A ROUGH SUPER-BROWNIAN MOTION

NICOLAS PERKOWSKI AND TOMMASO CORNELIS ROSATI

ABSTRACT. We study the scaling limit of a branching random walk in static random environment in dimension $d = 1, 2$ and show that it is given by a super-Brownian motion in a white noise potential. In dimension 1 we characterize the limit as the unique weak solution to the stochastic PDE:

$$\partial_t \mu = (\Delta + \xi)\mu + \sqrt{2\nu\mu}\tilde{\xi}$$

for independent space white noise ξ and space-time white noise $\tilde{\xi}$. In dimension 2 the study requires paracontrolled theory and the limit process is described via a martingale problem. In both dimensions we prove persistence of this rough version of the super-Brownian motion.

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INTRODUCTION

This work explores the large scale behavior of a branching random walk in a random environment (BRWRE). Such process is a particular kind of spatial branching process on $\mathbb{Z}^d$, in which the branching and killing rate of a particle depends on the value of a potential V in the position of the particle. In the model analyzed in this work, the dimension is restricted to $d = 1, 2$ and the potential is chosen at random on the lattice:

$$V(x) = \xi(x), \quad \text{with} \quad \{\xi(x)\}_{x \in \mathbb{Z}^d} \ \text{i.i.d.}, \ \ \xi(x) \sim \Phi$$

for a given probability distribution Φ (normalized via $\mathbb{E}\Phi = 0, \mathbb{E}\Phi^2 = 1$).

All particles behave independently of each other: To clarify the model it is convenient to describe the behavior of a particle X in this process via its jump rates:

$$X(t + \mathrm{d}s) \text{ given } X(t) \begin{cases} \text{Jumps to nearest neighbor at rate } \mathrm{d}s, \\ \text{Gives birth to a particle at rate } \xi(X(t))_+ \, \mathrm{d}s, \\ \text{Dies at rate } \xi(X(t))_- \, \mathrm{d}s. \end{cases}$$

After branching, the new and the old particle follow the same rule independently of each other.

(NP) MAX-PLANCK-INSTITUT FÜR MATHEMATIK IN DEN NATURWISSENSCHAFTEN LEIPZIG & HUMBOLDT-UNIVERSITÄT ZU BERLIN
(TR) HUMBOLDT-UNIVERSITÄT ZU BERLIN
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The BRWRE is used as a model for chemical reactions or biological processes, e.g. mutation, in a random medium. This model is especially interesting in relation to intermittency and localization [ZMRS87, GM90, ABMY00, GKS13], and other large times properties such as survival [BGK09, GMPV10].

Scaling limits of branching particle systems have been an active field of research since the early results by Dawson et al. and gave rise to the study of superprocesses (see [Eth00, DP12] for excellent introductions). This work follows the original setting and studies the behavior under diffusive scaling: Spatial increments $\Delta x \simeq 1/n$, temporal increments $\Delta t \simeq 1/n^2$. The particular nature of our problem requires us to couple the diffusive scaling with the scaling of the environment: This is done via an "averaging parameter" $\varrho \geq d/2$, while the noise is assumed to scale to space white noise (i.e. $\xi^n(x) \simeq n^{d/2}$).

The diffusive scaling of spatial branching processes in a random environment has already been studied, for example by Mytnik [Myt96]. As opposed to the current setting, the environment in Mytnik's work renews itself independently in time. Thus on large scales it behaves like space-time white noise, instead of space white noise. This has the advantage that the model is amenable to probabilistic martingale arguments, which are not available in the space white noise case that we investigate here. Therefore, we replace some of the probabilistic tools with arguments of a more analytic flavor. Nonetheless, at a purely formal level our limiting process is very similar to the one obtained by Mytnik, up to exchanging space-time with space white noise: See for example the SPDE representation (2) below. Moreover, our approach is reminiscent of the conditional log-laplace transform, that is conditional duality, appearing in later works by Crişan [Cri04], Mytnik and Xiong [MX07]. Notwithstanding these resemblances, we shall see later that some statistical properties of the two processes differ substantially.

At the heart of our study of the BRWRE lies the following observation. If $u(t,x)$ indicates the numbers of particles in position x at time t, then the conditional expectation given the realization of the random environment, $w(t,x) = \mathbb{E}[u(t,x)|\xi]$, solves a linear PDE with stochastic coefficients (SPDE), which is a discrete - in the sense that the spatial variable is restricted to a lattice - version of the parabolic Anderson model (PAM):

$$(1) \qquad \partial_t w(t,x) = \Delta w(t,x) + \xi(x) w(t,x), \qquad (t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}, \qquad w(0,x) = w_0(x).$$

The PAM has been object of study both in the discrete and in the continuous setting (see [Kön16] for an overview). In the latter case (ξ is space white noise) the SPDE is not solvable via Itô integration theory, which highlights once more the difference between the current setting and the work by Mytnik. In particular, in dimension $d = 2, 3$ the study of the continuous PAM requires special analytical and stochastic techniques in the spirit of rough paths, such as the theory of regularity structures [Hai14] or of paracontrolled distributions [GIP15]. In dimension $d = 1$ classical analytical techniques are sufficient. In dimension $d \geq 4$ no solution is expected to exist, because the equation is no longer *locally subcritical.* Local subcriticality is a notion that in the present context was introduced by Hairer [Hai14], and it means that on small scales the equation is well approximated by a linear equation with additive noise. The dependence of the subcriticality condition on the dimension is explained by the fact that white noise loses regularity as the dimension increases.

Moreover, in dimension $d = 2, 3$ certain functionals of the white noise need to be tamed with a technique called *renormalization,* with which we remove certain diverging singularities. In this work, we restrict to dimensions $d = 1, 2$ as this simplifies several calculations. At the level of the 2-dimensional BRWRE, the renormalization has the effect of slightly tilting the centered potential by considering instead an effective potential:

$$\xi^n_e(x) = \xi^n(x) - c_n, \qquad c_n \simeq \log(n),$$

which means that our system is out of criticality, albeit very slightly when confronted with the other orders of magnitude involved.

The special character of the noise and the analytic tools just highlighted allow, in a nutshell, to fix one realization of the environment - outside a nullset - and to derive a scaling limit for that single realization. Tightness of the measure-valued process then follows via a study of the associated martingale problem, whereas the uniqueness of the limit is shown by duality, which is similar to the case of classical super-Brownian motion (SBM), but different from the uniqueness proof in [Myt96], where duality is not available.

For "averaging parameter" $\varrho > d/2$ a law of large numbers holds: The process converges to the continuous PAM. Instead, for $\varrho = d/2$ one captures fluctuations from the branching mechanism and the limiting process can be characterized via duality or a martingale problem (see Theorem 2.13) and is referred to in this work as rough super-Brownian motion (rSBM). In dimension $d = 1$, following the analogous results for SBM by [KS88, Rei89], the rSBM admits a density which in turn solves the SPDE:

$$\partial_t \mu(t,x) = \Delta\mu(t,x) + \xi(x)\mu(t,x) + \sqrt{2\nu\mu(t,x)}\tilde{\xi}(t,x), \quad (t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}, \quad \mu(0,x) = \delta_0(x), \tag{2}$$

where $\tilde{\xi}$ is space-time white noise that is independent of the space white noise ξ, and where $\nu = \mathbb{E}\Phi^+$. The solution is weak both in the probabilistic and in the analytic sense (see Theorem 2.19 for a precise statement). This means that the last product represents a stochastic integral in the sense of Walsh [Wal86] and the space-time noise is constructed from the solution. Moreover, the product $\xi \cdot \mu$ is defined only upon testing with functions in the random domain of the Anderson Hamiltonian $\mathscr{H} = \Delta + \xi$, a random operator that was introduced by Fukushima-Nakao [FN77] in $d = 1$ and by Allez-Chouk [AC15] in $d = 2$, see also [Lab18] for $d = 3$. This notion of solution should be expected. Indeed, the fact that the solutions are probabilistically weak is due to the presence of the super-Brownian non-linearity, since even for the classical SBM the existence of probabilistically strong solutions is open. In addition, the expected local Hölder regularity in space of the solution is $C^{1/2-\varepsilon}_{\text{loc}}$, for any $\varepsilon > 0$, due to the presence of the space-time white noise. This means that the product $\xi \cdot \mu$ is a priori not well-defined. Although the path-wise theories of regularity structures and paracontrolled distributions aim exactly at finding spaces of functions where such products are well-defined, the presence of the singular non-linearity $\sqrt{2\nu\mu}$ makes this equation currently untreatable in such a framework (see however [CT19] for some progress on finite-dimensional rough path differential equations with square root nonlinearities).

One of the main motivations for this work was the aim to understand the SPDE (2) in $d = 1$ and the corresponding martingale problem in $d = 2$. For $\tilde{\xi} = 0$, equation (2) is just the PAM which we can only solve with pathwise methods, while for $\xi = 0$ we obtain the classical SBM, for which the existence of pathwise solutions is a long standing open problem and for which we only have probabilistic martingale techniques. So the challenge was to combine these two approaches, and the weak formulation based on the Anderson Hamiltonian allows us to do exactly that, we can transfer all the pathwise regularity analysis into the construction of the domain of $\mathscr{H}$ and then only use martingale analysis on the level of the process μ. A similar point of view was recently taken by Corwin-Tsai [CT18] who deal with the multiplicative linear stochastic heat equation driven by independent space and space-time white noises in $d = 1$ (where no paracontrolled analysis is required), and to a certain extent also in [GUZ18].

Coming back to the rSBM, we conclude this work with a proof of persistence of the process in dimension $d = 1, 2$. More precisely we even show that with positive probability we have $\mu(t, K) \to \infty$ for all compact sets $K \subset \mathbb{R}^d$ with non-empty interior. This is opposed to what happens for the classical SBM, where persistence holds only in dimension $d \geq 3$, whereas in dimensions $d = 1, 2$ the process dies out: See [Eth00, Section 2.7] and the references therein. Even more extreme is the case of SBM in a random, white in time, environment: Under the assumption of a heavy-tailed spatial correlation function Mytnik and Xiong [MX07] prove extinction in finite time in any dimension. Note also that in [Eth00, MX07] the process is started in the Lebesgue measure, whereas here we prove persistence if the initial value is a Dirac mass. Intuitively, this phenomenon can be explained by the presence of "very favorable regions" in the random environment: With positive probability, one particle survives long enough to reach a favorable region, and once it arrives there the mass grows exponentially.

Structure of the Work

After clarifying the notation and introducing some first analytical tools, we present the model for our branching process in random environment. In Assumption 2.1 we state the probabilistic requirements on the random environment. In the spirit of rough paths, the probabilistic assumptions allow us to fix a null set outside of which certain analytical conditions are satisfied, see Lemma 2.4 for details. We then introduce the model, although a rigorous construction of the (random) Markov process is a bit subtle because of the unbounded branching rates, and we postpone it to Section A of the Appendix. We also state the main results in Section 2, namely the law of large numbers (Theorem 2.10), the convergence to the rSBM (Theorem 2.13), the representation as an SPDE in dimension $d = 1$ (Theorem 2.19) and the persistence of the process (Theorem 2.21). We then proceed to the proofs. In Section 3 we study the discrete and continuous PAM both on the real line and on a box with Dirichlet boundary conditions. In the first case, we recall the results from [MP17] and adapt them to the current setting. In the second case, we introduce the techniques developed by Chouk and van Zuijlen [CvZ19] to study (paracontrolled) equations on boxes with Dirichlet boundary conditions. We extend these techniques to the lattice, mimicking the construction of [MP17]. The required stochastic calculations are postponed to Section C of the Appendix.

With these techniques at our disposal, we prove the convergence in distribution of the BRWRE in Section 4. First, we show tightness by using a mild martingale problem (see Remark 4.1) which fits well with our analytical tools. We then show the duality of the process to the SPDE (6). Eventually we use duality to deduce the uniqueness of the limit points of the BRWRE and thus we get its weak convergence.

Since our only way of constructing the rSBM is through this weak convergence, the parameter $\nu = \mathbb{E}\Phi_+$ in (2) must be in $(0, 1/2]$. In Section 4.2 we show how to recover all values of ν by mixing our process with a classical Dawson-Watanabe superprocess.

In Section 5 we derive some properties of the rough super-Brownian motion: We show that in $d = 1$ it is the weak solution to an SPDE, where the key point is that the random measure admits a density w.r.t. the Lebesgue measure, as proven in Lemma 5.1. We also show that the process survives with positive probability, which we do by relating it to the rSBM on a finite box with Dirichlet boundary conditions and by applying the spectral theory for the Anderson Hamiltonian on that box. To construct the rSBM with Dirichlet boundary conditions we need to study a modification of the BRWRE, where all particles that reach the boundary are killed. This process and its scaling limit are described in Section 5.3.

1. Notations

We define $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\iota = \sqrt{-1}$. We write $\mathbb{Z}^d_n$ for the lattice $\frac{1}{n}\mathbb{Z}^d$, for $n \in \mathbb{N}$, and since it is convenient we also set $\mathbb{Z}^d_\infty = \mathbb{R}^d$. Let us recall the basic constructions from [MP17], where paracontrolled distributions on lattices were developed. Define the Fourier transforms for $k, x \in \mathbb{R}^d$

$$\mathscr{F}_{\mathbb{R}^d}(f)(k) = \int_{\mathbb{R}^d} \mathrm{d}x\ f(x) e^{-2\pi\iota\langle x,k\rangle},\ \mathscr{F}^{-1}_{\mathbb{R}^d}(f)(x) = \int_{\mathbb{R}^d} \mathrm{d}k\ f(k) e^{2\pi\iota\langle x,k\rangle}$$

as well as for $x \in \mathbb{Z}^d_n, k \in \mathbb{T}^d_n$ (with $\mathbb{T}^d_n = n[-1/2, 1/2]^d /\!\sim$ being the n-dilatation of the torus $\mathbb{T}^d$ and "$\sim$" being the relation that glues two opposing edges):

$$\mathscr{F}_n(f)(k) = \frac{1}{n^d} \sum_{x \in \mathbb{Z}^d_n} f(x) e^{-2\pi\iota\langle x,k\rangle}, k \in \mathbb{T}^d_n,\ \mathscr{F}^{-1}_n(f)(x) = \int_{\mathbb{T}^d_n} \mathrm{d}k\ f(k) e^{2\pi\iota\langle x,k\rangle}.$$

Consider $\omega(x) = |x|^\sigma$ for some $\sigma \in (0,1)$. We then define $\mathscr{S}_\omega$ and $\mathscr{S}'_\omega$ as in [MP17, Definition 2.8]. Roughly speaking $\mathscr{S}_\omega$ is a subset of the usual Schwartz functions, and $\mathscr{S}'_\omega$ consists of *ultradistributions*, a generalization of Schwartz distributions with more permissive growth conditions for $|x| \to \infty$. We also introduce the space $\boldsymbol{\varrho}(\omega)$ of admissible weights as in [MP17, Definition 2.7]. For our purposes it suffices to know that for any $a \in \mathbb{R}_{\geq 0}, l \in \mathbb{R}$, the functions $p(a)$ and $e(l)$ belong to $\boldsymbol{\varrho}(\omega)$, where

$$p(a)(x) = (1+|x|)^{-a}, \qquad e(l)(x) = e^{-l|x|^\sigma}.$$

Moreover, we fix functions ϱ, χ in $\mathscr{S}_\omega$ supported in an annulus and a ball respectively, such that for $\varrho_{-1} = \chi$ and $\varrho_j(\cdot) = \varrho(2^{-j}\cdot)$, $j \in \mathbb{N}_0$, the sequence $\{\varrho_j\}_{j\geq -1}$ forms a dyadic partition of the unity. We also assume that $\mathrm{supp}(\chi), \mathrm{supp}(\varrho) \subset (-1/2, 1/2)^d$ and write $j_n \in \mathbb{N}$ for the smallest index such that $\mathrm{supp}(\varrho_j) \not\subseteq n[-1/2,1/2]^d$. For $j < j_n$ and $\varphi \colon \mathbb{Z}^d_n \to \mathbb{R}$ we define the *Littlewood-Paley blocks*

$$\Delta^n_j \varphi = \mathscr{F}^{-1}_n\big(\varrho_j \mathscr{F}_n(\varphi)\big), \qquad \Delta^n_{j_n} \mathscr{F}^{-1}_n\big((1-\sum_{-1\leq j<j_n} \varrho_j)\mathscr{F}_n(\varphi)\big)$$

and define for $\alpha \in \mathbb{R}$, $p,q \in [1,\infty]$ and $z \in \boldsymbol{\varrho}(\omega)$ the discrete weighted Besov spaces $B^\alpha_{p,q}(\mathbb{Z}^d_n, z)$ via the norm:

$$\|\varphi\|_{B^\alpha_{p,q}(\mathbb{Z}^d_n,z)} = \big\|\big(2^{j\alpha}\|\Delta^n_j\varphi\|_{L^p(\mathbb{Z}^d_n,z)}\big)_{j\leq j_n}\big\|_{\ell^q(\leq j_n)}$$

where $\|\varphi\|_{L^p(\mathbb{Z}^d_n,z)} = \big(\sum_{x\in\mathbb{Z}^d_n} n^{-d}|z(x)\varphi(x)|^p\big)^{1/p}$ and $\|\cdot\|_{\ell^q(\leq j_n)}$ is the classical ℓ^q norm with the sum truncated at the j_n-th term. We write $\mathscr{C}^\alpha(\mathbb{Z}^d_n, z)$ for $B^\alpha_{\infty,\infty}(\mathbb{Z}^d_n, z)$ and $\mathscr{C}^\alpha_p(\mathbb{Z}^d_n, z)$ for $B^\alpha_{p,\infty}(\mathbb{Z}^d_n, z)$. The same definitions and notations are assumed for the classical Besov spaces on the whole space $B^\alpha_{p,q}(\mathbb{R}^d, z)$, which are defined analogously (with $\Delta_j\varphi = \Delta^\infty_j\varphi = \mathscr{F}^{-1}_{\mathbb{R}^d}(\rho_j\mathscr{F}_{\mathbb{R}^d}\varphi)$ for all $j \geq -1$, and $j_\infty = \infty$). We also consider the extension operator $\mathscr{E}^n \colon B^\alpha_{p,q}(\mathbb{Z}^d_n, z) \to B^\alpha_{p,q}(\mathbb{R}^d, z)$ as in [MP17, Lemma 2.24]. We denote with $C^\infty_c(\mathbb{R}^d)$ the space of smooth and compactly supported functions and with $C_b(\mathbb{R}^d)$ the space of continuous and bounded functions.

Remark 1.1. *In the setting we just introduced, we can decompose the (in the continuous case a priori ill-posed) product of two distributions φ, ψ as:*

$$\varphi\cdot\psi = \varphi \varolessthan \psi + \varphi \odot \psi + \psi \varolessthan \varphi, \quad \varphi \varolessthan \psi = \sum_{1\leq i\leq j_n} \Delta^n_{<i-1}\varphi\Delta^n_i\psi, \quad \varphi\odot\psi = \sum_{\substack{|i-j|\leq 1\\ -1\leq i,j\leq j_n}} \Delta^n_i\varphi\Delta^n_j\psi$$

with $\Delta^n_{<i-1}\varphi = \sum_{-1\leq j<i-1}\Delta^n_j\varphi$. Here we explicitly allow the case $n = \infty$. To simplify the notation and because it will be clear from the context, we do not include n in the notation for $\varolessthan$ and $\odot$. We call $\varphi \varolessthan \psi$ the paraproduct, *and $\varphi\odot\psi$ the* resonant product.

Now we consider time-dependent functions. Fix an (arbitrary) time horizon $T > 0$ and assume we are given an increasing family of normed spaces $X = (X(t))_{t\in[0,T]}$ with decreasing norms ($X(t) \equiv X(0)$ is allowed). Usually we will use this to deal with time-dependent weights and take $X(t) = \mathscr{C}^\alpha(\mathbb{Z}^d_n, e(l+t))$ for some $\alpha, l \in \mathbb{R}$. We then write CX for the space of continuous functions $\varphi \colon [0,T] \to X(T)$ endowed with the supremum norm $\|\varphi\|_{CX} = \sup_{t\in[0,T]}\|\varphi(t)\|_{X(t)}$. For $\alpha \in (0,1)$ we sometimes quantify the time regularity via $C^\alpha X = \{f \in CX : \|f\|_{C^\alpha X} < \infty\}$, where

$$\|f\|_{C^\alpha X} = \|f\|_{CX} + \sup_{0\leq s<t\leq T}\frac{\|f(t)-f(s)\|_{X(t)}}{|t-s|^\alpha}.$$

To control a blowup of the norm of order $\gamma \in [0,1)$ as $t \to 0$ we also define the spaces $\mathscr{M}^\gamma X$ of functions $f \colon (0,T] \to X(T)$ with norm $\|\varphi\|_{\mathscr{M}^\gamma X} = \sup_{t\in(0,T]} t^\gamma\|\varphi(t)\|_{X(t)}$. Finally, we need the following parabolically scaled spaces

$$\mathscr{L}^{\gamma,\alpha}_p(\mathbb{Z}^d_n, e(l)) = \big\{f \in C([0,T], \mathscr{S}'_\omega) : f \in \mathscr{M}^\gamma\mathscr{C}^\alpha(\mathbb{Z}^d_n, e(l+\cdot)), t \mapsto t^\gamma f(t) \in C^{\alpha/2}L^p(\mathbb{Z}^d_n, e(l+\cdot))\big\}.$$

See [MP17, Definition 3.8] for these constructions.

We will write $\mathfrak{L}^n = \partial_t - \Delta^n$, where Δ^n is the discrete Laplacian (for $x, y \in \mathbb{Z}^d_n$ we say $x \sim y$ if $|x-y| = n^{-1}$):

$$\Delta^n \varphi(x) = \frac{1}{n^2} \sum_{y \sim x} (\varphi(y) - \varphi(x)),$$

and $\Delta^\infty = \Delta$ is the usual Laplacian. We stress that Δ^n without subscript always denotes the discrete Laplacian, while Δ^n_j always denotes a Littlewood-Paley block. The following estimates will be useful in the discussion ahead.

Lemma 1.2. *The estimates below hold uniformly over $n \in \mathbb{N} \cup \{\infty\}$ (recall that $\mathbb{Z}^d_\infty = \mathbb{R}^d$). Consider $z, z_1, z_2, z_3 \in \boldsymbol{\varrho}(\omega)$ and $\alpha, \beta \in \mathbb{R}$. We find that:*

$$\begin{aligned}
&\|\varphi \varolessthan \psi\|_{\mathscr{C}^\alpha_p(\mathbb{Z}^d_n; z_1 z_2)} \lesssim \|\varphi\|_{L^p(\mathbb{Z}^d_n; z_1)} \|\psi\|_{\mathscr{C}^\alpha(\mathbb{Z}^d_n; z_2)}, \\
&\|\varphi \varolessthan \psi\|_{\mathscr{C}^{\alpha+\beta}_p(\mathbb{Z}^d_n; z_1 z_2)} \lesssim \|\varphi\|_{\mathscr{C}^\beta_p(\mathbb{Z}^d_n; z_1)} \|\psi\|_{\mathscr{C}^\alpha(\mathbb{Z}^d_n; z_2)}, && \textit{if } \beta < 0, \\
&\|\varphi \odot \psi\|_{\mathscr{C}^{\alpha+\beta}_p(\mathbb{Z}^d_n; z_1 z_2)} \lesssim \|\varphi\|_{\mathscr{C}^\beta_p(\mathbb{Z}^d_n; z_1)} \|\psi\|_{\mathscr{C}^\alpha(\mathbb{Z}^d_n; z_2)} && \textit{if } \alpha+\beta > 0.
\end{aligned}$$

Similar bounds hold if we estimate ψ in a $\mathscr{C}_p$ Besov space and therefore φ in $\mathscr{C} = \mathscr{C}_\infty$. And for any $\gamma \in [0, 1), \varepsilon \in [0, 2\gamma] \cap [0, \alpha), 0 < \alpha < 2$ and any $\delta > 0$ we can bound:

$$\|\varphi\|_{\mathscr{L}^{\gamma-\varepsilon/2, \alpha-\varepsilon}_p(\mathbb{Z}^d_n; z)} \lesssim \|\varphi\|_{\mathscr{L}^{\gamma, \alpha}_p(\mathbb{Z}^d_n; z)}.$$

Moreover, for the operator $C_1(\varphi, \psi, \zeta) = (\varphi \varolessthan \psi) \odot \zeta - \varphi(\psi \odot \zeta)$ we have:

$$\|C_1(\varphi, \psi, \zeta)\|_{\mathscr{C}^{\beta+\gamma}_p(\mathbb{Z}^d_n; z_1 z_2 z_3)} \lesssim \|\varphi\|_{\mathscr{C}^\alpha_p(\mathbb{Z}^d_n; z_1)} \|\psi\|_{\mathscr{C}^\beta(\mathbb{Z}^d_n; z_2)} \|\zeta\|_{\mathscr{C}^\gamma(\mathbb{Z}^d_n; z_3)},$$

if $\beta+\gamma < 0$, $\alpha+\beta+\gamma > 0$.

Proof. The proof of the first three estimates is contained in [MP17, Lemma 4.2] and the fourth estimate comes from [MP17, Lemma 3.11]. In that lemma the case $\varepsilon = 2\gamma < \alpha$ is not included, but it follows by the same arguments (since [GP17, Lemma A.1] still applies in that case). The last estimate is provided by [MP17, Lemma 4.4]. □

For two functions $\psi, \varphi \colon \mathbb{R}^d \to \mathbb{R}$ we define $\langle \psi, \varphi \rangle = \int \mathrm{d}x \; \psi(x)\varphi(x)$. For two functions $\psi, \varphi \colon \mathbb{Z}^d_n \to \mathbb{R}$ we write:

$$\langle \psi, \varphi \rangle_n = \frac{1}{n^d} \sum_{x \in \mathbb{Z}^d_n} \psi(x)\varphi(x), \qquad (\psi, \varphi) = \sum_{x \in \mathbb{Z}^d_n} \psi(x)\varphi(x)$$

and whenever there is no danger of misunderstanding we write $\langle \psi, \varphi \rangle$ instead of $\langle \psi, \varphi \rangle_n$. We also use the following notation for convolutions:

$$\begin{aligned}
f * g(x) &= \int_{\mathbb{R}^d} \mathrm{d}y \; f(x-y)g(y), && \text{for } f, g \colon \mathbb{R}^d \to \mathbb{R}, \\
f *_n g(x) &= \frac{1}{n^d} \sum_{y \in \mathbb{Z}^d_n} f(x-y)g(y), && \text{for } f, g \colon \mathbb{Z}^d_n \to \mathbb{R}.
\end{aligned}$$

Moreover, for $f, g \colon \mathbb{R}^d \to \mathbb{R}$ if $\mathrm{supp}(\mathscr{F}_{\mathbb{R}^d} f), \mathrm{supp}(\mathscr{F}_{\mathbb{R}^d} g) \subset (-n/2, n/2)^d$, then $f * g = f *_n g$ and we will use the two notations without distinction.

Finally, for a metric space E we denote with $\mathbb{D}([0, T]; E)$ and $\mathbb{D}([0, +\infty); E)$ the Skorohod space equipped with the Skorohod topology (cf. [EK86, Section 3.5]).

We will also write $\mathscr{M}(\mathbb{R}^d)$ for the space of positive finite measures on $\mathbb{R}^d$ with the weak topology, which is a Polish space (cf. [DP12, Section 3]).

2. The Model

We consider a branching random walk in a random environment (BRWRE). This will be a process on the lattice $\mathbb{Z}^d_n$, for $n \in \mathbb{N}$ and dimension $d = 1, 2$, and we are interested in the limit $n \to \infty$. The evolution of this process depends on the environment it lives in. Therefore, we start by discussing the environment before introducing the Markov process.

A *deterministic environment* is a sequence $\{\xi^n\}_{n\in\mathbb{N}}$ of potentials on the lattice, i.e. functions $\xi^n \colon \mathbb{Z}^d_n \to \mathbb{R}$. The environment we will work with will be chosen randomly. A *random environment* is a sequence of probability spaces $(\Omega^{p,n}, \mathscr{F}^{p,n}, \mathbb{P}^{p,n})$ together with a sequence $\{\xi^n_p\}_{n\in\mathbb{N}}$ of measurable maps $\xi^n_p \colon \Omega^{p,n} \times \mathbb{Z}^d_n \to \mathbb{R}$.

Assumption 2.1 (Random Environment). *We assume that $\{\xi^n_p(x)\}_{x\in\mathbb{Z}^d_n}$ is a set of i.i.d random variables which satisfy:*

$$n^{-d/2}\xi^n_p(x) \sim \Phi, \qquad \forall n \in \mathbb{N} \tag{3}$$

for a probability distribution Φ on $\mathbb{R}$ which finite moments of every order and which satisfies

$$\mathbb{E}[\Phi] = 0, \quad \mathbb{E}[\Phi^2] = 1.$$

Remark 2.2. *For clarity, in this setting it follows that ξ^n_p converges in distribution to a white noise ξ_p on $\mathbb{R}^d$, in the sense that:*

$$\langle \xi^n_p, f\rangle_n = \frac{1}{n^d}\sum_{x\in\mathbb{Z}^d_n} \xi^n_p(x) f(x) \longrightarrow \xi_p(f)$$

for any continuous f with compact support.

To separate the randomness coming from the potential from that of the branching random walks it will be convenient to freeze the realization of ξ^n_p and to consider it as a deterministic environment. But of course we cannot expect to obtain reasonable scaling limits (or even a well defined branching random walk) for all deterministic environments. Therefore, we need to identify certain analytical properties that hold for typical realizations of random potentials satisfying Assumption 2.1. The reader only interested in random environments may skip the following assumption and use it as a black box, since by Lemma 2.4 below it is satisfied for random environments satisfying Assumption 2.1.

Assumption 2.3 (Deterministic environment). *Let ξ^n be a deterministic environment and let X^n be the solution to the equation $-\Delta^n X^n = \chi(D)\xi^n = \mathscr{F}^{-1}_n(\chi\mathscr{F}_n\xi^n)$ in the sense explained in* [MP17, Section 5.1]*, where χ is a smooth function equal to 1 outside of $(-1/4, 1/4)^d$ and equal to zero on $(-1/8, 1/8)^d$. Consider a regularity parameter*

$$\alpha \in (1, \tfrac{3}{2}) \text{ in } d = 1, \qquad \alpha \in (\tfrac{2}{3}, 1) \text{ in } d = 2.$$

We assume that the following holds:

(i) There exists $\xi \in \bigcap_{a>0} \mathscr{C}^{\alpha-2}(\mathbb{R}^d, p(a))$ such that for all $a > 0$:

$$\sup_n \|\xi^n\|_{\mathscr{C}^{\alpha-2}(\mathbb{Z}^d_n, p(a))} < +\infty \quad \text{and} \quad \mathscr{E}^n\xi^n \to \xi \text{ in } \mathscr{C}^{\alpha-2}(\mathbb{R}^d, p(a)).$$

(ii) For any $a, \varepsilon > 0$ we can bound:

$$\sup_n \|n^{-d/2}\xi^n_+\|_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}^d_n, p(a))} + \sup_n \|n^{-d/2}|\xi^n|\|_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}^d_n, p(a))} < +\infty$$

as well as for any $b > d/2$:

$$\sup_n \|n^{-d/2}\xi^n_+\|_{L^2(\mathbb{Z}^d_n, p(b))} < +\infty.$$

Moreover, there exists $\nu \geq 0$ such that the following convergences hold:

$$\mathscr{E}^n n^{-d/2}\xi^n_+ \to \nu, \qquad \mathscr{E}^n n^{-d/2}|\xi^n| \to 2\nu$$

in $\mathscr{C}^{-\varepsilon}(\mathbb{R}^d, p(a))$.

(iii) If $d=2$ there exists a sequence $c_n \in \mathbb{R}$ such that $n^{-d/2}c_n \to 0$ and distributions $X \in \bigcap_{a>0} \mathscr{C}^{\alpha}(\mathbb{R}^d, p(a))$ and $X \diamond \xi \in \bigcap_{a>0} \mathscr{C}^{2\alpha-2}(\mathbb{R}^d, p(a))$ which satisfy for all $a>0$:

$$\sup_n \|X^n\|_{\mathscr{C}^{\alpha}(\mathbb{Z}^d_n, p(a))} + \sup_n \|(X^n \odot \xi^n) - c_n\|_{\mathscr{C}^{2\alpha-2}(\mathbb{Z}^d_n, p(a))} < +\infty$$

and $\mathscr{E}^n X^n \to X$ *in* $\mathscr{C}^{\alpha}(\mathbb{R}^d, p(a))$ *and* $\mathscr{E}^n\big((X^n \odot \xi^n) - c_n\big) \to X \diamond \xi$ *in* $\mathscr{C}^{2\alpha-2}(\mathbb{R}^d, p(a))$.

We also say that $\xi \in \mathscr{S}'_\omega(\mathbb{R}^d)$ is a *deterministic environment satisfying Assumption 2.3* if there exists a sequence $\{\xi^n\}_{n\in\mathbb{N}}$ such that the conditions of Assumption 2.3 hold.

The next result establishes the connection between the probabilistic and the analytical conditions we have stated. To formulate it we need the following sequence of diverging constants:

$$\kappa_n = \int_{\mathbb{T}^2_n} \mathrm{d}k \, \frac{\chi(k)}{l^n(k)} \sim \log(n), \tag{4}$$

with l^n being the Fourier multiplier associated to the discrete Laplacian Δ^n.

Lemma 2.4. *Given a random environment $\{\bar{\xi}^n_p\}_{n\in\mathbb{N}}$ satisfying Assumption 2.1, there exists a probability space $(\Omega^p, \mathscr{F}^p, \mathbb{P}^p)$ supporting random variables $\{\xi^n_p\}_{n\in\mathbb{N}}$ such that $\bar{\xi}^n_p = \xi^n_p$ in distribution and such that $\{\xi^n_p(\omega^p, \cdot)\}_{n\in\mathbb{N}}$ is a deterministic environment satisfying Assumption 2.3 for all $\omega^p \in \Omega^p$. Moreover the sequence c_n in Assumption 2.3 can be chosen equal to κ_n (see Equation (4)) outside of a nullset. Similarly, ν is strictly positive and deterministic outside of a nullset and equals the expectation $\mathbb{E}[\Phi_+]$.*

Proof. The existence of such a probability space is provided by the Skorohod representation theorem. Indeed it is a consequence of Assumption 2.1 that all the convergences hold in the sense of distributions: The convergences in (i) and (iii) follow from Lemma B.2 if $d=1$ and from [MP17, Lemmata 5.3 and 5.5] if $d=2$ (where it is also shown that we can choose $c_n = \kappa_n$). The convergence in (ii) for $\nu = \mathbb{E}[\Phi_+]$ is shown in Lemma B.1. After changing the probability space the Skorohod representation theorem guarantees almost sure convergence, so setting $\xi^n, \xi, c^n, \nu = 0$ on a nullset we find the result for every ω^p. (There is a small subtlety in the application of the Skorohod representation theorem because $\mathscr{C}^{\gamma}(\mathbb{R}^d, p(a))$ is not separable and thus not Polish space, but we can always restrict our attention to the closure of smooth compactly supported functions in $\mathscr{C}^{\gamma}(\mathbb{R}^d, p(a))$, which is a closed separable subspace). □

Notation 2.5. *A sequence of random variables $\{\xi^n_p\}_{n\in\mathbb{N}}$ defined on a common probability space $(\Omega^p, \mathscr{F}^p, \mathbb{P}^p)$ which almost surely satisfies Assumption 2.3 is called a* controlled random environment*. By Lemma 2.4, for any random environment satisfying Assumption 2.1 we can find new random variables on a new probability space with the same distribution, which form a controlled random environment. For a given controlled random environment we encode the renormalization needed in dimension $d=2$ by introducing the effective potential:*

$$\xi^n_{p,e}(\omega^p, x) = \xi^n_p(\omega^p, x) - c_n(\omega^p) 1_{\{d=2\}}.$$

If we work with a deterministic environment we will write ξ^n_e for the effective potential, defined analogously. In addition, given a controlled random environment we define $\mathscr{H}^{\omega^p}$ as the random Anderson Hamiltonian defined on the random domain $\mathscr{D}_{\mathscr{H}^{\omega^p}}$ (see Lemma 3.5). If the environment is deterministic we write $\mathscr{H}, \mathscr{D}_{\mathscr{H}}$ instead.

We pass to the description of the particle system. This will be a (random) Markov process on the space $E = \big(\mathbb{N}_0^{\mathbb{Z}^d_n}\big)_0$ of functions $\eta \colon \mathbb{Z}^d_n \to \mathbb{N}_0$ with compact support, whose construction is discussed in detail in Appendix A. We define $\eta^{x\mapsto y}(z) = \eta(z) + (1_{\{y\}}(z) - 1_{\{x\}}(z)) 1_{\{\eta(x)\geq 1\}}$ and $\eta^{x\pm}(z) = (\eta(z) \pm 1_{\{x\}}(z))_+$. Moreover, $C_b(E)$ is the Banach space of continuous and bounded functions on E endowed with the discrete topology.

Definition 2.6. *Fix an "averaging parameter" $\varrho \geq 0$ and a controlled random environment ξ^n_p. Let $\mathbb{P}^n$ be the measure on $\Omega^p \times \mathbb{D}([0,+\infty); E)$ defined as the "semidirect product measure"*

$\mathbb{P}^p \ltimes \mathbb{P}^{\omega^p,n}$, *where for* $\omega^p \in \Omega^p$ *the measure* $\mathbb{P}^{\omega^p,n}$ *on* $\mathbb{D}([0,+\infty);E)$ *is the law under which the canonical process* $u^n_p(\omega^p,\cdot)$ *started in* $u^n_p(\omega^p,0) = \lfloor n^\varrho \rfloor 1_{\{0\}}(x)$ *is the Markov process associated to the generator*

$$\mathscr{L}^{n,\omega^p} : \mathscr{D}(\mathscr{L}^{n,\omega^p}) \to C_b(E)$$

defined via:

$$\begin{aligned}\mathscr{L}^{n,\omega^p}(F)(\eta) = \sum_{x\in\mathbb{Z}^d_n} \eta_x \cdot \Bigg[& \sum_{y\sim x} n^2\big(F(\eta^{x\mapsto y}) - F(\eta)\big) \\ & + (\xi^n_e)_+(\omega^p,x)[F(\eta^{x+}) - F(\eta)] + (\xi^n_e)_-(\omega^p,x)[F(\eta^{x-}) - F(\eta)]\Bigg]\end{aligned} \tag{5}$$

where the domain $\mathscr{D}(\mathscr{L}^{n,\omega^p})$ *is the set of all functions* $F \in C_b(E)$ *such that the right-hand side of Equation* (5) *lies in* $C_b(E)$. *To* u^n_p *we associate the process*

$$\mu^n_p(\omega^p,t,x) = n^d \lfloor n^\varrho \rfloor^{-1} u^n_p(\omega^p,t,x)$$

with the pairing

$$\mu^n_p(\omega^p,t)(\varphi) := \langle \mu^n_p(\omega^p,t), \varphi\rangle_n = \sum_{x\in\mathbb{Z}^d_n} \lfloor n^\varrho \rfloor^{-1} u^n_p(\omega^p,t,x)\varphi(x)$$

for any function $\varphi\colon \mathbb{Z}^d_n \to \mathbb{R}$. *Hence* μ^n_p *is a stochastic process with values in* $\mathbb{D}([0,+\infty);\mathscr{M}(\mathbb{R}^d))$, *with the law induced by* $\mathbb{P}^n$.

Remark 2.7. *Although not explicitly stated, it is part of the definition that* $\omega^p \mapsto \mathbb{P}^{\omega^p,n}(A)$ *is measurable for Borel sets* $A \in \mathscr{B}(\mathbb{D}([0,+\infty);E))$.

Remark 2.8. *Since all particles move and branch independently, we expect that for* $\varrho \to \infty$ *the law of large numbers applies and we obtain the expected value of the branching dynamics conditionally on the realization of the random environment. This is why we refer to* ϱ *as an averaging parameter.*

Notation 2.9. *In the terminology of stochastic processes in random media, we refer to* $\mathbb{P}^{\omega^p,n}$ *as the* quenched law *of the process* u^n_p *(or* μ^n_p*) given the noise* ξ^n_p. *We also call* $\mathbb{P}^n$ *the* total law. *Moreover, although clearly a deterministic environment is also a controlled random environment, we will naturally distinguish the case in which we deal with a deterministic environment by dropping all the subscripts* p *and the dependence on* ω^p *(we will then consider the processes* u^n *or* μ^n*).*

We can now state the main results of this work. We will first prove quenched versions of the convergence results: the total version is then an easy corollary. We start with a law of large numbers.

Theorem 2.10. *For any deterministic environment* $\{\xi^n\}_{n\in\mathbb{N}}$ *satisfying Assumption 2.3 and for averaging parameter* $\varrho > d/2$, *let* w *be the solution of PAM* (1) *with initial condition* $w(0,x) = \delta_0(x)$, *as constructed in Proposition 3.1 (cf. also Remark 3.2). The measure-valued process* μ^n *from Definition 2.6 converges to* w *in probability in the space* $\mathbb{D}([0,+\infty);\mathscr{M}(\mathbb{R}^d))$ *as* $n \to +\infty$.

Proof. The proof can be found in Section 4.1. □

If the averaging parameter takes the critical value $\varrho = d/2$, we see random fluctuations in the limit and we end up with the *rough super-Brownian motion*. As in the case of the classical super-Brownian motion, the limiting process can be characterized also via duality with the following equation:

$$\partial_t \varphi = \mathscr{H}\varphi - \frac{\kappa}{2}\varphi^2, \qquad \varphi(0) = \varphi_0, \tag{6}$$

for $\varphi_0 \in C^\infty_c(\mathbb{R}^d)$, $\varphi_0 \geq 0$. With some abuse of notation (since the equation above is not linear) we write $U_t\varphi_0 = \varphi(t)$. Since the following definition is set in continuous space, we slightly tweak the original definition and say that a distribution ξ is a deterministic random environment satisfying Assumption 2.3, if there exists a sequence ξ^n which satisfies Assumption 2.3 with such ξ.

Definition 2.11. *Let ξ be a deterministic environment satisfying Assumption 2.3, let $\kappa > 0$ and let μ be a stochastic process with values in $C([0,+\infty);\mathscr{M}(\mathbb{R}^d))$, such that $\mu(0) = \delta_0$. Write $\mathscr{F} = \{\mathscr{F}_t\}_{t\in[0,+\infty)}$ for the completed and right-continuous filtration generated by μ. We say that μ is a* rough super-Brownian motion with parameter κ *if it satisfies one of the three properties below:*

(i) For any $t \geq 0$ and $\varphi_0 \in C_c^\infty(\mathbb{R}^d), \varphi_0 \geq 0$ and for $U.\varphi_0$ the solution to Equation (6) with initial condition φ_0, the process

$$N_t^{\varphi_0}(s) = e^{-\langle \mu(s), U_{t-s}\varphi_0\rangle}, \qquad s \in [0,t],$$

is a bounded continuous $\mathscr{F}-$martingale.

(ii) For any $t \geq 0$ and $\varphi_0 \in C_c^\infty(\mathbb{R}^d)$ and $f \in C([0,t];\mathscr{C}^\zeta(\mathbb{R}^d, e(l)))$ for some $\zeta > 0$ and $l < -t$, and for φ_t solving

$$\partial_s \varphi_t + \mathscr{H}\varphi_t = f, \qquad s \in [0,t], \qquad \varphi_t(t) = \varphi_0,$$

it holds that

$$M_t^{\varphi_0,f}(s) := \langle \mu(s), \varphi_t(s)\rangle - \langle \mu(0), \varphi_t(0)\rangle - \int_0^s \mathrm{d}r\ \langle \mu(r), f(r)\rangle, \qquad s \in [0,t],$$

is a continuous square-integrable $\mathscr{F}-$martingale with quadratic variation

$$\langle M_t^{\varphi_0,f}\rangle_s = \kappa \int_0^s \mathrm{d}r\ \langle \mu(r), (\varphi_t)^2(r)\rangle.$$

(iii) For any $\varphi \in \mathscr{D}_{\mathscr{H}}$ the process:

$$L^\varphi(t) = \langle \mu(t), \varphi\rangle - \langle \mu(0), \varphi\rangle - \int_0^t \mathrm{d}r\ \langle \mu(r), \mathscr{H}\varphi\rangle, \qquad t \in [0,+\infty),$$

is a continuous $\mathscr{F}-$martingale, square-integrable on $[0,T]$ for all $T > 0$, with quadratic variation

$$\langle L^\varphi\rangle_t = \kappa \int_0^t \mathrm{d}r\ \langle \mu(r), \varphi^2\rangle.$$

Every one of the three properties above is sufficient to characterize the process uniquely.

Lemma 2.12. *The three conditions of Definition 2.11 are equivalent. Moreover, if μ is a rough super-Brownian motion with parameter κ, then the law of μ is unique.*

Proof. The proof can be found at the end of Section 4.1. □

Theorem 2.13. *Let $\{\xi^n\}_{n\in\mathbb{N}}$ be a deterministic environment satisfying Assumption 2.3 and let the averaging parameter $\varrho = d/2$. Then the sequence $\{\mu^n\}_{n\in\mathbb{N}}$ converges to the rough super-Brownian motion μ with parameter $\kappa = 2\nu$ and initial condition $\mu(0) = \delta_0$ in distribution in $\mathbb{D}([0,+\infty);\mathscr{M}(\mathbb{R}^d))$.*

Proof. The proof can be found at the end of Section 4.1. □

Remark 2.14. *Lemma 2.12 gives the uniqueness of the rough super-Brownian motion for all parameters $\kappa > 0$, but Theorem 2.13 provides existence conditional on having an environment which satisfies Assumption 2.3. Here a natural constraint $\nu \in (0, \frac{1}{2}]$ appears, because we should think of $\nu = \mathbb{E}[\Phi_+]$ for a centered random variable Φ with $\mathbb{E}[\Phi^2] = 1$. We can establish the existence of the rough super-Brownian motion for general parameters $\kappa > 0$ by adding a critical branching mechanism to the dynamics of μ^n, see Section 4.2 for details.*

Remark 2.15. *We restrict our attention to the Dirac delta initial condition for simplicity, but most of our arguments extend to initial conditions* $\mu \in \mathscr{M}(\mathbb{R}^d)$ *that satisfy* $\langle \mu, e(l) \rangle < \infty$ *for all* $l < 0$. *In this case only the construction of the initial value sequence* $\{\mu^n(0)\}_{n \in \mathbb{N}}$ *is more technical, because we need to come up with an approximation in terms of integer valued point measures (which we need as initial condition for the particle system). The canonical choice would be* $\mu^n = n^{-\varrho} \sum_{k \in \mathbb{Z}^d_n} \delta_k \lfloor n^\varrho \mu(Q_{1/n}(k)) \rfloor$, *where* $Q_{1/n}(k)$ *is a box with radius* $1/n$ *centered around* k. *But for* $\varrho \in [\frac{d}{2}, d)$ *and for absolutely continuous* μ *with bounded density this would give* $\mu^n \equiv 0$ *for all large* n. *A possible solution is to discretize* μ *on a coarser grid than* $\mathbb{Z}^d_n = \frac{1}{n}\mathbb{Z}^d$, *say on* $\frac{M(n)}{n}\mathbb{Z}^d$ *with* $M(n) \gg n^{1/2}$. *We also need that* $\|\mu^n\|_{\mathscr{C}^0_1(\mathbb{Z}^d_n, e(l))} \lesssim \mu^n(e(l)) \lesssim \mu(e(l))$ *for all* $l < 0$, *which can be verified by writing the discrete Littlewood-Paley blocks as discrete convolutions.*

The extension to $\mu \in \mathscr{M}(\mathbb{R}^d)$ *without the moment condition* $\mu(e(l)) < \infty$ *or even to measures with infinite mass seems more subtle and would need more significant adaptations of our arguments.*

The previous results describe the scaling behavior of the BRWRE conditionally on the environment, and we now pass to the unconditional statements. To a given random environment ξ^n_p satisfying Assumption 2.1 (not necessarily a *controlled* random environment) we associate a sequence of random variables in $\mathscr{S}'_\omega(\mathbb{R}^d)$ by defining $\xi^n_p(f) = n^{-d} \sum_x \xi^n_p(x) f(x)$. The sequence of measures $\overline{\mathbb{P}}^n = \mathbb{P}^{p,n} \ltimes \mathbb{P}^{\omega^p,n}$ on $\mathscr{S}'_\omega(\mathbb{R}^d) \times \mathbb{D}([0,+\infty); \mathscr{M}(\mathbb{R}^d))$ is then such that $\mathbb{P}^{p,n}$ is the law of ξ^n_p and $\mathbb{P}^{\omega^p,n}$ is the quenched law of the branching process μ^n_p given ξ^n_p (cf. Appendix A).

Corollary 2.16. *The sequence of measures* $\overline{\mathbb{P}}^n$ *converges weakly to the measure* $\overline{\mathbb{P}} = \mathbb{P}^p \ltimes \mathbb{P}^{\omega^p}$ *on* $\mathscr{S}'_\omega(\mathbb{R}^d) \times \mathbb{D}([0,+\infty); \mathscr{M}(\mathbb{R}^d))$, *where* $\mathbb{P}^p$ *is the law of the space white noise on* $\mathscr{S}'_\omega(\mathbb{R}^d)$, *and* $\mathbb{P}^{\omega^p}$ *is the quenched law of* μ_p *given* ξ_p *which is described by Theorem 2.10 if* $\varrho > d/2$ *or by Theorem 2.13 if* $\varrho = d/2$.

Proof. Consider a continuous bounded function $F\colon \mathscr{S}'_\omega(\mathbb{R}^d) \times \mathbb{D}([0,+\infty); \mathscr{M}(\mathbb{R}^d)) \to \mathbb{R}$. We need to prove convergence of:

$$\lim_n \mathbb{E}\big[F(\xi^n_p, \mu^n)\big] \to \mathbb{E}\big[F(\xi_p, \mu)\big].$$

Up to changing the probability space (which does not affect the law) we may assume that ξ^n_p is a controlled random environment. We condition on the noise, rewriting the left-hand side as

$$\mathbb{E}\big[F(\xi^n_p, \mu^n)\big] = \int \mathbb{E}^{\omega^p,n}\big[F(\xi^n_p(\omega^p), \mu^n)\big] \mathbb{P}^p(\mathrm{d}\,\omega^p).$$

Under the additional property of being a controlled random environment and for fixed $\omega^p \in \Omega^p$, the conditional law $\mathbb{P}^{\omega^p,n}$ on the space $\mathbb{D}([0,+\infty); \mathscr{M}(\mathbb{R}^d))$ converges weakly to the measure $\mathbb{P}^{\omega^p}$ given by Theorem 2.10 respectively Theorem 2.13, according to the value of ϱ. We can thus deduce the result by dominated convergence. □

For $\varrho > d/2$ the process of Corollary 2.16 is simply the continuous parabolic Anderson model. For $\varrho = d/2$ it is a new process, which we name as follows:

Definition 2.17. *For* $\varrho = d/2$ *we call the process* μ *of Corollary 2.16 an* SBM in static random environment *(of parameter* $\kappa > 0$*).*

In dimension $d = 1$ we characterize the process μ as the solution to the SPDE (2). First, we rigorously define solutions to such an equation.

Definition 2.18. *Consider dimension* $d = 1$, *a value* $\kappa > 0$, *and* $\pi \in \mathscr{M}(\mathbb{R})$. *A weak solution to the SPDE*

$$\partial_t \mu_p(t,x) = \mathscr{H}^{\omega^p} \mu_p(t,x) + \sqrt{\kappa \mu_p(t,x)}\,\tilde{\xi}(t,x), \qquad \mu_p(0) = \pi, \tag{7}$$

is a couple formed by a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ *and a random process*

$$\mu_p\colon \Omega \to C([0,+\infty); \mathscr{M}(\mathbb{R}))$$

such that $\Omega = \Omega^p \times \bar{\Omega}$ *and* $\mathbb{P}$ *is of the form* $\mathbb{P}^p \ltimes \mathbb{P}^{\omega^p}$ *with* $(\Omega^p, \mathbb{P}^p)$ *supporting a space white noise* ξ_p *and* $(\bar{\Omega}, \mathbb{P})$ *supporting a space-time white noise* $\tilde{\xi}$ *that is independent of* ξ*, and such that the following properties are fulfilled for almost all* $\omega^p \in \Omega^p$*:*

- *There exists a filtration* $\{\mathscr{F}_t^{\omega^p}\}_{t\in[0,T]}$ *on the space* $(\bar{\Omega}, \mathbb{P}^{\omega^p})$ *which satisfies the usual conditions and such that* $\mu_p(\omega^p, \cdot)$ *is adapted to the filtration and almost surely lies in* $L^p([0,T]; L^2(\mathbb{R}, e(l)))$ *for all* $p < 2$ *and* $l \in \mathbb{R}$*. Moreover, under* $\mathbb{P}^{\omega^p}$ *the process* $\tilde{\xi}(\omega^p, \cdot)$ *is a space-time white noise adapted to the same filtration.*
- *The random process* μ_p *satisfies for all* $\varphi \in \mathscr{D}_{\mathscr{H}^{\omega^p}}$*:*

$$\int_{\mathbb{R}} \mathrm{d}x \; \mu_p(\omega^p, t, x)\varphi(x) = \int_{\mathbb{R}} \varphi(x)\pi(dx) + \int_0^t \int_{\mathbb{R}} \mathrm{d}s \, \mathrm{d}x \; \mu_p(\omega^p, s, x)(\mathscr{H}^{\omega^p}\varphi)(x) + \int_0^t \int_{\mathbb{R}} \tilde{\xi}(\omega^p, \mathrm{d}s, \mathrm{d}x)\sqrt{\kappa\mu_p(\omega^p, s, x)}\varphi(x), \quad \forall t \geq 0,$$

  with the last integral understood in the sense of Walsh [Wal86].

Thus, we can state the existence and uniqueness of solutions to the above SPDE.

Theorem 2.19. *For* $\pi = \delta_0$ *and any* $\kappa > 0$ *there exists a weak solution* μ_p *to the SPDE* (7) *in the sense of the above equation. The law of* μ_p *as a random process on* $C([0, +\infty); \mathscr{M}(\mathbb{R}))$ *is unique and corresponds to a SBM in static random environment of parameter* κ*.*

Proof. The proof can be found at the end of Section 5.1. □

As a last result, we show that rSBM is persistent in dimension $d = 1, 2$.

Definition 2.20. *We say that a random process* $\mu \in C([0, +\infty); \mathscr{M}(\mathbb{R}^d))$ *is* super-exponentially persistent *if for any nonzero positive function* $\varphi \in C_c^\infty(\mathbb{R}^d)$ *and for all* $\lambda > 0$ *it holds that:*

$$\mathbb{P}\big(\lim_{t\to\infty} e^{-t\lambda}\langle \mu(t), \varphi\rangle = \infty\big) > 0$$

Theorem 2.21. *Let* μ_p *be an SBM in static random environment. Then for almost all* $\omega^p \in \Omega^p$ *the process* $\mu_p(\omega^p, \cdot)$ *is super-exponentially persistent.*

The result follows from Corollary 5.7 and the preceding discussion.

3. Discrete and Continuous PAM & Anderson hamiltonian

We discuss the existence of solutions to PAM (1) in the discrete and continuous setting and the interplay between the two.

Recall that the regularity parameter α from Assumption 2.3 satisfies:

$$\alpha \in (1, \tfrac{3}{2}) \text{ in } d = 1, \qquad \alpha \in (\tfrac{2}{3}, 1) \text{ in } d = 2. \tag{8}$$

3.1. Spatially Global Solutions.

Here we review some results from [MP17] regarding the solution of the PAM on the whole space (see also [HL15]), and regarding the convergence of lattice models to the PAM. We take an initial condition $w_0 \in \mathscr{C}_p^\zeta(\mathbb{R}^d, e(l))$ and a forcing $f \in \mathscr{M}^{\gamma_0}\mathscr{C}_p^{\alpha_0}(\mathbb{R}^d, e(l))$, and consider the generalized equation

$$\partial_t w = \Delta w + \xi w + f, \qquad w(0) = w_0 \tag{9}$$

and its discrete counterpart

$$\partial_t w^n = (\Delta^n + \xi_e^n) w^n + f^n, \qquad w^n(0) = w_0^n. \tag{10}$$

To motivate the constraints on the parameters appearing in the proposition below, let us first formally discuss the solution theory in $d = 1$. Under Assumption 2.3 it follows from the Schauder estimates in [MP17, Lemma 3.10] that the best regularity we can expect at a fixed time is $w(t) \in \mathscr{C}^{\alpha\wedge(\zeta+2)\wedge(\alpha_0+2)}(\mathbb{R}, e(k))$ for some $k \in \mathbb{R}$. In fact we lose a bit of regularity, so let $\vartheta < \alpha$ be "large enough" (we will see soon what we need from ϑ) and assume that $\zeta + 2 \geq \vartheta$

and $\alpha_0 + 2 \geq \vartheta$. Then we expect $w(t) \in \mathscr{C}^{\vartheta}(\mathbb{R}, e(k))$, and the Schauder estimates suggest the blow-up $\gamma = \max\{(\vartheta + \varepsilon - \zeta)_+/2, \gamma_0\}$ for some $\varepsilon > 0$, which has to be in $[0,1)$ to be locally integrable, so in particular $\gamma_0 \in [0,1)$. If $\vartheta + \alpha - 2 > 0$ (which is possible because in $d = 1$ we have $2\alpha - 2 > 0$), then the product $w(t)\xi$ is well defined and in $\mathscr{C}^{\alpha-2}(\mathbb{R}, e(k)p(a))$, so we can set up a Picard iteration. The loss of control in the weight (going from $e(k)$ to $e(k)p(a)$) is handled by introducing time-dependent weights so that $w(t) \in \mathscr{C}^{\vartheta}(\mathbb{R}^d, e(l+t))$. In the setting of singular SPDEs this idea was introduced by Hairer-Labbé [HL15], and it induces a small loss of regularity which explains why we only obtain regularity $\vartheta < \alpha$ for the solution and the additional $+\varepsilon/2$ in the blow-up γ.

In two dimensions the white noise is less regular and we no longer have $2\alpha - 2 > 0$, and therefore we have to use paracontrolled analysis to solve the equation. The solution lives in a space of *paracontrolled distributions*, and now we take $\vartheta > 0$ such that $\vartheta + 2\alpha - 2 > 0$.

and to solve the equation in that space we need additional regularity requirements for the initial condition w_0 and for the forcing f. More precisely, we need to be able to multiply $(P_t w_0)\xi$ and $\big(\int_0^t P_{t-s} f(s)\,\mathrm{d}s\big)\xi$, and therefore we require now also $\zeta{+}2{+}(\alpha{-}2) > 0$ and $\alpha_0{+}2{+}(\alpha{-}2) > 0$, i.e. $\zeta, \alpha_0 > -\alpha$.

We do not provide the details of the construction and refer to [MP17] instead, where the two-dimensional case is worked out (the one-dimensional case follows from similar, but much easier arguments).

Proposition 3.1. *Consider α as in (8), any $T > 0$, $p \in [1, +\infty]$, $l \in \mathbb{R}$ and $\vartheta, \zeta, \gamma_0, \alpha_0$ satisfying:*

$$(11)\quad \vartheta \in \begin{cases} (2{-}\alpha, \alpha), & d = 1, \\ (2{-}2\alpha, \alpha), & d = 2, \end{cases} \qquad \zeta > (\vartheta{-}2) \vee (-\alpha), \qquad \gamma_0 \in [0,1), \qquad \alpha_0 > (\vartheta{-}2) \vee (-\alpha),$$

and let $w_0^n \in \mathscr{C}_p^{\zeta}(\mathbb{Z}_n^d, e(l))$ and $f^n \in \mathscr{M}^{\gamma_0}\mathscr{C}_p^{\alpha_0}(\mathbb{Z}_n^d, e(l))$ such that

$$\mathscr{E}^n w_0^n \to w_0, \text{ in } \mathscr{C}_p^{\zeta}(\mathbb{R}^d, e(l)), \qquad \mathscr{E}^n f^n \to f \text{ in } \mathscr{M}^{\gamma_0}\mathscr{C}_p^{\alpha_0}(\mathbb{R}^d, e(l)).$$

Then under Assumption 2.3 there exist unique (paracontrolled) solutions w^n, w to Equation (10) and (9). Moreover, for all $\gamma > (\vartheta{-}\zeta)_+/2 \vee \gamma_0$ and for all $\hat{l} \geq l{+}T$, the sequence w^n is uniformly bounded in $\mathscr{L}_p^{\gamma,\vartheta}(\mathbb{Z}_n^d, e(\hat{l}))$:

$$\sup_n \|w^n\|_{\mathscr{L}_p^{\gamma,\vartheta}(\mathbb{Z}_n^d, e(\hat{l}))} \lesssim \sup_n \|w_0^n\|_{\mathscr{C}_p^{\zeta}(\mathbb{Z}_n^d, e(l))} + \sup_n \|f^n\|_{\mathscr{M}^{\gamma_0}\mathscr{C}_p^{\alpha_0}(\mathbb{Z}_n^d, e(l))},$$

where the proportionality constant depends on the time horizon T and the norms of the objects in Assumption 2.3. Moreover

$$\mathscr{E}^n w^n \to w \text{ in } \mathscr{L}_p^{\gamma,\vartheta}(\mathbb{R}^d, e(\hat{l})).$$

Remark 3.2. *For most applications the integrability parameter $p = +\infty$ is sufficient. In this work, $p < \infty$ is only required for the construction of the Green function associated to PAM. Indeed the Dirac measure δ_0 lies in $\mathscr{C}^{-d}(\mathbb{R}^d, e(l))$ for any $l \in \mathbb{R}$. This means that $\zeta = -d$, and in $d = 1$ we can choose ϑ small enough such that (11) holds, which allows us to solve the PAM (9) in dimension $d = 1$ with initial condition δ_0. But in $d = 2$ this is not sufficient, so we use instead that $\delta_0 \in \mathscr{C}_p^{d(1-p)/p}(\mathbb{R}^d, e(l))$ for $p \in [1, \infty]$ and any $l \in \mathbb{R}$, so that for $p \in [1,2)$ the conditions in (11) are satisfied.*

Notation 3.3. *We write*

$$t \mapsto T_t^n w_0^n + \int_0^t \mathrm{d}s\ T_{t-s}^n f_s^n, \qquad t \mapsto T_t w_0 + \int_0^t \mathrm{d}s\ T_{t-s} f_s$$

for the solution to Equation (10) and (9), respectively.

Proposition 3.1 can roughly speaking be interpreted as the convergence of the semigroup associated to the discrete Anderson hamiltonian $\mathscr{H}^n = \Delta^n {+} \xi_e^n$ to that of the continuous Anderson hamiltonian $\mathscr{H} = \Delta{+}\xi$, since formally $T_t^n = e^{t\mathscr{H}^n}$ and $T_t = e^{t\mathscr{H}}$. We are also interested in the martingale problem based on $\mathscr{H}$, and therefore we need to rigorously construct $\mathscr{H}$. This is a bit subtle, because smooth functions are not in its domain. In finite volume and $d = 1$ Fukushima-Nakao [FN77] use Dirichlet forms for the construction, while the two-dimensional case in finite volume is studied by Allez-Chouk [AC15], who use paracontrolled distributions and the resolvent equation. In infinite volume the resolvent equation is problematic though, because we expect the spectrum of $\mathscr{H}$ to be unbounded from above. Hairer-Labbé [HL18] suggest a construction based on spectral calculus by setting $\mathscr{H} = t^{-1}\log T_t$, but this gives insufficient information about the domain. And constructing an infinitesimal generator of the semigroup (T_t) is also quite subtle, since due to the time-dependent weights T_t maps $\mathscr{C}^\zeta(\mathbb{R}^d, e(l))$ to $\mathscr{C}^{\zeta\wedge\vartheta}(\mathbb{R}^d, e(l+t))$, and therefore it does not define a continuous semigroup in a Banach space. Therefore, we take an ad-hoc approach here which is sufficient for our purpose.

Let us first treat the case $d = 1$. Then $\xi \in \mathscr{C}^{\alpha-2}(\mathbb{R}, p(a))$ for all $a > 0$ by assumption, where $\alpha \in (1, \frac{3}{2})$. In particular, $\mathscr{H}u = (\Delta+\xi)u$ is well defined for all $u \in \mathscr{C}^\vartheta(\mathbb{R}, e(l))$ with $\vartheta > 2-\alpha$ and $l \in \mathbb{R}$, and $\mathscr{H}u \in \mathscr{C}^{\alpha-2}(\mathbb{R}, e(l)p(a))$. Our aim is to identify a subset of $\mathscr{C}^\vartheta(\mathbb{R}, e(l))$ on which $\mathscr{H}u$ is even a continuous function. We can do this by defining for $t > 0$

$$A_t u = \int_0^t T_s u \,\mathrm{d}s.$$

Then $A_t u \in \mathscr{C}^\vartheta(\mathbb{R}, e(l+t))$, and by definition

$$\mathscr{H} A_t u = \int_0^t \mathscr{H} T_s u \,\mathrm{d}s = \int_0^t \partial_s T_s u \,\mathrm{d}s = T_t u - u \in \mathscr{C}^\vartheta(\mathbb{R}, e(l+t)).$$

Moreover,

$$\lim_{n\to\infty} n(T_{1/n} - \mathrm{id})A_t u = \lim_{n\to\infty} n\bigg(\int_t^{t+1/n} T_s \,\mathrm{d}s - \int_0^{1/n} T_s u \,\mathrm{d}s\bigg) = T_t u - u = \mathscr{H} A_t u,$$

where the convergence is in $\mathscr{C}^\vartheta(\mathbb{R}, e(l+t+\varepsilon))$ for arbitrary $\varepsilon > 0$. Therefore, we define

$$\mathscr{D}_{\mathscr{H}} = \{A_t u : u \in \mathscr{C}^\vartheta(\mathbb{R}, e(l)), l \in \mathbb{R}, t \in [0, T]\}.$$

Since for $u \in \mathscr{C}^\vartheta(\mathbb{R}, e(l))$ the map $(t \mapsto T_t u)_{t\in[0,\varepsilon]}$ is continuous in $\mathscr{C}^\vartheta(\mathbb{R}, e(l+\varepsilon))$ we can find for all $u \in \mathscr{C}^\vartheta(\mathbb{R}, e(l))$ a sequence $\{u^m\}_{m\in\mathbb{N}} \subset \mathscr{D}_{\mathscr{H}}$ such that $\|u^m - u\|_{\mathscr{C}^\vartheta(\mathbb{R}, e(l+\varepsilon))} \to 0$ for all $\varepsilon > 0$. Indeed, it suffices to set $u^m = m^{-1} A_{m^{-1}} u$. The same construction also works for $\mathscr{H}^n$ instead of $\mathscr{H}$.

In the two-dimensional case $(\Delta+\xi)u$ would be well defined whenever $u \in \mathscr{C}^\beta(\mathbb{R}^2, e(l))$ with $\beta > 2-\alpha$ for $\alpha \in (\frac{2}{3}, 1)$. But in this space it seems impossible to find a domain that is mapped to continuous functions. And also $(\Delta+\xi)u$ is not the right object to look at, we have to take the renormalization into account and should think of $\mathscr{H} = \Delta+\xi-\infty$. So we first need an appropriate notion of paracontrolled distributions u for which can define $\mathscr{H}u$ as a distribution. As in Proposition 3.1 we let $\vartheta \in (2-2\alpha, \alpha)$.

Definition 3.4. *We say that u^n (resp. u) is* paracontrolled *if $u \in \mathscr{C}^\vartheta(\mathbb{R}^2, e(l))$ for some $l \in \mathbb{R}$, and*

$$u^\sharp = u - u \varolessthan X \in \mathscr{C}^{\alpha+\vartheta}(\mathbb{R}^2, e(l)),$$

where we recall that $X = (-\Delta)^{-1}\chi(D)\xi$ is defined in Assumption 2.3. For paracontrolled u we set

$$\mathscr{H}u = \Delta u + \xi \varolessthan u + u \varolessthan \xi + u^\sharp \odot \xi + C_1(u, X, \xi) + u(X \diamond \xi),$$

where C_1 is defined in Lemma 1.2. The same lemma also shows that $\mathscr{H}u$ is a well defined distribution in $\mathscr{C}^{\alpha-2}(\mathbb{R}^2, e(l)p(a))$.

The operator T_t maps paracontrolled distributions to paracontrolled distributions, and therefore the same arguments as in one dimension allow us to find a domain $\mathscr{D}_{\mathscr{H}}$ such that for all paracontrolled $u \in \mathscr{C}^\vartheta(\mathbb{R}^2, e(l))$ there exists a sequence $\{u^m\}_{m\in\mathbb{N}} \subset \mathscr{D}_{\mathscr{H}}$ with $\|u^m - u\|_{\mathscr{C}^\vartheta(\mathbb{R}^2, e(l+\varepsilon))} \to 0$ for all $\varepsilon > 0$. For general $u \in \mathscr{C}^\vartheta(\mathbb{R}^2, e(l))$ and $\varepsilon > 0$ we can find a paracontrolled $v \in$

$\mathscr{C}^\vartheta(\mathbb{R}^2, e(l))$ with $\|u-v\|_{\mathscr{C}^\vartheta(\mathbb{R}^2,e(l+\varepsilon))} < \varepsilon$, because $T_t u$ is paracontrolled for all $t > 0$ and converges to u in $\mathscr{C}^\vartheta(\mathbb{R}^2, e(l+\varepsilon))$ as $t \to 0$. Thus, we have established the following result:

Lemma 3.5. *Make Assumption 2.3 and let ϑ be as in Proposition 3.1. There exists a domain $\mathscr{D}_{\mathscr{H}} \subset \bigcup_{l\in\mathbb{R}} \mathscr{C}^\vartheta(\mathbb{R}^d, e(l))$ such that $\mathscr{H}u = \lim_n n(T_{1/n} - \mathrm{id})u \in \mathscr{C}^\vartheta(\mathbb{R}^d, e(l+\varepsilon))$ for all $u \in \mathscr{D}_{\mathscr{H}} \cap \mathscr{C}^\vartheta(\mathbb{R}^d, e(l))$ and $\varepsilon > 0$ and such that for all $u \in \mathscr{C}^\vartheta(\mathbb{R}^d, e(l))$ there exists a sequence $\{u^m\}_{m\in\mathbb{N}} \subset \mathscr{D}_{\mathscr{H}}$ with $\|u^m - u\|_{\mathscr{C}^\vartheta(\mathbb{R}^2,e(l+\varepsilon))} \to 0$ for all $\varepsilon > 0$. The same is true for the discrete operator $\mathscr{H}^n$ (with $\mathbb{R}^d$ replaced by $\mathbb{Z}^d_n$).*

3.2. **Bounded Domains with Dirichlet Boundary Conditions.** We will discuss the results of [CvZ19], in order to solve PAM with Dirichlet boundary conditions both on a discrete and a continuous box. We fix the size of the box to be an arbitrary $L \in \mathbb{N}$ and define $N = 2L$. The main result here will be the analog of Proposition 3.1 with Dirichlet boundary conditions. We study the equation:

$$\begin{aligned} \partial_t w(t,x) &= \Delta w(t,x) + \xi(x) w(t,x) + f(t,x), \qquad (t,x) \in (0,T)\times(0,L)^d, \\ w(0,x) &= w_0(x), \qquad w(t,x) = 0 \text{ on } (0,T]\times\partial[0,L]^d. \end{aligned} \tag{12}$$

We consider $n \in \mathbb{N}\cup\{\infty\}$, and for $n = \infty$ we find ourselves in the continuous case, which studied in [CvZ19]. We write Λ_n for the lattice $\frac{1}{n}(\mathbb{Z}^d \cap [0, Ln]^d)$ (resp. $\Lambda_\infty = [0, L]^d$ if $n = \infty$). Similarly, we call Θ_n the lattice $\frac{1}{n}(\mathbb{Z}^d \cap [-\frac{Nn}{2}, \frac{Nn}{2}]^d)/\sim$ with opposite boundaries identified (resp. $\mathbb{T}^d_N$ if $n = \infty$) and define the "dual lattice" (where the Fourier transform lives) $\Xi_n = \frac{1}{N}(\mathbb{Z}^d \cap [-\frac{Nn}{2}, \frac{Nn}{2}]^d)/\sim$, (resp. $\frac{1}{N}\mathbb{Z}^d$ if $n = \infty$) as well as $\Xi^+_n = \frac{1}{N}(\mathbb{Z}^d \cap [0, Ln]^d)$, (resp. $\frac{1}{N}\mathbb{N}^d_0$) and $\partial\Xi^+_n = \{k \in \Xi^+_n \;:\; k_i = 0 \text{ for some } i \in \{1,\dots,d\}\}$.

The idea of [CvZ19] in the case $n = \infty$ is to consider suitable even and odd extensions of functions on Λ_n to periodic functions on Θ_n, and then to work with the usual tools from periodic paracontrolled distributions on Θ_n. So for $u\colon \Lambda_n \to \mathbb{R}$ we define

$$\begin{aligned} \Pi_o u\colon \Theta_n \to \mathbb{R}, &\qquad \Pi_o u(\mathfrak{q}\circ x) = \prod \mathfrak{q}\cdot u(x), \\ \Pi_e u\colon \Theta_n \to \mathbb{R}, &\qquad \Pi_e u(\mathfrak{q}\circ x) = u(x), \end{aligned}$$

where $x \in \Lambda_n, \mathfrak{q} \in \{-1,1\}^d$ and we define the component-wise product $\mathfrak{q}\circ x = (\mathfrak{q}_i x_i)_{i=1,\dots,d}$ as well as the total product $\prod \mathfrak{q} = \prod_{i=1}^d \mathfrak{q}_i$. We can interpret a function on Θ_n as being defined on the whole $\mathbb{Z}^d_n$ by extending it periodically and thus in principle we would be in the same setting as in Section 3.1. But it is convenient to make use of the periodic structure, and to work with a discrete periodic Fourier transform, defined for $\varphi\colon \Theta_n \to \mathbb{R}$ by

$$\mathscr{F}_{\Theta_n}\varphi(k) = \frac{1}{n^d}\sum_{x\in\Theta_n} \varphi(x) e^{-2\pi\iota\langle x,k\rangle}, \quad k \in \Xi_n.$$

As in [CvZ19] we have a periodic, a Dirichlet and a Neumann basis, which we indicate with: $\{\mathfrak{e}_k\}_{k\in\Xi_n}, \{\mathfrak{d}_k\}_{k\in\Xi^+_n\setminus\partial\Xi^+_n}, \{\mathfrak{n}_k\}_{k\in\Xi^+_n}$ respectively. Here $\mathfrak{e}_k$ is the classical Fourier basis:

$$\mathfrak{e}_k(x) = \frac{e^{2\pi\iota\langle x,k\rangle}}{N^{\frac{d}{2}}}, \qquad \text{so that} \quad \mathscr{F}_{\Theta_n}\varphi(k) = N^{\frac{d}{2}}\langle \varphi, \mathfrak{e}_k\rangle, \quad k \in \Xi_n,$$

the Dirichlet basis consists of sine functions,

$$\mathfrak{d}_k(x) = \frac{1}{N^{\frac{d}{2}}}\prod_{i=1}^d 2\sin(2\pi k_i x_i), \qquad k \in \Xi^+_n \setminus \partial\Xi^+_n.$$

and the Neumann basis of cosine functions:

$$\mathfrak{n}_k(x) = \frac{1}{N^{\frac{d}{2}}}\prod_{i=1}^d 2^{1-1_{\{k_i=0\}}/2}\cos(2\pi k_i x_i), \qquad k \in \Xi^+_n.$$

We will not work with the explicit expressions for $\mathfrak{d}_k$ and $\mathfrak{n}_k$, and instead mostly rely on the following alternative characterization:

Remark 3.6. *For $k \in \Xi_n^+$ define $\nu_k = 2^{-\#\{i: k_i = 0\}/2}$. Then we have:*

$$\Pi_o \mathfrak{d}_k = \iota^d \sum_{\mathfrak{q} \in \{-1,1\}^d} \prod \mathfrak{q} \cdot \mathfrak{e}_{\mathfrak{q} \circ k}, \qquad \forall k \in \Xi_n^+ \setminus \partial \Xi_n^+,$$
$$\Pi_e \mathfrak{n}_k = \nu_k \sum_{\mathfrak{q} \in \{-1,1\}^d} \mathfrak{e}_{\mathfrak{q} \circ k}, \qquad \forall k \in \Xi_n^+.$$

Notation 3.7. *The following results will be stated for distributions. In the discrete case of course any distribution is a function. Thus for $\mathfrak{l} \in \{\mathfrak{d}, \mathfrak{n}\}$ and $n < \infty$ we write:*

$$\mathscr{S}'_{\mathfrak{l}}(\Lambda_n) = \text{ span } \{\mathfrak{l}_k\}_k.$$

For $n = \infty$ we define distributions via formal Fourier series:

$$\mathscr{S}'_{\mathfrak{l}}([0,L]^d) = \left\{ \sum_k \alpha_k \mathfrak{l}_k \ : \ |\alpha_k| \le C(1+|\kappa|^\gamma), \text{ for some } C, \gamma \ge 0 \right\}.$$

In both cases the range of k depends implicitly on the choice of $\mathfrak{l}$ (and n).

Now we want to introduce Littlewood-Paley blocks on the lattice, in order to control products between distributions on Λ_n uniformly in n. First, let us recall the notion of a Fourier multiplier. Consider a function $\sigma^+ \colon \Xi_n^+ \to \mathbb{R}$. Then for $\varphi \in \mathscr{S}'_{\mathfrak{l}}(\Lambda_n)$ we define:

$$\sigma^+(D)\varphi = \sum_k \sigma^+(k) \langle \varphi, \mathfrak{l}_k \rangle \mathfrak{l}_k.$$

Upon extending φ in an even or odd fashion we recover the classical notion of Fourier multiplier (namely on a torus: $\sigma(D)\varphi = \mathscr{F}^{-1}_{\Theta_n}(\sigma \mathscr{F}_{\Theta_n}\varphi)$),

$$\Pi_o\big(\sigma^+(D)\varphi\big) = \big(\Pi_e \sigma^+\big)(D)\Pi_o\varphi, \qquad \Pi_e\big(\sigma^+(D)\varphi\big) = \big(\Pi_e\sigma^+\big)(D)\Pi_e\varphi.$$

Remark 3.8. *We are particularly interested in radial Fourier multipliers σ. Since radial functions are even, we can replace both $\sigma^+, \Pi_e\sigma^+$ with σ.*

Fix a dyadic partition of the unity $\{\varrho_j\}_{j \ge -1}$ and j_n as in Section 1, so as to define for $\varphi \in \mathscr{S}'_{\mathfrak{l}}(\Lambda_n)$:

$$\Delta_j^n \varphi = \varrho_j(D)\varphi \text{ for } j < j_n, \qquad \Delta^n_{j_n}\varphi = \Bigg(1 - \sum_{-1 \le j < j_n} \varrho_j(D)\Bigg)\varphi.$$

In view of the previous calculations this is coherent with our original definition on the lattice, in the sense that:

$$\Pi_o\big(\Delta_j^n\varphi\big) = \Delta_j^n \Pi_o\varphi, \qquad \Pi_e\big(\Delta_j^n\varphi\big) = \Delta_j^n\Pi_e\varphi, \ \ -1 \le j \le j_n.$$

We then define Dirichlet and Neumann Besov spaces via the following norms:

$$\|u\|_{B^{\mathfrak{d},\alpha}_{p,q}(\Lambda_n)} = \|\Pi_o u\|_{B^\alpha_{p,q}(\Theta_n)} = \|(2^{\alpha j}\|\Delta_j \Pi_o u\|_{L^p(\Theta_n)})_j\|_{\ell^q(\le j_n)} \qquad u \in \operatorname{span}\{\mathfrak{d}_k\}_{k \in \Xi_n^+ \setminus \partial\Xi_n^+},$$
$$\|u\|_{B^{\mathfrak{n},\alpha}_{p,q}(\Lambda_n)} = \|\Pi_e u\|_{B^\alpha_{p,q}(\Theta_n)} = \|(2^{\alpha j}\|\Delta_j \Pi_e u\|_{L^p(\Theta_n)})_j\|_{\ell^q(\le j_n)}, \qquad u \in \operatorname{span}\{\mathfrak{n}_k\}_{k \in \Xi_n^+},$$

and for brevity we write $\mathscr{C}^\alpha_{\mathfrak{l},p}(\Lambda_n) = B^{\mathfrak{l},\alpha}_{p,\infty}(\Lambda_n)$ and $\mathscr{C}^\alpha_{\mathfrak{l}}(\Lambda_n) = B^{\mathfrak{l},\alpha}_{\infty,\infty}(\Lambda_n)$ for $\mathfrak{l} \in \{\mathfrak{n}, \mathfrak{d}\}$. We also write $\|u\|_{L^p_{\mathfrak{d}}(\Lambda_n)} = \|\Pi_o u\|_{L^p(\Theta_n)}$ and $\|u\|_{L^p_{\mathfrak{n}}(\Lambda_n)} = \|\Pi_e u\|_{L^p(\Theta_n)}$. Moreover, we do not explicitly include the lattice in the notation, whenever it is clear from the context on which lattice we are. The last ingredient for multiplication in our spaces is the following identity for the extension of products of functions:

$$\Pi_e(\varphi\psi) = \Pi_e\varphi\Pi_e\psi, \qquad \Pi_o(\varphi\psi) = \Pi_o\varphi\Pi_e\psi.$$

To solve equations with Dirichlet boundary conditions, we are interested in the Laplace operator with Dirichlet boundary conditions. For $n < \infty$ and $\varphi\colon \Lambda_n \to \mathbb{R}$ we define this operator as

$$\Delta^n_{\mathfrak{d}}\varphi(x) = n^2 \left(\sum_{\substack{y\sim x,\\ y\notin\partial\Lambda_n}} \big(\varphi(y) - \varphi(x)\big) + \sum_{\substack{y\sim x,\\ y\in\partial\Lambda_n}} -\varphi(x) \right) 1_{\{\Lambda_n\setminus\partial\Lambda_n\}}(x).$$

Define also the domain

$$\mathrm{Dom}(\Delta^n_{\mathfrak{d}}) = \{\varphi\colon \Lambda_n \to \mathbb{R} \ : \ \varphi = 0 \text{ on } \partial\Lambda_n\} = \mathrm{span}\{\mathfrak{d}_k\}_{k\in\Xi^+_n\setminus\partial\Xi^+_n}$$

and note that on this domain the identity $\Delta^n_{\mathfrak{d}}\varphi = \big(\Delta^n \Pi_o \varphi\big)|_{\Lambda_n}$ holds true.

A direct computation (Remark 3.6 in combination with [MP17, Section 3]) then shows that we can represent the Laplacian with Dirichlet boundary conditions as a Fourier multiplier:

$$\Delta^n_{\mathfrak{d}}\mathfrak{d}_k = l^n(k)\mathfrak{d}_k, \qquad l^n(k) = \sum_{j=1}^d 2n^2\big(\cos(2\pi k_j/n) - 1\big).$$

Note that l^n is an even function in k, so all the remarks from the previous discussion apply. The Laplacian with Neumann boundary conditions we simply define as

$$\Delta^n_{\mathfrak{n}}\varphi := \big(\Delta^n \Pi_e \varphi\big)|_{\Lambda_n}.$$

By the same argument as in the Dirichlet case, this as well can be represented via a Fourier multiplier, with the same l^n. We will use the following notations for the parabolic operators:

$$\mathfrak{L}^n_{\mathfrak{d}} = \partial_t - \Delta^n_{\mathfrak{d}}, \quad \mathfrak{L}^n_{\mathfrak{n}} = \partial_t - \Delta^n_{\mathfrak{n}}, \quad \mathfrak{L}^n = \partial_t - \Delta^n.$$

For $n = \infty$ we use the classical Laplacian: the boundary condition is encoded in the domain. We write $\Delta_{\mathfrak{l}}$ for the Laplacian on $\mathscr{S}'_{\mathfrak{l}}([0,L]^d)$. The next result follows from [MP17, Lemma 3.4] by even or odd extension.

Lemma 3.9. *For $\alpha \in \mathbb{R}, p \in [1,\infty], \delta \in [0,1]$ and $\mathfrak{l} \in \{\mathfrak{d}, \mathfrak{n}\}$ we can estimate:*

$$\|\Delta^n_{\mathfrak{l}}\varphi\|_{\mathscr{C}^{\alpha-2}_{\mathfrak{l},p}(\Lambda_n)} \lesssim \|\varphi\|_{\mathscr{C}^{\alpha}_{\mathfrak{l},p}(\Lambda_n)}, \qquad \|(\Delta^n_{\mathfrak{l}} - \Delta_{\mathfrak{l}})\varphi\|_{\mathscr{C}^{\alpha-2-\delta}_{\mathfrak{l},p}([0,L]^d)} \lesssim n^{-\delta}\|\varphi\|_{\mathscr{C}^{\alpha}_{\mathfrak{l},p}([0,L]^d)},$$

where we slightly abuse notation by defining $\Delta^n_{\mathfrak{l}}$ for distributions in $\mathscr{S}'_{\mathfrak{l}}([0,L]^d)$ via the same formula (which makes sense, because translations are well defined on distributions).

We introduce Dirichlet and Neumann extension operators as follows:

$$\mathscr{E}^n_{\mathfrak{d}}u = \mathscr{E}^n(\Pi_o u)\big|_{[0,L]^d}, \qquad \mathscr{E}^n_{\mathfrak{n}}u = \mathscr{E}^n(\Pi_e u)\big|_{[0,L]^d}, \qquad \text{for } n < \infty.$$

These functions are well-defined since for fixed n the extension $\mathscr{E}_n(\cdot)$ is a smooth function. Moreover a simple calculation shows that

$$\Pi_o(\mathscr{E}^n_{\mathfrak{d}}u) = \mathscr{E}^n(\Pi_o u), \qquad \Pi_e(\mathscr{E}^n_{\mathfrak{n}}u) = \mathscr{E}^n(\Pi_e u). \tag{13}$$

In the following we will use the notation $\Delta^n_{<i}\varphi = \sum_{j<i}\Delta^n_j\varphi$, and we introduce the parabolic spaces $\mathscr{L}^{\gamma,\alpha}_{\mathfrak{l},p}$ and $\mathscr{M}^{\gamma}\mathscr{C}^{\alpha}_{\mathfrak{l},p}$ with the same definitions as in Section 1, mutatis mutandis. The consistency between the lattice and the continuous space is then stated in terms of an extension property. Consider a Banach space $X_{\mathfrak{l}} \subset \mathscr{S}'_{\mathfrak{l}}([0,L]^d)$ (resp. $X_{\mathfrak{l}} \subset C([0,T], \mathscr{S}'_{\mathfrak{l}}([0,L]^d))$) for $\mathfrak{l} \in \{\mathfrak{d}, \mathfrak{n}\}$ which possesses discrete approximations $X^n_{\mathfrak{l}} \subset \mathscr{S}'_{\mathfrak{l}}(\Lambda_n)$ (resp. $X^n_{\mathfrak{l}} \subset C([0,T], \mathscr{S}'_{\mathfrak{l}}(\Lambda_n))$). Similarly, consider a functional F which has discrete approximations F^n: For concreteness let us write F^∞ instead of F and $X^\infty_{\mathfrak{l}}$ instead of $X_{\mathfrak{l}}$. In this setting, suppose that we are given a bound:

$$\|F^n(u_1,\dots,u_m)\|_{X^n_{\mathfrak{l}}} \lesssim \|u_1\|_{X^n_{1,\mathfrak{l}_1}} \cdots \|u_m\|_{X^n_{m,\mathfrak{l}_m}}, \forall n \in \mathbb{N}\cup\{\infty\}.$$

We then say that F satisfies the $(\mathscr{E})$-*Property* if

$$\|\mathscr{E}^n_{\mathfrak{l}}F^n(u_1,\dots,u_m) - F^\infty(\mathscr{E}^n_{\mathfrak{l}_1}u_1,\dots,\mathscr{E}^n_{\mathfrak{l}_m}u_m)\|_{X^\infty_{\mathfrak{l}}} \lesssim \varepsilon(n)c(\|u_1\|_{X^n_{1,\mathfrak{l}_1}},\dots,\|u_m\|_{X^n_{m,\mathfrak{l}_m}})$$

for some $\varepsilon(n) \to 0$ as $n \to \infty$ and a continuous function c.

We also need to consider a sort of space-time paraproduct, so let $\theta \in C_c^\infty((0,\infty))$ be such that $\int_0^\infty \theta(s)\,\mathrm{d}s = 1$ and define for $i \geq -1$ as in [MP17, Definition 4.5]

$$Q_i u(t) = \int_{-\infty}^t 2^{2i}\theta(2^{2i}(t-s))u(s\vee 0)\,\mathrm{d}s.$$

As in [GIP15, MP17] we silently identify u with $t \mapsto 1_{\{t>0\}}u(t)$ if $u \in \mathscr{M}^\gamma X$ for some X.

Lemma 3.10. *Consider $n \in \mathbb{N}\cup\{\infty\}$. We define the following paraproducts for $u, v, w \colon \Lambda_n \to \mathbb{R}$ (depending also on time in case of the parabolically scaled paraproduct $\prec\!\!\!\prec$):*

$$u \varolessthan v = \sum_{1\leq i\leq j_n} \Delta^n_{<i-1}u\ \Delta^n_i v,\ \ u \odot v = \sum_{|i-j|\leq 1} \Delta^n_j u\ \Delta^n_i v,\ \ u \prec\!\!\!\prec v = \sum_{1\leq i\leq j_n} Q_i(\Delta^n_{<i-1}u)\ \Delta^n_j v$$

as well as the following operators, which we call the "paracontrolled operators" because they form the backbone of paracontrolled analysis:

$$C_1(u,v,w) = (u\varolessthan v)\odot w - u\cdot(v\odot w),\ \ C_2(u,v) = u\prec\!\!\!\prec w - u\varolessthan w,\ \ C_3(u,v) = \mathfrak{L}^n_{\mathfrak{d}}(u\prec\!\!\!\prec v) - u\prec\!\!\!\prec \mathfrak{L}^n_{\mathfrak{n}}v.$$

For $p\in[1,+\infty], \gamma\in[0,1), \alpha,\beta,\delta\in\mathbb{R}$ we can bound such terms uniformly in n as follows:

$$\begin{aligned}
&\|u\varolessthan v\|_{\mathscr{C}^\alpha_{\mathfrak{d},p}} \lesssim \|u\|_{L^p_{\mathfrak{d}}}\|v\|_{\mathscr{C}^\alpha_{\mathfrak{n}}}, \qquad \|u\varolessthan v\|_{\mathscr{C}^{\alpha+\beta}_{\mathfrak{d},p}} \lesssim \|u\|_{\mathscr{C}^\beta_{\mathfrak{d},p}}\|v\|_{\mathscr{C}^\alpha_{\mathfrak{n}}}, && \text{if } \beta<0,\\
&\|u\odot v\|_{\mathscr{C}^{\alpha+\beta}_{\mathfrak{d},p}} \lesssim \|u\|_{\mathscr{C}^\beta_{\mathfrak{d},p}}\|v\|_{\mathscr{C}^\alpha_{\mathfrak{n}}}, && \text{if } \alpha+\beta>0,\\
&\|u\prec\!\!\!\prec v\|_{\mathscr{M}^\gamma\mathscr{C}^\alpha_p} \lesssim \|u\|_{\mathscr{M}^\gamma L^p_{\mathfrak{d}}}\|v\|_{C\mathscr{C}^\alpha_{\mathfrak{n}}}, \quad \|u\prec\!\!\!\prec v\|_{\mathscr{M}^\gamma\mathscr{C}^{\alpha+\beta}_{\mathfrak{d},p}} \lesssim \|u\|_{\mathscr{M}^\gamma\mathscr{C}^\beta_{\mathfrak{d},p}}\|v\|_{C\mathscr{C}^\alpha_{\mathfrak{n}}}, && \text{if } \beta<0,\\
&\|u\prec\!\!\!\prec v\|_{\mathscr{L}^{\gamma,\alpha}_{\mathfrak{d},p}} \lesssim \|u\|_{\mathscr{L}^{\gamma,\delta}_{\mathfrak{d},p}}(\|v\|_{C\mathscr{C}^\alpha_{\mathfrak{n}}} + \|\mathfrak{L}^n_{\mathfrak{n}}v\|_{C\mathscr{C}^{\alpha-2}_{\mathfrak{n}}}), && \text{if } \delta,\alpha\in(0,2).
\end{aligned}$$

And for the paracontrolled operators we find:

$$\begin{aligned}
&\|C_1(u,v,w)\|_{\mathscr{C}^{\alpha+\delta}_{\mathfrak{d},p}} \lesssim \|u\|_{\mathscr{C}^\beta_{\mathfrak{d},p}}\|v\|_{\mathscr{C}^\alpha_{\mathfrak{n}}}\|w\|_{\mathscr{C}^\delta_{\mathfrak{n}}}, && \text{if } \alpha+\beta+\delta>0,\ \alpha+\delta\neq 0,\\
&\|C_2(u,v)\|_{\mathscr{M}^\gamma\mathscr{C}^{\alpha+\beta}_{\mathfrak{d},p}} \lesssim \|u\|_{\mathscr{L}^{\gamma,\beta}_{\mathfrak{d},p}}\|v\|_{C\mathscr{C}^\alpha_{\mathfrak{n}}}, && \text{if } \beta\in(0,2),\\
&\|C_3(u,v)\|_{\mathscr{M}^\gamma\mathscr{C}^{\alpha+\beta-2}_{\mathfrak{d},p}} \lesssim \|u\|_{\mathscr{L}^{\gamma,\beta}_{\mathfrak{d},p}}\|v\|_{C\mathscr{C}^\alpha_{\mathfrak{n}}}, && \text{if } \beta\in(0,2).
\end{aligned}$$

and similar bounds hold if we consider $(\mathfrak{n},\mathfrak{n})$ or $(\mathfrak{n},\mathfrak{d})$ instead of $(\mathfrak{d},\mathfrak{n})$ boundary conditions, or if we move the integrability constant p from one function to the other. Moreover, all estimates satisfy the $(\mathscr{E})$-Property if the regularity on the left hand side is lowered by an arbitrary amount.

Proof. All the proofs follow via even or odd extension from [MP17, Lemmata 4.2, 4.3, 4.4, 4.7, 4.8, 4.9]. □

With the help of the above paraproduct estimates, we can solve PAM with Dirichlet boundary conditions (12). We essentially follow verbatim the construction of [MP17], except that things are slightly simpler now because we do not have to work with weights. Let us start with the analytical assumption on the stochastic data. A *deterministic Neumann environment* is just a sequence of functions $\xi^n \colon \Lambda_n \to \mathbb{R}$, for $n\in\mathbb{N}$.

In the following assumption we shift Λ_n to be centered around the origin and identify it with a subset of $[-L/2, L/2]^d$. This is convenient because later we want to interpret processes on Λ_n as "restrictions" of processes on $\mathbb{Z}^d_n$ to (large) boxes centered around the origin.

Assumption 3.11 (Deterministic Neumann environment)**.** *Let ξ^n be a deterministic Neumann environment and let X^n be the solution to the equation $-\Delta^n_{\mathfrak{n}} X^n_{\mathfrak{n}} = \chi(D)\xi^n$, where χ is the same cut-off function as in Assumption 2.3. Consider once more a regularity parameter*

$$\alpha\in(1,\tfrac{3}{2})\ \text{in } d=1, \qquad \alpha\in(\tfrac{2}{3},1)\ \text{in } d=2. \tag{14}$$

We assume that the following holds:

(i) There exists $\xi \in \bigcap_{a>0} \mathscr{C}_{\mathfrak{n}}^{\alpha-2}([-L/2, L/2]^d)$ *such that:*

$$\sup_n \|\xi^n\|_{\mathscr{C}_{\mathfrak{n}}^{\alpha-2}(\Lambda_n)} < +\infty \quad \text{and} \quad \mathscr{E}_{\mathfrak{n}}^n \xi^n \to \xi \text{ in } \mathscr{C}_{\mathfrak{n}}^{\alpha-2}([-L/2, L/2]^d).$$

(ii) For any $\varepsilon > 0$ *we can bound:*

$$\sup_n \|n^{-d/2}\xi_+^n\|_{\mathscr{C}_{\mathfrak{n}}^{-\varepsilon}(\Lambda_n)} + \sup_n \|n^{-d/2}|\xi^n|\|_{\mathscr{C}_{\mathfrak{n}}^{-\varepsilon}(\Lambda_n)} + \sup_n \|n^{-d/2}\xi_+^n\|_{L_{\mathfrak{n}}^2(\Lambda_n)} < +\infty$$

Moreover, there exists a $\nu \geq 0$ *such that*

$$\mathscr{E}_{\mathfrak{n}}^n n^{-d/2}\xi_+^n \to \nu, \quad \mathscr{E}_{\mathfrak{n}}^n n^{-d/2}|\xi^n| \to 2\nu \text{ in } \mathscr{C}_{\mathfrak{n}}^{-\varepsilon}(\Lambda_n).$$

(iii) If $d = 2$ *there exists a sequence* $c_n \in \mathbb{R}$ *such that* $n^{-d/2}c_n \to 0$ *and distributions* $X_{\mathfrak{n}}, X_{\mathfrak{n}} \diamond \xi$ *in* $\mathscr{C}_{\mathfrak{n}}^{\alpha}([-L/2, L/2]^d)$ *and* $\mathscr{C}_{\mathfrak{n}}^{2\alpha-2}([-L/2, L/2]^d)$ *respectively, sucht that:*

$$\sup_n \|X_{\mathfrak{n}}^n\|_{\mathscr{C}_{\mathfrak{n}}^{\alpha}(\Lambda_n)} + \sup_n \|(X_{\mathfrak{n}}^n \odot \xi^n) - c_n\|_{\mathscr{C}_{\mathfrak{n}}^{2\alpha-2}(\Lambda_n)} < +\infty$$

and $\mathscr{E}_{\mathfrak{n}}^n X_{\mathfrak{n}}^n \to X_{\mathfrak{n}}$ *in* $\mathscr{C}_{\mathfrak{n}}^{\alpha}([-L/2, L/2]^d)$, $\mathscr{E}_{\mathfrak{n}}^n\big((X_{\mathfrak{n}}^n \odot \xi^n) - c_n\big) \to X_{\mathfrak{n}} \diamond \xi$ *in* $\mathscr{C}_{\mathfrak{n}}^{2\alpha-2}([-L/2, L/2]^d)$.

Under this assumption we get the following "Dirichlet version" of Proposition 3.1.

Proposition 3.12. *Consider* α *as in* (14), *any* $T > 0$, $p \in [1, +\infty]$ *and* $\vartheta, \zeta, \gamma_0, \alpha_0$ *satisfying:*

$$(15)\quad \vartheta \in \begin{cases} (2-\alpha, \alpha), & d = 1, \\ (2-2\alpha, \alpha), & d = 2, \end{cases} \qquad \zeta > (\vartheta-2) \vee (-\alpha), \qquad \gamma_0 \in [0, 1), \qquad \alpha_0 > (\vartheta-2) \vee (-\alpha),$$

and let $w_0^n \in \mathscr{C}_{\mathfrak{d},p}^{\zeta}(\Lambda_n)$ *and* $f^n \in \mathscr{M}^{\gamma_0}\mathscr{C}_{\mathfrak{d},p}^{\alpha_0}(\Lambda_n)$ *such that*

$$\mathscr{E}^n w_0^n \to w_0 \text{ in } \mathscr{C}_{\mathfrak{d},p}^{\zeta}([-L/2, L/2]^d), \qquad \mathscr{E}^n f^n \to f \text{ in } \mathscr{M}^{\gamma_0}\mathscr{C}_{\mathfrak{d},p}^{\alpha_0}([-L/2, L/2]^d).$$

Let $w^n \colon [0, T] \times \Lambda_n \to \mathbb{R}$ *be the unique solution to the finite-dimensional linear ODE:*

$$(16)\qquad \partial_t w^n = (\Delta_{\mathfrak{d}}^n + \xi_e^n)w^n + f^n, \qquad w^n(0) = w_0^n, \qquad w(t, x) = 0 \ \ \forall (t, x) \in (0, T] \times \partial\Lambda_n.$$

Then under Assumption 3.11 there exist a unique (paracontrolled in $d = 2$*) solution* w *to the equation*

$$(17)\quad \partial_t w = \Delta_{\mathfrak{d}} w + \xi w + f, \qquad w(0) = w_0, \qquad w(t, x) = 0 \ \ \forall (t, x) \in (0, T] \times \partial[-L/2, L/2]^d,$$

and for all $\gamma > (\vartheta-\zeta)_+/2 \vee \gamma_0$ *the sequence* w^n *is uniformly bounded in* $\mathscr{L}_{\mathfrak{d},p}^{\gamma,\vartheta}(\Lambda_n)$*:*

$$\sup_n \|w^n\|_{\mathscr{L}_{\mathfrak{d},p}^{\gamma,\vartheta}(\Lambda_n)} \lesssim \sup_n \|w_0^n\|_{\mathscr{C}_{\mathfrak{d},p}^{\zeta}(\Lambda_n)} + \sup_n \|f^n\|_{\mathscr{M}^{\gamma_0}\mathscr{C}_{\mathfrak{d},p}^{\alpha_0}(\Lambda_n)},$$

where the proportionality constant depends on the time horizon T *and the magnitude of the norms in Assumption 3.11. Moreover,*

$$\mathscr{E}^n w^n \to w \text{ in } \mathscr{L}_{\mathfrak{d},p}^{\gamma,\vartheta}([-L/2, L/2]^d).$$

Proof. Note that solving Equation (16) (resp. (17)) is equivalent to solving on the discrete (resp. continuous) torus Θ_n the equation:

$$\partial_t \tilde{w}^n = \Delta^n \tilde{w}^n + \Pi_e(\xi_e^n)\tilde{w}^n + \Pi_o f, \qquad \tilde{w}^n(0) = \Pi_o w_0,$$

and then restricting the solution to the cube Λ_n, i.e. $w^n = \tilde{w}^n|_{\Lambda_n}$, and $\tilde{w}^n = \Pi_o w^n$. In view of Assumption 3.11 and the estimates of Lemma 3.10 this equation can be solved via Schauder estimates and (in dimension $d = 2$) paracontrolled theory following the arguments of [MP17] (without considering weights). From the arguments of the same article we can also deduce the convergence of the extensions. □

As on the full space, we also use the Anderson hamiltonian $\mathscr{H}_{\mathfrak{d}}$ with Dirichlet boundary conditions. The domain and spectral decomposition for this operator are constructed in [CvZ19] with the help of the resolvent equation. Unlike $\mathscr{H}$ the operator $\mathscr{H}_{\mathfrak{d}}$ has a compact resolvent and thus a discrete spectrum that is bounded from above, and based on that we easily obtain the following result:

Notation 3.13. *We write* $\mathscr{H}^n_{\mathfrak{d}}, \mathscr{H}_{\mathfrak{d}}$ *for the operators* $\Delta^n_{\mathfrak{d}}+\xi^n-c_n 1_{\{d=2\}}$ *and* $\Delta_{\mathfrak{d}}+\xi-\infty 1_{\{d=2\}}$ *respectively. Furthermore, we write:*

$$t\mapsto T^{n,\mathfrak{d}}_t u^n_0+\int_0^t \mathrm{d}s\; T^{n,\mathfrak{d}}_{t-s}f^n_s, \qquad t\mapsto T^{\mathfrak{d}}_t u_0+\int_0^t \mathrm{d}s\; T^{\mathfrak{d}}_{t-s}f_s$$

for the solutions to Equation (16) *and* (17) *respectively. Then we have* $T^{n,\mathfrak{d}}_t = e^{t\mathscr{H}^n_{\mathfrak{d}}}$ *and* $T^{\mathfrak{d}}_t = e^{t\mathscr{H}_{\mathfrak{d}}}$.

4. The Rough Super-Brownian Motion

4.1. **Scaling Limit of Branching Random Walks in Random Environment.** In this section we consider a deterministic environment, that is a sequence $\{\xi^n\}_{n\in\mathbb{N}}$ satisfying Assumption 2.3, to which we associate the Markov process μ^n as in Definition 2.6: Our aim is to prove that the sequence μ^n is weakly converging, the limit depending on the value of ϱ. This section is divided in two parts. First, we prove a tightness result for the sequence μ^n in $\mathbb{D}([0,T];\mathscr{M}(\mathbb{R}^d))$ for $\varrho\geq d/2$. Then, we prove uniqueness in law of the limit points and thus deduce the weak convergence of the sequence. Recall that for $\mu\in\mathscr{M}(\mathbb{R}^d)$ and $\varphi\in C_b(\mathbb{R}^d)$ we use both the notation $\langle\mu,\varphi\rangle$ and $\mu(\varphi)$ for the integration of φ against the measure μ.

Remark 4.1. *For any* $\varphi\in L^\infty(\mathbb{Z}^d_n;e(l))$, *for some* $l\in\mathbb{R}$*:*

$$M^{n,\varphi}_t(s)=\mu^n(s)(T^n_{t-s}\varphi)-T^n_t\varphi(0) \tag{18}$$

is a centered martingale on $[0,t]$ *with predictable quadratic variation given by:*

$$\langle M^{n,\varphi}_t\rangle_s=\int_0^s \mu^n(r)\big(n^{-\varrho}|\nabla^n T^n_{t-r}\varphi|^2+n^{-\varrho}|\xi^n_e|(T^n_{t-r}\varphi)^2\big)\,\mathrm{d}r.$$

Sketch of proof. This follows from the definition of the generator of the process, by using Dynkin's formula. We first apply the martingale problem to $\mu\mapsto F_K(\mu(\varphi))$, where $F_K(x)=(x\wedge K)\vee(-K)$. Sending $K\to\infty$ and using that the solution to the discrete PAM with compactly supported initial condition is in $C([0,T],L^\infty(\mathbb{Z}^d_n,e(-k)))$ for all $k,T>0$ by Proposition 3.1, we obtain that

$$L^{n,\varphi}_t=\mu^n_t(\varphi)-\mu^n_0(\varphi)-\int_0^t \mu^n_s(\mathscr{H}^n\varphi)\,\mathrm{d}s, \qquad t\geq 0,$$

is a martingale with predictable quadratic variation

$$\langle L^{n,\varphi}\rangle_t=\int_0^t \mu^n(r)\big(n^{-\varrho}|\nabla^n\varphi|^2+n^{-\varrho}|\xi^n_e|\varphi^2\big)\,\mathrm{d}s.$$

This extends to time-dependent functions by an approximation argument (via time discretization), for which the martingale becomes

$$\mu^n_t(\varphi(t))-\mu^n_0(\varphi(0))-\int_0^t \mu^n_s(\partial_s\varphi(s)+\mathscr{H}^n\varphi(s))\,\mathrm{d}s.$$

Now it suffices to use that $\partial_s T^n_{t-s}\varphi=-\mathscr{H}^n T^n_{t-s}\varphi$. □

For the remainder of this section we assume that $\varrho\geq d/2$. To prove the tightness of the measure-valued process we use the following auxiliary result, which gives the tightness of the real-valued processes $\{t\mapsto\mu^n(t)(\varphi)\}_{n\in\mathbb{N}}$.

Lemma 4.2. *For any* $l\in\mathbb{R}$ *and* $\varphi\in C^\infty(\mathbb{R}^d,e(l))$ *the processes* $\{t\mapsto\mu^n(t)(\varphi)\}_{n\in\mathbb{N}}$ *form a tight sequence in* $\mathbb{D}([0,+\infty);\mathbb{R})$.

Proof. Choose $0<\vartheta<2$ according as in Proposition 3.1. In the following computation $k\in\mathbb{R}$ may change from line to line, but it is uniformly bounded for $l\in\mathbb{R}$ and $T>0$ varying in a bounded set.

We apply [EK86, Theorem 3.8.8]. For this purpose, let $(\mathscr{F}^n_t)_{t\geq 0}$ be the natural filtration induced by μ^n and let us start by bounding the following conditional expectation for $0 \leq t \leq t{+}h \leq T$:

$$
\begin{aligned}
\mathbb{E}\big[|\mu^n(t{+}h)(\varphi){-}\mu^n(t)(\varphi)|^2|\mathscr{F}^n_t\big] &= \mathbb{E}\big[|M^{n,\varphi}_{t+h}(t{+}h){-}M^{n,\varphi}_{t+h}(t){+}\mu^n(t)(T^n_h\varphi{-}\varphi)|^2|\mathscr{F}^n_t\big]\\
&\lesssim_\varphi \mathbb{E}\bigg[\int_t^{t+h} \mathrm{d}r\ \mu^n(r)\big(n^{-\varrho}|\nabla^n T^n_{t+h-r}\varphi|^2 + n^{-\varrho}|\xi^n_e|(T^n_{t+h-r}\varphi)^2\big)\bigg|\mathscr{F}^n_t\bigg] + h^\vartheta|\mu^n(t)(e^{k|x|^\sigma})|^2\\
&= \int_t^{t+h} \mathrm{d}r\ \mu^n(t)\Big(T^n_{r-t}\big(n^{-\varrho}|\nabla^n T^n_{t+h-r}\varphi|^2 + n^{-\varrho}|\xi^n_e|(T^n_{t+h-r}\varphi)^2\big)\Big) + h^\vartheta|\mu^n(t)(e^{k|x|^\sigma})|^2\\
&\lesssim \int_t^{t+h} \mathrm{d}r\ \mu^n(t)\big(e^{k|x|^\sigma} + (r{-}t)^{-\zeta}e^{k|x|^\sigma}\big) + h^\vartheta|\mu^n(t)(e^{k|x|^\sigma})|^2\\
&\lesssim h^{1-\zeta}\mu^n(t)(e^{k|x|^\sigma}) + h^\vartheta|\mu^n(t)(e^{k|x|^\sigma})|^2 \qquad (19)
\end{aligned}
$$

for any $\zeta > 0$. Here we have first used that, applying Proposition 3.1 together with the results of Lemmata E.1, E.2, E.3, as well as the fourth estimate in Lemma 1.2, the term $n^{-\varrho}|\nabla^n T^n_{t+h-r}\varphi|^2$ converges to zero in $\mathscr{C}^{\tilde\vartheta}(\mathbb{Z}^d_n, e(2(l{+}t{+}h{-}r)))$ for $0 < \tilde\vartheta < \vartheta{-}1{+}\varrho/2$ (we can choose ϑ sufficiently large so that the latter quantity is positive). Thus Proposition 3.1 gives the bound for $T^n_{r-t}(n^{-\varrho}|\nabla^n T^n_{t+h-r}\varphi|^2)$. Moreover, since according to Assumption 2.3 for $\varrho \geq 2/d$ the term $n^{-\varrho}|\xi^n_e|$ is bounded in $\mathscr{C}^{-\varepsilon}(\mathbb{Z}^d_n, p(a))$ whenever $\varepsilon > 0$, it follows with the same arguments as before that the quantity $s \mapsto T^n_s(n^{-\varrho}|\xi^n_e|(T^n_{t+h-r}\varphi)^2)$ is bounded in $\mathscr{M}^\zeta\mathscr{C}^{2\zeta-\varepsilon}(\mathbb{Z}^d_n, e(k))$ for any $\varepsilon/2 < \zeta < 1$. As for the last addend, we simply used that $s \mapsto T^n_s\varphi \in \mathscr{L}^\vartheta(\mathbb{Z}^d_n, e(l))$.

To apply [EK86, Theorem 3.8.8] we have to multiply two increments of $\mu^n(\varphi)$ on $[t{-}h, h]$ and on $[t, t{+}h]$. We use the previous computation to bound:

$$
\begin{aligned}
&\mathbb{E}\big[(|\mu^n(t{+}h)(\varphi){-}\mu^n(t)(\varphi)| \wedge 1)^2(|\mu^n(t)(\varphi){-}\mu^n(t{-}h)(\varphi)| \wedge 1)^2\big]\\
&\qquad \leq \mathbb{E}\big[|\mu^n(t{+}h)(\varphi){-}\mu^n(t)(\varphi)|^2|\mu^n(t)(\varphi){-}\mu^n(t{-}h)(\varphi)|\big]\\
&\qquad \lesssim \mathbb{E}\Big[\big(h^{1-\zeta}\mu^n(t)(e^{k|x|^\sigma}) + h^\vartheta|\mu^n(t)(e^{k|x|^\sigma})|^2\big)|\mu^n(t)(\varphi){-}\mu^n(t{-}h)(\varphi)|\Big]. \qquad (20)
\end{aligned}
$$

Note that we voluntarily dropped the square in the second term. Now we treat one addend at a time. For the first one we compute

$$
\begin{aligned}
&\mathbb{E}\Big[\mu^n(t)(e^{k|x|^\sigma})|\mu^n(t)(\varphi){-}\mu^n(t{-}h)(\varphi)|\Big]\\
&\quad \leq \mathbb{E}\Big[\big(|\mu^n(t)(e^{k|x|^\sigma}){-}\mu^n(t{-}h)(e^{k|x|^\sigma})|{+}\mu^n(t{-}h)(e^{k|x|^\sigma})\big)|\mu^n(t)(\varphi){-}\mu^n(t{-}h)(\varphi)|\Big]\\
&\quad \lesssim (h^{(1-\zeta)}{+}h^\vartheta){+}(h^{(1-\zeta)/2}{+}h^{\vartheta/2}) \lesssim h^{(1-\zeta)/2}{+}h^{\vartheta/2}
\end{aligned}
$$

by the Cauchy-Schwarz inequality together with (19) and the moment bound for $\mu^n_t(e^{k|x|^\sigma})$ that is shown in Lemma D.1. As for the second term in (20), we similarly bound:

$$
\begin{aligned}
&\mathbb{E}\bigg[|\mu^n(t)(e^{k|x|^\sigma})|^2|\mu^n(t)(\varphi){-}\mu^n(t{-}h)(\varphi)|\bigg]\\
&\qquad \lesssim \mathbb{E}\big[|\mu^n(t)(e^{k|x|^\sigma})|^4\big]^{1/2}\mathbb{E}\big[|\mu^n(t)(\varphi){-}\mu^n(t{-}h)(\varphi)|^2\big]^{1/2} \lesssim h^{(1-\zeta)/2}{+}h^{\vartheta/2}.
\end{aligned}
$$

Together with Young's inequality for products, this yields the following bound for the expression on the left hand side of (20):

$$
\mathbb{E}\big[(|\mu^n(t{+}h)(\varphi){-}\mu^n(t)(\varphi)| \wedge 1)^2(|\mu^n(t)(\varphi){-}\mu^n(t{-}h)(\varphi)| \wedge 1)^2\big] \lesssim h^{3(1-\zeta)/2}{+}h^{3\vartheta/2}.
$$

Since $\vartheta > \frac{2}{3}$ and $\zeta > 0$ is arbitrary, the right hand side is $\lesssim h^\theta$ for some $\theta > 1$.

Hence we can apply [EK86, Theorem 3.8.8] with $\beta = 4$, which in turn implies that the tightness criterion of Theorem 3.8.6 (b) of the same book is satisfied. This concludes the proof of tightness for $\{t \mapsto \mu^n(t)(\varphi)\}_{n\in\mathbb{N}}$. □

As a consequence, we find tightness of the process μ^n in the space of measures.

Corollary 4.3. *The processes $\{t \mapsto \mu^n(t)\}_{n\in\mathbb{N}}$ form a tight sequence in $\mathbb{D}([0,+\infty);\mathscr{M}(\mathbb{R}^d))$.*

Proof. We apply Jakubowski's criterion [DP12, Theorem 3.6.4]. We first need to verify the compact containment condition. For that purpose note that for all $l > 0$ and $R > 0$ the set

$$K_R = \{\mu \in \mathscr{M}(\mathbb{R}^d) \mid \mu(e^{l|x|^\sigma}) \leq R\}$$

is compact in $\mathscr{M}(\mathbb{R}^d)$. Since the processes $\mu^n(e^{l|x|^\sigma})$ are tight by Lemma 4.2, we find for all $l, T, \varepsilon > 0$ an $R(\varepsilon)$ such that

$$\sup_n \mathbb{P}\bigg(\sup_{t\in[0,T]} \mu^n(t)(e^{l|x|^\sigma}) \geq R(\varepsilon)\bigg) \leq \varepsilon,$$

which is the required compact containment condition.

Second we note that the space $C_c^\infty(\mathbb{R}^d)$ is closed under addition and the maps $\mu \mapsto \{\mu(\varphi)\}_{\varphi\in C_c^\infty(\mathbb{R}^d)}$ separate points in $\mathscr{M}(\mathbb{R}^d)$. Since Lemma 4.2 shows that $t \mapsto \mu^n(t)(\varphi)$ is tight for any $\varphi \in C_c^\infty(\mathbb{R}^d)$, we can conclude. □

Lemma 4.4. *Any limit point of the sequence $\{t \mapsto \mu^n(t)\}_{n\in\mathbb{N}}$ is supported in the space of continuous function $C([0,+\infty);\mathscr{M}(\mathbb{R}^d))$. Furthermore any such limit point μ satisfies condition (ii) of Definition 2.11 with*

$$\kappa = \begin{cases} 0, & \text{if } \varrho > d/2, \\ 2\nu, & \text{if } \varrho = d/2. \end{cases}$$

Proof. Step 1. We show the continuity of an arbitrary limit point μ. Consider $\varphi \in C_c^\infty(\mathbb{R}^d)$. We prove that the one-dimensional projection $t \mapsto \langle \mu(t), \varphi\rangle$ is continuous almost surely. Choosing a countable separating set of smooth functions the continuity of μ follows. Note that for $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $T > 0$ we get $\|(T_t^n\varphi)_{t\in[0,T]}\|_{\mathscr{L}^\vartheta(\mathbb{Z}_n^d)} \lesssim \|\varphi\|_{\mathscr{C}^\varsigma(\mathbb{Z}_n^d, e(-T))}$ from Proposition 3.1.

Now we apply a Burkholder-Davis-Gundy inequality that bounds càdlàg martingales in terms of their predictable quadratic variation and the supremum of their jumps (Lemma B.1 of [MW17]): for any $p \geq 2$ and $0 \leq t \leq t+h \leq T$ we have

$$\begin{aligned} \mathbb{E}\big[|\mu^n(t+h)(\varphi)-\mu^n(t)(\varphi)|^p\big] &\lesssim \mathbb{E}\big[|\mu^n(t+h)(\varphi)-\mu^n(t)(T_h^n\varphi)|^p\big]+\mathbb{E}\big[|\mu^n(t)(T_h^n\varphi-\varphi)|^p\big] \\ &= \mathbb{E}[|M_{t+h}^{n,\varphi}(t+h) - M_{t+h}^{n,\varphi}(t)|^p]+\mathbb{E}\big[|\mu^n(t)(T_h^n\varphi-\varphi)|^p\big] \\ &\lesssim \mathbb{E}\bigg[\bigg|\int_t^{t+h} dr\ \mu^n(r)\big(n^{-\varrho}|\nabla^n T_{t+h-r}^n\varphi|^2 + n^{-\varrho}|\xi_e^n|(T_{t+h-r}^n\varphi)^2\big)\bigg|^{p/2}\bigg] \\ &\quad + \mathbb{E}\big[\sup_{t\leq r\leq t+h} |\Delta_r M_{t+h}^{n,\varphi}(r)|^p\big] + \|\varphi\|_{\mathscr{C}^\vartheta(\mathbb{Z}_n^d, e(-T))}^p |h|^{p\vartheta/2}\mathbb{E}[|\mu^n(t)(1)|^p], \end{aligned}$$

where $\Delta_r M = M(r) - M(r-)$ is the jump at time r. Since the functions $T_t^n\varphi$ are bounded uniformly in $n \in \mathbb{N}$ and $t \in [0,T]$, we can estimate the jump term by $n^{-p\varrho}$, up to a multiplicative constant. The expectation in the last addend is controlled with Lemma D.1. We are left with the most complicated term, for which we estimate

$$\begin{aligned}\mathbb{E}\Big[\Big|\int_t^{t+h} \mathrm{d}r\ \mu^n(r)\big(n^{-\varrho}|\nabla^n T^n_{t+h-r}\varphi|^2+n^{-\varrho}|\xi^n_e|(T^n_{t+h-r}\varphi)^2\big)\Big|^{p/2}\Big] \\ \lesssim |h|^{p/2-1}\int_t^{t+h}\mathrm{d}r\ \mathbb{E}\big[\big|\mu^n(r)\big(n^{-\varrho}|\nabla^n T^n_{t+h-r}\varphi|^2+n^{-\varrho}|\xi^n_e|(T^n_{t+h-r}\varphi)^2\big)\big|^{p/2}\big] \\ \lesssim |h|^{p/2-1}\int_t^{t+h}\mathrm{d}r\ r^{-\gamma}\lesssim |h|^{p/2-\gamma}\end{aligned}$$

for any $\gamma\in(0,1)$, where in the last step we applied the second estimate of Lemma D.1. Passing to the limit (with Fatou's inequality), we find $\mathbb{E}\big[\big|\mu(t+h)(\varphi)-\mu(t)(\varphi)\big|^p\big]\lesssim |h|^{p/2-\gamma}$ for arbitrarily small $\gamma>0$. It thus follows from Kolmogorov's continuity criterion that this process is almost surely continuous.

Step 2. We fix a limit point μ and study the required martingale property. For f,φ_0 as required, observe that $\varphi_0^n=\varphi_0|_{\mathbb{Z}^d_n}$ is uniformly bounded in $\mathscr{C}^{\zeta_0}(\mathbb{Z}^d_n;e(l))$ for any $\zeta_0>0$ and $l\in\mathbb{R}$, and similarly $f^n=f|_{\mathbb{Z}^d_n}$ is uniformly bounded in $C([0,t];\mathscr{C}^{\zeta}(\mathbb{Z}^d_n))$, with an application of Lemma E.1. Hence by Proposition 3.1 the discrete solutions φ^n_t to

$$\partial_s\varphi^n_t+\mathscr{H}^n\varphi^n_t=f^n,\qquad \varphi^n_t(t)=\varphi^n_0$$

converge in $\mathscr{L}^{\vartheta}(\mathbb{R}^d,e(l))$ to φ_t, up to choosing a possibly larger l. At the discrete level we find that

$$M^{\varphi_0,f,n}_t(s):=\langle\mu^n(s),\varphi^n_t(s)\rangle-\int_0^s \mathrm{d}r\ \langle\mu^n(r),f^n(r)\rangle,\qquad s\in[0,t]$$

is a square-integrable martingale. Moreover this martingale is bounded in L^2 uniformly over n, since the second moment can be bounded via the initial value and the predictable quadratic variation by

$$\mathbb{E}\Big[\sup_{s\le t}|M^{\varphi_0,f,n}_t|^2(s)\Big]\lesssim|\langle\mu^n(0),\varphi^n_t(0)\rangle|^2+\int_0^t\mathrm{d}r\ T^n_r\big(n^{-\varrho}|\nabla^n\varphi^n_t(r)|^2+n^{-\varrho}|\xi^n|(\varphi^n_t(r))^2\big)$$

and the latter quantity is uniformly bounded in n. This is sufficient to deduce that $M^{\varphi_0,f}_t$ is a martingale w.r.t. its own filtration, but we want to prove that in fact it is a $\mathscr{F}-$martingale. Since by assumption $M^{\varphi_0,f,n}_t$ converges to the continuous process $M^{\varphi_0,f}_t$, we get from [EK86, Theorem 3.7.8] that for $0\le s\le r\le t$ and for bounded and continuous $\Phi\colon\mathbb{D}([0,s];\mathscr{M})\to\mathbb{R}$

$$\mathbb{E}[\Phi(\mu|_{[0,s]})(M^{\varphi_0,f}_t(r)-M^{\varphi^0,f}_t(s))]=\lim_n\mathbb{E}[\Phi(\mu^n|_{[0,s]})(M^{\varphi_0,f,n}_t(r)-M^{\varphi^0,f,n}_t(s))]=0.$$

From here we easily deduce the martingale property of $M^{\varphi_0,f}_t$.

Step 3. We show that $M^{\varphi_0,f}_t$ has the correct quadratic variation, which should be given as the limit of

$$\langle M^{\varphi_0,f,n}_t\rangle_s=\int_0^s\mathrm{d}r\ \mu^n(r)\big(n^{-\varrho}|\nabla^n\varphi^n_t(r)|^2+n^{-\varrho}|\xi^n|(\varphi^n_t(r))^2\big).$$

We only treat the case $\varrho=d/2$, the case $\varrho>d/2$ follows by similar but easier arguments because then we can use Lemma E.2 to gain some regularity from the factor $n^{d/2-\varrho}$, so that then $\|n^{-\varrho}|\xi^n|\|_{\mathscr{C}^{\varepsilon}(\mathbb{Z}^d_n,p(a))}$ converges to zero for some $\varepsilon>0$ and for all $a>0$.

First we assume, leaving the proof for later, that for any sequence $\{\psi^n\}_{n\in\mathbb{N}}$ with $\lim_n\|\psi^n\|_{\mathscr{C}^{-\varepsilon}(\mathbb{R}^d,p(a))}=0$ for some $a>0$ and all $\varepsilon>0$ the following convergence holds true:

$$\mathbb{E}\Big[\sup_{s\le t}\Big|\int_0^s\mathrm{d}r\ \mu^n(r)\big(\psi^n\cdot(\varphi^n_t(r))^2\big)\Big|^2\Big]\longrightarrow 0. \tag{21}$$

By Assumption 2.3 we can apply this result to $\psi^n = n^{-\varrho}|\xi^n|-2\nu$, and deduce that along a subsequence we have the following weak convergence in $\mathbb{D}([0,t];\mathbb{R})$:

$$\big(M_t^{\varphi_0,f,n}\big)^2 - \langle M_t^{\varphi_0,f,n}\rangle \longrightarrow \big(M_t^{\varphi_0,f}\big)^2 - \int_0^{\cdot} \mathrm{d}r\ \mu(r)\big(2\nu(\varphi_t)^2(r)\big).$$

Note also that the limit lies in $C([0,t];\mathbb{R})$. If the martingales on the left-hand side are uniformly bounded in L^2 we can deduce as before that the limit is an L^2-martingale, and conclude that

$$\langle M_t^{\varphi_0,f}\rangle_s = \int_0^s \mathrm{d}r\ \mu(r)\big(2\nu(\varphi_t)^2(r)\big).$$

As for the uniform bound in L^2, note that it follows from Lemma D.1 that

$$\sup_n \sup_{0\le s\le t} \mathbb{E}\big[|M_t^{\varphi_0,f,n}|^4(s)\big] < +\infty.$$

For the quadratic variation term we can estimate:

$$\mathbb{E}\big[|\langle M_t^{\varphi_0,f,n}\rangle_s|^2\big] \le s\int_0^s \mathrm{d}r\ \mathbb{E}\big[\big|\mu^n(r)\big(n^{-\varrho}|\nabla^n\varphi_t^n(r)|^2 + n^{-\varrho}|\xi^n|(\varphi_t^n(r))^2\big)\big|^2\big],$$

and the right hand side can be bounded via the second estimate of Lemma D.1.

Thus, we are left with proving Equation (21). By introducing the martingale from Equation (18) we find that

$$\begin{aligned}\mathbb{E}\big[|\mu^n(r)\big(\psi^n(\varphi_t^n(r))^2\big)|^2\big] &\lesssim |T_r^n\big[\psi^n(\varphi_t^n(r))^2\big]|^2(0)\\ &+\int_0^r \mathrm{d}q\ T_q^n\big(n^{-\varrho}\big|\nabla^n\big[T_{r-q}^n[\psi^n(\varphi_t^n(r))^2]\big]\big|^2 + n^{-\varrho}|\xi^n|(T_{r-q}^n[\psi^n(\varphi_t^n(r))^2])^2\big)(0).\end{aligned}$$

We start with the first term. For any sufficiently small $\varepsilon>0$ and some $l>0$ as well as for $\vartheta\in(0,\alpha)$ (cf. Proposition 3.1), we have that

$$\|T_q[\psi^n(\varphi_t^n(r))^2]\|_{\mathscr{C}^{\vartheta}(\mathbb{Z}_n^d;e(l))} \lesssim q^{-(\vartheta+\varepsilon)/2}\|\psi^n\|_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}_n^d;p(a))}.$$

It follows that we can bound:

$$|T_r^n\big[\psi^n(\varphi_t^n(r))^2\big]|^2(0) \lesssim r^{-2\varepsilon}\|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}_n^d;p(a))}.$$

Now we pass to the first term in the integral. Let us assume that $1-d/4<\vartheta<1-\varepsilon$, since we can take ε small enough such that the two bounds are feasible. We then apply Lemmata 1.2, E.2, E.3 to obtain that:

$$\big\|\big|n^{-d/4}\nabla^n\big[T_{r-q}^n[\psi^n(\varphi_t^n(r))^2]\big]\big|^2\big\|_{\mathscr{C}^{\vartheta-1+d/4}(\mathbb{Z}_n^d;e(2l))} \lesssim (r-q)^{-(\vartheta+\varepsilon)}\|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}_n^d;p(a))},$$

so that we can overall estimate:

$$\begin{aligned}&\int_0^r \mathrm{d}q\ T_q^n\big(n^{-\varrho}\big|\nabla^n\big[T_{r-q}^n[\psi^n(\varphi_t^n(r))^2]\big]\big|^2\big)(0)\\ &\qquad\lesssim \|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}_n^d;p(a))}\int_0^r \mathrm{d}q\ (r-q)^{-(\vartheta+\varepsilon)} \lesssim \|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}_n^d;p(a))}.\end{aligned}$$

Following the same steps, in view of Assumption 2.3, we can treat similarly the second term in the integral (we now use the same parameter ε both for the regularity of $n^{-\varrho}|\xi^n|$ and of ψ^n):

$$\|n^{-\varrho}|\xi^n|(T_q^n[\psi^n(\varphi_t^n(r))^2])^2\|_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}_n^d;e(2l)p(a))} \lesssim q^{-(\vartheta+\varepsilon)}\|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}_n^d;p(a))}$$

so that we can estimate:

$$\begin{aligned}\int_0^r \mathrm{d}q\ T_q^n(n^{-\varrho}|\xi^n|(T_q^n[\psi^n(\varphi_t^n(r))^2])^2)(0) &\lesssim \|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}_n^d;p(a))}\int_0^r \mathrm{d}q\ (r-q)^{-(\vartheta+\varepsilon)}q^{-2\varepsilon}\\ &\lesssim r^{1-\vartheta-3\varepsilon}\|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}_n^d;p(a))} \lesssim r^{-2\varepsilon}\|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}_n^d;p(a))},\end{aligned}$$

since $1-\vartheta > \varepsilon$. Overall, we conclude that

$$\mathbb{E}\big[|\mu^n(r)\big(\psi^n(\varphi^n_t(r))^2\big)|^2\big] \lesssim r^{-2\varepsilon}\|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}^d_n;p(a))}.$$

Integrating over r proves (21) . □

Our first main result, the law of large numbers, is now an easy consequence.

Proof of Theorem 2.10. Recall that now we assume $\varrho > d/2$. In view of Corollary 4.3 we can assume that along a subsequence $\mu^{n_k} \Rightarrow \mu$ in distribution in the space $\mathbb{D}([0,+\infty);\mathscr{M}(\mathbb{R}^d))$. To show that $\mu^n \Rightarrow w$ it thus suffices to prove that $\mu = w$. And indeed the previous lemma shows that for $\varphi \in C^\infty_c(\mathbb{R}^d)$ the process $s \mapsto \mu(s)(T_{t-s}\varphi) - T_t\varphi(0)$ is a continuous square-integrable martingale with vanishing quadratic variation. Hence it is constantly zero and thus $\mu(t)(\varphi) = T_t\varphi(0) = (T_t\delta_0)(\varphi)$ almost surely for each fixed $t \geq 0$. Note that $T_\cdot\delta_0$ is well-defined, as explained in Remark 3.2. Since μ is continuous, the identity holds almost surely for all $t > 0$. The identity $\mu(t) = T_t\delta_0$ then follows by choosing a countable separating set of smooth functions in $C^\infty_c(\mathbb{R}^d)$. □

Now we pass to the case $\varrho = d/2$. To deduce weak convergence of the sequence μ^n we have to complete the last step of our program, namely prove that the distribution of the limit points is uniquely characterized. This is the content of the next results.

First, we introduce a duality principle for the Laplace transform of our measure-valued process. For this reason we have to study Equation (6). We will consider mild solutions, i.e. φ solves (6) if and only if

$$\varphi(t) = T_t\varphi_0 - \frac{\kappa}{2}\int_0^t \mathrm{d}s\; T_{t-s}(\varphi(s)^2)$$

We shall denote such solution via $\varphi(t) = U_t\varphi_0$, which is justified by the following existence and uniqueness result:

Proposition 4.5. *Let $T,\kappa > 0$, $l_0 < -T$ and $\varphi_0 \in C^\infty(\mathbb{R}^d, e(l_0))$ with $\varphi_0 \geq 0$. For $l = l_0 + T$ and ϑ as in Proposition 3.1 there is a unique mild solution $\varphi \in \mathscr{L}^\vartheta(\mathbb{R}^d, e(l))$ to Equation (6):*

$$\partial_t\varphi = \mathscr{H}\varphi - \frac{\kappa}{2}\varphi^2, \qquad \varphi(0) = \varphi_0.$$

We write $U_t\varphi_0 := \varphi(t)$ and we have the following bounds:

$$0 \leq U_t\varphi_0 \leq T_t\varphi_0, \qquad \|\{U_t\varphi_0\}_{t\in[0,T]}\|_{\mathscr{L}^\vartheta(\mathbb{R}^d,e(l))} \lesssim e^{C\|\{T_t\varphi_0\}_{t\in[0,T]}\|_{CL^\infty(\mathbb{R}^d,e(l))}}.$$

Proof. We define the map $\mathscr{I}(\psi) = \varphi$, where φ is the solution to

$$\partial_t\varphi = \big(\mathscr{H} - \frac{\kappa}{2}\psi\big)\varphi, \qquad \varphi(0) = \varphi_0.$$

If $l_0 < -T$, then $(T_t\varphi_0)_{t\in[0,T]} \in \mathscr{L}^\vartheta(\mathbb{R}^d, e(l))$ for $l = l_0 + T$, and thus a slight adaptation of the arguments leading to Proposition 3.1 shows that $\mathscr{I}$ satisfies

$$\mathscr{I}\colon \mathscr{L}^\vartheta(\mathbb{R}^d, e(l)) \to \mathscr{L}^\vartheta(\mathbb{R}^d, e(l)), \qquad \|\mathscr{I}(\psi)\|_{\mathscr{L}^\vartheta(\mathbb{R}^d,e(l))} \lesssim e^{C\|\psi\|_{CL^\infty(\mathbb{R}^d,e(l))}}$$

for some $C > 0$. Moreover, for positive ψ this map satisfies the a priori bound:

$$0 \leq \mathscr{I}(\psi)(t) \leq T_t\varphi_0,$$

so in particular $\|\mathscr{I}(\psi)\|_{CL^\infty(\mathbb{R}^d,e(l))} \leq \|\{T_t\varphi_0\}_{t\in[0,T]}\|_{CL^\infty(\mathbb{R}^d,e(l))}$. We define $\varphi^0 = T_t\varphi_0$ and then iteratively $\varphi^m = \mathscr{I}(\varphi^{m-1})$ for $m \geq 1$. Hence our a priori bounds guarantee that

$$\sup_m \|\varphi^m\|_{\mathscr{L}^\vartheta(\mathbb{R}^d,e(l))} \lesssim e^{C\|\{T_t\varphi_0\}_{t\in[0,T]}\|_{CL^\infty(\mathbb{R}^d,e(l))}}.$$

By compact embedding of $\mathscr{L}^\vartheta(\mathbb{R}^d, e(l)) \subset \mathscr{L}^\zeta(\mathbb{R}^d, e(l'))$ for $\zeta < \vartheta$, $l' < l$ we obtain convergence of a subsequence in the latter space. The regularity ensures that the limit point is indeed a solution to Equation (6). The uniqueness of such a fixed-point follows from the fact that the difference $z=\varphi-\psi$ of two solutions φ and ψ solves the well posed linear equation:

$$\partial_t z = \big(\mathscr{H}+\frac{\kappa}{2}(\varphi+\psi)\big)z, \qquad z(0)=0,$$

and thus $z=0$. □

We proceed by proving some implications between the properties $(i)-(iii)$ of Definition 2.11.

Lemma 4.6. *In Definition 2.11 the following implications hold between the three properties:*

$$(ii)\Rightarrow(i), \qquad\qquad (ii)\Leftrightarrow(iii).$$

Proof. $(ii)\Rightarrow(i)$: Consider $U_{\cdot}\varphi_0$ as in point (i) of Definition 2.11, which is well defined in view of Proposition 4.5. An application of Itô's formula together with property (ii) guarantees that for any $F\in C^2(\mathbb{R})$, and for $f(r)=\frac{\kappa}{2}(U_{t-r}\varphi_0)^2$:

$$\begin{aligned}F(\langle\mu(t),\varphi_0\rangle) =& F(\langle\mu(s),U_{t-s}\varphi_0\rangle)+\int_s^t \mathrm{d}r\ F'(\langle\mu(r),U_{t-r}\varphi_0\rangle)\langle\mu(r),f(r)\rangle\\ &+\frac{1}{2}\int_s^t F''(\langle\mu(r),U_{t-r}\varphi_0\rangle)\,\mathrm{d}\langle M_t^{\varphi_0,f}\rangle_r+\int_s^t F'(\langle\mu(r),U_{t-r}\varphi_0\rangle)\,\mathrm{d}M_t^{\varphi_0,f}(r),\end{aligned}$$

where $\mathrm{d}\langle M_t^{\varphi_0,f}\rangle_r=\langle\mu(r),\kappa(U_{t-r}\varphi_0)^2\rangle\,\mathrm{d}r$. Since the function $F(x)=e^{-x}$ is bounded for positive x, we deduce property (i) from this.

$(ii)\Rightarrow(iii)$: Let $\varphi\in\mathscr{D}_{\mathscr{H}}$ and $t>0$ and let $0=t_0^n\le t_1^n\le\ldots\le t_n^n=t$, $n\in\mathbb{N}$, be a sequence of partitions of $[0,t]$ with $\max_{k\le n-1}\Delta_k^n:=\max_{k\le n-1}(t_{k+1}^n-t_k^n)\to 0$. Then

$$\begin{aligned}\langle\mu(t),\varphi\rangle-\langle\mu(0),\varphi\rangle &= \sum_{k=0}^{n-1}\big[\big(\langle\mu(t_{k+1}^n),\varphi\rangle-\langle\mu(t_k^n),T_{\Delta_k^n}\varphi\rangle\big)+\langle\mu(t_k^n),T_{\Delta_k^n}\varphi-\varphi\rangle\big]\\ &=\sum_{k=0}^{n-1}\big[\big(M^{\varphi,0}_{t_{k+1}^n}(t_{k+1}^n)-M^{\varphi,0}_{t_{k+1}^n}(t_k^n)\big)+\Delta_k^n\langle\mu(t_k^n),\frac{T_{\Delta_k^n}\varphi-\varphi}{\Delta_k^n}\rangle\big].\end{aligned}$$

We start by studying the second term on the right hand side:

$$\begin{aligned}\sum_{k=0}^{n-1}\Delta_k^n\langle\mu(t_k^n),\frac{T_{\Delta_k^n}\varphi-\varphi}{\Delta_k^n}\rangle &= \sum_{k=0}^{n-1}\Delta_k^n\langle\mu(t_k^n),\frac{T_{\Delta_k^n}\varphi-\varphi}{\Delta_k^n}-\mathscr{H}\varphi\rangle+\sum_{k=0}^{n-1}\Delta_k^n\langle\mu(t_k^n),\mathscr{H}\varphi\rangle\\ &=: R_n+\sum_{k=0}^{n-1}\Delta_k^n\langle\mu(t_k^n),\mathscr{H}\varphi\rangle.\end{aligned}$$

By continuity of μ the second term on the right hand side converges almost surely to the Riemann integral $\int_0^t\langle\mu(r),\mathscr{H}\varphi\rangle\,\mathrm{d}r$. Moreover, from the characterization (ii) we get $\mathbb{E}[\mu(s)(\psi)]=\langle\mu(0),T_s\psi\rangle$ and

$$\mathbb{E}[\mu(s)(\mathscr{H}\varphi)^2]\lesssim\langle\mu(0),(T_s(\mathscr{H}\varphi))^2\rangle+\int_0^s\mathrm{d}r\,\langle T_r,(T_{s-r}\mathscr{H}\varphi)^2\rangle,$$

which is uniformly bounded in $s\in[0,t]$. So the sequence is uniformly integrable and converges also in L^1 and not just almost surely. Moreover,

$$\mathbb{E}[|R_n|]\lesssim\sum_{k=0}^{n-1}\Delta_k^n\big\langle\mu_0,T_{t_k^n}(|(\Delta_k^n)^{-1}(T_{\Delta_k^n}\varphi-\varphi)-\mathscr{H}\varphi|)\big\rangle,$$

and since $\max_{k\le n-1}(\Delta_k^n)^{-1}(T_{\Delta_k^n}\varphi-\varphi)$ converges to $\mathscr{H}\varphi$ in $\mathscr{C}^\vartheta(\mathbb{R}^d,e(l))$ for some $l\in\mathbb{R}$ and $\vartheta>0$ (so in particular uniformly), it follows from Proposition 3.1 and the assumption $\langle\mu_0,e(l)\rangle<\infty$

for all $l \in \mathbb{R}$ that $\mathbb{E}[|R_n|] \to 0$. Thus, we showed that

$$L_t^{\varphi} = \langle \mu(t), \varphi \rangle - \langle \mu(0), \varphi \rangle - \int_0^t \langle \mu(r), \mathscr{H}\varphi \rangle \, \mathrm{d}r = \lim_{n\to\infty} \sum_{k=0}^{n-1} \big(M^{\varphi,0}_{t^n_{k+1}}(t^n_{k+1}) - M^{\varphi,0}_{t^n_{k+1}}(t^n_k) \big),$$

and the convergence is in L^1. By taking partitions that contain $s \in [0,t)$ and using the martingale property of $M_r^{\varphi,0}$ we get $\mathbb{E}[L^{\varphi}(t)|\mathscr{F}_s] = L^{\varphi}(s)$, i.e. L^{φ} is a martingale. By the same arguments that we used to show the uniform integrability above, $L^{\varphi}(t)$ is square integrable for all $t > 0$. To derive the quadratic variation we use again a sequence of partitions containing $s \in [0,t)$ and obtain

$$\begin{aligned} \mathbb{E}\big[L^{\varphi}(t)^2 - L^{\varphi}(s)^2 \big| \mathscr{F}_s\big] &= \mathbb{E}\big[(L^{\varphi}(t) - L^{\varphi}(s))^2 \big| \mathscr{F}_s\big] \\ &= \lim_{n\to\infty} \sum_{k: t^n_{k+1} > s} \mathbb{E}\big[\big(M^{\varphi,0}_{t^n_{k+1}}(t^n_{k+1}) - M^{\varphi,0}_{t^n_{k+1}}(t^n_k)\big)^2 \big| \mathscr{F}_s\big] \\ &= \lim_{n\to\infty} \sum_{k: t^n_{k+1} > s} \mathbb{E}\Big[\kappa \int_{t^n_k}^{t^n_{k+1}} \mathrm{d}r \, \langle \mu(r), (T_{t^n_{k+1}-r}\varphi)^2 \rangle \Big| \mathscr{F}_s\Big] \\ &= \mathbb{E}\Big[\kappa \int_s^t \mathrm{d}r \, \langle \mu(r), \varphi^2 \rangle \Big| \mathscr{F}_s\Big]. \end{aligned}$$

Since the process $\kappa \int_0^{\cdot} \mathrm{d}r \, \langle \mu(r), \varphi^2 \rangle$ is increasing and predictable, it must be equal to $\langle L^{\varphi} \rangle$.

(iii) ⇒ *(ii)*: Let $t \geq 0$, $\varphi_0 \in \mathscr{D}_{\mathscr{H}}$, and let $f \colon [0,t] \to \mathscr{D}_{\mathscr{H}}$ be a piecewise constant function (in time). We write φ for the solution to the backward equation

$$(\partial_s + \mathscr{H})\varphi = f, \qquad \varphi(t) = \varphi_0,$$

which is given by $\varphi(s) = T_{t-s}\varphi_0 + \int_s^t T_{r-s} f(r) \, \mathrm{d}r$. Note that by assumption $\varphi(r) \in \mathscr{D}_{\mathscr{H}}$ for all $r \leq t$. For $0 \leq s \leq t$, let $0 = t^n_0 \leq t^n_1 \leq \ldots \leq t^n_n = s$, $n \in \mathbb{N}$, be a sequence of partitions of $[0,s]$ with $\max_{k \leq n-1} \Delta^n_k := \max_{k\leq n-1}(t^n_{k+1} - t^n_k) \to 0$. Similarly to the computation in the step "*(i)* ⇒ *(ii)*" we can decompose:

$$\begin{aligned} \langle \mu(s), \varphi(s) \rangle - \langle \mu(0), \varphi(0) \rangle &= \sum_{k=0}^{n-1} \big[\big(\langle \mu(t^n_{k+1}), \varphi(t^n_{k+1}) \rangle - \langle \mu(t^n_k), \varphi(t^n_{k+1}) \rangle \big) - \langle \mu(t^n_k), \varphi(t^n_k) - \varphi(t^n_{k+1}) \rangle \big] \\ &= \sum_{k=0}^{n-1} \Big[L^{\varphi(t^n_{k+1})}(t^n_{k+1}) - L^{\varphi(t^n_{k+1})}(t^n_k) + \int_{t^n_k}^{t^n_{k+1}} \mathrm{d}r \, \langle \mu(r), f(r) \rangle \Big] + R_n, \end{aligned}$$

with

$$\begin{aligned} R_n = \sum_{k=0}^{n-1} \int_{t^n_k}^{t^n_{k+1}} \mathrm{d}r \Big[&\langle \mu(r), \mathscr{H}\varphi(t^n_{k+1}) \rangle - \langle \mu(t^n_k), (\Delta^n_k)^{-1}(T_{\Delta^n_k} - \mathrm{id})\varphi(t^n_{k+1}) \rangle \\ &+ \langle \mu(t^n_k), T_{r-t^n_k} f(r) \rangle - \langle \mu(r), f(r) \rangle \Big]. \end{aligned}$$

By similar arguments as in the step *(ii)* ⇒ *(iii)* we see that R_n converges to zero in L^1, and therefore $s \mapsto \langle \mu(s), \varphi(s) \rangle - \langle \mu(0), \varphi(0) \rangle - \int_0^s \mathrm{d}r \, \langle \mu(r), f(r) \rangle$ is a martingale. Square integrability and the right form of the quadratic variation are shown again by similar arguments as before.

By density of $\mathscr{D}_{\mathscr{H}}$ it follows that $M_t^{\varphi_0, f}$ is a martingale on $[0,t]$ with the required quadratic variation for any $\varphi_0 \in C_c^{\infty}(\mathbb{R}^d)$ and $f \in C([0,t]; \mathscr{C}^{\zeta}(\mathbb{R}^d))$ for $\zeta > 0$. This concludes the proof. □

Characterization *(i)* of Definition 2.11 enables us to deduce the uniqueness in law and then to conclude the proof of the equivalence of the different characterizations in Definition 2.11.

Proof of Lemma 2.12. First, we claim that property (i) of Definition 2.11 gives uniqueness in law for the stochastic process μ. Indeed, we have for $0 \le s \le t$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi \ge 0$

$$\mathbb{E}\big[e^{-\langle \mu(t),\varphi\rangle}\big|\mathscr{F}_s\big] = e^{-\langle \mu(s), U_{t-s}\varphi\rangle}.$$

For $s=0$ we can use the Laplace transform and the linearity of $\varphi \mapsto \langle \mu(t),\varphi\rangle$ to deduce that the law of $(\langle \mu(t),\varphi_1\rangle,\dots,\langle \mu(t),\varphi_n\rangle)$ is uniquely determined by (i) whenever $\varphi_1,\dots,\varphi_n \in C_c^\infty(\mathbb{R}^d)$ are positive functions. By density of $C_c^\infty(\mathbb{R}^d)$ this shows that the law of $\mu(t)$ is unique. We then see inductively that the finite-dimensional distributions of $\mu = \{\mu(t)\}_{t\ge 0}$ are unique, and thus that the law of μ is unique.

It remains to show the implication $(i) \Rightarrow (ii)$ to conclude the proof of the equivalence of the characterizations in Definition 2.11. But this is now immediate, because we showed in Lemma 4.4 that there exists a process satisfying (ii), and in Lemma 4.6 we showed that then it must also satisfy (i). And since we just saw that there is uniqueness in law for processes satisfying (i) and since property (ii) only depends on the law and it holds for one process satisfying (i), it must hold for all processes satisfying (i) (strictly speaking Lemma 4.4 only gives the existence for $\kappa = 2\nu \in (0,1]$, but see Section 4.2 below for general κ). □

Now the convergence of the sequence $\{\mu^n\}_{n\in\mathbb{N}}$ is an easy consequence:

Proof of Theorem 2.10. This follows from the characterization of the limit points from Lemma 4.4 together with the uniqueness result from Lemma 2.12.

□

4.2. **Mixing with a classical Superprocess.** In Section 4.1 we constructed the rSBM of parameter $\kappa = 2\nu$, for ν defined via Assumption 2.1. This leads to the restriction $\nu \in (0,\frac{1}{2}]$. This section is devoted to constructing the rSBM for arbitrary $\kappa > 0$. We do so by means of an interpolation between the rSBM and a Dawson-Watanabe superprocesses (cf. [Eth00, Chapter 1]). Let Ψ be the generating function of a discrete finite positive measure $\Psi(s) = \sum_{k\ge 0} p_k s^k$ and ξ_p^n a controlled random environment associated to a parameter $\nu = \mathbb{E}[\Phi_+]$. We consider the quenched generator:

$$\mathscr{L}_\Psi^{n,\omega^p}(F)(\eta) = \sum_{x\in\mathbb{Z}_n^d} \eta_x \cdot \bigg[\sum_{y\sim x} n^2\big(F(\eta^{x\mapsto y}) - F(\eta)\big)(\xi_e^n)_+(\omega^p,x)[F(\eta^{x;1}) - F(\eta)] \\ + (\xi_e^n)_-(\omega^p,x)[F(\eta^{x;-1}) - F(\eta)] + n^{\varrho}\sum_{k\ge 0} p_k[F(\eta^{x;(k-1)}) - F(\eta)]\bigg]$$

with the notation $\eta^{x;k}(y) = (\eta(y) + k1_{\{x\}}(y))_+$, for $k \ge -1$. The rigorous derivation of this operator as the generator of a Markov process follows analogously to the results in Section A.

Assumption 4.7 (On the Moment generating function)**.** *We study the process associated to the generator $\mathscr{L}_\Psi^{n,\omega^p}$ under the assumption that $\Psi'(1) = 1$ (critical branching, i.e. the expected number of offsprings in one branching/killing event is* 1*) and we write $\sigma^2 = \Psi''(1)$ for the variance of the offspring distribution.*

Now we introduce the associated process. The construction of the process $\bar{u}^n$ is analogous to the case without Ψ, which is treated in Appendix A.

Definition 4.8. *Let $\varrho \ge d/2$ and let Ψ be a moment generating function satisfying the previous assumptions. Consider a controlled random environment ξ_p^n associated to a parameter $\nu \in (0,\frac{1}{2}]$. Let $\mathbb{P}^n = \mathbb{P}^p \ltimes \mathbb{P}^{n,\omega^p}$ be the measure on $\Omega^p \times \mathbb{D}([0,+\infty);E)$ such that for fixed $\omega^p \in \Omega^p$, under the measure $\mathbb{P}^{n,\omega^p}$ the canonical process on $\mathbb{D}([0,+\infty);E)$ is the Markov process $\bar{u}_p^n(\omega^p,\cdot)$ started in $\bar{u}_p^n(0) = \lfloor n^{\varrho}\rfloor 1_{\{0\}}(x)$ associated to the generator $\mathscr{L}_\Psi^{\omega^p,n}$ defined as above. To $\bar{u}_p^n$ we associate* *the measure valued process*

$$\langle \bar{\mu}_p^n(\omega^p, t), \varphi \rangle = \sum_{x \in \mathbb{Z}_n^d} \bar{u}_p^n(\omega^p, t, x)\varphi(x)\lfloor n^{\varrho} \rfloor^{-1}$$

for any bounded $\varphi\colon \mathbb{Z}_n^d \to \mathbb{R}$. *With this definition* $\bar{\mu}_p^n$ *takes values in* $\Omega^p \times \mathbb{D}([0,T]; \mathscr{M}(\mathbb{R}^d))$ *with the law induced by* $\mathbb{P}^n$.

Remark 4.9. *As in Remark 4.1 we see that for* $\varphi \in L^\infty(\mathbb{Z}_n^d, e(l))$ *with* $l \in \mathbb{R}$ *the process* $\bar{M}_t^{n,\varphi}(s) := \bar{\mu}^n(s)(T_{t-s}^n\varphi) - T_t^n\varphi(0)$ *is a martingale with predictable quadratic variation:*

$$\langle \bar{M}_t^{n,\varphi} \rangle_s = \int_0^s \mathrm{d}r \; \bar{\mu}^n(r)\big(n^{-\varrho}|\nabla^n T_{t-r}^n\varphi|^2 + (n^{-\varrho}|\xi_e^n| + \sigma^2)(T_{t-r}^n\varphi)^2\big).$$

In view of this Remark, we can follow the discussion of Section 4.1 to deduce the following result (cf. Corollary 2.16).

Proposition 4.10. *The sequence of measures* $\mathbb{P}^n$ *as in Definition 4.8 converge weakly as measures on* $\Omega^p \times \mathbb{D}([0,T]; \mathscr{M}(\mathbb{R}^d))$ *to the measure* $\mathbb{P}^p \times \mathbb{P}^{\omega^p}$ *associated to a rSBM of parameter* $\kappa = 1_{\{\varrho = \frac{d}{2}\}} 2\nu + \sigma^2$, *in the sense of Theorem 2.13 and Corollary 2.16. In short, we write* $\overline{\mu}_p^n \to \overline{\mu}_p$.

In particular the rSBM is also the scaling limit of critical branching random walks whose branching rates are perturbed by small random potentials.

5. Properties of the Rough Super-Brownian Motion

5.1. Scaling Limit as SPDE in d=1.

In this section we characterize the rough super-Brownian motion in dimension $d = 1$ as the solution to the SPDE (7):

$$\partial_t \mu_p(t,x) = \mathscr{H}^{\omega^p}\mu_p(t,x) + \sqrt{\kappa\mu_p(t,x)}\,\tilde{\xi}(t,x), \qquad (t,x) \in [0,T] \times \mathbb{R}$$

in the sense of Definition 2.18. The first result in this direction states that the random measure μ_p admits a density with respect to the Lebesgue measure.

Lemma 5.1. *Let* μ *be a one-dimensional rough super-Brownian motion of parameter* ν. *For any* $\beta < 1/2$, $p \in [1, 2/(\beta+1))$ *and* $l \in \mathbb{R}$, *we have:*

$$\mathbb{E}\big[\|\mu\|^p_{L^p([0,T]; B^\beta_{2,2}(\mathbb{R}, e(l)))}\big] < \infty.$$

Proof. Let $t > 0$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$. By point (ii) of Definition 2.11 the process $M_t^\varphi(s) = \langle \mu(s), T_{t-s}\varphi\rangle - \langle \mu(0), T_t\varphi\rangle$, $s \in [0,t]$, is a continuous square-integrable martingale with quadratic variation $\langle M_t^\varphi \rangle_s = \int_0^s \langle \mu(r), (T_{t-r}\varphi)^2\rangle$. With the help of the moment estimates of Lemma D.1, which by Fatou's lemma also hold for the limit μ of the $\{\mu^n\}$, we can extend this martingale property to $\varphi \in \mathscr{C}^\vartheta(\mathbb{R}, e(k))$ for arbitrary $k \in \mathbb{R}$ and $\vartheta > 0$. In particular, for such φ we get

$$\mathbb{E}[\langle \mu(t), \varphi\rangle^2] \lesssim \mathbb{E}\Big[\int_0^t \langle \mu(r), (T_{t-r}\varphi)^2\rangle \,\mathrm{d}r\Big] + (T_t\varphi)^2(0)a = \int_0^t T_r((T_{t-r}\varphi)^2)(0)\,\mathrm{d}r + (T_t\varphi)^2(0),$$

and thus for $\varphi = K_j(\cdot - x)$

$$\mathbb{E}\big[\|\mu(t)\|^2_{B^\beta_{2,2}(e(l))}\big] = \sum_j 2^{2j\beta}\int \mathbb{E}[\langle \mu(t), K_j(x-\cdot)\rangle^2]e^{-2l|x|^\sigma}\,\mathrm{d}x$$

$$\lesssim \sum_j 2^{2j\beta}\int \Big[\int_0^t T_r((T_{t-r}K_j(x-\cdot))^2)(0)\,\mathrm{d}r + (T_tK_j(x-\cdot))^2(0)\Big]e^{-2l|x|^\sigma}\,\mathrm{d}x. \tag{22}$$

We start by proving that for any $k > 0$ we can bound $\|K_j(x-\cdot)\|_{\mathscr{C}_1^\alpha(\mathbb{R}, e(k))} \lesssim 2^{j\alpha}e^{-k|x|^\sigma}$. Indeed, using that K_i is an even function and writing $\tilde{K}_{i-j} = 2^{(i-j)d}K_0(2^{i-j}\cdot) * K_0$ if $i, j \geq 0$ and appropriately adapted if $i=-1$ or $j=-1$:

$$
\begin{aligned}
\|\Delta_i(K_j(x-\cdot))e(k)\|_{L^1(\mathbb{R})} &= 1_{\{|i-j|\leq 1\}}\int_{\mathbb{R}^d}|K_i * K_j(x-y)|e^{-k|y|^\sigma}\,\mathrm{d}y \\
&= 1_{\{|i-j|\leq 1\}}\int_{\mathbb{R}^d}|\tilde{K}_{i-j}(y)|e^{-k|x-2^{-j}y|^\sigma}\,\mathrm{d}y \\
&\lesssim 1_{\{|i-j|\leq 1\}}\int_{\mathbb{R}^d}|\tilde{K}_{i-j}(y)|e^{k|2^{-j}y|^\sigma-k|x|^\sigma}\,\mathrm{d}y \\
&\lesssim 1_{\{|i-j|\leq 1\}}e^{-k|x|^\sigma},
\end{aligned}
$$

where in the last step we used that $|\tilde{K}_{i-j}(y)|\lesssim e^{-2k|y|^\sigma}$ and $2^{-j\sigma}\leq 2^\sigma<2$.

Now, for $\zeta<0$ satisfying the assumptions of Proposition 3.1 and for $p\in[1,\infty]$ and sufficiently small $\varepsilon>0$:

$$
\|T_sK_j(x-\cdot)\|_{\mathscr{C}^\varepsilon_p(\mathbb{R},e(k+s))}\lesssim\|T_sK_j(x-\cdot)\|_{\mathscr{C}_1^{1-\frac{1}{p}+\varepsilon}(\mathbb{R},e(k+s))}\lesssim 2^{j\zeta}s^{(\zeta-1+\frac{1}{p}-2\varepsilon)/2}e^{-k|x|^\sigma}.
$$

To control the first term on the right hand side of (22), we apply this with $p=2$ and obtain for $t\in[0,T]$ and $\zeta>-1/2$

$$
\begin{aligned}
\int_0^t T_r((T_{t-r}K_j(x-\cdot))^2)(0)\,\mathrm{d}r &\lesssim \int_0^t\|T_r((T_{t-r}K_j(x-\cdot))^2)\|_{\mathscr{C}^\varepsilon_\infty(\mathbb{R},e(2k+T))}\,\mathrm{d}r \\
&\lesssim \int_0^t\|T_r((T_{t-r}K_j(x-\cdot))^2)\|_{\mathscr{C}_1^{1+\varepsilon}(\mathbb{R},e(2k+T))}\,\mathrm{d}r \\
&\lesssim \int_0^t r^{-\frac{1+2\varepsilon}{2}}\|(T_{t-r}K_j(x-\cdot))^2\|_{\mathscr{C}^\varepsilon_1(\mathbb{R},e(2k))}\,\mathrm{d}r \\
&\lesssim \int_0^t r^{-\frac{1+2\varepsilon}{2}}\|T_{t-r}K_j(x-\cdot)\|^2_{\mathscr{C}^\varepsilon_2(\mathbb{R},e(k))}\,\mathrm{d}r \\
&\lesssim \int_0^t r^{-\frac{1+2\varepsilon}{2}}(2^{j\zeta}(t-r)^{(\zeta-\frac{1}{2}-2\varepsilon)/2}e^{-k|x|^\sigma})^2\,\mathrm{d}r \\
&\simeq 2^{2j\zeta}e^{-2k|x|^\sigma}t^{1-\frac{1+2\varepsilon}{2}+\zeta-\frac{1}{2}-2\varepsilon}=2^{2j\zeta}e^{-2k|x|^\sigma}t^{\zeta-3\varepsilon},
\end{aligned}
$$

where we used that $\int_0^t r^{-\alpha}(t-r)^{-\beta}\,\mathrm{d}r\simeq t^{1-\alpha-\beta}$ for $\alpha,\beta<1$. The second term on the right hand side of (22) is bounded by

$$
\begin{aligned}
(T_tK_j(x-\cdot))^2(0) &\lesssim \|(T_tK_j(x-\cdot))^2\|_{\mathscr{C}^\varepsilon_\infty(\mathbb{R},e(2k+2T))} \\
&\lesssim \|T_tK_j(x-\cdot)\|^2_{\mathscr{C}^\varepsilon_\infty(\mathbb{R},e(k+T))} \\
&\lesssim 2^{2j\zeta}t^{\zeta-1-2\varepsilon}e^{-2k|x|^\sigma}.
\end{aligned}
$$

Note that this estimate is much worse than the first one (because $t\in[0,T]$ is bounded above). We plug both those estimates into (22) and set $\zeta=-\beta-\varepsilon$ and $k>-l$ to obtain for $\beta<1/2$ and for $l\in\mathbb{R}$

$$
\mathbb{E}\big[\|\mu(t)\|^2_{B^\beta_{2,2}(e(l))}\big]\lesssim t^{-\beta-1-3\varepsilon}.
$$

So finally for $p\in[1,2)$

$$
\mathbb{E}\big[\|\mu\|^p_{L^p([0,T];B^\beta_{2,2}(\mathbb{R},e(l)))}\big]=\int_0^T\mathbb{E}\big[\|\mu(t)\|^p_{B^\beta_{2,2}(e(l))}\big]\,\mathrm{d}t\lesssim\int_0^T t^{(-\beta-1-3\varepsilon)\frac{p}{2}}\,\mathrm{d}t,
$$

and now it suffices to note that there exists $\varepsilon>0$ with $(-\beta-1-3\varepsilon)\frac{p}{2}<-1$ if and only if $p<2/(\beta+1)$.

□

Corollary 5.2. *In the setting of Proposition 5.1 we have almost surely* $\sqrt{\mu}\in L^2([0,T];L^2(\mathbb{R},e(l)))$ *for all* $T>0$ *and* $l\in\mathbb{R}$.

Proof of Theorem 2.19. We follow the same approach as Konno and Shiga [KS88]. First, consider a probability space $(\Omega^p, \mathscr{F}^p, \mathbb{P}^p)$ supporting a sequence of controlled random environments. If $\kappa \in (0,1)$ we additionally assume that $\nu = \kappa/2$, for ν as in Assumption 2.1, and let μ_p be the limit of the discrete processes μ_p^n as derived in Theorem 2.13 and Corollary 2.16. If $\kappa > 1$ we consider the process $\bar{\mu}_p^n$ constructed in Section 4.2 for an appropriate moment generating function Ψ, such that $\sigma^2 = \Psi''(1) = \kappa - 2\nu_0$, and for some $\nu_0 \in (0, 1/2)$ and a random environment ξ^n satisfying Assumption 2.1 with $\nu = \nu_0$. We then work with its limit $\bar{\mu}_p$ described in Proposition 4.10.

In both cases, we have constructed a process, which we denote with μ_p, on the space

$$(\Omega^p \times \mathbb{D}([0,T]; \mathscr{M}(\mathbb{R})), \mathscr{F}, \mathbb{P}^p \ltimes \mathbb{P}^{\omega^p}),$$

with $\mathscr{F}$ being the product sigma algebra. Enlarging the probability space, we can moreover assume that the process is defined on

$$(\Omega^p \times \bar{\Omega}, \mathscr{F}^p \otimes \bar{\mathscr{F}}, \mathbb{P}^p \ltimes \bar{\mathbb{P}}^{\omega^p})$$

such that the probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ supports a space-time white noise $\bar{\xi}$ which is independent of ξ. More precisely, we are given a map

$$\overline{\xi} : \Omega^p \times \overline{\Omega} \to \mathscr{S}'(\mathbb{R}^d \times [0,T])$$

which has the law of space-time white noise and does not depend on Ω^p, i.e. $\overline{\xi}(\omega^p, \overline{\omega}) = \overline{\xi}(\overline{\omega})$.

For $\omega^p \in \Omega^p$ let $\{\mathscr{F}_t^{\omega^p}\}_{t\in[0,T]}$ be the usual augmentation of the (random) filtration generated by $\mu(\omega^p, \cdot)$ and $\bar{\xi}$. For almost all $\omega^p \in \Omega^p$ the collection of martingales

$$t \mapsto L^{\varphi}(\omega^p, t), \qquad t \in [0,T], \quad \varphi \in \mathscr{D}_{\mathscr{H}^{\omega^p}},$$

defines a (random) worthy orthogonal martingale measure $M(\omega^p, \mathrm{d}t, \mathrm{d}x)$ in the sense of [Wal86], with quadratic variation

$$Q(A \times B \times [s,t]) = \int_s^t \mu(r)(A \cap B)\, \mathrm{d}r$$

for all Borel sets $A, B \subset \mathbb{R}$ (first we define $Q(\varphi \times \psi \times [s,t]) = \int_s^t \langle \mu(r), \varphi\psi \rangle \,\mathrm{d}r$ for $\varphi, \psi \in \mathscr{D}_{\mathscr{H}^{\omega^p}}$, then we use Lemma 5.1 with $p = 1$ and $\beta \in (0, 1/2)$ to extend the quadratic variation and the martingales to indicator functions of Borel sets). We can thus build a space-time white noise $\tilde{\xi}$ by defining for $\varphi \in L^2([0,T] \times \mathbb{R})$:

$$\begin{aligned}\int_{[0,T]\times\mathbb{R}} \tilde{\xi}(\omega^p, \mathrm{d}s, \mathrm{d}x)\varphi(s,x) := &\int_{[0,T]\times\mathbb{R}} M(\omega^p, \mathrm{d}s, \mathrm{d}x) \frac{\varphi(s,x)}{\sqrt{\mu(\omega^p, s, x)}} 1_{\{\mu(\omega^p,s,x)>0\}} \\ &+ \int_{[0,T]\times\mathbb{R}} \bar{\xi}(\mathrm{d}s, \mathrm{d}x)\varphi(s,x) 1_{\{\mu(\omega^p,s,x)=0\}}.\end{aligned}$$

By taking conditional expectations with respect to ξ^p we see that $\tilde{\xi}$ and ξ^p are independent.

Moreover, it is straightforward to see that any solution to the SPDE is a rSBM of parameter $\nu = \kappa/2$. Uniqueness in law of the latter then implies uniqueness in law of the solution to the SPDE. □

5.2. **Persistence.** In this section we study the persistence of the SBM in static random environment μ_p and we prove Theorem 2.21, i.e. that μ_p is super-exponentially persistent. We work with a slightly modified controlled random environment $\{\xi_p^n\}_{n\in\mathbb{N}}$ which we build in Lemma 5.9 and assume that μ_p is given as limit of branching random walks, as in Corollary 2.16. Since persistence is a property that only depends on the distribution and there is uniqueness in law for μ_p, this assumption does not restrict the validity of our arguments.

In the next section we rigorously construct for $L > 0$ a killed SBM in static random environment μ_p^L, where particles are killed once they leave the box $(-L/2, L/2)^d$, and we couple μ_p^L with μ_p so that almost surely $\mu_p^L \le \mu_p$ for all $L \in 2\mathbb{N}$ (see Proposition 5.19 and Corollary 5.20).

In this section we prove that given a nonzero positive $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $\lambda > 0$, for almost all ω^p there exists $L = L(\omega^p)$ with

$$\mathbb{P}\big(\lim_{t\to\infty} e^{-t\lambda}\langle \mu_p^L(\omega^p, t, \cdot), \varphi\rangle = \infty\big) > 0. \tag{23}$$

This implies Theorem 2.21.

The reason for working with μ_p^L is that the spectrum of the Anderson Hamiltonian on the space of bounded volume $(-L/2, L/2)^d$ is discrete, and its highest eigenvalue almost surely becomes bigger than λ for $L \to \infty$. Given this information, the proof of (23) follows from a simple martingale convergence argument, see Corollary 5.7 below.

Remark 5.3. *For simplicity we only treat the case of (killed) rSBM with parameter $\nu \in (0, 1/2]$. For greater values of ν we additionally need to mix the process with a Dawson-Watanabe superprocess as in Section 4.2, after which we can follow the same arguments to show persistence.*

Let us write $\lambda(\omega^p, L)$ for the largest eigenvalue of the Anderson Hamiltonian $\mathscr{H}_{\mathfrak{d},L}^{\omega^p}$ with Dirichlet boundary conditions on $(-L/2, L/2)^d$. We will use the following results.

Lemma 5.4 (Lemmata 2.3 and 4.1, [Che14]). *In dimension $d = 1$ there exists a constant $c_1 > 0$ such that for almost all $\omega^p \in \Omega^p$:*

$$\lim_{L\to+\infty} \frac{\lambda(\omega^p, L)}{\log(L)^{2/3}} = c_1.$$

Lemma 5.5 (Theorem 10.1, [CvZ19]). *In dimension $d = 2$ there exists a constant $c_2 > 0$ such that for almost all $\omega^p \in \Omega^p$:*

$$\lim_{L\to+\infty} \frac{\lambda(\omega^p, L)}{\log(L)} = c_2.$$

Lemma 5.6. *The operator $\mathscr{H}_{\mathfrak{d},L}^{\omega^p}$ admits an eigenfunction $e_{\lambda(\omega^p,L)}$ associated to $\lambda(\omega^p, L)$, such that $e_{\lambda(\omega^p,L)}(x) > 0$ for all $x \in (-\frac{L}{2}, \frac{L}{2})^d$.*

Proof. For $\varphi, \psi \in L^2((-\frac{L}{2}, \frac{L}{2})^d)$ we write $\psi \geq \varphi$ if $\psi(x) - \varphi(x) \geq 0$ for Lebesgue-almost all x and we write $\psi \gg \varphi$ if $\psi(x) - \varphi(x) > 0$ for Lebesgue-almost all x. By the strong maximum principle of [CFG17, Theorem 5.1] (which easily extends to our setting, see Remark 5.2 of the same paper) we know that for the semigroup $T_t^{\mathfrak{d},L,\omega^p} = e^{t\mathscr{H}_{\mathfrak{d},L}^{\omega^p}}$ of the PAM we have $T_t^{\mathfrak{d},L,\omega^p}\varphi \gg 0$ whenever $\varphi \geq 0$ and $\varphi \neq 0$; we even get $T_t^{\mathfrak{d},L,\omega^p}\varphi(x) > 0$ for all x in the interior $(-\frac{L}{2}, \frac{L}{2})^d$. So by a consequence of the Krein-Rutman theorem, see [Dei85, Theorem 19.3], there exists an eigenfunction $e_{\lambda(\omega^p,L)} \gg 0$. And since $e_{\lambda(\omega^p,L)} = e^{-t\lambda(\omega^p,L)}T_t^{\mathfrak{d},L,\omega^p}e_{\lambda(\omega^p,L)}$, we have $e_{\lambda(\omega^p,L)}(x) > 0$ for all $x \in (-\frac{L}{2}, \frac{L}{2})^d$. □

These results allow us to conclude the following.

Corollary 5.7. *Let $d \leq 2$ and $\lambda > 0$ and let μ_p be an SBM in static random environment, coupled for all $L \in 2\mathbb{N}$ to a killed SBM in static random environment μ_p^L on $[-\frac{L}{2}, \frac{L}{2}]^d$ with $\mu_p^L \leq \mu_p$ (as described in Corollary 5.20). For almost all $\omega^p \in \Omega^p$a there exists an $L_0(\omega^p) > 0$ such that for all $L \geq L_0(\omega^p)$ the killed SBM $\mu_p^L(\omega^p, \cdot)$ satisfies (23). In particular, for almost all $\omega^p \in \Omega^p$ the process $\mu_p(\omega^p, \cdot)$ is super-exponentially persistent.*

Proof. In view of Lemmas 5.4 and 5.5, for almost all $\omega^p \in \Omega^p$ we can choose $L_0(\omega^p)$ such that the largest eigenvalue of the Anderson Hamiltonian $\lambda(\omega^p, L)$ is bigger than λ for all $L \geq L_0(\omega^p)$. Now we fix ω^p such that the above holds true and thus drop the index p (i.e.: we will use a purely deterministic argument). We also fix some $L \geq L_0(\omega^p)$ and write λ_1 instead of $\lambda(\omega^p, L)$ for the largest eigenvalue. Finally, let e_1 be the strictly positive eigenfunction with $\|e_1\|_{L^2((-\frac{L}{2},\frac{L}{2})^d)} = 1$ associated to λ_1. By Proposition 5.19 we find for $0 \leq s < t$:

$$\mathbb{E}[\langle \mu^L(t), e_1\rangle | \mathscr{F}_s] = \langle \mu^L(t), T_{t-s}^{\mathfrak{d}} e_1\rangle = \langle \mu^L(t), e^{(t-s)\lambda_1} e_1\rangle,$$

and thus the process $E(t) = \langle \mu^L, e^{-\lambda_1 t} e_1 \rangle$, $t \geq 0$, is a martingale. Moreover, the variance of this martingale is bounded uniformly in t. Indeed:

$$\mathbb{E}\big[|E(t) - E(0)|^2\big] \simeq \int_0^t \mathrm{d}r\ T_r^{\mathfrak{d}}((e^{-\lambda_1 r} e_1)^2)(0) \lesssim \int_0^t \mathrm{d}r\ e^{-\lambda_1 r} \lesssim 1,$$

where we used that as a consequence of Proposition 3.12 we have $e_1 \in \mathscr{C}^{\vartheta}((-\frac{L}{2}, \frac{L}{2})^d)$ for some admissible $\vartheta > 0$, and therefore

$$T_r^{\mathfrak{d}}((e^{-\lambda_1 r} e_1)^2)(0) \leq \|e_1\|_\infty e^{-\lambda_1 r} T_r^{\mathfrak{d}}(e^{-\lambda_1 r} e_1)(0) = \|e_1\|_\infty e^{-\lambda_1 r} e_1(0) \lesssim e^{-\lambda_1 r}.$$

It follows that $E(t)$ converges almost surely and in L^2 to a random variable $E(\infty) \geq 0$ as $t \to \infty$, and since $\mathbb{E}[E(\infty)] = E(0) = e_1(0) > 0$ we know that $E(\infty)$ is strictly positive with positive probability. For $\varphi \geq 0$ nonzero with support in $[-L/2, L/2]^d$ we get by projecting on the eigenspaces:

$$e^{-\lambda_1 t} \langle \mu^L(t), \varphi \rangle \to \langle e_1, \varphi \rangle X, \quad \text{as } t \to \infty,$$

so that we get from the strict positivity of e_1 and from the fact that $\lambda_1 > \lambda$

$$\mathbb{P}\big(\lim_{t \to \infty} e^{-\lambda t} \langle \mu^L(t), \varphi \rangle = \infty\big) \geq \mathbb{P}(X > 0) > 0.$$

□

Remark 5.8. *The connection of extinction or persistence to the largest eigenvalue of the Hamiltonian in a branching particle system is reminiscent of conditions appearing in the theory of multi-type Galton-Watson processes: See for example* [Har02, Section 2.7]. *The above proof via the martingale argument can be traced back at least to Everett and Ulam, as explained in* [Har51, Theorem 7b].

5.3. **Killed rSBM.** Here we introduce the killed rSBM on a box of size $L \in 2\mathbb{N}$, and we couple it with the "usual" rSBM. The killed rSBM will also be the scaling limit of branching random walks, but now we kill all particles that leave the box. Throughout this section we work under the assumption that $\varrho = d/2$.

In Section 3.2 we considered the PAM on a box with Dirichlet Boundary conditions. Recall that to simplify the calculations, we chose the box to be of the form $[0, L]^d$ first, but later shifted it to $[-L/2, L/2]^d$. Here we will only consider $[-L/2, L/2]^d$ and the associated lattice approximation $\Lambda_n^L = \{x \in \mathbb{Z}_n^d \ :\ x \in [-L/2, L/2]^d\}$.

Let us start by showing that a random environment gives rise to deterministic environments satisfying both Assumption 2.3 and Assumption 3.11.

Lemma 5.9. *Given a random environment $\{\{\overline{\xi}_p^n(x)\}_{x \in \mathbb{Z}_n^d}\}_{n \in \mathbb{N}}$ satisfying Assumption 2.1, there exists a probability space $(\Omega^p, \mathscr{F}^p, \mathbb{P}^p)$ supporting random variables $\{\{\xi_p^n(x)\}_{x \in \mathbb{Z}_n^d}\}_{n \in \mathbb{N}}$ such that for all $n \in \mathbb{N}$ we have $\overline{\xi}^n(\cdot) = \xi^n(\cdot)$ in distribution. In addition, there exists a null set $N \subset \Omega^p$ such that for any $\omega^p \notin N$:*

(1) $\{\{\xi_p^n(\omega^p, x)\}_{x \in \mathbb{Z}_n^d}\}_{n \in \mathbb{N}}$ is a deterministic environment satisfying Assumption 2.3 with $c_n = \kappa_n$ (cf. Equation (4)*) and $\nu = \mathbb{E}[(\Phi)_+]$.*

(2) for all $L \in 2\mathbb{N}$ the restriction $\{\{\xi_p^n(\omega_p, x)\}_{x \in \Lambda_n^L}\}_{n \in \mathbb{N}}$ of the above sequence to Λ_n^L is a deterministic Neumann environment satisfying Assumption 3.11 with $c_n = \kappa_n$ and $\nu = \mathbb{E}[(\Phi)_+]$.

and such that $\xi^n = c_n = \nu = 0$ on the null set N.

Proof. The proof of the first point is already contained in Lemma 2.4. The proof of the second statement follows from Proposition C.1 and Lemma C.2. □

Notation 5.10. *We call a sequence of random variables $\{\xi_p^n\}_{n \in \mathbb{N}}$ which satisfy conditions (1) and (2) of Lemma 5.9 a* controlled random Neumann environment*, and we define the effective potential via:*

$$\xi_e^n(\omega^p, x) = \xi_p^n(\omega^p, x) - c_n(\omega^p) 1_{\{d=2\}}.$$

Let us introduce the discrete space $E^L = \{\eta \in \mathbb{N}_0^{\Lambda_n^L} \ : \ \eta(x) = 0, \forall x \in \partial\Lambda_n^L\}$

Definition 5.11. *Fix an averaging parameter $\varrho \geq d/2$ and a controlled random Neumann environment ξ_p^n. Let u_p^n be the (random) Markov process from Definition 2.6. The process u_p^n corresponds to an (unlabelled) BRWRE, which can be constructed from an underlying labelled BRWRE X_p^n (see Remark A.3 for a rigorous definition).*

We build $X_p^{n,L}$ by killing (i.e. setting to the cemetery state Δ) a particle in X_p^n as soon as it reaches the boundary $\partial\Lambda_n^L$. We then build the process $u_p^{n,L}$ taking values in E^L by disregarding the labels of the process $X_p^{n,L}$. That is, if at time t, $X_p^{n,L}(t)$ has $N(t)$ particles at positions $Y_p^n(t,1), \dots, Y_p^n(t, N(t))$, then:

$$u_p^{n,L}(t,x) = \#\{i \in \{1,\dots,N(t)\} \ : \ Y_p^n(t,i) = x\}.$$

The following result is now easy to verify.

Lemma 5.12. *For any $\omega^p \in \Omega^p$ the process $u_p^{n,L}(\omega^p, \cdot)$ is a Markov process taking values in $\mathbb{D}([0,+\infty); E^L)$, associated to the generator $\mathscr{L}_L^{n,\omega^p} : C_b(E^L) \to C_b(E^L)$ defined via:*

$$\mathscr{L}_L^{n,\omega^p}(F)(\eta) = \sum_{x \in \Lambda_n^L \setminus \partial\Lambda_n^L} \eta_x \cdot \Bigg[\sum_{x \sim y} n^2 (F(\eta^{x \mapsto y}) - F(\eta)) + (\xi_e^n)_+(\omega^p, x)[F(\eta^{x+}) - F(\eta)] + (\xi_e^n)_-(\omega^p, x)[F(\eta^{x-}) - F(\eta)] \Bigg],$$

where for $\eta \in E^L$ we define $\eta^{x \mapsto y}(z) = \eta(z) - 1_{\{z=x\}} + 1_{\{z=y,\ y \notin \partial\Lambda_n^L\}}$ and $\eta^{x\pm}$ is defined in the same way as for $\eta \in E = (\mathbb{N}_0^{\mathbb{Z}_n^d})_0$.

Remark 5.13. *Extending $u_p^{n,L}$ by 0 to $\mathbb{Z}_n^d \setminus \Lambda_n^L$, we find:*

$$u_p^{n,L}(\omega^p, t, x) \leq u_p^n(\omega^p, t, x), \qquad \forall (\omega^p, t, x) \in \Omega^p \times \mathbb{R}_{\geq 0} \times \mathbb{R}^d.$$

Consider now the random measure associated to this process:

$$\mu_p^{n,L}(\omega^p, t)(\varphi) = \sum_{x \in \Lambda_n^L} \lfloor n^{-\varrho} \rfloor u_p^{n,L}(\omega^p, t, x) \varphi(x), \tag{24}$$

and fix one realization of the noise ω^p. As before, we drop all dependence on the index p to underline that we are working with a deterministic (Neumann) environment. We also write $\mathscr{M}([-L/2, L/2]^d)$ for the set of all finite positive measure on $[-L/2, L/2]^d$.

Remark 5.14. *When studying the convergence of the process $\mu^{n,L}$, special care has to be taken with regard to what happens on the boundary of the box. Indeed a function $\varphi \in C^\infty([-L/2, L/2]^d)$ (i.e. smooth in the interior with all derivatives continuous on the entire box) is not smooth in the scale of spaces $B_{p,q}^{\mathfrak{l},\alpha}$ for $\mathfrak{l} \in \{\mathfrak{d}, \mathfrak{n}\}$, since it does not satisfy the required boundary conditions: a priori it only lies in the above space for $\alpha = 0$ and any value of p, q.*

For this reason we consider a weaker kind of convergence for the processes $\mu^{n,L}$ than one might expect. We write

$$\mathscr{M}_0^L = \big(\mathscr{M}((-L/2, L/2)^d), \tau_v\big)$$

of finite positive measures on $(-L/2, L/2)^d$ endowed with the vague topology τ_v (cf. [DP12, Section 3]), i.e. $\mu^n \to \mu$ in $\mathscr{M}_0^L$ if

$$\mu^n(\varphi) \to \mu(\varphi), \qquad \forall \varphi \in X$$

where X can be chosen to be either the space $C_c^\infty((-L/2, L/2)^d)$ or the space $C_0((-L/2, L/2)^d)$ of continuous functions which vanish on the boundary of the box (the latter is a Banach space, when endowed with the uniform norm). The reason why this topology is convenient is that sets of the form $K_R \subset \mathscr{M}_0^L$

$$K_R = \{\mu \in \mathscr{M}_0^L \ : \ \mu(1) \leq R\}$$

are compact. In this setting it is also important to remark the following embedding, which follows from a short calculation.

Remark 5.15. *For $\alpha > 0$ there is a continuous (in the sense of Banach spaces) embedding*

$$\mathscr{C}_{\mathfrak{d}}^{\alpha}([-L/2, L/2]^d) \hookrightarrow C_0((-L/2, L/2)^d).$$

Now we can pass to study the convergence of the killed process.

Lemma 5.16. *We can bound the mass of the killed process locally uniformly in time:*

$$\sup_n \mathbb{E}[\sup_{t\in[0,T]} \mu^{n,L}(t)(1)] < +\infty,$$

as well as the mass of the semigroup:

$$\sup_n \sup_{t\in[0,T]} \|T_t^{n,\mathfrak{d}} 1\|_\infty < +\infty.$$

Proof. The first bound follows from comparison with the process on the whole real line. The second bound follows from the first. The second bound follows from Proposition 3.12 because the antisymmetric extension of 1 is in L^∞: we have $|\Pi_o 1(\cdot)| \equiv 1$. □

Lemma 5.17. *The sequence $\{t \mapsto \mu^{n,L}(t)\}_{n\in\mathbb{N}}$ is tight in the space $\mathbb{D}(\mathbb{R}_{\geq 0}; \mathscr{M}_0^L)$. Any limit point μ^L lies in $C(\mathbb{R}_{\geq 0}; \mathscr{M}_0^L)$.*

Proof. We want to apply Jakubowski's tightness criterion [DP12, Theorem 3.6.4]. The sequence $\mu^{n,L}$ satisfies the compact containment condition in view of Lemma 5.16. The tightness thus follows if we prove that the sequence $\{t \mapsto \mu^n(t)(\varphi)\}_{n\in\mathbb{N}}$ is tight in $\mathbb{D}([0,T];\mathbb{R})$ for any $\varphi \in C_c^\infty((-L/2, L/2)^d)$. Here we follow the calculation of Lemma 4.2, using the results from Section 3.2 on the PAM with Dirichlet boundary conditions. The continuity of the limit points is shown as in Lemma 4.4. □

We will characterize the limit points of $\{\mu^{n,L}\}_{n\in\mathbb{N}}$ in a similar way as the rough super-Brownian motion, and for that purpose we need to solve the following equation:

Lemma 5.18. *For $T > 0$ and $\varphi_0 \in C_c^\infty((-L/2, L/2)^d)$ with $\varphi_0 \geq 0$ and ϑ as in Proposition 3.12, there exists a unique (paracontrolled in $d = 2$) solution $\varphi \in \mathscr{L}_{\mathfrak{d}}^{\vartheta}([-L/2, L/2]^d)$ to*

$$\partial_t \varphi = \mathscr{H}_{\mathfrak{d}}\varphi - \nu\varphi^2, \quad \varphi(0) = \varphi_0, \quad \varphi(t, x) = 0, \ \forall (t, x) \in (0, T] \times \partial[-L/2, L/2]^d, \tag{25}$$

and the following bounds hold:

$$0 \leq \varphi(t) \leq T_t^{\mathfrak{d}}\varphi_0, \qquad \|\varphi\|_{\mathscr{L}_{\mathfrak{d}}^{\vartheta}([-L/2,L/2]^d)} \lesssim e^{C\|\{T_t^{\mathfrak{d}}\varphi_0\}_{t\in[0,T]}\|_{CL^\infty([-L/2,L/2]^d)}}.$$

The proof is analogous to the one of Proposition 4.5, except that here we do not need to consider weights. As in Section 4.1 we thus arrive at the following description of the limit points of $\{\mu^{n,L}\}_{n\in\mathbb{N}}$:

Proposition 5.19. *For any deterministic Neumann environment $\{\xi^n\}_{n\in\mathbb{N}}$ satisfying Assumption 3.11 there exists $\mu^L \in C(\mathbb{R}_{\geq 0}; \mathscr{M}_0^L)$ such that for $\varrho = d/2$ we have $\mu^{n,L} \to \mu^L$ in distribution in $\mathbb{D}(\mathbb{R}_{\geq 0}; \mathscr{M}_0^L)$. The process μ^L is the unique (in law) process in $C(\mathbb{R}_{\geq 0}; \mathscr{M}_0^L)$ which satisfies one (and then all) of the following equivalent properties with $\mathscr{F} = \{\mathscr{F}_t\}_{t\geq 0}$ being the usual augmentation of the filtration generated by μ^L.*

(i) For any $t \geq 0$ and $\varphi_0 \in C_c^\infty((-L/2, L/2)^d), \varphi_0 \geq 0$ and for $U_t^{\mathfrak{d}}\varphi_0$ the solution to Equation (25) *with initial condition φ_0 the process*

$$N_t^{\varphi_0}(s) = e^{-\langle \mu(s), U_{t-s}^{\mathfrak{d}}\varphi_0 \rangle}, \qquad s \in [0, t]$$

is a bounded continuous $\mathscr{F}-$martingale.

(ii) For any $t \geq 0$ and $\varphi_0 \in C_c^\infty((-L/2, L/2)^d)$ and $f \in C([0, t]; C_0((-L/2, L/2)^d))$, and for φ_t solving

$$\partial_s \varphi_t + \mathscr{H}_{\mathfrak{d}}\varphi_t = f, \qquad s \in [0, t], \qquad \varphi_t(t) = \varphi_0$$

it holds that

$$M_t^{\varphi_0, f}(s) := \langle \mu^L(s), \varphi_t(s) \rangle - \langle \delta_0, \varphi_t(0) \rangle - \int_0^s \mathrm{d}r \ \langle \mu^L(r), f(r) \rangle, \qquad s \in [0, t]$$

is a continuous square-integrable $\mathscr{F}-$martingale with quadratic variation

$$\langle M_t^{\varphi_0, f} \rangle_s = 2\nu \int_0^s \mathrm{d}r \ \langle \mu^L(r), (\varphi_t)^2(r) \rangle.$$

(iii) For any $\varphi \in \mathscr{D}_{\mathscr{H}_{\mathfrak{d}}}$ the process:

$$L^\varphi(t) = \langle \mu^L(t), \varphi \rangle - \langle \delta_0, \varphi \rangle - \int_0^t \mathrm{d}r \ \langle \mu^L(r), \mathscr{H}_{\mathfrak{d}}\varphi \rangle, \qquad t \in [0, T]$$

is a continuous $\mathscr{F}-$martingale, square-integrable on $[0, T]$ for all $T > 0$, with quadratic variation

$$\langle L^\varphi \rangle_t = 2\nu \int_0^t \mathrm{d}r \ \langle \mu^L(r), \varphi^2 \rangle.$$

Proof. The proof is almost identical to the one of Theorem 2.13. The main difference is that here we only test against functions with zero boundary conditions and thus use the results from Section 3.2. □

We call the above process the killed rough super-Brownian motion (killed rSBM) on $(-\frac{L}{2}, \frac{L}{2})^d$. Note that we can interpret the killed rSBM as an element of $C(\mathbb{R}_{\geq 0}; \mathscr{M}(\mathbb{R}^d))$ by extending it by zero, i.e. $\mu^L(t, A) = \mu^L(t, A \cap (-L/2, L/2)^d)$ for any measurable $A \subset \mathbb{R}^d$. This allows us to couple infinitely many killed rSBMs with a rSBM on $\mathbb{R}^d$ so that they are ordered in the natural way.

Corollary 5.20. *For any deterministic Neumann environment $\{\xi^n\}_{n \in \mathbb{N}}$ satisfying conclusions (1) and (2) of Lemma 5.9 there exists a process $(\mu, \mu^2, \mu^4, \dots) \in C(\mathbb{R}_{\geq 0}; \mathscr{M}(\mathbb{R}^d))^{\mathbb{N}}$ (equipped with the product topology) such that μ is an rSBM and μ^L is a killed rSBM for all $L \in 2\mathbb{N}$ (all associated to the environment $\{\xi^n\}_{n \in \mathbb{N}}$), and such that almost surely*

$$\mu^2(t, A) \leq \mu^4(t, A) \leq \cdots \leq \mu(t, A) \tag{26}$$

for all $t \geq 0$ and all Borel sets $A \subset \mathbb{R}^d$.

Proof. The construction (24) of μ^n and $\mu^{n,L}$ based on the labelled particle system gives us a coupling $(\mu^n, \mu^{n,2}, \mu^{n,4}, \dots)$ such that almost surely

$$\mu^{n,2}(t, A) \leq \mu^{n,4}(t, A) \leq \cdots \leq \mu^n(t, A)$$

for all $t \geq 0$ and all Borel sets $A \subset \mathbb{R}^d$, where as above we extend $\mu^{n,L}$ to $\mathbb{R}^d$ by setting it to zero outside of $(-\frac{L}{2}, \frac{L}{2})^d$. By Theorem 2.13 respectively Proposition 5.19 we get tightness of the finite-dimensional projections $(\mu^n, \mu^{n,2}, \dots, \mu^{n,L})$ for $L \in 2\mathbb{N}$, and this gives us tightness of the whole sequence in the product topology. Moreover, for any subsequential limit $(\mu, \mu^2, \mu^4, \dots)$ we know that μ is an rSBM and μ^L is a killed rSBM on $(-\frac{L}{2}, \frac{L}{2})^d$.

It is however a little subtle to obtain the ordering (26), because we only showed tightness in the vague topology on $\mathscr{M}_0^L$ for the $\mu^{n,L}$ component. So we introduce suitable cut-off functions to show that the ordering is preserved along any (subsequential) limit: Let $\chi^m \in C_c^\infty((-L/2, L/2)^d)$,

$\chi^m \geq 0$ such that $\chi^m = 1$ on a sequence of compact sets K^m which increase to $(-L/2, L/2)^d$ as $m \to \infty$. Note that on compact sets the sequence $\mu^{n,L}$ converges weakly (and not just vaguely). We then estimate (in view of Remark 5.13) for $\varphi \in C_b(\mathbb{R}^d)$ with $\varphi \geq 0$:

$$\langle \mu^L(t), \varphi \rangle = \lim_{m\to\infty} \langle \mu^L(t), \varphi \cdot \chi^m \rangle = \lim_{m\to\infty} \lim_{n\to\infty} \langle \mu^{n,L}(t), \varphi \cdot \chi^m \rangle \leq \lim_{m\to\infty} \langle \mu(t), \varphi \cdot \chi^m \rangle = \langle \mu(t), \varphi \rangle,$$

and similarly we get $\langle \mu^L(t), \varphi \rangle \leq \langle \mu^{L'}(t), \varphi \rangle$ for $L \leq L'$. Since a signed measure that has a positive integral against every positive continuous function must be positive, our claim follows. □

Appendix A. Construction of the Markov Process

This section is dedicated to a rigorous construction of the BRWRE. For simplicity and without loss of generality we will work with $n = 1$. Since the space $\mathbb{N}_0^{\mathbb{Z}^d}$ is harder to deal with and we do not need it, we consider the countable subspace $E = \big(\mathbb{N}_0^{\mathbb{Z}^d}\big)_0$ of functions $\eta\colon \mathbb{Z}^d \to \mathbb{N}_0$ with $\eta(x) = 0$, except for finitely many $x \in \mathbb{Z}^d$. We endow E with the following distance:

$$d(\eta, \eta') = \sum_{x\in\mathbb{Z}^d} |\eta(x) - \eta'(x)|,$$

under which E is a discrete and hence locally compact separable metric space. Recall that we are given a probability space $(\Omega^p, \mathscr{F}^p, \mathbb{P}^p)$ with a random potential $\xi_p\colon \Omega^p \times \mathbb{Z}^d \to \mathbb{R}$. Recall also that we write $x \sim y$ with $x, y \in \mathbb{Z}^d$, $|x - y| = 1$, and $\eta^{x\mapsto y}(z) = \eta(z) + (1_{\{y\}}(z) - 1_{\{x\}}(z))1_{\{\eta(x)\geq 1\}}$ and $\eta^{x\pm}(z) = (\eta(z) \pm 1_{\{x\}}(z))_+$, as well as $(\cdot)_+ = \max\{0, \cdot\}$ and $(\cdot)_- = \max\{0, -\cdot\}$. Furthermore, let $C_b(E)$ be the Banach space of continuous and bounded functions on E, endowed with the supremum norm.

Lemma A.1. *Assume that for any $\omega^p \in \Omega^p$ the potential $\xi_p(\omega^p)$ is uniformly bounded and consider $\pi \in E$. There exists a unique probability measure $\mathbb{P}_\pi$ on $\Omega = \Omega^p \times \mathbb{D}([0, +\infty); E)$ endowed with the product sigma algebra, such that $\mathbb{P}_\pi$ is of the form $\mathbb{P}^p \ltimes \mathbb{P}_\pi^{\omega^p}$, with $\mathbb{P}_\pi^{\omega^p}$ being the unique measure on $\mathbb{D}([0, +\infty); E)$ under which the canonical process u is a Markov jump process with $u(0) = \pi$ whose generator is given by $\mathscr{L}^{\omega^p}\colon \mathscr{D}(\mathscr{L}^{\omega^p}) \to C_b(E)$:*

$$\begin{aligned}\mathscr{L}^{\omega^p}(F)(\eta) = \sum_{x\in\mathbb{Z}^d} \eta_x \cdot \Big[\sum_{y\sim x} \big(F(\eta^{x\mapsto y}) - F(\eta)\big) \\ + (\xi_p)_+(\omega^p, x)[F(\eta^{x+}) - F(\eta)] + (\xi_p)_-(\omega^p, x)[F(\eta^{x-}) - F(\eta)]\Big],\end{aligned} \tag{27}$$

where the domain $\mathscr{D}(\mathscr{L}^{\omega^p})$ is the set of functions $F \in C_b(E)$ such that the right-hand side of Equation (27) lies in $C_b(E)$.

Proof. First, let us construct the process for fixed $\omega^p \in \Omega^p$. This follows via a classical construction. Indeed, let us consider the modified generator

$$\begin{aligned}\mathscr{L}_b^{\omega^p}(F)(\eta) &= \sum_{x\in\mathbb{Z}^d} \frac{\eta_x}{\sum_{z\in\mathbb{Z}^d} \eta_z (2d + |\xi_p|(\omega^p, z))} \\ &\times \Big[\sum_{y\sim x} \big(F(\eta^{x\mapsto y}) - F(\eta)\big) + (\xi_p)_+(\omega^p, x)[F(\eta^{x+}) - F(\eta)] + (\xi_p)_-(\omega^p, x)[F(\eta^{x-}) - F(\eta)]\Big].\end{aligned}$$

This is the generator associated to a discrete time Markov chain Y on E. We turn this Markov chain to a Markov jump process u as in [EK86, Equation (4.2.3)], with

$$\lambda(\eta) = \sum_{x\in\mathbb{Z}^d} \eta_x (2d + |\xi_p|(\omega^p, x)).$$

In order for this process to be defined for all times we need to verify that

$$\sum_{k\in\mathbb{N}} \frac{1}{\lambda(Y_k)} = +\infty, \text{ a.s..}$$

This is the case, since by assumption ξ_p is bounded and thus

$$\sum_{k\in\mathbb{N}} \frac{1}{\lambda(Y_k)} \gtrsim \sum_{k\in\mathbb{N}} \frac{1}{\sum_x Y_k(x)} \geq \sum_{k\in\mathbb{N}} \frac{1}{c+k} = +\infty$$

with $c = \sum_x \pi(x)$. It follows via classical calculations that $\mathscr{L}^{\omega^p}$ is the generator associated to the process u. This allows us to define for fixed ω^p the unique measure $\kappa(\omega^p, \cdot)$ on $\mathbb{D}([0, +\infty); E)$, which is associated to the law of the process we just described. To define the measure $\mathbb{P}_\pi$ we have to show that κ is a Markov kernel. This amounts to proving measurability in the ω^p coordinate. But κ depends continuously on ξ_p, which we can verify for example by coupling the processes for ξ_p and $\tilde{\xi}_p$ through a construction based on Poisson jumps at rate $K > \|\xi_p\|_\infty, \|\tilde{\xi}_p\|_\infty$ and then rejecting the jumps if an independent uniform $[0, K]$ variable is not in $[0, |\xi_p(x)|]$ respectively in $[0, |\tilde{\xi}_p(x)|]$. Since ξ_p is measurable in ω^p, also κ is measurable in ω^p.

□

In the previous result, we have constructed the random Markov process under the assumption that the random potential is bounded. Now we extend the result to allow sub-polynomial growth.

Lemma A.2. *Assume that for all $\omega^p \in \Omega^p$ the potential $\xi_p(\omega^p)$ lies in $\bigcap_{a>0} L^\infty(\mathbb{Z}^d, p(a))$ and consider $\pi \in E$. There exists a unique probability measure $\mathbb{P}_\pi$ on $\Omega = \Omega^p \times \mathbb{D}([0, +\infty); E)$ endowed with the product sigma algebra, such that $\mathbb{P}_\pi$ is of the form $\mathbb{P}^p \ltimes \mathbb{P}^{\omega^p}_\pi$, with $\mathbb{P}^{\omega^p}_\pi$ being the unique measure on $\mathbb{D}([0, +\infty); E)$ under which the canonical process u is a Markov jump process with $u(0) = \pi$ whose generator is given by $\mathscr{L}^{\omega^p} \colon \mathscr{D}(\mathscr{L}^{\omega^p}) \to C_b(E)$, with $\mathscr{L}^{\omega^p}$ and $\mathscr{D}(\mathscr{L}^{\omega^p})$ defined as in the previous result.*

Proof. Let us fix $\omega^p \in \Omega^p$. Consider the Markov jump processes u^k started in π with generator $\mathscr{L}^{\omega^p,k}$ associated to the uniformly bounded noise $\xi^k_p(x) = (\xi_p(x) \wedge k) \vee (-k)$ whose existence follows from the previous result. The sequence $\{u^k\}_{k\in\mathbb{N}}$ is tight (this follows as in Lemma 4.2 and Corollary 4.3, keeping n fixed but letting k vary) and converges weakly to a Markov process u. Indeed, for $k, R \in \mathbb{N}$ let τ^k_R be the first time with $\mathrm{supp}(u^k(\tau^k_R)) \not\subset Q(R)$, where $Q(R)$ is the square of radius R around the origin, and let τ_R be the corresponding exit time for u. Then we get for all $k, l > \max_{x\in Q(R)} |\xi_p(x)|$, for all $T > 0$, and all $F \in C_b(\mathbb{D}([0, T]; E))$:

$$\mathbb{E}[F((u^k(t))_{t\in[0,T]})1_{\{\tau^k_R\leq T\}}] = \mathbb{E}[F((u^l(t))_{t\in[0,T]})1_{\{\tau^l_R\leq T\}}] = \mathbb{E}[F((u(t))_{t\in[0,T]})1_{\{\tau_R\leq T\}}],$$

where we used that the exit time τ_R is continuous because E is a discrete space. Moreover, from the tightness of $\{u^k\}_{k\in\mathbb{N}}$ it follows that for all $\varepsilon > 0$ and $T > 0$ there exists $R \in \mathbb{N}$ with $\mathbb{P}(\tau^k_R \leq T) < \varepsilon$. This proves the uniqueness in law and that u is the limit (rather than subsequential limit) of $\{u^k\}_{k\in\mathbb{N}}$. Similarly we get the Markov property of u from the Markov property of the $\{u^k\}_{k\in\mathbb{N}}$ and from the convergence of the transition function of u^k as $k \to \infty$.

It remains to verify that $\mathscr{L}^{\omega^p}$ is the generator of u. But for large enough R we have $\mathbb{P}^{\omega^p}_\pi(\tau_R \leq h) = O(h^2)$ as $h \to 0^+$, because on the event $\{\tau_R \leq h\}$ at least two transitions must have happened (recall that π is compactly supported). We can thus compute for any $F \in C_b(E)$:

$$\mathbb{E}^{\omega^p}_\pi\big[F(u(h))\big] = \mathbb{E}^{\omega^p}_\pi\big[F(u^k(h))\big] + O(h^2).$$

The result on the generator then follows from the previous lemma. As before, we now have a constructed a collection of probability measures $\kappa(\omega^p, \cdot)$ as the limit of the Markov kernels $\kappa^k(\omega^p, \cdot)$. Since measurability is preserved when passing to the limit, we can again construct the measure $\mathbb{P}_\pi$ on the whole space Ω.

□

We introduced the BRWRE without keeping track of the individual particles (all particles are identical and only their position matters). Sometimes it is also useful to consider a labelled process, which distinguishes individual particles and lives in a much larger (although still countable) space and which can be constructed using similar arguments to the unlabelled case. We thus introduce the space $E_{\mathrm{lab}} = \bigsqcup_{m\in\mathbb{N}}(\mathbb{Z}^d \cup \{\Delta\})^m$, where $\bigsqcup$ denotes the disjoint union, endowed with the discrete topology. Here Δ is a cemetery state. Moreover, for $\eta \in E_{\mathrm{lab}}$ we write $\dim(\eta) = m$ if $\eta \in (\mathbb{Z}^d \cup \{\Delta\})^m$.

Remark A.3. *Assume that for all $\omega^p \in \Omega^p$ the potential $\xi_p(\omega^p)$ lies in $\bigcap_{a>0} L^\infty(\mathbb{Z}^d, p(a))$ and consider an initial condition $X(0) \in E_{\mathrm{lab}}$. We can construct a (random) Markov jump process X on E_{lab} via the following generator:*

$$\mathscr{L}^{\omega^p}_{\mathrm{lab}}(F)(\eta) = \sum_{i=1}^{\dim(\eta)} 1_{\{\mathbb{Z}^d\}}(\eta_i)\Big[\sum_{y\sim\eta_i}\big(F(\eta^{i\mapsto y})-F(\eta)\big) \\ + (\xi^n)_+(\omega^p,\eta_i)\big(F(\eta^{i,+})-F(\eta)\big)+(\xi^n)_-(\omega^p,\eta_i)\big(F(\eta^{i,-})-F(\eta)\big)\Big],$$

where $\eta^{i\mapsto y}_j = \eta_j(1-1_{\{i\}}(j))+y1_{\{i\}}(j)$ and $\eta^{i,+}_j = \eta_j 1_{[0,\dim(\eta)]}(j)+\eta_i 1_{\{\dim(\eta)+1\}}(j)$ as well as $\eta^{i,-}_j = \eta_j(1-1_{\{i\}}(j))+\Delta 1_{\{i\}}(j)$, and where F is such that the right hand-side is bounded. We then find that the process

$$u(t,x) = \#\{i \in \{i,\dots,\dim(X(t))\} \;:\; X_i(t) = x\}$$

has the same law $\mathbb{P}^{\omega^p}_\pi$ as the process introduced in Lemma A.2, with $\pi(x) = u(0,x)$.

Appendix B. Some Estimates for the Random Noise

In this section we prove parts of Lemma 2.4, i.e. that a random environemnt satisfying Assumption 2.1 gives rise to a deterministic environment satisfying Assumption 2.3.

Lemma B.1. *Let $a,\varepsilon,q > 0$ and $b > d/2$. Under Assumption 2.1 we can bound*

$$\sup_n \Big[\mathbb{E}\|n^{-d/2}(\xi^n_p)_+\|^q_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}^d_n,p(a))} + \mathbb{E}\|n^{-d/2}(\xi^n_p)_+\|^2_{L^2(\mathbb{Z}^d_n,p(b))}\Big] < +\infty,$$

and the same holds if we replace $(\xi^n_p)_+$ with $|\xi^n_p|$. Furthermore, for $\nu = \mathbb{E}[\Phi_+]$, the following convergences hold true in distribution in $\mathscr{C}^{-\varepsilon}(\mathbb{R}^d, p(a))$:

$$\mathscr{E}^n n^{-d/2}(\xi^n_p)_+ \longrightarrow \nu, \qquad \mathscr{E}^n n^{-d/2}|\xi^n_p| \longrightarrow 2\nu.$$

Proof. We prove the result only for $(\xi^n_p)_+$, since then we can treat $(\xi^n_p)_-$ by considering $-\xi^n_p$ (note that $-\Phi$ is still a centered distribution). Start by noting that

$$\mathbb{E}[\|n^{-d/2}(\xi^n_p)_+\|^q_{L^q(\mathbb{Z}^d_n,p(a))}] = \sum_{x\in\mathbb{Z}^d_n} n^{-d}\mathbb{E}[|n^{-d/2}(\xi^n_p)_+|^q]|p(a)(x)|^q \lesssim \mathbb{E}[|\Phi|^q]\int_{\mathbb{R}^d}(1+|y|)^{-aq}\,\mathrm{d}y,$$

which is finite whenever $aq > d$. From here the uniform bound on the expectations follows by Besov embedding.

Now we pass to the convergence result. The uniform bound guarantees tightness of the sequence $\mathscr{E}^n n^{-d/2}(\xi^n_p)_+$ in $\mathscr{C}^{-\varepsilon}(\mathbb{R}^d, p(a))$, for any $\varepsilon, a > 0$ and we are left with proving that the weak limit is ν. Using the spatial independence of ξ^n_p we can compute for any $\varphi \in \mathscr{S}_\omega$ with $\mathrm{supp}(\mathscr{F}_{\mathbb{R}^d}\varphi) \subset n\mathbb{T}^d$:

$$\mathbb{E}\big[\langle\mathscr{E}^n n^{-d/2}(\xi^n_p)_+ - \nu, \varphi\rangle^2\big] = \mathbb{E}\Big[\Big(\sum_{x\in\mathbb{Z}^d_n}\varphi(x)n^{-d}(n^{-d/2}(\xi^n_p(x))_+ - \nu)\Big)^2\Big] \\ \simeq n^{-2d}\sum_{x\in\mathbb{Z}^d_n}\varphi(x)^2 = O(n^{-d}).$$

This proves that ν is indeed the limit. □

The following result is a simpler variant of [MP17, Lemma 5.5] for the case $d=1$.

Lemma B.2. *Fix ξ^n satisfying Assumption 2.1, $d=1$, $a,q>0$ and $\alpha<2-d/2$. We have:*

$$\sup_n \mathbb{E}\big[\|\xi_p^n\|^q_{\mathscr{C}^{\alpha-2}(\mathbb{Z}_n^d,p(a))}\big]<+\infty, \qquad \mathscr{E}^n\xi_p^n\to\xi_p,$$

where ξ_p is a white noise on $\mathbb{R}$ and the convergence holds in distribution in $\mathscr{C}^{\alpha-2}(\mathbb{R}^d,p(a))$.

Proof. The uniform bound follows along the lines of [MP17, Lemma 5.5]. This guarantees the tightness of the sequence. Convergence follows by an application of the central limit theorem. □

Appendix C. Stochastic Estimates under Dirichlet Bundary Conditions

The following bounds are essentially an adaptation of [CGP17, Section 4.2] to the Dirichlet boundary condition setting (see also [CvZ19] for the spatially continuous setting). Our aim is to prove the following proposition. Fix a box of size $L\in\mathbb{N}$ and assume the box is given by $[0,L]^d$ (the same results hold for any integer translation of this box, mutatis mutandis). Recall the notation and constructions from Section 3.2 as well as the definition of κ_n from Equation (4).

Proposition C.1. *Fix a sequence ξ_p^n satisfying Assumption 2.1 and consider the restriction $\{\{\xi_p^n(x)\}_{x\in\Lambda_n}\}_{n\in\mathbb{N}}$. The following bounds and convergences (all of which are to be understood in distribution) hold true for any $\alpha<2-d/2$:*

- *Let ξ_p be space white noise on $[0,L]^d$, then:*
$$\sup_n \mathbb{E}[\|\xi_p^n\|^q_{\mathscr{C}_{\mathfrak{n}}^{\alpha-2}(\Lambda_n)}]<+\infty, \qquad \mathscr{E}^n_{\mathfrak{n}}\xi_p^n\to\xi_p \text{ in } \mathscr{C}_{\mathfrak{n}}^{\alpha-2}([0,L]^d).$$
- *In dimension $d=2$, for $X_{\mathfrak{n}}^n=\frac{\chi(D)}{l^n(D)}\xi_p^n$ and $X_{\mathfrak{n}}=\Delta_{\mathfrak{n}}^{-1}\chi(D)\xi_p$ (with the same χ as in Equation (4)) we have:*
$$\sup_n \mathbb{E}[\|X_{\mathfrak{n}}^n\|_{\mathscr{C}_{\mathfrak{n}}^{\alpha}(\Lambda_n)}+\|(X_{\mathfrak{n}}^n\odot\xi_p^n)-\kappa_n\|_{\mathscr{C}_{\mathfrak{n}}^{2\alpha-2}(\Lambda_n)}]<+\infty$$
and there exists a random distribution $X_{\mathfrak{n}}\diamond\xi_p\in\mathscr{C}_{\mathfrak{n}}^{2\alpha-2}([0,L]^d)$ such that
$$\mathscr{E}^n_{\mathfrak{n}}X_{\mathfrak{n}}^n\to X_{\mathfrak{n}} \text{ in } \mathscr{C}_{\mathfrak{n}}^{\alpha}([0,L]^d), \qquad \mathscr{E}^n_{\mathfrak{n}}(X_{\mathfrak{n}}^n\odot\xi_p^n-\kappa_n)\to X_{\mathfrak{n}}\diamond\xi_p \text{ in } \mathscr{C}_{\mathfrak{n}}^{2\alpha-2}([0,L]^d).$$

Proof. Step 1. First, we establish all uniform bounds. We will derive only bounds in spaces of the kind $B^{\mathfrak{n},\beta}_{p,p}(\Lambda_n)$ for appropriate β and any p sufficiently large. The results on the Hölder scale then follow by Besov embedding. Also, in order to avoid confusion and for clarity, we will omit the subindices $p,\mathfrak{n}$ in the noise terms.

Recall that with $N=2L$ we have $\Theta_n=\frac{1}{n}(\mathbb{Z}^d\cap[-\frac{Nn}{2},\frac{Nn}{2}]^d)/\!\sim$ with opposite boundaries identified (resp. $\mathbb{T}^d_N$ if $n=\infty$) and $\Xi_n=\frac{1}{N}(\mathbb{Z}^d\cap[-\frac{Nn}{2},\frac{Nn}{2}]^d)/\!\sim$, (resp. $\frac{1}{N}\mathbb{Z}^d$ if $n=\infty$) as well as $\Xi_n^+=\frac{1}{N}(\mathbb{Z}^d\cap[0,Ln]^d)$, (resp. $\frac{1}{N}\mathbb{N}_0^d$). We write sums as discrete integrals against scaled measures with the following definitions:

$$\int_{\Theta_n}\mathrm{d}x\ f(x)=\sum_{x\in\Theta_n}\frac{f(x)}{n^d},\quad \int_{\Xi_n}\mathrm{d}k\ f(k)=\sum_{k\in\Xi_n}\frac{f(k)}{N^d},\quad \int_{\{-1,1\}^d}\mathrm{d}\mathfrak{q}\ f(\mathfrak{q})=\sum_{\mathfrak{q}\in\{-1,1\}^d}f(\mathfrak{q}).$$

For $k_1,k_2\in\Xi_n$ and $\mathfrak{q}_1,\mathfrak{q}_2\in\{-1,1\}^d$ we moreover adopt the notation: $k_{[12]}=k_1+k_2,\mathfrak{q}_{[12]}=\mathfrak{q}_1+\mathfrak{q}_2$ and $(\mathfrak{q}\circ k)_{[12]}=\mathfrak{q}_1\circ k_1+\mathfrak{q}_2\circ k_2$. Let us start with the first estimate. We have:

$$\mathbb{E}[\|\xi^n\|^p_{B^{\alpha-2,\mathfrak{n}}_{p,p}(\Lambda_n)}]=\sum_{-1\le j\le j_n}2^{(\alpha-2)jp}\int_{\Theta_n}\mathrm{d}x\ \mathbb{E}[|\Delta_j\Pi_e\xi^n|^p(x)]$$

Since the integral over Θ_n is bounded, it suffices to derive a uniform bound for $\mathbb{E}[|\Delta_j \Pi_e \xi^n|^p(x)]$ in n, x. Let us thus rewrite:

$$\begin{aligned}\Delta_j \Pi_e \xi^n(x) &= \int_{\Xi_n} \mathrm{d}k \; e^{2\pi\iota\langle x,k\rangle} \varrho_j(k) \mathscr{F}_{\Theta_n} \Pi_e \xi^n(k) \\ &= \int_{\Xi_n^+} \mathrm{d}k \; \int_{\{-1,1\}^d} \mathrm{d}\mathfrak{q} \; \nu_k N^{\frac{d}{2}} e^{2\pi\iota\langle x, \mathfrak{q}\circ k\rangle} \varrho_j(k) \langle \xi^n, \mathfrak{n}_k\rangle = \Pi_e \Delta_j \xi^n(x),\end{aligned}$$

where we have used that $\mathscr{F}_{\Theta_n}\varphi(k) = N^{\frac{d}{2}}\langle \varphi, \mathfrak{e}_k\rangle$. By enumerating Ξ_n^+ we can consider the integral as a sum of martingale increments. Indeed, a simple calculation shows that the $\{\langle \xi^n, \mathfrak{n}_k\rangle\}_k$ are independent centered normal random variables with variance 1. Hence we can estimate for $p \geq 2$ with the discrete time Burkholder-Davis-Gundy inequality and Minkowski's inequality:

$$\mathbb{E}[|\Delta_j \Pi_e \xi^n(x)|^p] \lesssim \mathbb{E}\left[\left|\int_{\Xi_n^+} \mathrm{d}k \; \varrho_j^2(k) \langle \xi^n, \mathfrak{n}_k\rangle^2\right|^{\frac{p}{2}}\right] \lesssim \left(\int_{\Xi_n^+} \mathrm{d}k \; \varrho_j^2(k) \mathbb{E}[\langle \xi^n, \mathfrak{n}_k\rangle^p]^{\frac{2}{p}}\right)^{\frac{p}{2}} \lesssim 2^{jdp/2}.$$

This provides an estimate on the regularity of the required order. The bound for X^n follows along the same lines.

Let us pass to a bound for the resonant product. Here we first compute:

$$\begin{aligned}&\Delta_j \Pi_e(\xi^n \odot X^n)(x) \\ &\quad= \int_{(\{-1,1\}^d \times \Xi_n^+)^2} \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k_{12} \; N^d \nu_{k_1}\nu_{k_2} e^{2\pi\iota\langle x, (\mathfrak{q}\circ k)_{[12]}\rangle} \cdot \\ &\qquad\qquad \cdot \varrho_j((\mathfrak{q}\circ k)_{[12]}) \psi_0(k_1,k_2) \frac{\chi(k_2)}{l^n(k_2)} \langle \xi^n, \mathfrak{n}_{k_1}\rangle \langle \xi^n, \mathfrak{n}_{k_2}\rangle \\ &\quad= \int_{(\{-1,1\}^d \times \Xi_n^+)^2} \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k_{12} \; 1_{\{k_1\neq k_2\}N^d\nu_{k_1}\nu_{k_2}} e^{2\pi\iota\langle x, (\mathfrak{q}\circ k)_{[12]}\rangle} \cdot \\ &\qquad\qquad \cdot \varrho_j((\mathfrak{q}\circ k)_{[12]}) \psi_0(k_1,k_2) \frac{\chi(k_2)}{l^n(k_2)} \langle \xi^n, \mathfrak{n}_{k_1}\rangle \langle \xi^n, \mathfrak{n}_{k_2}\rangle + \mathrm{Diag}\end{aligned}$$

where Diag indicates the integral over the set $\{k_1 = k_2\}$. The first term can be bounded by generalizing the martingale inequality argument we used for $\Delta_j \Pi_e \xi^n(x)$ to multiple discrete stochastic integrals, see [CGP17, Proposition 4.3]. We can thus bound for arbitrary $\ell \in \mathbb{N}$:

$$\begin{aligned}&\mathbb{E}[|\Delta_j(\Pi_e(\xi^n \odot X^n)(x) - \kappa_n)|^p] \\ &\lesssim \left[\int \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k_{12} \left|\varrho_j((\mathfrak{q}\circ k)_{[12]}) \psi_0(k_1,k_2) \frac{\chi(k_2)}{l^n(k_2)}\right|^2\right]^{\frac{p}{2}} \mathbb{E}[\langle \xi^n, \mathfrak{n}_\ell\rangle^p]^2 + \mathbb{E}[|\,\mathrm{Diag} - 1_{\{j=-1\}}\kappa_n|^p].\end{aligned}$$

For the first term on the right hand side we have:

$$\begin{aligned}&\int_{(\{-1,1\}^d \times \Xi_n^+)^2} \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k_{12} \left|\varrho_j((\mathfrak{q}\circ k)_{[12]}) \psi_0(k_1,k_2) \frac{\chi(k_2)}{l^n(k_2)}\right|^2 \\ &\quad= \int_{\Xi_n^2} \mathrm{d}k_{12} \left|\varrho_j(k_{[12]}) \psi_0(k_1,k_2) \frac{\chi(k_2)}{l^n(k_2)}\right|^2 \\ &\quad\lesssim \sum_{i \geq j-\ell} \int_{\Xi_n^2} \mathrm{d}k_{12} \; 1_{\{|k_1+k_2|\sim 2^j\}} 1_{\{|k_2|\sim 2^i\}} 2^{-4i} \lesssim \sum_{i\geq j-\ell} 2^{jd} 2^{i(d-4)} \lesssim 2^{2j(d-2)},\end{aligned}$$

which is of the required order (and we used that $d<4$). Let us pass to the diagonal term. We first smuggle in the expectation of Diag:

$$\mathbb{E}[|\operatorname{Diag}-\mathbb{E}[\operatorname{Diag}]|^p]=\mathbb{E}\left[\left|\int_{\Xi_n^+\times(\{-1,1\}^d)^2}\mathrm{d}\mathfrak{q}_{12}\,\mathrm{d}k\;\nu_k^2 e^{2\pi\iota\langle x,\mathfrak{q}_{[12]}\circ k\rangle}\varrho_j(\mathfrak{q}_{[12]}\circ k)\frac{\chi(k)}{l^n(k)}\eta(k)\right|^p\right],$$

where we have lost the factor N^d due to the normalization of the integral in k and $\eta(k)=\langle\xi^n,\mathfrak{n}_k\rangle^2-\mathbb{E}[\langle\xi^n,\mathfrak{n}_k\rangle^2]=\langle\xi^n,\mathfrak{n}_k\rangle^2-1$ is sequence of centered i.i.d random variables. Therefore, we can use the same martingale argument as above to bound the integral by:

$$\begin{aligned}\mathbb{E}[|\operatorname{Diag}-\mathbb{E}[\operatorname{Diag}]|^p]&\lesssim\left(\int_{\Xi_n^+}\mathrm{d}k\left|\int_{(\{-1,1\}^d)^2}\mathrm{d}\mathfrak{q}_{12}\;\varrho_j(\mathfrak{q}_{[12]}\circ k)\right|^2\left|\frac{\chi(k)}{l^n(k)}\right|^2\mathbb{E}[|\eta(k)|^p]^{\frac{2}{p}}\right)^{\frac{p}{2}}\\&\lesssim\left(\int_{x\in\mathbb{R}^d:|x|\gtrsim 2^j}\frac{1}{|x|^4}\,\mathrm{d}x\right)^{p/2}\lesssim 2^{j(d-4)}=2^{j(\frac{d}{2}-2)}\end{aligned}$$

whenever $d<4$, which is even better than the bound for the off-diagonal terms. We are hence left with a last, deterministic term:

$$\int_{\Xi_n^+\times(\{-1,1\}^d)^2}\mathrm{d}\mathfrak{q}_{12}\,\mathrm{d}k\;\nu_k^2 e^{2\pi\iota\langle x,\mathfrak{q}_{[12]}\circ k\rangle}\varrho_j(\mathfrak{q}_{[12]}\circ k)\frac{\chi(k)}{l^n(k)}\quad-\quad 1_{\{j=-1\}}\kappa_n.$$

We split up this sum in different terms according to the relative value of $\mathfrak{q}_1,\mathfrak{q}_2$. If $\mathfrak{q}_1=-\mathfrak{q}_2$ (there are 2^d such terms) the sum does not depend on x and it disappears for $j\geq 0$. Let us assume $j=-1$: We are then left with the constant:

$$2^d\int_{\Xi_n^+}\mathrm{d}k\;\nu_k^2\frac{\chi(k)}{l^n(k)}-\kappa_n=\int_{\Xi_n}\mathrm{d}k\;\frac{\chi(k)}{l^n(k)}-k_n$$

Note that the sum on the left-hand side diverges logarithmically in n and we now show how to renormalize with κ_n. Let us recall the definition of κ_n and to clarify our computation let us also introduce an auxiliary constant $\bar{\kappa}_n$ (it differs fron the previous integrand only on the boundary):

$$\kappa_n=\int_{\mathbb{T}_n^d}\mathrm{d}k\;\frac{\chi(k)}{l^n(k)},\qquad\bar{\kappa}_n=\int_{\Xi_n}\mathrm{d}k\;\bar{\nu}_k^2\frac{\chi(k)}{l^n(k)},$$

where $\bar{\nu}_k=2^{-\#\{i:\,k_i=\pm n\}/2}$. For $x\in\mathbb{R}^d,r\geq 0$, let us indicate with $Q_r^n(x)\subseteq\mathbb{T}_n^d$ the box $Q_r^n(x)=\{y\in\mathbb{T}_n^d\colon|y-x|_\infty\leq r/2\}$ ($|\cdot|_\infty$ being the maximum of the component-wise distances in $\mathbb{T}_n^d$). Then note that we can bound uniformly over n and N:

$$\begin{aligned}|\kappa_n-\bar{\kappa}_n|&=\left|\int_{\mathbb{T}_n^d}\mathrm{d}k\;\frac{\chi(k)}{l^n(k)}-\int_{\Xi_n}\mathrm{d}k\;\bar{\nu}_k^2\frac{\chi(k)}{l^n(k)}\right|=\left|\sum_{k\in\Xi_n}\int_{Q^n_{\frac{1}{N}}(k)}\mathrm{d}k'\;\frac{\chi(k+k')}{l^n(k+k')}-\frac{\chi(k)}{l^n(k)}\right|\\&\lesssim\frac{1}{N}\left(1+\frac{1}{N^d}\sum_{k\in\Xi_n}\sup_{\vartheta\in Q_{\frac{1}{N}}(k)}\frac{\chi(k)}{(l^n(\vartheta))^2}|\nabla l^n(\vartheta)|\right)\lesssim\frac{1}{N}\left(1+\frac{1}{N^d}\sum_{k\in\frac{1}{N}\mathbb{Z}^d}\frac{\chi(k)}{|k|^3}\right)\lesssim\frac{1}{N},\end{aligned}$$

where we have used that $d=2$, $|l^n(\vartheta)|\gtrsim|\vartheta|^2$ on $[-n/2,n/2]^d$ as well as $|\nabla l^n(\vartheta)|\lesssim|\theta|$ on $[-n/2,n/2]^d$. Similar calculations show that the difference converges: $\lim_{n\to\infty}\kappa_n-\bar{\kappa}_n\in\mathbb{R}$. We are now able to estimate:

$$\left|\int_{\Xi_n}\mathrm{d}k\;\frac{\chi(k)}{l^n(k)}-\kappa_n\right|\lesssim 1+|\bar{\kappa}_n-\kappa_n|\lesssim 1$$

where we used that the sum on the boundary $\partial\Xi_n$ converges to zero and is thus uniformly bounded in n. For the same reason, the above difference converges to the limit $\lim_{n\to\infty}\bar{\kappa}_n-\kappa_n\in\mathbb{R}$.

For all other possibilities of $\mathfrak{q}_1,\mathfrak{q}_2$ we will show convergence in a distributional sense. If $\mathfrak{q}_1=\mathfrak{q}_2$ we have:

$$\left|\int_{\Xi_n^+}\mathrm{d}k\;\nu_k^2 e^{2\pi\iota\langle x,2\mathfrak{q}_1 k\rangle}\varrho_j(2k)\frac{\chi(k)}{l^n(k)}\right|\lesssim 2^{j(d-2)}.$$

Finally, if only one of the two components of $\mathfrak{q}_1, \mathfrak{q}_2$ differs (let us suppose it is the first one) we find (with $x = (x_1, x_2)$ and $k = (k_1, k_2)$):

$$\left| \int_{\Xi_n^+} \mathrm{d}k \; \nu_k^2 e^{2\pi\iota 2x_2k_2} \varrho_j(2k_2) \frac{\chi(k)}{l^n(k)} \right| \lesssim \left(\sum_{k_1 \geq 1} \frac{1}{|k_1|^{2\theta}} \right) \left(\sum_{k_2 \geq 1} \frac{\varrho_j(2k_2)}{|k_2|^{2(1-\theta)}} \right) \lesssim 2^{j\varepsilon}$$

for any $\varepsilon > 0$, up to choosing $\theta \in (1/2, 1)$ sufficiently close to $1/2$.

Step 2. Now we briefly address the convergence in distribution. Clearly the previous calculations and compact embeddings of Hölder-Besov spaces guarantee tightness of the sequences $\xi_p^n, X_{\mathfrak{n}}^n$ and $X_{\mathfrak{n}}^n \odot \xi_p^n - \kappa_n$ in the respective spaces for any $\alpha < 2 - d/2$. We have to uniquely identify the distribution of any limit point. For $\xi_p, X_{\mathfrak{n}}^n$ the limit points are Gaussian and uniquely identified as white noise ξ_p and $\Delta_{\mathfrak{n}}^{-1} \chi(D) \xi_p$ respectively. The resonant product requires more care, but we can use the same arguments as in [MP17, Section 5.1]. □

Lemma C.2. *In the same setting as above, we have that for any $\varepsilon > 0$ and $\nu = \mathbb{E}[(\Phi)_+]$:*

$$\sup_n \mathbb{E}[\| n^{-d/2} (\xi_p^n)_+ \|_{\mathscr{C}_{\mathfrak{n}}^{-\varepsilon}(\Lambda_n)} + \| n^{-d/2} (\xi_p^n)_+ \|_{L^2(\Lambda_n)}] < +\infty,$$

as well as:

$$\mathscr{E}_{\mathfrak{n}}^n n^{-d/2} (\xi^n)_+ \to \nu \text{ in } \mathscr{C}_{\mathfrak{n}}^{-\varepsilon}([0, L]^d).$$

Proof. This result is analogous to Lemma B.1 □

Appendix D. Moment Estimates

Here we derive uniform bounds for the moments of the processes $\{\mu^n\}_{n \in \mathbb{N}}$. As a convention, in the following we will write $\mathbb{E}$ and $\mathbb{P}$ for the expectation and the probability under the distribution of u^n. For different initial conditions $\eta \in E$ we will write $\mathbb{E}_\eta, \mathbb{P}_\eta$.

Lemma D.1. *Let for all $n \in \mathbb{N}$ the process $\{\mu^n(t)\}_{t \geq 0}$ be as in Definition 2.6 and consider $\varphi^n \colon \mathbb{Z}_n^d \to \mathbb{R}$ with $\varphi^n \geq 0$ and $q, T > 0$. If for all $n \in \mathbb{N}$ we have $\varphi^n = \varphi|_{\mathbb{Z}_n^d}$ with $\varphi \in \mathscr{C}^2(\mathbb{R}^d, e(l))$ for some $l \in \mathbb{R}$, then*

$$\sup_n \sup_{t \in [0,T]} \mathbb{E}\big[|\mu^n(t)(\varphi^n)|^q \big] < +\infty.$$

If for all $\varepsilon > 0$ there exists an $l \in \mathbb{R}$ such that $\sup_n \|\varphi^n\|_{\mathscr{C}^{-\varepsilon}(\mathbb{R}^d, e(l))} < +\infty$, we can bound for any $\gamma \in (0,1)$:

$$\sup_n \sup_{t \in [0,T]} t^\gamma \mathbb{E}\big[|\mu^n(t)(\varphi^n)|^q \big] < +\infty.$$

Proof. We prove the second estimate (for uniformly bounded $\|\varphi^n\|_{\mathscr{C}^{-\varepsilon}(\mathbb{R}^d, e(l))}$), since the first estimate is similar but easier (Lemma E.1 below controls $\|\varphi^n\|_{\mathscr{C}^\vartheta(\mathbb{Z}_n^d, e(l))}$ for all $\vartheta < 2$ in that case). Also, we assume without loss of generality that $q \geq 2$. As usual, we use the convention of freely increasing the value of l in the exponential weight. Let us start by recalling that $\mathbb{E}\big[\mu^n(t)(\varphi^n)\big] = T_t^n \varphi^n(0)$. Moreover, via the assumption on the regularity, Proposition 3.1 guarantees that for any $\gamma \in (0,1)$ there exists $\varepsilon = \varepsilon(\gamma, q) > 0$ such that

$$\sup_n \| t \mapsto T_t^n \varphi^n \|_{\mathscr{L}^{\gamma/q, \varepsilon}(\mathbb{Z}_n^d, e(l))} < +\infty.$$

By the triangle inequality it thus suffices to prove that for any $\gamma > 0$:

$$\sup_n \sup_{t \in [0,T]} t^\gamma \mathbb{E}\big[|\mu^n(t)(\varphi^n) - T_t^n \varphi^n(0)|^q \big] < +\infty.$$

Note that we can interpret the particle system u^n as the superposition of $\lfloor n^\varrho \rfloor$ independent particle systems, each started with one particle in zero; we write $u^n = u_1^n + \cdots + u_{\lfloor n^\varrho \rfloor}^n$. To lighten the notation we assume that $n^\varrho \in \mathbb{N}$. We then apply Rosenthal's inequality, [Pet95, Theorem 2.9] (recall that $q \geq 2$) and obtain

$$\begin{aligned}
&\mathbb{E}\big[|\mu^n(t)(\varphi^n) - T^n_t \varphi^n(0)|^q\big] \\
&\quad = \mathbb{E}\Big[\Big|\sum_{k=1}^{n^\varrho} \big[n^{-\varrho}(u^n_k(t),\varphi^n) - n^{-\varrho} T^n_t\varphi^n(0)\big]\Big|^q\Big] \\
&\quad \lesssim n^{-\varrho q}\sum_{k=1}^{n^\varrho} \mathbb{E}\big[|(u^n_k(t),\varphi^n) - T^n_t\varphi^n(0)|^q\big] + n^{-\varrho q}\Big(\sum_{k=1}^{n^\varrho}\mathbb{E}\big[|(u^n_k(t),\varphi^n) - T^n_t\varphi^n(0)|^2\big]\Big)^{\frac{q}{2}} \\
&\quad \lesssim n^{-\varrho(q-1)}\mathbb{E}\big[|(u^n_1(t),\varphi^n)|^q\big] + \big(n^{-\varrho}\mathbb{E}\big[|(u^n_1(t),\varphi^n)|^2\big]\big)^{q/2} + n^{-\frac{\varrho q}{2}} t^\gamma \|t \mapsto T^n_t\varphi^n\|^q_{\mathscr{L}^{\gamma/q,\varepsilon}(\mathbb{Z}^d_n, e(l))}
\end{aligned}$$

for the same $\varepsilon > 0$ and $l \in \mathbb{R}$ as above. The two scaled expectations are of the same form, in the second term we simply have $q = 2$. To control them, we define for $p \in \mathbb{N}$ the map

$$m^{p,n}_{\varphi^n}(t,x) = n^{\varrho(1-p)}\mathbb{E}_{1_{\{x\}}}\big[|(u^n_1(t),\varphi^n)|^p\big].$$

As a consequence of Kolmogorov's backward equation each $m^{p,n}_{\varphi^n}$ solves the discrete PDE (see also Equation (2.4) in [ABMY00]):

$$\partial_t m^{p,n}_{\varphi^n}(t,x) = \mathscr{H}^n m^{p,n}_{\varphi^n}(t,x) + n^{-\varrho}(\xi^n_e)_+(x)\sum_{i=1}^{p-1}\binom{p}{i} m^{i,n}_{\varphi^n}(t,x) m^{p-i,n}_{\varphi^n}(t,x),$$

with initial condition $m^{p,n}_{\varphi^n}(0,x) = n^{\varrho(1-p)}|\varphi^n(x)|^p$. We claim that this equation has a unique (paracontrolled in $d = 2$) solution $m^{p,n}_{\varphi^n}$, such that for all $\gamma > 0$ there exists $\varepsilon = \varepsilon(\gamma,p) > 0$ with $\sup_n \|m^{n,p}_{\varphi^n}\|_{\mathscr{L}^{\gamma,\varepsilon}(\mathbb{Z}^d_n, e(l))} < \infty$. Once this is shown, the proof is complete. We proceed by induction over p. For $p = 1$ we simply have $m^{n,1}_{\varphi^n}(t,x) = T^n_t\varphi^n(x)$. For $p \geq 2$ we use that by Lemma E.2 we have $\|n^{\varrho(1-p)}|\varphi^n(x)|^p\|_{\mathscr{C}^\kappa(\mathbb{Z}^d_n, e(l))} \to 0$ for some $\kappa > 0$ and we assume that the induction hypothesis holds for all $p' < p$. Since it suffices to prove the bound for small $\gamma > 0$, we may assume also that $\kappa > \gamma$. We choose then $\gamma' < \gamma$ such that for some $\varepsilon(\gamma', p) > 0$:

$$\sup_n \Big\|\sum_{i=1}^{p-1} m^{i,n}_{\varphi^n} m^{p-i,n}_{\varphi^n}\Big\|_{\mathscr{M}^{\gamma'}\mathscr{C}^{\varepsilon(\gamma',p)}(\mathbb{Z}^d_n, e(l))} < +\infty.$$

Since by Assumption 2.3 $\|n^{-\varrho}(\xi^n_e)_+\|_{\mathscr{C}^{-\varepsilon}(\mathbb{Z}^d_n, p(a))}$ is uniformly bounded in n for all $\varepsilon, a > 0$, the above bound is sufficient to control the product:

$$\sup_n \Big\|n^{-\varrho}(\xi^n_e)_+ \sum_{i=1}^{p-1} m^{i,n}_{\varphi^n} m^{p-i,n}_{\varphi^n}\Big\|_{\mathscr{M}^{\gamma'}\mathscr{C}^{-\varepsilon}(\mathbb{Z}^d_n, e(l))} < +\infty.$$

Now the claimed bound for $m^{n,p}_{\varphi^n}$ follows from an application of Proposition 3.1. For non-integer q we simply use interpolation between the bounds for $p < q < p'$ with $p, p' \in \mathbb{N}$.

□

Appendix E. Besov Spaces

Here we prove some results concerning discrete and continuous Besov spaces. First, we show that restricting a function to the lattice preserves its regularity.

Lemma E.1. *Let $\varphi \in \mathscr{C}^\alpha(\mathbb{R}^d)$ for $\alpha \in \mathbb{R}_{>0} \setminus \mathbb{N}$. Then $\varphi|_{\mathbb{Z}^d_n} \in \mathscr{C}^\alpha(\mathbb{Z}^d_n)$ and*

$$\sup_{n\in\mathbb{N}} \|\varphi|_{\mathbb{Z}^d_n}\|_{\mathscr{C}^\alpha(\mathbb{Z}^d_n)} \lesssim \|\varphi\|_{\mathscr{C}^\alpha(\mathbb{R}^d)}.$$

For the extension of $\varphi|_{\mathbb{Z}^d_n}$ we have $\mathscr{E}^n(\varphi|_{\mathbb{Z}^d_n}) \to \varphi$ in $\mathscr{C}^\beta(\mathbb{R}^d)$ for all $\beta < \alpha$.

Proof. Let us call $\varphi^n = \varphi|_{\mathbb{Z}_n^d}$. We have to estimate the norm $\|\Delta_j^n \varphi^n\|_{L^\infty(\mathbb{Z}_n^d)}$, and for that purpose we consider the cases $j < j_n$ and $j = j_n$ separately. In the first case we have for $x \in \mathbb{Z}_n^d$

$$\Delta_j^n \varphi^n(x) = K_j * \varphi(x) = \Delta_j \varphi(x)$$

where we used that since $\operatorname{supp}(\varrho_j) \subset n(-1/2, 1/2)^d$ the discrete and the continuous convolutions coincide. Therefore:

$$\|\Delta_j^n \varphi\|_{L^\infty(\mathbb{Z}_n^d)} \le \|\Delta_j \varphi\|_{L^\infty(\mathbb{R}^d)} \le 2^{j\alpha} \|\varphi\|_{\mathscr{C}^\alpha}.$$

As for $j = j_n$ we have $\varrho_{j_n}^n(\cdot) = 1 - \chi(2^{-j_n} \cdot)$ where $\chi \in \mathscr{S}_\omega$ is one of the two functions generating the dyadic partition of unity, a symmetric smooth function such that $\chi = 1$ in a ball around the origin. By construction we have $\varrho_{j_n}^n(x) \equiv 1$ for x near the boundary of $n(-1/2, 1/2)^d$, and therefore $\operatorname{supp}(\chi(2^{-j_n} \cdot)) \subset n(-1/2, 1/2)^d$. Let us define $\psi_n = \mathscr{F}_n^{-1} \chi(2^{-j_n} \cdot) = \mathscr{F}_{\mathbb{R}^d}^{-1} \chi(2^{-j_n} \cdot)$. Then

$$\sum_{x \in \mathbb{Z}_n^d} n^{-d} \psi_n(x) = \mathscr{F}_n \psi_n(0) = \chi(2^{-j_n} \cdot 0) = 1,$$

and for every monomial M of strictly positive degree we have, since ψ_n is an even function,

$$\sum_{x \in \mathbb{Z}_n^d} n^{-d} \psi_n(x) M(x) = (\psi_n *_n M)(0) = (\psi_n * M)(0) = \mathscr{F}_{\mathbb{R}^d}^{-1}(\chi(2^{-j_n} \cdot) \mathscr{F}_{\mathbb{R}^d} M)(0) = M(0),$$

where we used that the Fourier transform of a polynomial is supported in 0. Thus we get for $x \in \mathbb{Z}_n^d$ with the usual multi-index notation:

$$\Delta_{j_n}^n \varphi^n(x) = \varphi(x) - (\psi_n *_n \varphi)(x) = -\psi_n *_n \Big(\varphi(\cdot) - \varphi(x) - \sum_{1 \le |k| \le \lfloor \alpha \rfloor} \frac{1}{k!} \partial^k \varphi(x) (\cdot - x)^k \Big)(x),$$

and as above we can replace the discrete convolution $*_n$ with a convolution on $\mathbb{R}^d$. Moreover, since $\varphi \in \mathscr{C}^\alpha(\mathbb{R}^d)$ and $\alpha > 0$ is not an integer, we can estimate

$$\Big\| \varphi(\cdot) - \sum_{0 \le |k| \le \lfloor \alpha \rfloor} \frac{1}{k!} \partial^k \varphi(x) (\cdot - x)^{\otimes k} \Big\|_{L^\infty(\mathbb{R}^d)} \lesssim |y|^\alpha \|\varphi\|_{\mathscr{C}^\alpha(\mathbb{R}^d)},$$

and from here the estimate for the convolution holds by a scaling argument. The convergence then follows by interpolation. □

The following result shows that multiplying a function on $\mathbb{Z}_n^d$ by $n^{-\kappa}$ for some $\kappa > 0$ gains regularity and gives convergence to zero under a uniform bound for the norm.

Lemma E.2. *Consider $z \in \boldsymbol{\varrho}(\omega)$ and $p \in [1, \infty], \alpha \in \mathbb{R}$ and a sequence of functions $f^n \in \mathscr{C}_p^\alpha(\mathbb{Z}_n^d, z)$ with uniformly bounded norm:*

$$\sup_n \|f^n\|_{\mathscr{C}_p^\alpha(\mathbb{Z}_n^d, z)} < +\infty.$$

Then for any $\kappa > 0$ the sequence $n^{-\kappa} f^n$ is bounded in $\mathscr{C}_p^{\alpha+\kappa}(\mathbb{Z}_n^d, z)$:

$$\sup_n \|n^{-\kappa} f^n\|_{\mathscr{C}_p^{\alpha+\kappa}(\mathbb{Z}_n^d, z)} \lesssim \sup_n \|f^n\|_{\mathscr{C}_p^\alpha(\mathbb{Z}_n^d, z)}$$

and $n^{-\kappa} \mathscr{E}^n f^n$ converges to zero in $\mathscr{C}_p^\beta(\mathbb{R}^d, z)$ for any $\beta < \alpha + \kappa$.

Proof. This is a simple consequence of the definition of the Besov spaces on $\mathbb{Z}_n^d$. Indeed we have to consider only the Littlewood-Paley blocks up to an order $j_n \simeq \log_2(n)$. Hence for $j \le j_n$ and $\varepsilon \ge 0$:

$$2^{j(\alpha+\kappa-\varepsilon)} n^{-\kappa} \lesssim 2^{j\alpha} n^{-\varepsilon}.$$

Thus the claim follows from the definition of the Besov norm. □

Now we study the action of discrete gradients. We write $\mathscr{C}_p^\alpha(\mathbb{Z}_n^d, z; \mathbb{R}^d)$ for the space of maps $\varphi \colon \mathbb{Z}_n^d \to \mathbb{R}^d$ such that each component lies in $\mathscr{C}_p^\alpha(\mathbb{Z}_n^d, z)$ with the naturally induced norm.

Lemma E.3 ([MP17], Lemma 3.4). *The discrete gradient* $(\nabla^n\varphi)_i(x) = n(\varphi(x+e_i/n)-\varphi(x))$ *for* $i = 1,\dots,d$ *(with* $\{e_i\}_i$ *being the standard basis in* $\mathbb{R}^d$*) and the discrete Laplacian*

$$\Delta^n\varphi(x) = n^2\sum_{i=1}^{d}(\varphi(x+e_i/n)-2\varphi(x)+\varphi(x-e_i/n))$$

are bounded linear maps

$$\nabla^n\colon \mathscr{C}_p^\alpha(\mathbb{Z}_n^d,z)\to\mathscr{C}_p^{\alpha-1}(\mathbb{Z}_n^d,z;\mathbb{R}^d), \qquad \|\nabla^n\varphi\|_{\mathscr{C}_p^{\alpha-1}(\mathbb{Z}_n^d,z;\mathbb{R}^d)}\lesssim\|\varphi\|_{\mathscr{C}_p^\alpha(\mathbb{Z}_n^d,z)},$$
$$\Delta^n\colon \mathscr{C}_p^\alpha(\mathbb{Z}_n^d,z)\to\mathscr{C}_p^{\alpha-2}(\mathbb{Z}_n^d,z), \qquad \|\Delta^n\varphi\|_{\mathscr{C}_p^{\alpha-2}(\mathbb{Z}_n^d,z)}\lesssim\|\varphi\|_{\mathscr{C}_p^\alpha(\mathbb{Z}_n^d,z)},$$

for all $\alpha\in\mathbb{R}$ *and* $p\in[1,\infty]$*, where the two estimates hold uniformly in* $n\in\mathbb{N}$.

Proof. The only nontrivial statement is that the bounds hold uniformly in n. For Δ^n (and more generally for generators of symmetric random walks) this is shown in [MP17, Lemma 3.4]. The argument for the gradient ∇^n is essentially the same but slightly easier. □

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