Canard cycles in generic slow-fast systems on the two-torus

How many ducks can dance on the torus?

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Definitions

Definition

- Consider planar slow-fast system of the following form:

$$\begin{cases} \dot{x} = f(x, y, \varepsilon), \\ \dot{y} = \varepsilon g(x, y, \varepsilon), \end{cases} \qquad \varepsilon \in (\mathbb{R}, 0). \tag{1}$$

- Variables: x is a fast variable, and y is a slow one, ε is a small parameter.
- Slow curve is a set $M := \{(x, y) \mid f(x, y, 0) = 0\}$.
- *Duck* (or *canard*) solutions are solutions, whose phase curves contain an arc of length bounded away from 0 uniformly in ε, that keeps close to the *unstable* part of the slow curve
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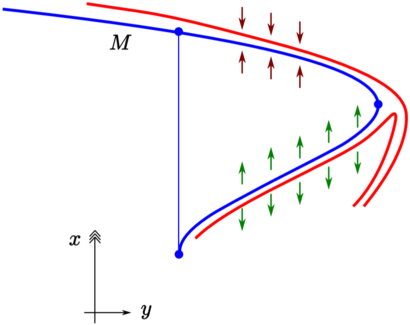
How canards born

- Consider *true slow curves*, which are invariant curves appearing near attracting and repelling parts of the slow curve.
- Canard born when they coincide.
- It occurs in presence of additional parameter.



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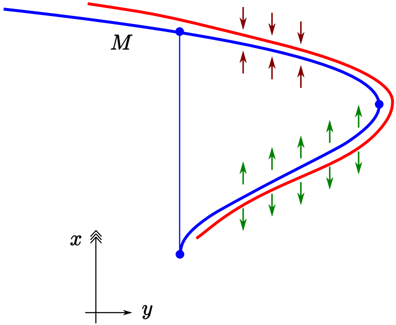

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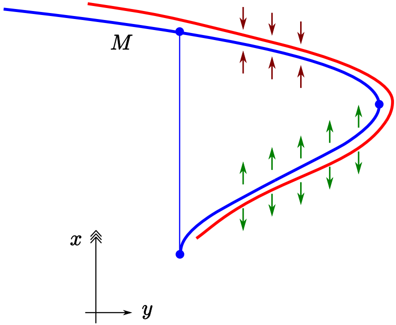

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Ducks on the torus: introduction

- Consider slow-fast system on the two-torus
- Pick a point far from M
- Consider its trajectory in forward time
- Reverse the time
- When ε decreases, L moves down, and R moves up.
- For some ε, we've got canard cycle.

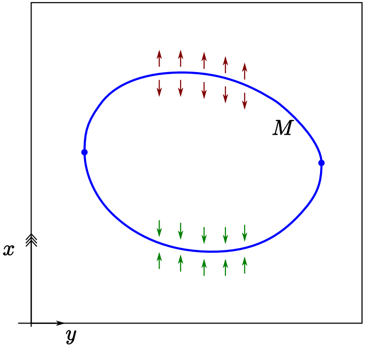

Figure: Ducks on the torus (Yu. S. Ilyashenko, J. Guckenheimer, 2001)

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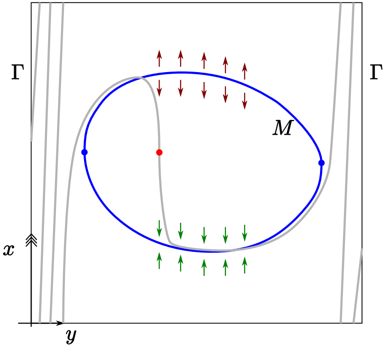

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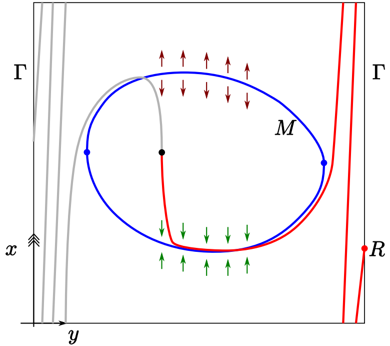

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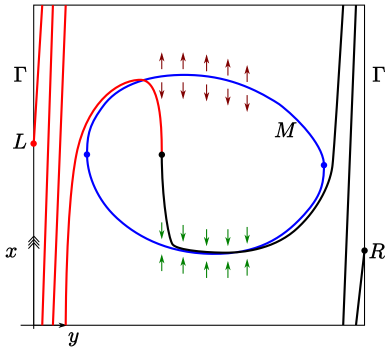

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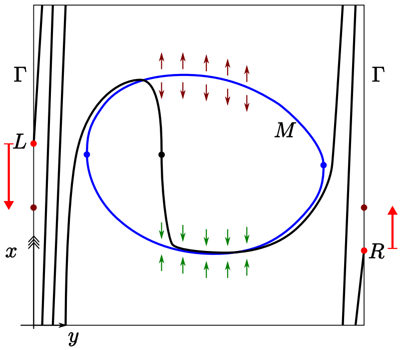

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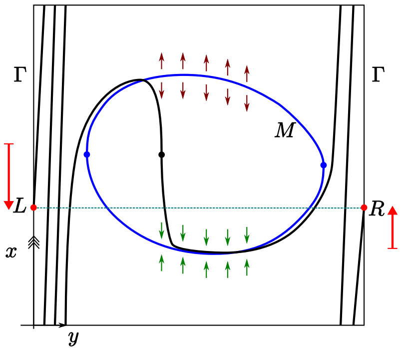

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Main results: the structure of ducky area

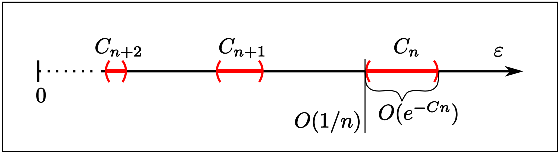

Figure: Intervals C_n: the ducks live here

- There exists a sequence of intervals $\{C_n\}_{n=1}^{\infty}$ such that for every $\varepsilon \in C_n$ the system has *attracting* canard cycles.
- Intervals C_n are exponentially small.
- They accumulate to 0.
- Their density is 0 near $\varepsilon = 0$.
- The number of canard cycles is bounded by the number of folds of M.

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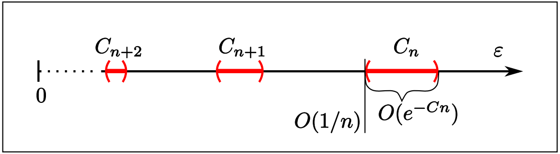

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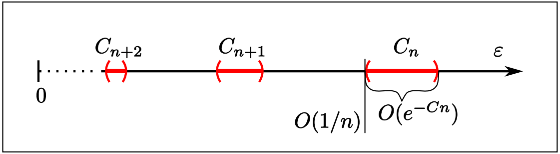

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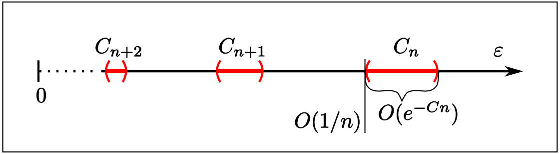

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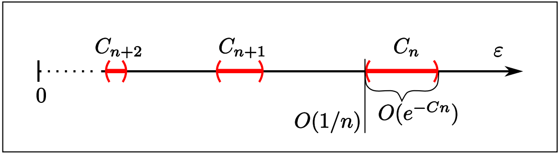

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Theorem 1 (Upper estimate for the number of canards)

Consider slow-fast system on the two-torus, i.e. $(x, y) \in \mathbb{T}^2$*, and the speed of the slow motion is bounded away from zero (*$g > 0$*). Assume* M *is connected nondegenerate curve with* $2N$ *fold points,* $N < \infty$*, and some additional nondegenericity assumptions hold. Then there exists number* $0 < K \leq N$*, such that the following assertions hold:*

- *There exists a sequence* $\{C_n\}_{n=1}^{\infty}$ *of intervals on the ray* $\{\varepsilon > 0\}$*, accumulating to 0, such that for every* $\varepsilon \in C_n$*, the system has exactly* $2K$ *canard cycles (*K *attracting and* K *repelling).*
- *For any* $\varepsilon > 0$ *small enough, the number of limit cycles that make one rotation along y-axis is bounded by* $2K$*.*
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Illustrations for main result: $N = K = 2$

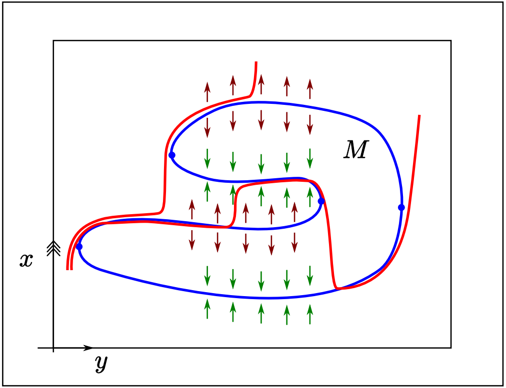

Figure: Simple example: two pair of folds, two attracting ducks (two repelling ducks are not shown)

How many ducks can dance on the torus? (2)

Remark

It follows from theorem 1, that for convex slow curve, there exists exactly one pair of canard cycles

Remark

The number K of canard cycles can be effectively computed without intergration of the system.

Theorem 2 (Sharp estimate for K)

For every $N > 0$ there exists an open set in the space of slow-fast systems on the two-torus for which the number of canard cycles reaches its maximum: $K = N$.

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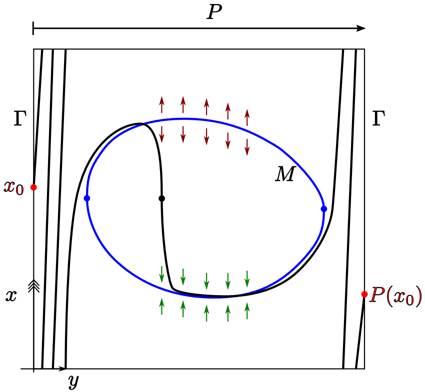
Sketch of the proof: Poincaré map

- Consider Poincaré map $P : \Gamma \to \Gamma$.
- Use Rolle's lemma to estimate number of fixed points.
- Fixed points located near roots of $P'(x) = 1$ correspond to canard cycles.



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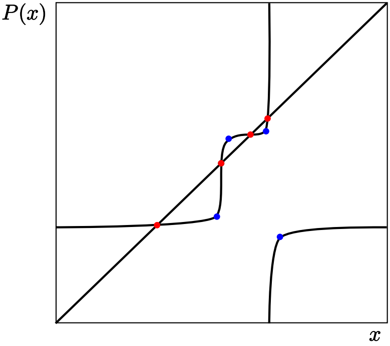

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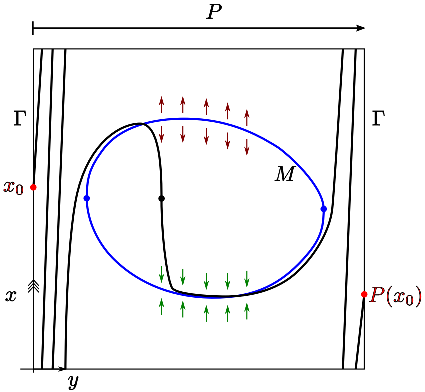
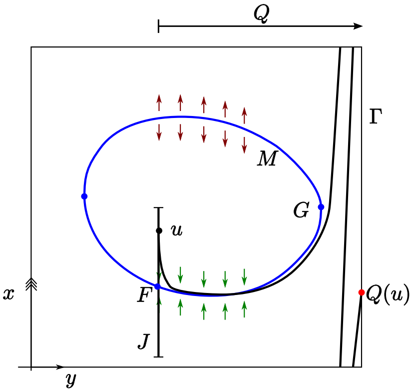


Figure: Poincaré map

Main technical result

- Consider interval J and Poincaré map $Q : J \to \Gamma$.
- $\log Q'(u) = \frac{1}{\varepsilon} \int_{FG} f'_x(x, y, 0)\, dx + o(\frac{1}{\varepsilon})$
- Main obstacle: dynamics near the jump point.
- Solution: Distortion Lemma (Denjoy-Schwartz).



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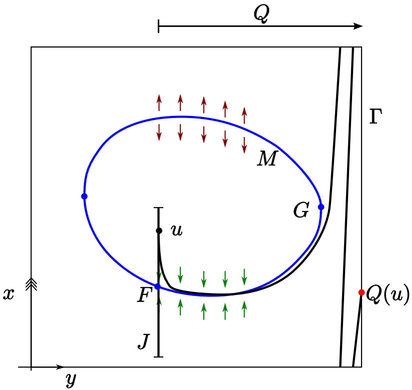

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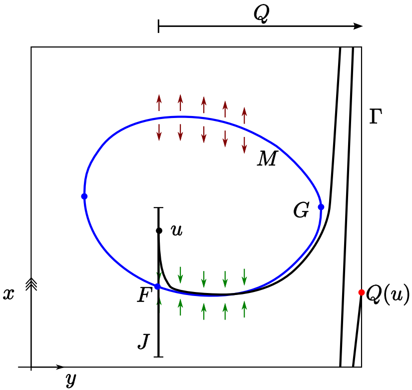

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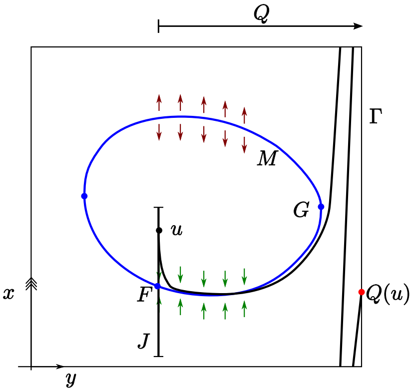

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The duck farm

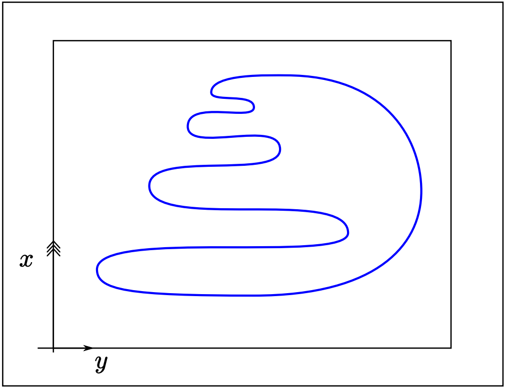

Figure: The construction of open set of slow-fast systems with maximal number of canard cycles. E.g. $K = N = 4$ on the figure.

References

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- `http://ilya.schurov.com/math`

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