P.D.E.s with Discontinuous Hysteresis

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Plan

- Hysteresis
  - Hysteresis Operators
  - Discontinuous Hysteresis
  - P.D.E.s with Hysteresis
- Ferromagnetic Hysteresis
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What is Hysteresis?

Hysteresis occurs in *plasticity*, *ferromagnetism*, *ferroelectricity*, *superconductivity*, *undercooling*, *shape memory*, *porous media filtration*, and so on.

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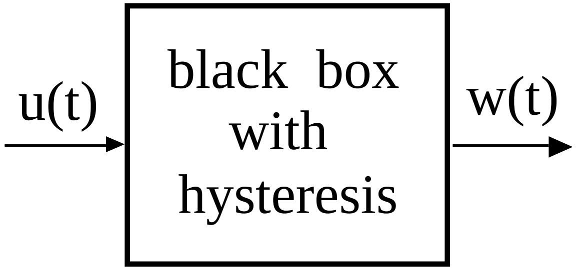

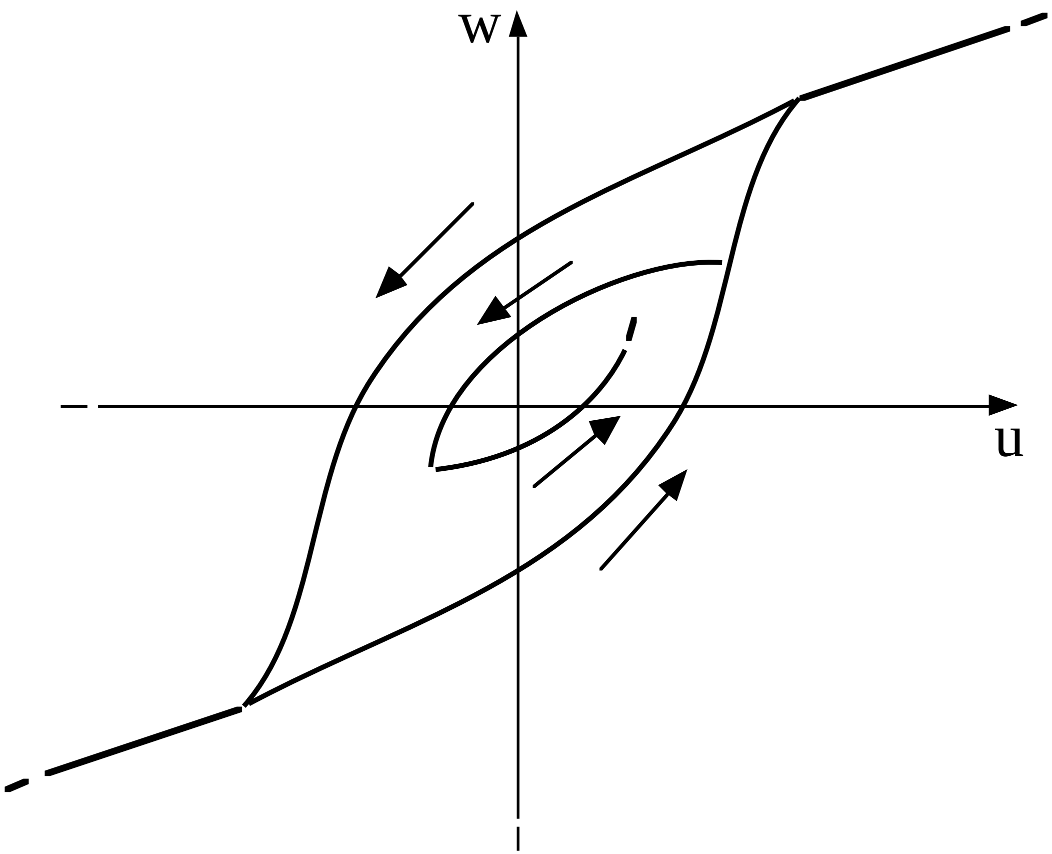

Hysteresis = Rate-Independent Memory

Memory: $w(t)$ depends on the previous evolution of u, and on the initial state:

$$w(t) = \left[\mathcal{F}(u, w^0)\right](t) \qquad \forall t \in [0, T].$$

Rate-Independence: For any increasing diffeomorphism $\varphi : [0, T] \to [0, T]$,

$$\mathcal{F}(\cdot, w^0) : u \mapsto w \qquad \Rightarrow \qquad \mathcal{F}(\cdot, w^0) : u \circ \varphi \mapsto w \circ \varphi.$$

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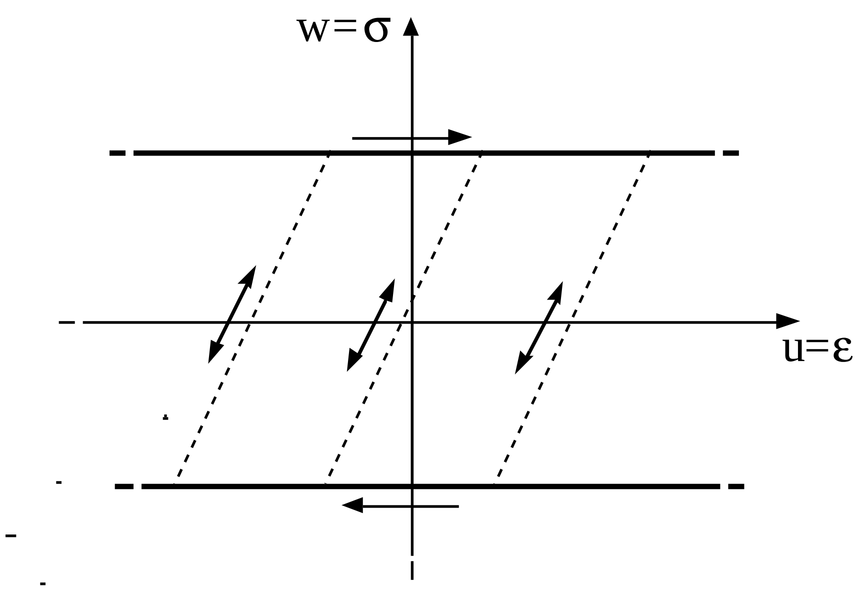
Continuous hysteresis operators are typically constructed via

(i) definition for piecewise monotone inputs,

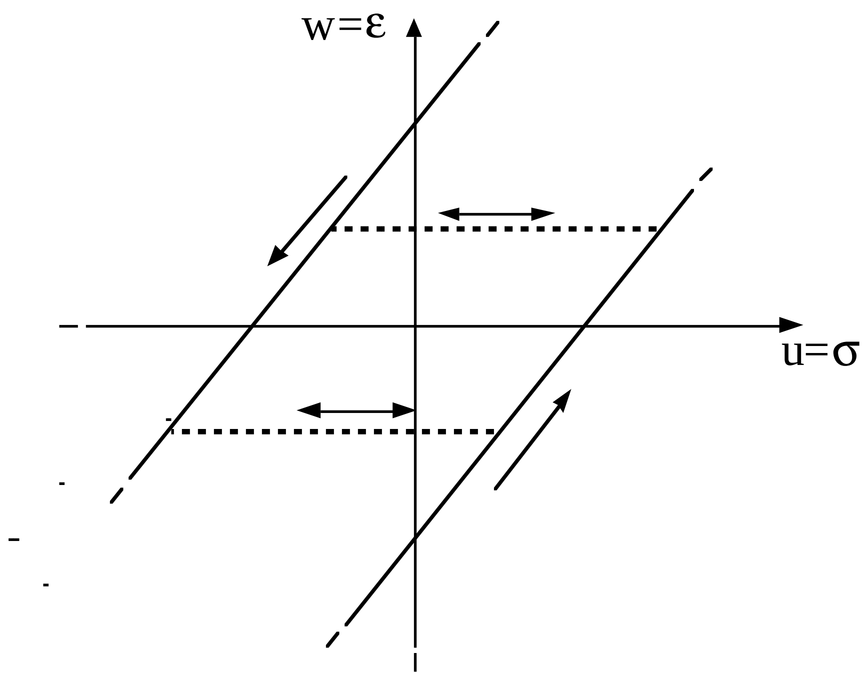
(ii) derivation of a property of uniform continuity,

(iii) extention by continuity to a Banach space (e.g., $C^0([0, T])$).

Examples (from elasto-plasticity)



Stop



Play

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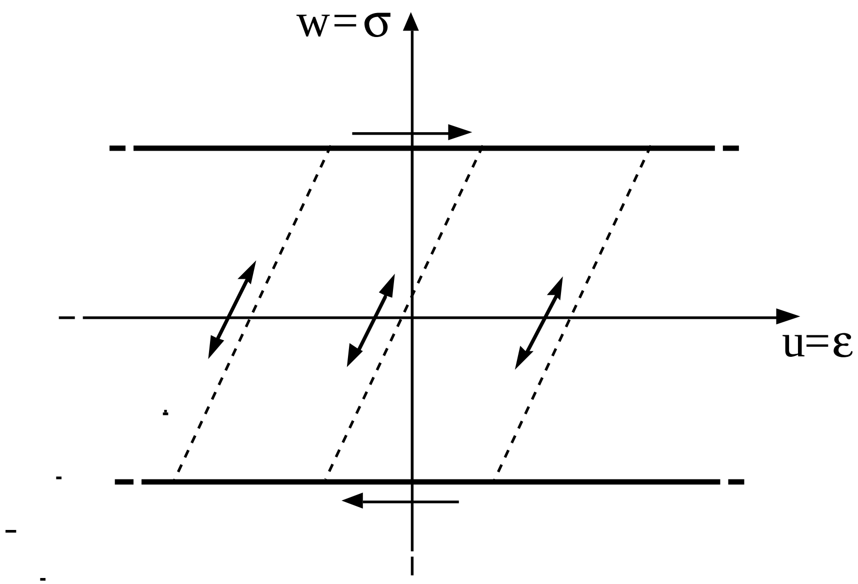
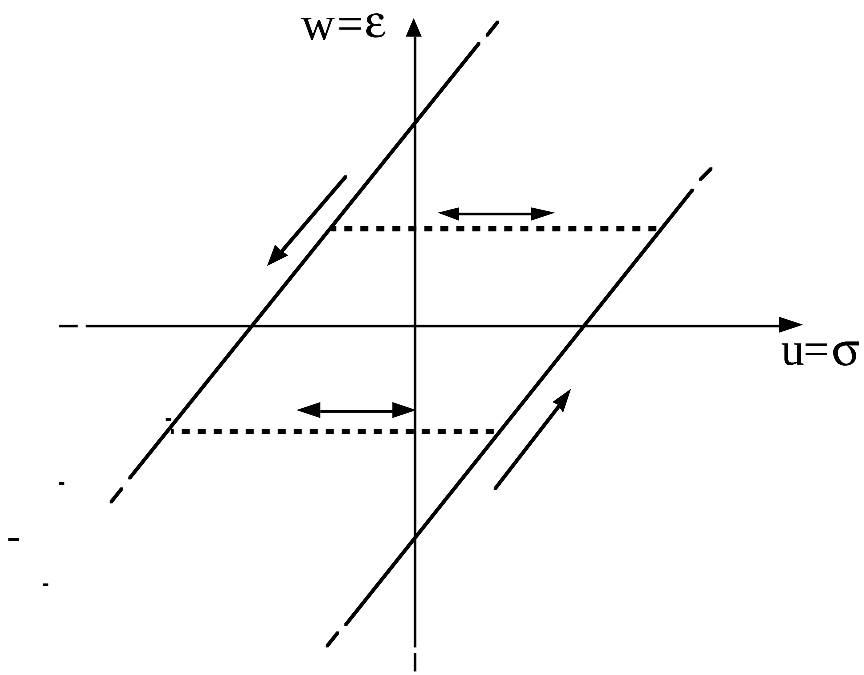


Stop

$$w' + \text{sign}^{-1}(w) \ni u'$$



Play

$$\text{sign}(w') + w \ni u$$

The large class of **Prandtl-Ishlinskiĭ models** is obtained by composing stops and plays.

Another Example: Duhem's Model

Let $g_1, g_2 \in C^1(\mathbf{R}^2)$. $\forall u \in W^{1,1}(0,T)$, $\forall w^0 \in \mathbf{R}$,

$$\begin{cases} \dfrac{dw}{dt} = g_1(u,w)\left(\dfrac{du}{dt}\right)^+ - g_2(u,w)\left(\dfrac{du}{dt}\right)^- & \text{a.e. in }]0,T[\\ w(0) = w^0. \end{cases}$$

This Cauchy problem defines a continuous operator

$$\mathcal{F} : W^{1,1}(0,T) \to W^{1,1}(0,T) : u \mapsto w.$$

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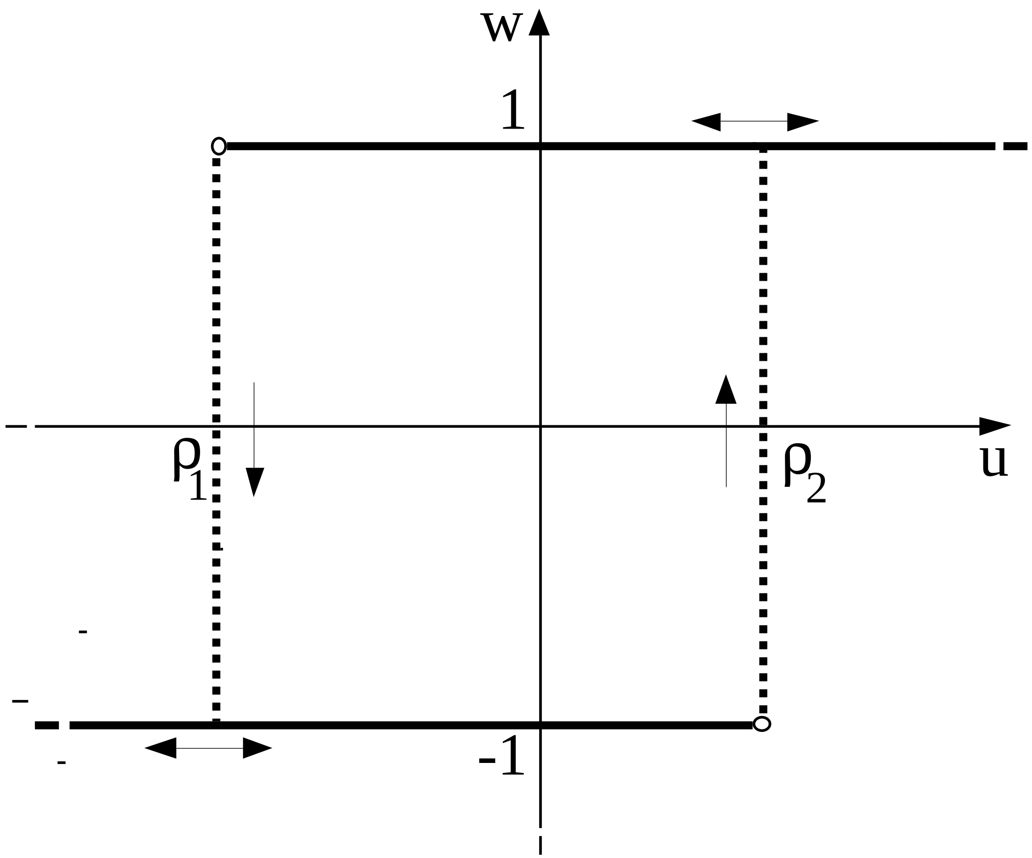
By the irreversibility of hysteresis, $dt \geq 0$; the O.D.E. is then equivalent to

$$\left.\begin{aligned} \frac{dw}{du} &= g_1(u,w) \quad \text{if } u \nearrow \\ \frac{dw}{du} &= g_2(u,w) \quad \text{if } u \searrow \end{aligned}\right\} \Rightarrow \qquad \mathcal{F} \text{ is rate-independent.}$$

This formulation can be modified, to confine (u,w) to a subset of $\mathbf{R}^2$.

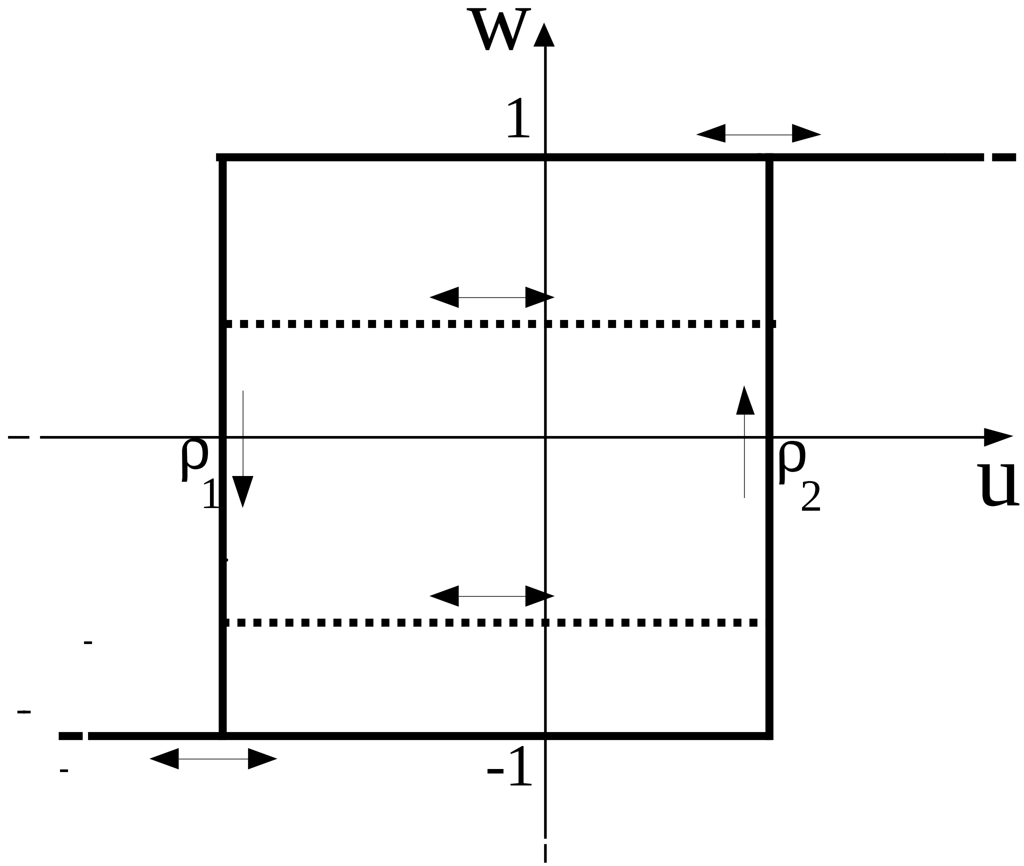
Discontinuous Hysteresis

Relays. For any pair $\rho := (\rho_1, \rho_2) \in \mathbf{R}^2$ $(\rho_1 < \rho_2)$, we define the *relay operator* h_ρ:



The operator $h_\rho : C^0([0,T]) \times \{-1,1\} \to BV(0,T)$ is not closed.

In connection with P.D.E.s, it is of interest to deal with the closure of h_ρ.
Its graph invades the whole rectangle $[\rho_1, \rho_2] \times [-1, 1]$.

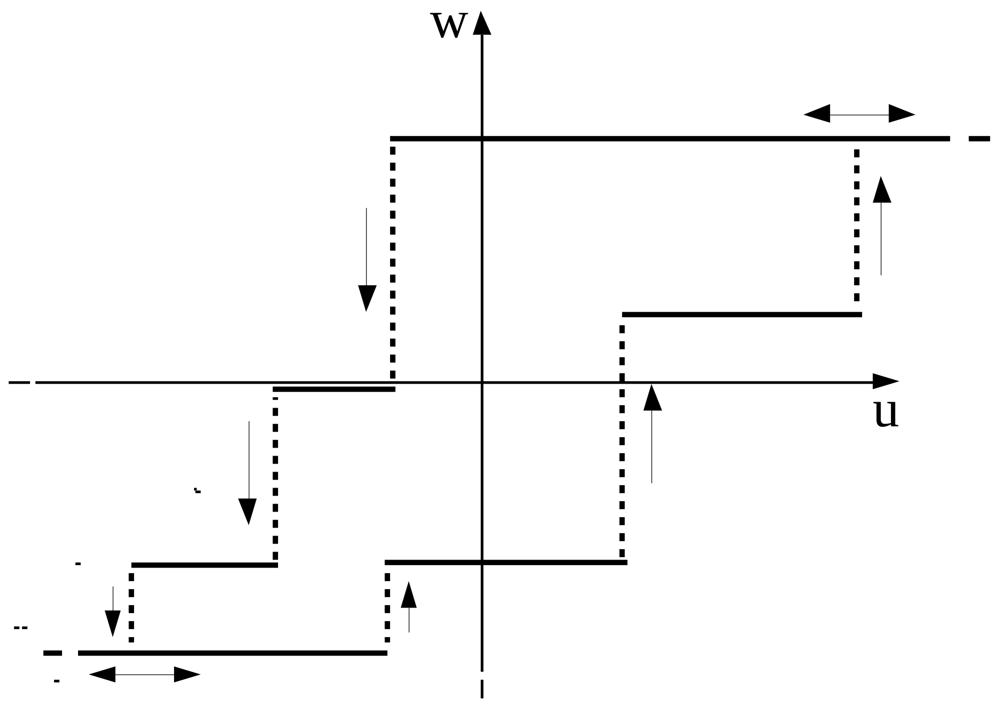


At variance with stops and plays,
relays **cannot** be represented by variational inequalities.

Preisach's Model (1935)

Linear combination of delayed relays with different thresholds and *the same input:*

$$\mathcal{H}_\mu(u, \{\xi_\rho\}) := \int_{\rho_1 < \rho_2} h_\rho(u, \xi_\rho)\, d\mu(\rho) \qquad \text{in } [0, T].$$



Under natural hypotheses on the Preisach measure μ,

$\mathcal{H}_\mu$ operates and is continuous in $C^0([0, T])$.

P.D.E.s with Hysteresis

$\mathcal{F}$: (possibly discontinuous) hysteresis operator, A: elliptic operator.

$$\frac{\partial}{\partial t}[u + \mathcal{F}(u)] + Au = f \qquad \text{quasilinear parabolic} \tag{1}$$

$$\frac{\partial u}{\partial t} + Au + \mathcal{F}(u) = f \qquad \text{semilinear parabolic} \tag{2}$$

$$\frac{\partial \mathcal{F}(u)}{\partial t} + \vec{v} \cdot \nabla u = f \qquad 1^{\text{st}}\text{-order quasilinear hyperbolic} \tag{3}$$

$$\frac{\partial^2}{\partial t^2}[u + \mathcal{F}(u)] + Au = f \qquad 2^{\text{nd}}\text{-order quasilinear hyperbolic.} \tag{4}$$

(Discontinuous $\mathcal{F} \Rightarrow$ moving fronts, i.e., *free boundaries.)*

Initial- and boundary-value problems associated with (1), (2), (3) are well-posed.

(4) is considered below.

Hysteresis and Monotonicity

The standard L^2-monotonicity,

$$\int_0^T [\mathcal{F}(u_1) - \mathcal{F}(u_2)](u_1 - u_2)dt \geq 0 \qquad \forall u_1, u_2 \in C^0([0,T])$$

is too strong for hysteresis operators.

Piecewise monotonicity looks appropriate:

$$\begin{cases} \forall u \in C^0([0,T]), \forall [t_1, t_2] \subset [0,T], \\ \text{if } u \text{ is nondecreasing (nonincreasing, resp.) in } [t_1, t_2], \text{ then} \\ \text{if } \mathcal{F}(u) \text{ is also nondecreasing (nonincreasing, resp.) in } [t_1, t_2]. \end{cases}$$

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Hence

$$u, \mathcal{F}(u) \in W^{1,1}(0,T) \quad \Rightarrow \quad \frac{d\mathcal{F}(u)}{dt}\,\frac{du}{dt} \geq 0 \quad \text{a.e. in }]0,T[.$$

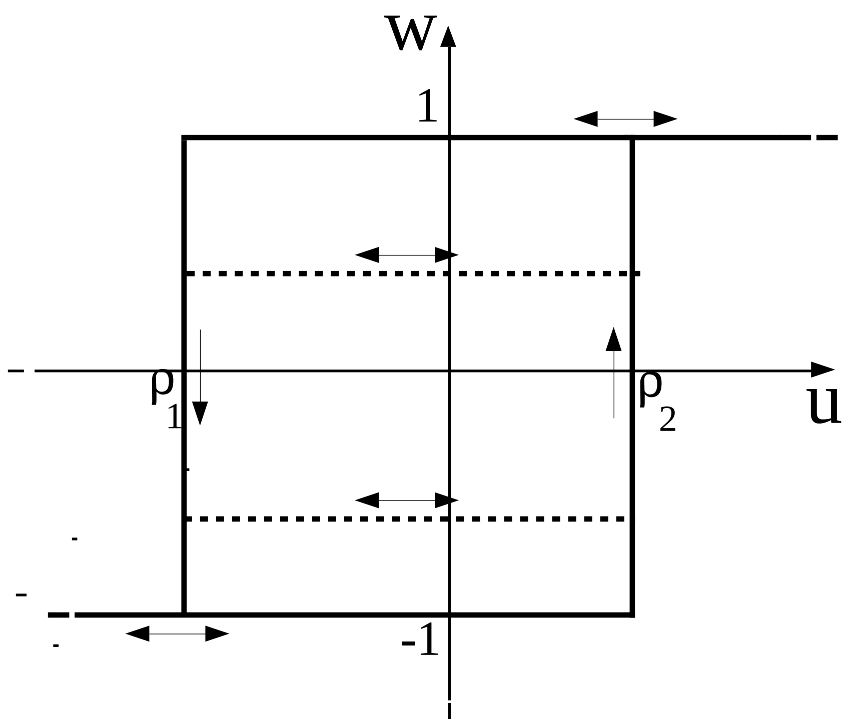
This means that *hysteresis branches* are nondecreasing.

For several P.D.E.s with hysteresis, this property is at the basis of a priori estimates.

However, to pass to the limit in the hysteresis operator, this property is of no use.

Ferromagnetic Hysteresis

Scalar Problem



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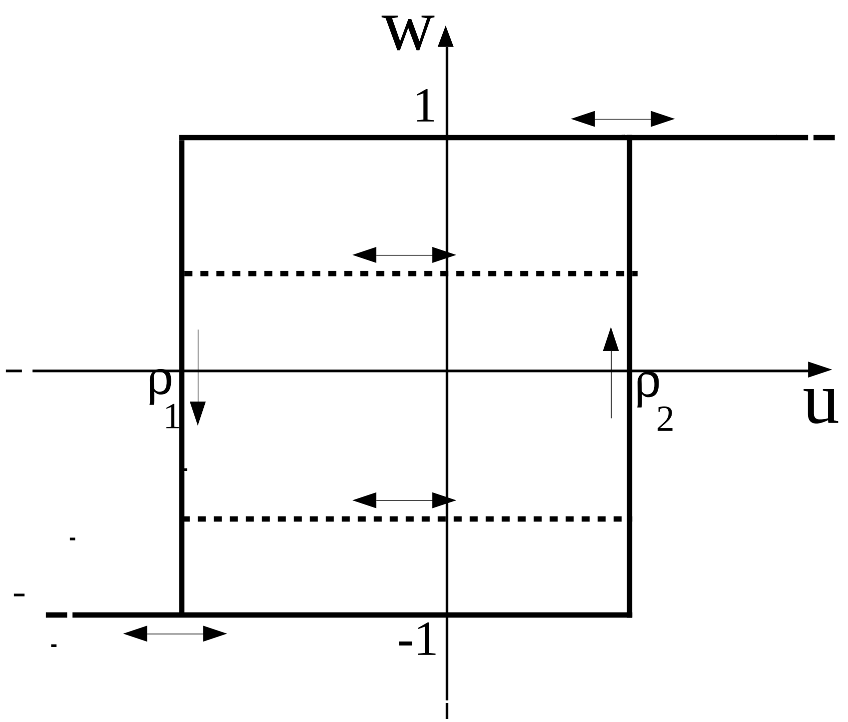

(i) *Confinement condition:*

$$\begin{cases} |w| \leq 1 \\ (w-1)(u-\rho_2) \geq 0 \\ (w+1)(u-\rho_1) \geq 0 \end{cases}$$

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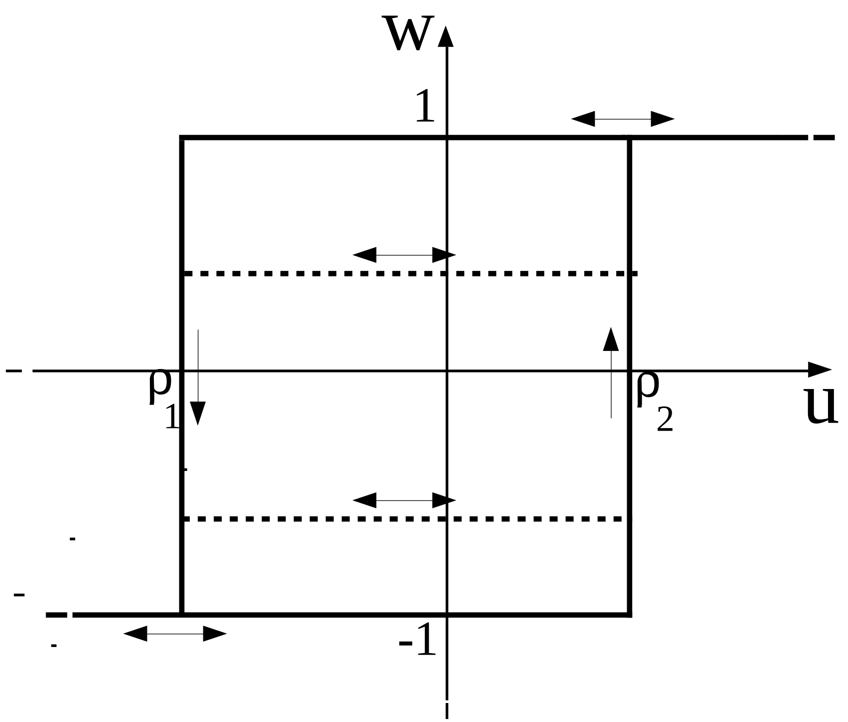

(i) *Confinement condition:*

$$\begin{cases} |w| \leq 1 \\ (w-1)(u-\rho_2) \geq 0 \\ (w+1)(u-\rho_1) \geq 0 \end{cases}$$

(ii) *Dissipation condition:*

$$\begin{cases} \displaystyle\int_0^t u\,dw \geq \int_0^t \left[\rho_2(dw)^+ - \rho_1(dw)^-\right] \\ \displaystyle = \frac{\rho_2-\rho_1}{2}\int_0^t |dw| + \frac{\rho_2+\rho_1}{2} w\Big|_0^y \\ =: \Psi_\rho(w,t) \qquad \forall t \end{cases}$$

Scalar Quasilinear Hyperbolic Equation with Hysteresis

Data: $u^0, w^0 \in L^2(\Omega), \quad F \in L^2\big(0,T; H^{-1}(\Omega)\big).$

Problem 1. *To find* $U \in H^1(Q)$ *and* $w \in L^\infty(Q)$ *such that* $\dfrac{\partial w}{\partial t} \in C^0(\bar{Q})'$

$$\frac{\partial}{\partial t}(u+w) - \Delta U = F \qquad \textit{in } H^{-1}(Q) \qquad \left(u := \frac{\partial U}{\partial t}\right)$$

$$|w| \le 1, \qquad \begin{cases} (w-1)(u-\rho_2) \ge 0 \\ (w+1)(u-\rho_1) \ge 0 \end{cases} \qquad \textit{a.e. in } Q$$

$$\frac{1}{2}\int_\Omega \left[u(x,t)^2 - u^0(x)^2 + |\nabla U(x,t)|^2\right] dx + \int_\Omega \Psi_\rho(w(x,\cdot),t) \le \int_0^t \langle F, u\rangle \, d\tau$$

$$\gamma_0 U = 0 \qquad \textit{on } (\Omega \times \{0\}) \cup (\partial\Omega \times]0,T[)$$

$$(u+w)|_{t=0} = u^0 + w^0 \qquad \textit{in } \Omega.$$

Theorem. $F \in L^1_t(L^2_x) + W^{1,1}_t(H^{-1}_x) \;\;\Rightarrow\;\; \exists$ *solution* (U, w)*:*

$$U \in W^{1,\infty}\big(0,T; L^2(\Omega)\big) \cap L^\infty\big(0,T; H^1_0(\Omega)\big).$$

This can be extended to the Preisach model. The argument is based upon:

(i) approximation via implicit time-discretization,

(ii) derivation of a priori estimates; in particular, by the dissipation condition,

$$\Big\|\frac{\partial w_m}{\partial t}\Big\|_{C^0(\bar{Q})'} = \int_\Omega dx \int_0^T |dw_m| \le \text{Constant},$$

(iii) passage to the limit by compactness and lower semicontinuity.

A. V.: *Quasi-linear hyperbolic equations with hysteresis.*
Ann. Inst. H. Poincaré. Nonlinear Analysis, **19** (2002), 451-476

The argument also uses the following *compensated compactness* result.

Lemma 1. *If*

$$z_m \to z \qquad \textit{weakly in } L^2(Q) \cap H^{-1}\big(0, T; H^1(\Omega)\big)$$

$$w_m \to w \qquad \textit{weakly star in } L^\infty(Q)$$

$$\left\| \frac{\partial w_m}{\partial t} \right\|_{L^1(Q)} \leq \text{Constant}$$

then

$$\iint_Q w_m z_m \, dxdt \to \iint_Q wz \, dxdt.$$

Vector Problem — Maxwell-Ohm's Equations (in Gauss units)

$$c\nabla\times\vec{H} = 4\pi\vec{J} + \frac{\partial\vec{D}}{\partial t} \qquad c\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad (\nabla\times := \text{curl})$$

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\cdot\vec{D} = 4\pi\hat{\rho} \qquad (\nabla\cdot := \text{div})$$

Ohm's law: $\vec{J} = \sigma\vec{E} + \vec{J}_e$ Dielectric relation: $\vec{D} = \epsilon\vec{E}$

$$\Rightarrow \qquad \epsilon\frac{\partial^2\vec{B}}{\partial t^2} + 4\pi\sigma\frac{\partial\vec{B}}{\partial t} + c^2\nabla\times\nabla\times\vec{H} = 4\pi c\nabla\times\vec{J}_e \quad (\text{:datum})$$

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In ferr<u>i</u>magnetic insulators: $\sigma = 0 \quad \longrightarrow$ quasilinear hyperbolic

In ferr<u>o</u>magnetic metals: $\epsilon\frac{\partial^2\vec{B}}{\partial t^2} \ll 4\pi\sigma\frac{\partial\vec{B}}{\partial t} \quad \longrightarrow$ quasilinear parabolic

For quasi-static processes: $c\nabla\times\vec{H} = 4\pi\vec{J} \qquad \nabla\cdot\vec{B} = 0.$

Constitutive Law $\vec{H} \mapsto \vec{M}$

Vector Relay. Each "magnetic element" is characterized by

(i) a magnetization direction $\vec{\theta} \in S^2$,

(ii) a pair of thresholds $\rho := (\rho_1, \rho_2) \in \mathcal{P}$.

The vector relay $\vec{h}_{(\rho,\vec{\theta})}$ is defined in terms of the scalar relay $\vec{h}_\rho$ as follows:

$$\vec{h}_{(\rho,\vec{\theta})}(\vec{H}) := h_\rho(\vec{H} \cdot \vec{\theta})\vec{\theta} \qquad \forall (\rho, \vec{\theta}) \in \mathcal{P} \times S^2.$$

A vector relay may represent the behaviour of a strongly anisotropic crystal having crystallographic orientation $\vec{\theta}$.

Each of the 3 P.D.E. systems above
(i.e., hyperbolic, parabolic, quasistationary evolution)
can be coupled with the constitutive law

$$\vec{M}(x,t) = \left[\vec{h}_{(\rho(x),\vec{\theta}(x))}\big(\vec{H}(x,\cdot)\big)\right](t) \qquad \text{pointwise in } Q.$$

Each of these 3 problems has a weak solution.

A. V.: *Maxwell's equations with vector hysteresis.*
Archive Rat. Mech. Anal. **175** (2005) 1–38

The argument also uses the following *compensated compactness* result.

Lemma 2. *If*

$$\vec{u}_m \to \vec{u} \qquad \textit{weakly in } L^2\big(\mathbf{R}^3\times]0,T[\big)^3 \cap H^{-1}\big(0,T;L^2_{\mathrm{rot}}(\mathbf{R}^3)^3\big)$$

$$\vec{z}_m \to \vec{z} \qquad \textit{weakly star in } L^\infty\big(\mathbf{R}^3\times]0,T[\big)^3$$

$$\|\vec{z}_m\|_{L^1(\mathbf{R}^3;BV(0,T)^3)} \le \text{Constant}$$

$$\nabla\cdot(\vec{u}_m + \vec{z}_m) = 0 \qquad \textit{in } \mathcal{D}'\big(\mathbf{R}^3\times]0,T[\big), \forall m,$$

then

$$\limsup_{m\to\infty} \iint_{\mathcal{B}\times]0,T[} \vec{z}_m\cdot\vec{u}_m\,dxdt \le \iint_{\mathcal{B}\times]0,T[} \vec{z}\cdot\vec{u}\,dxdt \qquad \forall \text{ ball } \mathcal{B}\subset\mathbf{R}^3.$$