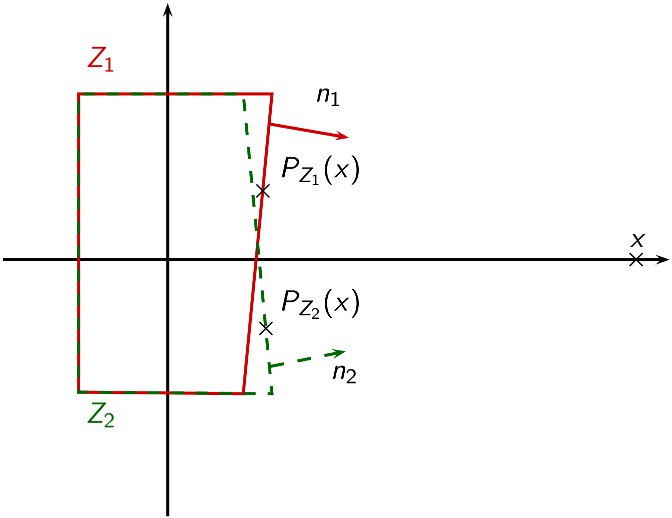
then there exists a unique solution to the quasivariational sweeping process on BV.



Z_1
$n_1 = \lambda(n-1,-1)^T$
$(1,1)^T$



Z_1
n_1
$P_{Z_1}(x)$
$x = (n, 0)^T$



Z_1
n_1
$P_{Z_1}(x)$
x
$P_{Z_2}(x)$
n_2
Z_2

A lemma on the projection

Lemma (Krejčí and R., 2010)

Assume $Z : \mathcal{R} \to 2^X$ satisfies the following conditions. $Z(r) \subset X$ is nonempty, closed and convex for all $r \in \mathcal{R}$ and there exist functions $j : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$ and $\psi : [0, \infty) \to [0, \infty)$ such that

$$|n_r - n_s| \leq j(r, s) + \psi(|x - y|)$$

for all $x \in \partial Z(r)$, $y \in \partial Z(s)$ and $n_r \in \partial I_{Z(r)}(x)$, $n_s \in \partial I_{Z(s)}(y)$. Then for all $u \in X$ it holds

$$|P_r(u) - P_s(u)| \leq \\ d_H(r, s) + \min\{d(u, Z(r)), d(u, Z(s))\}\left(j(r, s) + \psi(d_H(r, s))\right).$$

Strategy of the proof

Global strategy:

- Decompose BV function in intervalls with very small jumps $|u(t) - u(t+)| \leq \varepsilon$ and a (finite) number of larger jumps $\varepsilon \leq |u(t) - u(t+)| \leq c$.
- For both parts use Banach's contraction principle.

On the intervalls:

- Approximate function of bounded variation with step functions
- Step functions $\rightarrow$ sequence of projections

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Rate independence

Definition

An operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathrm{Map}(0, T; X) \to \mathrm{Map}(0, T; X)$ is called rate independent if

$$\mathcal{A}(u \circ \phi) = \mathcal{A}(u) \circ \phi$$

for all $\phi : [0, T] \to [0, T]$ monotone increasing, surjective sucht that $u \circ \phi \in \mathcal{D}(\mathcal{A})$.

Precise assumption for the result by Brokate, Krejčí and Schnabel (2004)

Let there exist $C > 0$ such that $0 \in Z(r) \subset B_C(0)$ for all $r \in \mathcal{R}$. Moreover assume that the partial Fréchet derivatives $\partial_r M(r,x) \in \mathcal{R}'$ and $\partial_x M(r,x) \in X$ exist for every $r \in \mathcal{R}$ and every $x \in X \setminus \{0\}$. We denote $B(r,x) = \frac{1}{2}M^2(r,x)$. The maps

$$
\begin{aligned}
J(r,x) &= \partial_x B(r,x) = M(r,x)\partial_x M(r,x) : X \times \mathcal{R} \to X \,, \\
K(r,x) &= \partial_r B(r,x) = M(r,x)\partial_r M(r,x) : X \times \mathcal{R} \to \mathcal{R}'
\end{aligned}
$$

allow continuous extensions to $x = 0$. Furthermore, there exist constants K_0, C_J, C_K such that for all $x, y \in B_C(0)$, $r, s \in \mathcal{R}$ it holds

$$
\begin{aligned}
\|K(r,x)\|_{\mathcal{R}'} &\leq K_0 \,, \\
|J(r,x) - J(s,y)| &\leq C_J \left(|x-y| + \|r-s\|_{\mathcal{R}} \right) \,, \\
\|K(r,x) - K(s,y)\|_{\mathcal{R}'} &\leq C_K \left(|x-y| + \|r-s\|_{\mathcal{R}} \right) \,.
\end{aligned}
$$

Precise formulation of the continuity result on BV (Krejčí and R., 2010)

Let the smoothness hypothesis hold. Then there exist constants $\alpha, \beta, \gamma > 0$ depending only on C, C_J, C_K, K_0 such that for all $u, v \in BV_L(0, T; X)$, $r, s \in BV_L(0, T; \mathcal{R})$, $x_0 \in Z(r(0))$, $y_0 \in Z(s(0))$, the solutions ξ, η corresponding to (u, r, x_0), (v, s, y_0), respectively, satisfy the inequality

$$\begin{aligned} &\mathrm{Var}(\xi - \eta) + C|B(r(T), x(T)) - B(s(T), y(T))| \\ &\quad \leq \quad \alpha \exp(\beta V)\left(\mathrm{Var}(r-s) + \mathrm{Var}(u-v)\right) \\ &\qquad\quad + \gamma \exp(\beta V)(1+V)\left(|x_0 - y_0| + \|u-v\|_\infty + (1+W)\|r-s\|_\infty\right), \end{aligned}$$

where $\|\cdot\|_\infty$ denotes the sup-norm, and

$$\begin{aligned} V = V(r,s,u,v) &:= \mathrm{Var}(r) + \mathrm{Var}(s) + \mathrm{Var}(u) + \mathrm{Var}(v), \\ W = W(r,s,u,v) &:= \|r\|_\infty + \|s\|_\infty + \|u\|_\infty + \|v\|_\infty . \end{aligned}$$

Precise formulation of the existence result on BV (R., 2010)

Let $u \in BV_L^{c_u}(0,T;X)$, $g \in BV_L^{c_g}(0,T;C^1_{\omega,\gamma}(X \times X;\mathcal{R}))$ and $x_0 \in Z(g(0,u(0),u(0)-x_0))$. Assume that the smoothness hypothesis holds,

$$\begin{aligned} \delta := CK_0\gamma &< 1 \qquad \text{and} \\ CK_0|c_g| + (1+CK_0\omega)|c_u| &\leq \frac{(1-\delta)^2}{C_J C(1+\delta)} \end{aligned}$$

hold. Then there exists a unique solution to the quasivariational sweeping process.