

The Baum–Connes and the Farrell–Jones Conjectures in K - and L -Theory

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Summary. We give a survey of the meaning, status and applications of the Baum–Connes Conjecture about the topological K -theory of the reduced group C^* -algebra and the Farrell–Jones Conjecture about the algebraic K - and L -theory of the group ring of a (discrete) group G .

Keywords: K - and L -groups of group rings and group C^* -algebras, Baum–Connes Conjecture, Farrell–Jones Conjecture.

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Introduction

This survey article is devoted to the Baum–Connes Conjecture about the topological K -theory of the reduced group C^* -algebra and the Farrell–Jones Conjecture about the algebraic K - and L -theory of the group ring of a discrete group G . We will present a unified approach to these conjectures hoping that it will stimulate further interactions and exchange of methods and ideas between algebraic and geometric topology on the one side and non-commutative geometry on the other.

Each of the above mentioned conjectures has already been proven for astonishingly large classes of groups using a variety of different methods coming from operator theory, controlled topology and homotopy theory. Methods have been developed for this purpose which turned out to be fruitful in other contexts. The conjectures imply many other well-known and important conjectures. Examples are the Borel Conjecture about the topological rigidity of closed aspherical manifolds, the Novikov Conjecture about the homotopy invariance of higher signatures, the stable Gromov–Lawson–Rosenberg Conjecture about the existence of Riemannian metrics with positive scalar curvature and the Kadison Conjecture about idempotents in the reduced C^* -algebra of a torsionfree discrete group G .

Formulation of the Conjectures

The Baum–Connes and Farrell–Jones Conjectures predict that for every discrete group G the following so called “assembly maps” are isomorphisms.

$$\begin{aligned}
 K_n^G(E_{\mathcal{FAN}}(G)) &\rightarrow K_n(C_r^*(G)) ; \\
 H_n^G(E_{\text{VEY}}(G); \mathbf{K}_R) &\rightarrow K_n(RG) ; \\
 H_n^G(E_{\text{VEY}}(G); \mathbf{L}_R^{(-\infty)}) &\rightarrow L_n^{(-\infty)}(RG) .
 \end{aligned}$$

Here the targets are the groups one would like to understand, namely the topological K -groups of the reduced group C^* -algebra in the Baum–Connes case and the algebraic K - or L -groups of the group ring RG for R an associative ring with unit.

In each case the source is a G -homology theory evaluated on a certain classifying space. In the Baum–Connes Conjecture the G -homology theory is equivariant topological K -theory and the classifying space $E_{\mathcal{F},1,N}(G)$ is the classifying space of the family of finite subgroups, which is often called the classifying space for proper G -actions and denoted $\underline{E}G$ in the literature. In the Farrell–Jones Conjecture the G -homology theory is given by a certain K - or L -theory spectrum over the orbit category, and the classifying space $E_{\mathcal{V}Cyc}(G)$ is the one associated to the family of virtually cyclic subgroups. The conjectures say that these assembly maps are isomorphisms.

These conjectures were stated in [28, conjecture 3.15 on page 254] and [111, 1.6 on page 257]. Our formulations differ from the original ones, but are equivalent. In the case of the Farrell–Jones Conjecture we slightly generalize the original conjecture by allowing arbitrary coefficient rings instead of \mathbb{Z} . At the time of writing no counterexample to the Baum–Connes Conjecture 59 or the Farrell–Jones Conjecture 58 is known to the authors.

One can apply methods from algebraic topology such as spectral sequences and Chern characters to the sources of the assembly maps. In this sense the sources are much more accessible than the targets. The conjectures hence lead to very concrete calculations. Probably even more important is the structural insight: to what extent do the target groups show a homological behaviour. These aspects can be treated completely analogously in the Baum–Connes and the Farrell–Jones setting.

However, the conjectures are not merely computational tools. Their importance comes from the fact that the assembly maps have geometric interpretations in terms of indices in the Baum–Connes case and in terms of surgery theory in the Farrell–Jones case. These interpretations are the key ingredient in applications and the reason that the Baum–Connes and Farrell–Jones Conjectures imply so many other conjectures in non-commutative geometry, geometric topology and algebra.

A User’s Guide

We have tried to write the text in a way such that one can read small units independently from the rest. Moreover, a reader who may only be interested in the Baum–Connes Conjecture or only in the Farrell–Jones Conjecture for K -theory or for L -theory can ignore the other parts. But we emphasize again that one basic idea of this paper is to explain the parallel treatment of these conjectures.

A reader without much prior knowledge about the Baum–Connes Conjecture or the Farrell–Jones Conjecture should begin with Chapt. 2.2. There, the special case of a torsionfree group is treated, since the formulation of the conjectures is less technical in this case and there are already many interesting applications. The applications themselves however, are not needed later. A more experienced reader may pass directly to Chapt. 2.3.

Other (survey) articles on the Farrell–Jones Conjecture and the Baum–Connes Conjecture are [111, 128, 147, 225, 307].

Notations and Conventions

Here is a briefing on our main notational conventions. Details are of course discussed in the text. The columns in the following table contain our notation for: the spectra, their associated homology theory, the right hand side of the corresponding assembly maps, the functor from groupoids to spectra and finally the G -homology theory associated to these spectra valued functors.

\mathbf{BU}	$K_n(X)$	$K_n(C_r^*G)$	\mathbf{K}^{top}	$H_n^G(X; \mathbf{K}^{\text{top}})$
$\mathbf{K}(R)$	$H_n(X; \mathbf{K}(R))$	$K_n(RG)$	\mathbf{K}_R	$H_n^G(X; \mathbf{K}_R)$
$\mathbf{L}^{(j)}(R)$	$H_n(X; \mathbf{L}^{(j)}(R))$	$L_n^{(j)}(RG)$	$\mathbf{L}_R^{(j)}$	$H_n^G(X; \mathbf{L}_R^{(j)})$

We would like to stress that \mathbf{K} without any further decoration will always refer to the non-connective K -theory spectrum. $\mathbf{L}^{(j)}$ will always refer to quadratic L -theory with decoration j . For a C^* - or Banach algebra A the symbol $K_n(A)$ has two possible interpretations but we will mean the topological K -theory.

A ring is always an associative ring with unit, and ring homomorphisms are always unital. Modules are left modules. We will always work in the category of compactly generated spaces, compare [295] and [330, I.4]. For our conventions concerning spectra see Sect. 2.7.2. Spectra are denoted with boldface letters such as \mathbf{E} .

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The Conjectures in the Torsion Free Case

In this chapter we discuss the Baum–Connes and Farrell–Jones Conjectures in the case of a torsion free group. Their formulation is less technical than in the general case, but already in the torsion free case there are many interesting and illuminating conclusions. In fact some of the most important consequences of the conjectures, like for example the Borel Conjecture (see Conjecture 27) or the Kadison Conjecture (see Conjecture 39), refer exclusively to the torsion free case. On the other hand in the long run the general case, involving groups with

torsion, seems to be unavoidable. The general formulation yields a clearer and more complete picture, and furthermore there are proofs of the conjectures for torsion free groups, where in intermediate steps of the proof it is essential to have the general formulation available (compare Sect. 2.8.9).

The statement of the general case and further applications will be presented in the next chapter. The reader may actually skip this chapter and start immediately with Chapt. 2.3.

We have put some effort into dealing with coefficient rings R other than the integers. A topologist may a priori be interested only in the case $R = \mathbb{Z}$ but other cases are interesting for algebraists and also do occur in computations for integral group rings.

2.2.1 Algebraic K -Theory – Low Dimensions

A ring R is always understood to be associative with unit. We denote by $K_n(R)$ the algebraic K -group of R for $n \in \mathbb{Z}$. In particular $K_0(R)$ is the Grothendieck group of finitely generated projective R -modules and elements in $K_1(R)$ can be represented by automorphisms of such modules. In this section we are mostly interested in the K -groups $K_n(R)$ with $n \leq 1$. For definitions of these groups we refer to [221, 266, 286, 299, 323] for $n = 0, 1$ and to [22] and [268] for $n \leq 1$.

For a ring R and a group G we denote by

$$A_0 = K_0(i): K_0(R) \rightarrow K_0(RG)$$

the map induced by the natural inclusion $i: R \rightarrow RG$. Sending $(g, [P]) \in G \times K_0(R)$ to the class of the RG -automorphism

$$R[G] \otimes_R P \rightarrow R[G] \otimes_R P, \quad u \otimes x \mapsto ug^{-1} \otimes x$$

defines a map $\Phi: G_{\text{ab}} \otimes_{\mathbb{Z}} K_0(R) \rightarrow K_1(RG)$, where G_{ab} denotes the abelianized group. We set

$$A_1 = \Phi \oplus K_1(i): G_{\text{ab}} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) \rightarrow K_1(RG).$$

We recall the notion of a regular ring. We think of modules as left modules unless stated explicitly differently. Recall that R is *Noetherian* if any submodule of a finitely generated R -module is again finitely generated. It is called *regular* if it is Noetherian and any R -module has a finite-dimensional projective resolution. Any principal ideal domain such as \mathbb{Z} or a field is regular.

The Farrell–Jones Conjecture about algebraic K -theory implies for a torsion free group the following conjecture about the low dimensional K -theory groups.

1

Conjecture 1: (*The Farrell–Jones Conjecture for Low Dimensional K -theory and Torsion Free Groups.*) Let G be a torsion free group and let R be a regular ring. Then

$$K_n(RG) = 0 \quad \text{for } n \leq -1$$

and the maps

$$K_0(R) \xrightarrow{A_0} K_0(RG) \quad \text{and}$$

$$G_{\text{ab}} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) \xrightarrow{A_1} K_1(RG)$$

are both isomorphisms.

Every regular ring satisfies $K_n(R) = 0$ for $n \leq -1$ [268, 5.3.30 on page 295] and hence the first statement is equivalent to $K_n(i): K_n(R) \rightarrow K_n(RG)$ being an isomorphism for $n \leq -1$. In Remark 15 below we explain why we impose the regularity assumption on the ring R .

For a regular ring R and a group G we define $\text{Wh}_1^R(G)$ as the cokernel of the map A_1 and $\text{Wh}_0^R(G)$ as the cokernel of the map A_0 . In the important case where $R = \mathbb{Z}$ the group $\text{Wh}_1^{\mathbb{Z}}(G)$ coincides with the classical *Whitehead group* $\text{Wh}(G)$ which is the quotient of $K_1(\mathbb{Z}G)$ by the subgroup consisting of the classes of the units $\pm g \in (\mathbb{Z}G)^{\text{inv}}$ for $g \in G$. Moreover for every ring R we define the *reduced algebraic K -groups* $\tilde{K}_n(R)$ as the cokernel of the natural map $K_n(\mathbb{Z}) \rightarrow K_n(R)$. Obviously $\text{Wh}_0^{\mathbb{Z}}(G) = \tilde{K}_0(\mathbb{Z}G)$.

Lemma 2 The map A_0 is always injective. If R is commutative and the natural map $\mathbb{Z} \rightarrow K_0(R), 1 \mapsto [R]$ is an isomorphism, then the map A_1 is injective.

2

Proof The augmentation $\varepsilon: RG \rightarrow R$, which maps each group element g to 1, yields a retraction for the inclusion $i: R \rightarrow RG$ and hence induces a retraction for A_0 . If the map $\mathbb{Z} \rightarrow K_0(R), 1 \mapsto [R]$ induces an isomorphism and R is commutative, then we have the commutative diagram

$$\begin{array}{ccc}
 G_{\text{ab}} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) & \xrightarrow{A_1} & K_1(RG) \\
 \cong \downarrow & & \downarrow \\
 & & K_1(RG_{\text{ab}}) \\
 & & \downarrow (\det, K_1(\varepsilon)) \\
 G_{\text{ab}} \oplus K_1(R) & \longrightarrow & RG_{\text{ab}}^{\text{inv}} \oplus K_1(R),
 \end{array}$$

where the upper vertical arrow on the right is induced from the map $G \rightarrow G_{\text{ab}}$ to the abelianization. Since RG_{ab} is a commutative ring we have the determinant $\det: K_1(RG_{\text{ab}}) \rightarrow (RG_{\text{ab}})^{\text{inv}}$. The lower horizontal arrow is induced from the obvious inclusion of G_{ab} into the invertible elements of the group ring RG_{ab} and in particular injective.

In the special case $R = \mathbb{Z}$ Conjecture 1 above is equivalent to the following conjecture.

3 **Conjecture 3:** (*Vanishing of Low Dimensional K-theory for Torsionfree Groups and Integral Coefficients.*) For every torsion free group G we have

$$K_n(\mathbb{Z}G) = 0 \quad \text{for } n \leq -1, \quad \tilde{K}_0(\mathbb{Z}G) = 0 \quad \text{and} \quad \text{Wh}(G) = 0.$$

4 **Remark 4:** (*Torsionfree is Necessary.*) In general $\tilde{K}_0(\mathbb{Z}G)$ and $\text{Wh}(G)$ do not vanish for finite groups. For example $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/23]) \cong \mathbb{Z}/3$ [221, pp. 29,30] and $\text{Wh}(\mathbb{Z}/p) \cong \mathbb{Z}^{\frac{p-3}{2}}$ for p an odd prime [70, 11.5 on page 45]. This shows that the assumption that G is torsion free is crucial in the formulation of Conjecture 1 above.

For more information on $\tilde{K}_0(\mathbb{Z}G)$ and Whitehead groups of finite groups see for instance [22, chapt. XI], [79, 220, 231, 232] and [299].

2.2.2 Applications I

We will now explain the geometric relevance of the groups whose vanishing is predicted by Conjecture 3.

The s-Cobordism Theorem and the Poincaré Conjecture

The Whitehead group $\text{Wh}(G)$ plays a key role if one studies manifolds because of the so called s-Cobordism Theorem. In order to state it, we explain the notion of an h-cobordism.

Manifold always means smooth manifold unless it is explicitly stated differently. We say that W or more precisely $(W; M^-, f^-, M^+, f^+)$ is an n -dimensional *cobordism* over M^- if W is a compact n -dimensional manifold together with the following: a disjoint decomposition of its boundary ∂W into two closed $(n-1)$ -dimensional manifolds $\partial^- W$ and $\partial^+ W$, two closed $(n-1)$ -dimensional manifolds M^- and M^+ and diffeomorphisms $f^-: M^- \rightarrow \partial^- W$ and $f^+: M^+ \rightarrow \partial^+ W$. The cobordism is called an *h-cobordism* if the inclusions $i^-: \partial^- W \rightarrow W$ and $i^+: \partial^+ W \rightarrow W$ are both homotopy equivalences. Two cobordisms $(W; M^-, f^-, M^+, f^+)$ and $(W'; M^-, f'^-, M'^+, f'^+)$ over M^- are *diffeomorphic relative M^-* if there is a diffeomorphism $F: W \rightarrow W'$ with $F \circ f^- = f'^-$. We call a cobordism over M^- *trivial*, if it is diffeomorphic relative M^- to the trivial h-cobordism given by the cylinder $M^- \times [0, 1]$ together with the obvious inclusions of $M^- \times \{0\}$ and $M^- \times \{1\}$. Note that “trivial” implies in particular that M^- and M^+ are diffeomorphic.

The question whether a given h-cobordism is trivial is decided by the Whitehead torsion $\tau(W; M^-) \in \text{Wh}(G)$ where $G = \pi_1(M^-)$. For the details of the definition of $\tau(W; M^-)$ the reader should consult [70, 220] or chapt. 2 in [200]. Compare also [266].

Theorem 5: (*s-Cobordism Theorem.*) Let M^- be a closed connected oriented manifold of dimension $n \geq 5$ with fundamental group $G = \pi_1(M^-)$. Then

5

- (i) An h-cobordism W over M^- is trivial if and only if its Whitehead torsion $\tau(W, M^-) \in \text{Wh}(G)$ vanishes.
- (ii) Assigning to an h-cobordism over M^- its Whitehead torsion yields a bijection from the diffeomorphism classes relative M^- of h-cobordisms over M^- to the Whitehead group $\text{Wh}(G)$.

The s-Cobordism Theorem is due to Barden, Mazur and Stallings. There are also topological and PL-versions. Proofs can be found for instance in [173], [176, Essay III], [200] and [272, pp. 87–90].

The s-Cobordism Theorem tells us that the vanishing of the Whitehead group (as predicted in Conjecture 3 for torsion free groups) has the following geometric interpretation.

Consequence 6 For a finitely presented group G the vanishing of the Whitehead group $\text{Wh}(G)$ is equivalent to the statement that each h-cobordism over a closed connected manifold M^- of dimension $\dim(M^-) \geq 5$ with fundamental group $\pi_1(M^-) \cong G$ is trivial.

6

Knowing that all h-cobordisms over a given manifold are trivial is a strong and useful statement. In order to illustrate this we would like to discuss the case where the fundamental group is trivial.

Since the ring \mathbb{Z} has a Gaussian algorithm, the determinant induces an isomorphism $K_1(\mathbb{Z}) \xrightarrow{\cong} \{\pm 1\}$ (compare [268, theorem 2.3.2]) and the Whitehead group $\text{Wh}(\{1\})$ of the trivial group vanishes. Hence any h-cobordism over a simply connected closed manifold of dimension ≥ 5 is trivial. As a consequence one obtains the Poincaré Conjecture for high dimensional manifolds.

Theorem 7: (*Poincaré Conjecture.*) Suppose $n \geq 5$. If the closed manifold M is homotopy equivalent to the sphere S^n , then it is homeomorphic to S^n .

7

Proof We only give the proof for $\dim(M) \geq 6$. Let $f: M \rightarrow S^n$ be a homotopy equivalence. Let $D_-^n \subset M$ and $D_+^n \subset M$ be two disjoint embedded disks. Let W be the complement of the interior of the two disks in M . Then W turns out to be a simply connected h-cobordism over ∂D_-^n . Hence we can find a diffeomorphism

$$F: (\partial D_-^n \times [0, 1]; \partial D_-^n \times \{0\}, \partial D_-^n \times \{1\}) \rightarrow (W; \partial D_-^n, \partial D_+^n),$$

which is the identity on $\partial D_-^n = \partial D_-^n \times \{0\}$ and induces some (unknown) diffeomorphism $f^+: \partial D_-^n \times \{1\} \rightarrow \partial D_+^n$. By the *Alexander trick* one can extend

$f^+ : \partial D_-^n = \partial D_-^n \times \{1\} \rightarrow \partial D_+^n$ to a homeomorphism $\overline{f^+} : D_-^n \rightarrow D_+^n$. Namely, any homeomorphism $f : S^{n-1} \rightarrow S^{n-1}$ extends to a homeomorphism $\overline{f} : D^n \rightarrow D^n$ by sending $t \cdot x$ for $t \in [0, 1]$ and $x \in S^{n-1}$ to $t \cdot f(x)$. Now define a homeomorphism $h : D_-^n \times \{0\} \cup_{i_-} \partial D_-^n \times [0, 1] \cup_{i_+} D_+^n \times \{1\} \rightarrow M$ for the canonical inclusions $i_k : \partial D_-^n \times \{k\} \rightarrow \partial D_-^n \times [0, 1]$ for $k = 0, 1$ by $h|_{D_-^n \times \{0\}} = \text{id}$, $h|_{\partial D_-^n \times [0, 1]} = F$ and $h|_{D_+^n \times \{1\}} = \overline{f^+}$. Since the source of h is obviously homeomorphic to S^n , Theorem 7 follows.

The Poincaré Conjecture (see Theorem 7) is at the time of writing known in all dimensions except dimension 3. It is essential in its formulation that one concludes M to be homeomorphic (as opposed to diffeomorphic) to S^n . The Alexander trick does not work differentiably. There are *exotic spheres*, i.e. smooth manifolds which are homeomorphic but not diffeomorphic to S^n [218].

More information about the Poincaré Conjecture, the Whitehead torsion and the s-Cobordism Theorem can be found for instance in [50, 70, 86, 131, 132, 141, 173, 200, 219, 220, 266] and [272].

Finiteness Obstructions

We now discuss the geometric relevance of $\tilde{K}_0(\mathbb{Z}G)$.

Let X be a CW-complex. It is called *finite* if it consists of finitely many cells. It is called *finitely dominated* if there is a finite CW-complex Y together with maps $i : X \rightarrow Y$ and $r : Y \rightarrow X$ such that $r \circ i$ is homotopic to the identity on X . The fundamental group of a finitely dominated CW-complex is always finitely presented.

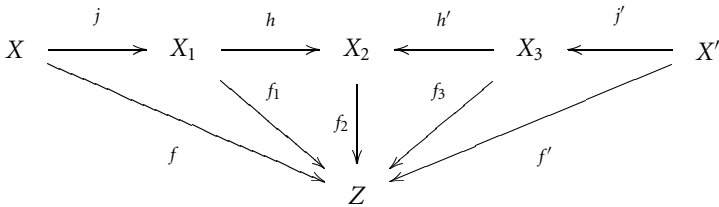
While studying existence problems for spaces with prescribed properties (like for example group actions), it happens occasionally that it is relatively easy to construct a finitely dominated CW-complex within a given homotopy type, whereas it is not at all clear whether one can also find a homotopy equivalent *finite* CW-complex. *Wall's finiteness obstruction*, a certain obstruction element $\tilde{\sigma}(X) \in \tilde{K}_0(\mathbb{Z}\pi_1(X))$, decides the question.

8

Theorem 8: (*Properties of the Finiteness Obstruction.*) Let X be a finitely dominated CW-complex with fundamental group $\pi = \pi_1(X)$.

- (i) The space X is homotopy equivalent to a finite CW-complex if and only if $\tilde{\sigma}(X) = 0 \in \tilde{K}_0(\mathbb{Z}\pi)$.
- (ii) Every element in $\tilde{K}_0(\mathbb{Z}G)$ can be realized as the finiteness obstruction $\tilde{\sigma}(X)$ of a finitely dominated CW-complex X with $G = \pi_1(X)$, provided that G is finitely presented.
- (iii) Let Z be a space such that $G = \pi_1(Z)$ is finitely presented. Then there is a bijection between $\tilde{K}_0(\mathbb{Z}G)$ and the set of equivalence classes of maps $f : X \rightarrow Z$ with X finitely dominated under the equivalence relation explained below.

The equivalence relation in (iii) is defined as follows: Two maps $f: X \rightarrow Z$ and $f': X' \rightarrow Z$ with X and X' finitely dominated are equivalent if there exists a commutative diagram



where h and h' are homotopy equivalences and j and j' are inclusions of sub-complexes for which X_1 , respectively X_3 , is obtained from X , respectively X' , by attaching a finite number of cells.

The vanishing of $\tilde{K}_0(\mathbb{Z}G)$ as predicted in Conjecture 3 for torsion free groups hence has the following interpretation.

Consequence 9 For a finitely presented group G the vanishing of $\tilde{K}_0(\mathbb{Z}G)$ is equivalent to the statement that any finitely dominated CW -complex X with $G \cong \pi_1(X)$ is homotopy equivalent to a finite CW -complex.

9

For more information about the finiteness obstruction we refer for instance to [125, 126, 196, 224, 257, 266, 308, 317] and [318].

Negative K -Groups and Bounded h -Cobordisms

One possible geometric interpretation of negative K -groups is in terms of bounded h -cobordisms. Another interpretation will be explained in Sect. 2.2.4 below.

We consider *manifolds W parametrized over \mathbb{R}^k* , i.e. manifolds which are equipped with a surjective proper map $p: W \rightarrow \mathbb{R}^k$. We will always assume that the fundamental group(oid) is bounded, compare [239, definition 1.3]. A map $f: W \rightarrow W'$ between two manifolds parametrized over \mathbb{R}^k is *bounded* if $\{p' \circ f(x) - p(x) \mid x \in W\}$ is a bounded subset of \mathbb{R}^k .

A *bounded cobordism* $(W; M^-, f^-, M^+, f^+)$ is defined just as in Sect. 2.2.2 but compact manifolds are replaced by manifolds parametrized over \mathbb{R}^k and the parametrization for M^\pm is given by $p_W \circ f^\pm$. If we assume that the inclusions $i^\pm: \partial^\pm W \rightarrow W$ are homotopy equivalences, then there exist deformations $r^\pm: W \times I \rightarrow W$, $(x, t) \mapsto r_t^\pm(x)$ such that $r_0^\pm = \text{id}_W$ and $r_1^\pm(W) \subset \partial^\pm W$.

A bounded cobordism is called a *bounded h -cobordism* if the inclusions i^\pm are homotopy equivalences and additionally the deformations can be chosen such that the two sets

$$S^\pm = \{p_W \circ r_t^\pm(x) - p_W \circ r_1^\pm(x) \mid x \in W, t \in [0, 1]\}$$

are bounded subsets of \mathbb{R}^k .

The following theorem (compare [239] and [327, appendix]) contains the s-Cobordism Theorem 5 as a special case, gives another interpretation of elements in $\tilde{K}_0(\mathbb{Z}\pi)$ and explains one aspect of the geometric relevance of negative K -groups.

10 **Theorem 10:** (*Bounded h-Cobordism Theorem.*) Suppose that M^- is parametrized over \mathbb{R}^k and satisfies $\dim M^- \geq 5$. Let π be its fundamental group(oid). Equivalence classes of bounded h-cobordisms over M^- modulo bounded diffeomorphism relative M^- correspond bijectively to elements in $\kappa_{1-k}(\pi)$, where

$$\kappa_{1-k}(\pi) = \begin{cases} \text{Wh}(\pi) & \text{if } k = 0, \\ \tilde{K}_0(\mathbb{Z}\pi) & \text{if } k = 1, \\ K_{1-k}(\mathbb{Z}\pi) & \text{if } k \geq 2. \end{cases}$$

More information about negative K -groups can be found for instance in [8, 22, 57, 58, 113, 213, 238, 239, 252, 259, 268] and [327, appendix].

2.2.3 Algebraic K -Theory – All Dimensions

So far we only considered the K -theory groups in dimensions ≤ 1 . We now want to explain how Conjecture 1 generalizes to higher algebraic K -theory. For the definition of higher algebraic K -theory groups and the (connective) K -theory spectrum see [35, 52, 158, 249, 268, 292, 315] and [323]. We would like to stress that for us $\mathbf{K}(R)$ will always denote the *non-connective algebraic K -theory spectrum* for which $K_n(R) = \pi_n(\mathbf{K}(R))$ holds for all $n \in \mathbb{Z}$. For its definition see [52, 194] and [237].

The Farrell–Jones Conjecture for algebraic K -theory reduces for a torsion free group to the following conjecture.

11 **Conjecture 11:** (*Farrell–Jones Conjecture for Torsion Free Groups and K -theory.*) Let G be a torsion free group. Let R be a regular ring. Then the assembly map

$$H_n(BG; \mathbf{K}(R)) \rightarrow K_n(RG)$$

is an isomorphism for $n \in \mathbb{Z}$.

Here $H_n(-; \mathbf{K}(R))$ denotes the homology theory which is associated to the spectrum $\mathbf{K}(R)$. It has the property that $H_n(\text{pt}; \mathbf{K}(R))$ is $K_n(R)$ for $n \in \mathbb{Z}$, where here and elsewhere pt denotes the space consisting of one point. The space BG is the *classifying space of the group G* , which up to homotopy is characterized by the property that it is a CW -complex with $\pi_1(BG) \cong G$ whose universal covering is contractible. The technical details of the construction of $H_n(-; \mathbf{K}(R))$ and the assembly map will be explained in a more general setting in Sect. 2.3.1.

The point of Conjecture 11 is that on the right-hand side of the assembly map we have the group $K_n(RG)$ we are interested in, whereas the left-hand side is

a homology theory and hence much easier to compute. For every homology theory associated to a spectrum we have the Atiyah–Hirzebruch spectral sequence, which in our case has $E_{p,q}^2 = H_p(BG; K_q(R))$ and converges to $H_{p+q}(BG; \mathbf{K}(R))$.

If R is regular, then the negative K -groups of R vanish and the spectral sequence lives in the first quadrant. Evaluating the spectral sequence for $n = p + q \leq 1$ shows that Conjecture 11 above implies Conjecture 1.

Remark 12: (*Rational Computation.*) Rationally an Atiyah–Hirzebruch spectral sequence collapses always and the homological Chern character gives an isomorphism

12

$$\text{ch}: \bigoplus_{p+q=n} H_p(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} (K_q(R) \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\cong} H_n(BG; \mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The Atiyah–Hirzebruch spectral sequence and the Chern character will be discussed in a much more general setting in Chapt. 2.9.

Remark 13: (*Separation of Variables.*) We see that the left-hand side of the isomorphism in the previous remark consists of a group homology part and a part which is the rationalized K -theory of R . (Something similar happens before we rationalize at the level of spectra: The left hand side of Conjecture 11 can be interpreted as the homotopy groups of the spectrum $BG_+ \wedge \mathbf{K}(R)$.) So essentially Conjecture 11 predicts that the K -theory of RG is built up out of two independent parts: the K -theory of R and the group homology of G . We call this principle *separation of variables*. This principle also applies to other theories such as algebraic L -theory or topological K -theory. See also Remark 179.

13

Remark 14: (*K -theory of the Coefficients.*) Note that Conjecture 11 can only help us to explicitly compute the K -groups of RG in cases where we know enough about the K -groups of R . We obtain no new information about the K -theory of R itself. However, already for very simple rings the computation of their algebraic K -theory groups is an extremely hard problem.

14

It is known that the groups $K_n(\mathbb{Z})$ are finitely generated abelian groups [248]. Due to Borel [39] we know that

$$K_n(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } n = 0 ; \\ \mathbb{Q} & \text{if } n = 4k + 1 \text{ with } k \geq 1 ; \\ 0 & \text{otherwise .} \end{cases}$$

Since \mathbb{Z} is regular we know that $K_n(\mathbb{Z})$ vanishes for $n \leq -1$. Moreover, $K_0(\mathbb{Z}) \cong \mathbb{Z}$ and $K_1(\mathbb{Z}) \cong \{\pm 1\}$, where the isomorphisms are given by the rank and the determinant. One also knows that $K_2(\mathbb{Z}) \cong \mathbb{Z}/2$, $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ [189] and $K_4(\mathbb{Z}) \cong$

0 [264]. Finite fields belong to the few rings where one has a complete and explicit knowledge of all K -theory groups [247]. We refer the reader for example to [177, 226, 265, 322] and Soulé's article in [193] for more information about the algebraic K -theory of the integers or more generally of rings of integers in number fields.

Because of Borel's calculation the left hand side of the isomorphism described in Remark 12 specializes for $R = \mathbb{Z}$ to

$$H_n(BG; \mathbb{Q}) \oplus \bigoplus_{k=1}^{\infty} H_{n-(4k+1)}(BG; \mathbb{Q}) \quad (2.1)$$

and Conjecture 11 predicts that this group is isomorphic to $K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Next we discuss the case where the group G is infinite cyclic.

15 Remark 15: (*Bass–Heller–Swan Decomposition.*) The so called *Bass–Heller–Swan-decomposition*, also known as the *Fundamental Theorem of algebraic K -theory*, computes the algebraic K -groups of $R[\mathbb{Z}]$ in terms of the algebraic K -groups and Nil-groups of R :

$$K_n(R[\mathbb{Z}]) \cong K_{n-1}(R) \oplus K_n(R) \oplus NK_n(R) \oplus NK_n(R) .$$

Here the group $NK_n(R)$ is defined as the cokernel of the split injection $K_n(R) \rightarrow K_n(R[t])$. It can be identified with the cokernel of the split injection $K_{n-1}(R) \rightarrow K_{n-1}(\mathcal{N}il(R))$. Here $K_n(\mathcal{N}il(R))$ denotes the K -theory of the exact category of nilpotent endomorphisms of finitely generated projective R -modules. For negative n it is defined with the help of Bass' notion of a contracting functor [22] (see also [57]). The groups are known as *Nil-groups* and often denoted $\text{Nil}_{n-1}(R)$.

For proofs of these facts and more information the reader should consult [22, chapt. XII], [25], [135, theorem on page 236], [249, corollary in §6 on page 38], [268, theorems 3.3.3 and 5.3.30], [292, theorem 9.8] and [300, theorem 10.1].

If we iterate and use $R[\mathbb{Z}^n] = R[\mathbb{Z}^{n-1}][\mathbb{Z}]$ we see that a computation of $K_n(RG)$ must in general take into account information about $K_i(R)$ for all $i \leq n$. In particular we see that it is important to formulate Conjecture 11 with the non-connective K -theory spectrum.

Since S^1 is a model for $B\mathbb{Z}$, we get an isomorphism

$$H_n(B\mathbb{Z}; \mathbf{K}(R)) \cong K_{n-1}(R) \oplus K_n(R)$$

and hence Conjecture 11 predicts

$$K_n(R[\mathbb{Z}]) \cong K_{n-1}(R) \oplus K_n(R) .$$

This explains why in the formulation of Conjecture 11 the condition that R is regular appears. It guarantees that $NK_n(R) = 0$ [268, theorem 5.3.30 on page 295]. There are weaker conditions which imply that $NK_n(R) = 0$ but “regular” has the

advantage that R regular implies that $R[t]$ and $R[\mathbb{Z}] = R[t^{\pm 1}]$ are again regular, compare the discussion in sect. 2 in [23].

The Nil-terms $NK_n(R)$ seem to be hard to compute. For instance $NK_1(R)$ either vanishes or is infinitely generated as an abelian group [95]. In Sect. 2.5.2 we will discuss the Isomorphism Conjecture for NK -groups. For more information about Nil-groups see for instance [73, 74, 146, 324] and [325].

Applications II

The Relation to Pseudo-Isotopy Theory

Let I denote the unit interval $[0, 1]$. A topological *pseudoisotopy* of a compact manifold M is a homeomorphism $h: M \times I \rightarrow M \times I$, which restricted to $M \times \{0\} \cup \partial M \times I$ is the obvious inclusion. The space $P(M)$ of pseudoisotopies is the (simplicial) group of all such homeomorphisms. Pseudoisotopies play an important role if one tries to understand the homotopy type of the space $\text{Top}(M)$ of self-homeomorphisms of a manifold M . We will see below in Sect. 2.2.6 how the results about pseudoisotopies discussed in this section combined with surgery theory lead to quite explicit results about the homotopy groups of $\text{Top}(M)$.

There is a stabilization map $P(M) \rightarrow P(M \times I)$ given by crossing a pseudoisotopy with the identity on the interval I and the stable pseudoisotopy space is defined as $\mathcal{P}(M) = \text{colim}_k P(M \times I^k)$. In fact $\mathcal{P}(-)$ can be extended to a functor on all spaces [144]. The natural inclusion $P(M) \rightarrow \mathcal{P}(M)$ induces an isomorphism on the i -th homotopy group if the dimension of M is large compared to i , see [43] and [157].

Waldhausen [314, 315] defines the algebraic K -theory of spaces functor $\mathbf{A}(X)$ and the functor $\mathbf{Wh}^{PL}(X)$ from spaces to spectra (or infinite loop spaces) and a fibration sequence

$$X_+ \wedge \mathbf{A}(\text{pt}) \rightarrow \mathbf{A}(X) \rightarrow \mathbf{Wh}^{PL}(X).$$

Here $X_+ \wedge \mathbf{A}(\text{pt}) \rightarrow \mathbf{A}(X)$ is an assembly map, which can be compared to the algebraic K -theory assembly map that appears in Conjecture 11 via a commutative diagram

$$\begin{array}{ccc} H_n(X; \mathbf{A}(\text{pt})) & \longrightarrow & \pi_n(\mathbf{A}(X)) \\ \downarrow & & \downarrow \\ H_n(B\pi_1(X); \mathbf{K}(\mathbb{Z})) & \longrightarrow & K_n(\mathbb{Z}\pi_1(X)). \end{array}$$

In the case where $X \simeq BG$ is aspherical the vertical maps induce isomorphisms after rationalization for $n \geq 1$, compare [314, proposition 2.2]. Since $\Omega^2 \mathbf{Wh}^{PL}(X) \simeq \mathcal{P}(X)$ (a guided tour through the literature concerning this and related results can

be found in [91, sect. 9]), Conjecture 11 implies rational vanishing results for the groups $\pi_n(\mathcal{P}(M))$ if M is an aspherical manifold. Compare also Remark 111.

16 Consequence 16 Suppose M is a closed aspherical manifold and Conjecture 11 holds for $R = \mathbb{Z}$ and $G = \pi_1(M)$, then for all $n \geq 0$

$$\pi_n(\mathcal{P}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = 0 .$$

Similarly as above one defines smooth pseudoisotopies and the space of stable smooth pseudoisotopies $\mathcal{P}^{\text{Diff}}(M)$. There is also a smooth version of the Whitehead space $\text{Wh}^{\text{Diff}}(X)$ and $\Omega^2 \text{Wh}^{\text{Diff}}(M) \simeq \mathcal{P}^{\text{Diff}}(M)$. Again there is a close relation to A -theory via the natural splitting $A(X) \simeq \Sigma^\infty(X_+) \vee \text{Wh}^{\text{Diff}}(X)$, see [316]. Here $\Sigma^\infty(X_+)$ denotes the suspension spectrum associated to X_+ . Using this one can split off an assembly map $H_n(X; \text{Wh}^{\text{Diff}}(\text{pt})) \rightarrow \pi_n(\text{Wh}^{\text{Diff}}(X))$ from the A -theory assembly map. Since for every space $\pi_n(\Sigma^\infty(X_+)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n(X; \mathbb{Q})$ Conjecture 11 combined with the rational computation in (2.1) yields the following result.

17 Consequence 17 Suppose M is a closed aspherical manifold and Conjecture 11 holds for $R = \mathbb{Z}$ and $G = \pi_1(M)$. Then for $n \geq 0$ we have

$$\pi_n(\mathcal{P}^{\text{Diff}}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{k=1}^{\infty} H_{n-4k+1}(M; \mathbb{Q}) .$$

Observe that the fundamental difference between the smooth and the topological case occurs already when G is the trivial group.

Negative K -Groups and Bounded Pseudo-Isotopies

We briefly explain a further geometric interpretation of negative K -groups, which parallels the discussion of bounded h-cobordisms in Sect. 2.2.2.

Let $p: M \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ denote the natural projection. The space $P_b(M; \mathbb{R}^k)$ of bounded pseudoisotopies is the space of all self-homeomorphisms $h: M \times \mathbb{R}^k \times I \rightarrow M \times \mathbb{R}^k \times I$ such that restricted to $M \times \mathbb{R}^k \times \{0\}$ the map h is the inclusion and such that h is bounded, i.e. the set $\{p \circ h(y) - p(y) \mid y \in M \times \mathbb{R}^k \times I\}$ is a bounded subset of \mathbb{R}^k . There is again a stabilization map $P_b(M; \mathbb{R}^k) \rightarrow P_b(M \times I; \mathbb{R}^k)$ and a stable bounded pseudoisotopy space $\mathcal{P}_b(M; \mathbb{R}^k) = \text{colim}_j P_b(M \times I^j; \mathbb{R}^k)$. There is a homotopy equivalence $\mathcal{P}_b(M; \mathbb{R}^k) \rightarrow \Omega \mathcal{P}_b(M; \mathbb{R}^{k+1})$ [144, appendix II] and hence the sequence of spaces $\mathcal{P}_b(M; \mathbb{R}^k)$ for $k = 0, 1, \dots$ is an Ω -spectrum $\mathbf{P}(M)$. Analogously one defines the differentiable bounded pseudoisotopies $\mathcal{P}_b^{\text{diff}}(M; \mathbb{R}^k)$ and an Ω -spectrum $\mathbf{P}^{\text{diff}}(M)$. The negative homotopy groups of these spectra have an interpretation in terms of low and negative dimensional K -groups. In terms of unstable homotopy groups this is explained in the following theorem which is closely related to Theorem 10 about bounded h-cobordisms.

Theorem 18: (*Negative Homotopy Groups of Pseudoisotopies.*) Let $G = \pi_1(M)$. Suppose n and k are such that $n + k \geq 0$, then for $k \geq 1$ there are isomorphisms

$$\pi_{n+k}(\mathcal{P}_b(M; \mathbb{R}^k)) = \begin{cases} \text{Wh}(G) & \text{if } n = -1, \\ \tilde{K}_0(\mathbb{Z}G) & \text{if } n = -2, \\ K_{n+2}(\mathbb{Z}G) & \text{if } n < -2. \end{cases}$$

The same result holds in the differentiable case.

Note that Conjecture 11 predicts that these groups vanish if G is torsionfree. The result above is due to Anderson and Hsiang [8] and is also discussed in [327, appendix].

L-Theory

We now move on to the L -theoretic version of the Farrell–Jones Conjecture. We will still stick to the case where the group is torsion free. The conjecture is obtained by replacing K -theory and the K -theory spectrum in Conjecture 11 by 4-periodic L -theory and the L -theory spectrum $L^{(-\infty)}(R)$. Explanations will follow below.

Conjecture 19: (*Farrell–Jones Conjecture for Torsion Free Groups and L-theory.*) Let G be a torsion free group and let R be a ring with involution. Then the assembly map

$$H_n(BG; \mathbf{L}^{(-\infty)}(R)) \rightarrow L_n^{(-\infty)}(RG)$$

is an isomorphism for $n \in \mathbb{Z}$.

To a ring with involution one can associate (decorated) symmetric or quadratic algebraic L -groups, compare [44, 45, 256, 259] and [332]. We will exclusively deal with the quadratic algebraic L -groups and denote them by $L_n^{(j)}(R)$. Here $n \in \mathbb{Z}$ and $j \in \{-\infty\} \sqcup \{j \in \mathbb{Z} \mid j \leq 2\}$ is the so called *decoration*. The decorations $j = 0, 1$ correspond to the decorations p, h and $j = 2$ is related to the decoration s appearing in the literature. Decorations will be discussed in Remark 21 below. The L -groups $L_n^{(j)}(R)$ are 4-periodic, i.e. $L_n^{(j)}(R) \cong L_{n+4}^{(j)}(R)$ for $n \in \mathbb{Z}$.

If we are given an involution $r \mapsto \bar{r}$ on a ring R , we will always equip RG with the involution that extends the given one and satisfies $\bar{g} = g^{-1}$. On \mathbb{Z}, \mathbb{Q} and \mathbb{R} we always use the trivial involution and on \mathbb{C} the complex conjugation.

One can construct an L -theory spectrum $\mathbf{L}^{(j)}(R)$ such that $\pi_n(\mathbf{L}^{(j)}(R)) = L_n^{(j)}(R)$, compare [258, § 13]. Above and in the sequel $H_n(-; \mathbf{L}^{(j)}(R))$ denotes the homology theory which is associated to this spectrum. In particular we have $H_n(\text{pt}; \mathbf{L}^{(j)}(R)) = L_n^{(j)}(R)$. We postpone the discussion of the assembly map to Sect. 2.3.1 where we will construct it in greater generality.

20 Remark 20: (*The Coefficients in the L-theory Case.*) In contrast to K -theory (compare Remark 14) the L -theory of the most interesting coefficient ring $R = \mathbb{Z}$ is well known. The groups $L_n^{(j)}(\mathbb{Z})$ for fixed n and varying $j \in \{-\infty\} \sqcup \{j \in \mathbb{Z} \mid j \leq 2\}$ are all naturally isomorphic (compare Proposition 23 below) and we have $L_0^{(j)}(\mathbb{Z}) \cong \mathbb{Z}$ and $L_2^{(j)}(\mathbb{Z}) \cong \mathbb{Z}/2$, where the isomorphisms are given by the signature divided by 8 and the Arf invariant, and $L_1^{(j)}(\mathbb{Z}) = L_3^{(j)}(\mathbb{Z}) = 0$, see [41, chapt. III], [256, proposition 4.3.1 on page 419].

21 Remark 21: (*Decorations.*) L -groups are designed as obstruction groups for surgery problems. The decoration reflects what kind of surgery problem one is interested in. All L -groups can be described as cobordism classes of suitable quadratic Poincaré chain complexes. If one works with chain complexes of finitely generated free based R -modules and requires that the torsion of the Poincaré chain homotopy equivalence vanishes in $\tilde{K}_1(R)$, then the corresponding L -groups are denoted $L_n^{(2)}(R)$. If one drops the torsion condition, one obtains $L_n^{(1)}(R)$, which is usually denoted $L^h(R)$. If one works with finitely generated projective modules, one obtains $L^{(0)}(R)$, which is also known as $L^p(R)$.

The L -groups with negative decorations can be defined inductively via the Shaneson splitting, compare Remark 26 below. Assuming that the L -groups with decorations j have already been defined one sets

$$L_{n-1}^{<j-1>}(R) = \text{coker}(L_n^{<j>}(R) \rightarrow L_n^{<j>}(R[\mathbb{Z}])) .$$

Compare [259, definition 17.1 on page 145]. Alternatively these groups can be obtained via a process which is in the spirit of Sects. 2.2.2 and 2.2.4. One can define them as L -theory groups of suitable categories of modules parametrized over \mathbb{R}^k . For details the reader could consult [55, sect. 4]. There are forgetful maps $L_n^{(j+1)}(R) \rightarrow L_n^{(j)}(R)$. The group $L_n^{(-\infty)}(R)$ is defined as the colimit over these maps. For more information see [254, 259].

For group rings we also define $L_n^s(RG)$ similar to $L_n^{(2)}(RG)$ but we require the torsion to lie in $\text{im } A_1 \subset \tilde{K}_1(RG)$, where A_1 is the map defined in Sect. 2.2.1. Observe that $L_n^s(RG)$ really depends on the pair (R, G) and differs in general from $L_n^{(2)}(RG)$.

22 Remark 22: (*The Interplay of K- and L-theory.*) For $j \leq 1$ there are forgetful maps $L_n^{(j+1)}(R) \rightarrow L_n^{(j)}(R)$ which sit inside the following sequence, which is known as the *Rothenberg sequence* [256, proposition 1.10.1 on page 104], [259, 17.2].

$$\begin{aligned} \dots \rightarrow L_n^{(j+1)}(R) \rightarrow L_n^{(j)}(R) \rightarrow \widehat{H}^n(\mathbb{Z}/2; \tilde{K}_j(R)) \\ \rightarrow L_{n-1}^{(j+1)}(R) \rightarrow L_{n-1}^{(j)}(R) \rightarrow \dots \end{aligned} \tag{2.2}$$

Here $\widehat{H}^n(\mathbb{Z}/2; \widetilde{K}_j(R))$ is the Tate-cohomology of the group $\mathbb{Z}/2$ with coefficients in the $\mathbb{Z}[\mathbb{Z}/2]$ -module $\widetilde{K}_j(R)$. The involution on $\widetilde{K}_j(R)$ comes from the involution on R . There is a similar sequence relating $L_n^s(RG)$ and $L_n^h(RG)$, where the third term is the $\mathbb{Z}/2$ -Tate-cohomology of $\text{Wh}_1^R(G)$. Note that Tate-cohomology groups of the group $\mathbb{Z}/2$ are always annihilated by multiplication with 2. In particular $L_n^{(j)}(R) \left[\frac{1}{2} \right] = L_n^{(j)}(R) \otimes_{\mathbb{Z}} \mathbb{Z} \left[\frac{1}{2} \right]$ is always independent of j .

Let us formulate explicitly what we obtain from the above sequences for $R = \mathbb{Z}G$.

Proposition 23 Let G be a torsion free group, then Conjecture 3 about the vanishing of $\text{Wh}(G)$, $\widetilde{K}_0(\mathbb{Z}G)$ and $K_{-i}(\mathbb{Z}G)$ for $i \geq 1$ implies that for fixed n and varying $j \in \{-\infty\} \cup \{j \in \mathbb{Z} \mid j \leq 1\}$ the L -groups $L_n^{(j)}(\mathbb{Z}G)$ are all naturally isomorphic and moreover $L_n^{(1)}(\mathbb{Z}G) = L_n^h(\mathbb{Z}G) \cong L_n^s(\mathbb{Z}G)$. 23

Remark 24: (*Rational Computation.*) As in the K -theory case we have an Atiyah–Hirzebruch spectral sequence: 24

$$E_{p,q}^2 = H_p(BG; L_q^{(-\infty)}(R)) \Rightarrow H_{p+q}(BG; \mathbf{L}^{(-\infty)}(R)) .$$

Rationally this spectral sequence collapses and the homological Chern character gives for $n \in \mathbb{Z}$ an isomorphism

$$\text{ch}: \bigoplus_{p+q=n} H_p(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} (L_q^{(-\infty)}(R) \otimes_{\mathbb{Z}} \mathbb{Q}) \xrightarrow{\cong} H_n(BG; \mathbf{L}^{(-\infty)}(R)) \otimes_{\mathbb{Z}} \mathbb{Q} . \quad (2.3)$$

In particular we obtain in the case $R = \mathbb{Z}$ from Remark 20 for all $n \in \mathbb{Z}$ and all decorations j an isomorphism

$$\text{ch}: \bigoplus_{k=0}^{\infty} H_{n-4k}(BG; \mathbb{Q}) \xrightarrow{\cong} H_n(BG; \mathbf{L}^{(j)}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} . \quad (2.4)$$

This spectral sequence and the Chern character above will be discussed in a much more general setting in Chapt. 2.9.

Remark 25: (*Torsion Free is Necessary.*) If G is finite, $R = \mathbb{Z}$ and $n = 0$, then the rationalized left hand side of the assembly equals \mathbb{Q} , whereas the right hand side is isomorphic to the rationalization of the real representation ring. Since the group homology of a finite group vanishes rationally except in dimension 0, the previous remark shows that we need to assume the group to be torsion free in Conjecture 19. 25

26 **Remark 26:** (*Shaneson splitting.*) The Bass–Heller–Swan decomposition in K -theory (see Remark 15) has the following analogue for the algebraic L -groups, which is known as the *Shaneson splitting* [284]

$$L_n^{(j)}(R[\mathbb{Z}]) \cong L_{n-1}^{(j-1)}(R) \oplus L_n^{(j)}(R). \quad (2.5)$$

Here for the decoration $j = -\infty$ one has to interpret $j-1$ as $-\infty$. Since S^1 is a model for $B\mathbb{Z}$, we get an isomorphisms

$$H_n(B\mathbb{Z}; \mathbf{L}^{(j)}(R)) \cong L_{n-1}^{(j)}(R) \oplus L_n^{(j)}(R).$$

This explains why in the formulation of the L -theoretic Farrell–Jones Conjecture for torsion free groups (see Conjecture 19) we use the decoration $j = -\infty$.

As long as one deals with torsion free groups and one believes in the low dimensional part of the K -theoretic Farrell–Jones Conjecture (predicting the vanishing of $\text{Wh}(G)$, $\tilde{K}_0(\mathbb{Z}G)$ and of the negative K -groups, see Conjecture 3) there is no difference between the various decorations j , compare Proposition 23. But as soon as one allows torsion in G , the decorations make a difference and it indeed turns out that if one replaces the decoration $j = -\infty$ by $j = s, h$ or p there are counterexamples for the L -theoretic version of Conjecture 58 even for $R = \mathbb{Z}$ [123].

Even though in the above Shaneson splitting (2.5) there are no terms analogous to the Nil-terms in Remark 15 such Nil-phenomena do also occur in L -theory, as soon as one considers amalgamated free products. The corresponding groups are the UNil-groups. They vanish if one inverts 2 [49]. For more information about the UNil-groups we refer to [15, 46, 47, 74, 77, 96, 260].

2.2.6 Applications III

The Borel Conjecture

One of the driving forces for the development of the Farrell–Jones Conjectures is still the following topological rigidity conjecture about closed aspherical manifolds, compare [107]. Recall that a manifold, or more generally a CW -complex, is called *aspherical* if its universal covering is contractible. An aspherical CW -complex X with $\pi_1(X) = G$ is a model for the classifying space BG . If X is an aspherical manifold and hence finite dimensional, then G is necessarily torsionfree.

27 **Conjecture 27:** (*Borel Conjecture.*) Let $f: M \rightarrow N$ be a homotopy equivalence of aspherical closed topological manifolds. Then f is homotopic to a homeomorphism. In particular two closed aspherical manifolds with isomorphic fundamental groups are homeomorphic.

Closely related to the Borel Conjecture is the conjecture that each aspherical finitely dominated Poincaré complex is homotopy equivalent to a closed topological man-

ifold. The Borel Conjecture 27 is false in the smooth category, i.e. if one replaces topological manifold by smooth manifold and homeomorphism by diffeomorphism [106].

Using surgery theory one can show that in dimensions ≥ 5 the Borel Conjecture is implied by the K -theoretic vanishing Conjecture 3 combined with the L -theoretic Farrell–Jones Conjecture.

Theorem 28: *(The Farrell–Jones Conjecture Implies the Borel Conjecture.)* Let G be a torsion free group. If $\text{Wh}(G)$, $\tilde{K}_0(\mathbb{Z}G)$ and all the groups $K_{-i}(\mathbb{Z}G)$ with $i \geq 1$ vanish and if the assembly map

$$H_n(BG; \mathbf{L}^{(-\infty)}(\mathbb{Z})) \rightarrow L_n^{(-\infty)}(\mathbb{Z}G)$$

is an isomorphism for all n , then the Borel Conjecture holds for all orientable manifolds of dimension ≥ 5 whose fundamental group is G .

The Borel Conjecture 27 can be reformulated in the language of surgery theory to the statement that the topological structure set $\mathcal{S}^{\text{top}}(M)$ of an aspherical closed topological manifold M consists of a single point. This set is the set of equivalence classes of homotopy equivalences $f: M' \rightarrow M$ with a topological closed manifold as source and M as target under the equivalence relation, for which $f_0: M_0 \rightarrow M$ and $f_1: M_1 \rightarrow M$ are equivalent if there is a homeomorphism $g: M_0 \rightarrow M_1$ such that $f_1 \circ g$ and f_0 are homotopic.

The surgery sequence of a closed orientable topological manifold M of dimension $n \geq 5$ is the exact sequence

$$\begin{aligned} \dots \rightarrow \mathcal{N}_{n+1}(M \times [0, 1], M \times \{0, 1\}) &\xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial} \mathcal{S}^{\text{top}}(M) \\ &\xrightarrow{\eta} \mathcal{N}_n(M) \xrightarrow{\sigma} L_n^s(\mathbb{Z}\pi_1(M)), \end{aligned}$$

which extends infinitely to the left. It is the basic tool for the classification of topological manifolds. (There is also a smooth version of it.) The map σ appearing in the sequence sends a normal map of degree one to its surgery obstruction. This map can be identified with the version of the L -theory assembly map where one works with the 1-connected cover $L^s(\mathbb{Z})\langle 1 \rangle$ of $L^s(\mathbb{Z})$. The map $H_k(M; L^s(\mathbb{Z})\langle 1 \rangle) \rightarrow H_k(M; L^s(\mathbb{Z}))$ is injective for $k = n$ and an isomorphism for $k > n$. Because of the K -theoretic assumptions we can replace the s -decoration with the $\langle -\infty \rangle$ -decoration, compare Proposition 23. Therefore the Farrell–Jones Conjecture 19 implies that the maps $\sigma: \mathcal{N}_n(M) \rightarrow L_n^s(\mathbb{Z}\pi_1(M))$ and $\mathcal{N}_{n+1}(M \times [0, 1], M \times \{0, 1\}) \xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi_1(M))$ are injective respectively bijective and thus by the surgery sequence that $\mathcal{S}^{\text{top}}(M)$ is a point and hence the Borel Conjecture 27 holds for M . More details can be found e.g. in [127, pages 17,18,28], [258, chapt. 18].

For more information about surgery theory we refer for instance to [41, 44, 45, 121, 122, 167, 178, 253, 293, 294], and [320].

Automorphisms of Manifolds

If one additionally also assumes the Farrell–Jones Conjectures for higher K -theory, one can combine the surgery theoretic results with the results about pseudoisotopies from Sect. 2.2.4 to obtain the following results.

29 **Theorem 29:** (*Homotopy Groups of Top(M).*) Let M be an orientable closed aspherical manifold of dimension > 10 with fundamental group G . Suppose the L -theory assembly map

$$H_n(BG; \mathbf{L}^{(-\infty)}(\mathbb{Z})) \rightarrow L_n^{(-\infty)}(\mathbb{Z}G)$$

is an isomorphism for all n and suppose the K -theory assembly map

$$H_n(BG; \mathbf{K}(\mathbb{Z})) \rightarrow K_n(\mathbb{Z}G)$$

is an isomorphism for $n \leq 1$ and a rational isomorphism for $n \geq 2$. Then for $1 \leq i \leq (\dim M - 7)/3$ one has

$$\pi_i(\text{Top}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \text{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1, \\ 0 & \text{if } i > 1. \end{cases}$$

In the differentiable case one additionally needs to study involutions on the higher K -theory groups. The corresponding result reads:

30 **Theorem 30:** (*Homotopy Groups of Diff(M).*) Let M be an orientable closed aspherical differentiable manifold of dimension > 10 with fundamental group G . Then under the same assumptions as in Theorem 29 we have for $1 \leq i \leq (\dim M - 7)/3$

$$\pi_i(\text{Diff}(M)) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \text{center}(G) \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } i = 1; \\ \bigoplus_{j=1}^{\infty} H_{(i+1)-4j}(M; \mathbb{Q}) & \text{if } i > 1 \text{ and } \dim M \text{ odd}; \\ 0 & \text{if } i > 1 \text{ and } \dim M \text{ even}. \end{cases}$$

See for instance [97], [109, sect. 2] and [120, lecture 5]. For a modern survey on automorphisms of manifolds we refer to [329].

2.2.7 The Baum–Connes Conjecture in the Torsion Free Case

We denote by $K_*(Y)$ the complex K -homology of a topological space Y and by $K_*(C_r^*(G))$ the (topological) K -theory of the reduced group C^* -algebra. More explanations will follow below.

31 **Conjecture 31:** (*Baum–Connes Conjecture for Torsion Free Groups.*) Let G be a torsion free group. Then the Baum–Connes assembly map

$$K_n(BG) \rightarrow K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

Complex K -homology $K_*(Y)$ is the homology theory associated to the topological (complex) K -theory spectrum \mathbf{K}^{top} (which is often denoted \mathbf{BU}) and could also be written as $K_*(Y) = H_*(Y; \mathbf{K}^{\text{top}})$. The cohomology theory associated to the spectrum \mathbf{K}^{top} is the well known complex K -theory defined in terms of complex vector bundles. Complex K -homology is a 2-periodic theory, i.e. $K_n(Y) \cong K_{n+2}(Y)$.

Also the *topological K -groups* $K_n(A)$ of a C^* -algebra A are 2-periodic. Whereas $K_0(A)$ coincides with the algebraically defined K_0 -group, the other groups $K_n(A)$ take the topology of the C^* -algebra A into account, for instance $K_n(A) = \pi_{n-1}(GL(A))$ for $n \geq 1$.

Let $\mathcal{B}(l^2(G))$ denote the bounded linear operators on the Hilbert space $l^2(G)$ whose orthonormal basis is G . The *reduced complex group C^* -algebra* $C_r^*(G)$ is the closure in the norm topology of the image of the regular representation $\mathbb{C}G \rightarrow \mathcal{B}(l^2(G))$, which sends an element $u \in \mathbb{C}G$ to the (left) G -equivariant bounded operator $l^2(G) \rightarrow l^2(G)$ given by right multiplication with u . In particular one has natural inclusions

$$\mathbb{C}G \subseteq C_r^*(G) \subseteq \mathcal{B}(l^2(G))^G \subseteq \mathcal{B}(l^2(G)).$$

It is essential to use the reduced group C^* -algebra in the Baum–Connes Conjecture, there are counterexamples for the version with the maximal group C^* -algebra, compare Sect. 2.5.1. For information about C^* -algebras and their topological K -theory we refer for instance to [37, 71, 80, 154, 188, 228, 279] and [321].

Remark 32: (*The Coefficients in the Case of Topological K -theory.*) If we specialize to the trivial group $G = \{1\}$, then the complex reduced group C^* -algebra reduces to $C_r^*(G) = \mathbb{C}$ and the topological K -theory is well known: by periodicity it suffices to know that $K_0(\mathbb{C}) \cong \mathbb{Z}$, where the homomorphism is given by the dimension, and $K_1(\mathbb{C}) = 0$. Correspondingly we have $K_q(\text{pt}) = \mathbb{Z}$ for q even and $K_q(\text{pt}) = 0$ for odd q .

32

Remark 33: (*Rational Computation.*) There is an Atiyah–Hirzebruch spectral sequence which converges to $K_{p+q}(BG)$ and whose E^2 -term is $E_{p,q}^2 = H_p(BG; K_q(\text{pt}))$. Rationally this spectral sequence collapses and the homological Chern character gives an isomorphism for $n \in \mathbb{Z}$

33

$$\begin{aligned} \text{ch}: \bigoplus_{k \in \mathbb{Z}} H_{n-2k}(BG; \mathbb{Q}) &= \bigoplus_{p+q=n} H_p(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} (K_q(\mathbb{C}) \otimes_{\mathbb{Z}} \mathbb{Q}) \\ &\xrightarrow{\cong} K_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q}. \end{aligned} \quad (2.6)$$

Remark 34: (*Torsionfree is Necessary.*) In the case where G is a finite group the reduced group C^* -algebra $C_r^*(G)$ coincides with the complex group ring $\mathbb{C}G$ and $K_0(C_r^*(G))$ coincides with the complex representation ring of G . Since the group

34

homology of a finite group vanishes rationally except in dimension 0, the previous remark shows that we need to assume the group to be torsion free in Conjecture 31.

35 Remark 35: (*Bass–Heller–Swan–Decomposition for Topological K -theory.*) There is an analogue of the Bass–Heller–Swan decomposition in algebraic K -theory (see Remark 15) or of the Shaneson splitting in L -theory (see Remark 26) for topological K -theory. Namely we have

$$K_n(C_r^*(G \times \mathbb{Z})) \cong K_n(C_r^*(G)) \oplus K_{n-1}(C_r^*(G)),$$

see [243, theorem 3.1 on page 151] or more generally [244, theorem 18 on page 632]. This is consistent with the obvious isomorphism

$$K_n(B(G \times \mathbb{Z})) = K_n(BG \times S^1) \cong K_{n-1}(BG) \oplus K_n(BG).$$

Notice that here in contrast to the algebraic K -theory no Nil-terms occur (see Remark 15) and that there is no analogue of the complications in algebraic L -theory coming from the varying decorations (see Remark 26). This absence of Nil-terms or decorations is the reason why in the final formulation of the Baum–Connes Conjecture it suffices to deal with the family of finite subgroups, whereas in the algebraic K - and L -theory case one must consider the larger and harder to handle family of virtually cyclic subgroups. This in some sense makes the computation of topological K -theory of reduced group C^* -algebras easier than the computation of $K_n(\mathbb{Z}G)$ or $L_n(\mathbb{Z}G)$.

36 Remark 36: (*Real Version of the Baum–Connes Conjecture.*) There is an obvious *real version of the Baum–Connes Conjecture*. It says that for a torsion free group the real assembly map

$$KO_n(BG) \rightarrow KO_n(C_r^*(G; \mathbb{R}))$$

is bijective for $n \in \mathbb{Z}$. We will discuss in Sect. 2.5.1 below that this real version of the Baum–Connes Conjecture is implied by the complex version Conjecture 31.

Here $KO_n(C_r^*(G; \mathbb{R}))$ is the topological K -theory of the *real reduced group C^* -algebra* $C_r^*(G; \mathbb{R})$. We use KO instead of K as a reminder that we work with real C^* -algebras. The topological real K -theory $KO_*(Y)$ is the homology theory associated to the spectrum \mathbf{BO} , whose associated cohomology theory is given in terms of real vector bundles. Both, topological K -theory of a real C^* -algebra and KO -homology of a space are 8-periodic and $KO_n(\text{pt}) = K_n(\mathbb{R})$ is \mathbb{Z} , if $n = 0, 4$ (8), is $\mathbb{Z}/2$ if $n = 1, 2$ (8) and is 0 if $n = 3, 5, 6, 7$ (8).

More information about the K -theory of real C^* -algebras can be found in [281].

Applications IV

We now discuss some consequences of the Baum–Connes Conjecture for Torsion Free Groups 31.

The Trace Conjecture in the Torsion Free Case

The assembly map appearing in the Baum–Connes Conjecture has an interpretation in terms of index theory. This is important for geometric applications. It is of the same significance as the interpretation of the L -theoretic assembly map as the map σ appearing in the exact surgery sequence discussed in Sect. 2.2.5. We proceed to explain this.

An element $\eta \in K_0(BG)$ can be represented by a pair (M, P^*) consisting of a cocompact free proper smooth G -manifold M with Riemannian metric together with an elliptic G -complex P^* of differential operators of order 1 on M [29]. To such a pair one can assign an index $\text{ind}_{C_r^*(G)}(M, P^*)$ in $K_0(C_r^*(G))$ [223] which is the image of η under the assembly map $K_0(BG) \rightarrow K_0(C_r^*(G))$ appearing in Conjecture 31. With this interpretation the surjectivity of the assembly map for a torsion free group says that any element in $K_0(C_r^*(G))$ can be realized as an index. This allows to apply index theorems to get interesting information.

Here is a prototype of such an argument. The *standard trace*

$$\text{tr}_{C_r^*(G)} : C_r^*(G) \rightarrow \mathbb{C} \tag{2.7}$$

sends an element $f \in C_r^*(G) \subseteq \mathcal{B}(l^2(G))$ to $\langle f(1), 1 \rangle_{l^2(G)}$. Applying the trace to idempotent matrices yields a homomorphism

$$\text{tr}_{C_r^*(G)} : K_0(C_r^*(G)) \rightarrow \mathbb{R} .$$

Let $\text{pr}: BG \rightarrow \text{pt}$ be the projection. For a group G the following diagram commutes

$$\begin{array}{ccccc}
 K_0(BG) & \xrightarrow{A} & K_0(C_r^*(G)) & \xrightarrow{\text{tr}_{C_r^*(G)}} & \mathbb{R} \\
 \downarrow K_0(\text{pr}) & & \cong & & \uparrow i \\
 K_0(\text{pt}) & \xrightarrow{\cong} & K_0(\mathbb{C}) & \xrightarrow{\text{tr}_{\mathbb{C}}} & \mathbb{Z} .
 \end{array} \tag{2.8}$$

Here $i: \mathbb{Z} \rightarrow \mathbb{R}$ is the inclusion and A is the assembly map. This non-trivial statement follows from Atiyah’s L^2 -index theorem [12]. Atiyah’s theorem says that the L^2 -index $\text{tr}_{C_r^*(G)} \circ A(\eta)$ of an element $\eta \in K_0(BG)$, which is represented by a pair (M, P^*) , agrees with the ordinary index of $(G \backslash M; G \backslash P^*)$, which is $\text{tr}_{\mathbb{C}} \circ K_0(\text{pr})(\eta) \in \mathbb{Z}$.

The following conjecture is taken from [27, page 21].

37 **Conjecture 37:** (*Trace Conjecture for Torsion Free Groups.*) For a torsion free group G the image of

$$\mathrm{tr}_{C_r^*(G)}: K_0(C_r^*(G)) \rightarrow \mathbb{R}$$

consists of the integers.

The commutativity of diagram (2.8) above implies

38 **Consequence 38** The surjectivity of the Baum–Connes assembly map

$$K_0(BG) \rightarrow K_0(C_r^*(G))$$

implies Conjecture 37, the Trace Conjecture for Torsion Free Groups.

The Kadison Conjecture

39 **Conjecture 39:** (*Kadison Conjecture.*) If G is a torsion free group, then the only idempotent elements in $C_r^*(G)$ are 0 and 1.

40 **Lemma 40** The Trace Conjecture for Torsion Free Groups 37 implies the Kadison Conjecture 39.

Proof Assume that $p \in C_r^*(G)$ is an idempotent different from 0 or 1. From p one can construct a non-trivial projection $q \in C_r^*(G)$, i.e. $q^2 = q$, $q^* = q$, with $\mathrm{im}(p) = \mathrm{im}(q)$ and hence with $0 < q < 1$. Since the standard trace $\mathrm{tr}_{C_r^*(G)}$ is faithful, we conclude $\mathrm{tr}_{C_r^*(G)}(q) \in \mathbb{R}$ with $0 < \mathrm{tr}_{C_r^*(G)}(q) < 1$. Since $\mathrm{tr}_{C_r^*(G)}(q)$ is by definition the image of the element $[\mathrm{im}(q)] \in K_0(C_r^*(G))$ under $\mathrm{tr}_{C_r^*(G)}: K_0(C_r^*(G)) \rightarrow \mathbb{R}$, we get a contradiction to the assumption $\mathrm{im}(\mathrm{tr}_{C_r^*(G)}) \subseteq \mathbb{Z}$.

Recall that a ring R is called an *integral domain* if it has no non-trivial zero-divisors, i.e. if $r, s \in R$ satisfy $rs = 0$, then r or s is 0. Obviously the Kadison Conjecture 39 implies for $R \subseteq \mathbb{C}$ the following.

41 **Conjecture 41:** (*Idempotent Conjecture.*) Let R be an integral domain and let G be a torsion free group. Then the only idempotents in RG are 0 and 1.

The statement in the conjecture above is a purely algebraic statement. If $R = \mathbb{C}$, it is by the arguments above related to questions about operator algebras, and thus methods from operator algebras can be used to attack it.

Other Related Conjectures

We would now like to mention several conjectures which are not directly implied by the Baum–Connes or Farrell–Jones Conjectures, but which are closely related to the Kadison Conjecture and the Idempotent Conjecture mentioned above.

The next conjecture is also called the *Kaplansky Conjecture*.

Conjecture 42: (*Zero-Divisor-Conjecture.*) Let R be an integral domain and G be a torsion free group. Then RG is an integral domain. **42**

Obviously the Zero-Divisor-Conjecture 42 implies the Idempotent Conjecture 41. The Zero-Divisor-Conjecture for $R = \mathbb{Q}$ is implied by the following version of the Atiyah Conjecture (see [202, lemma 10.5 and lemma 10.15]).

Conjecture 43: (*Atiyah-Conjecture for Torsion Free Groups.*) Let G be a torsion free group and let M be a closed Riemannian manifold. Let $\overline{M} \rightarrow M$ be a regular covering with G as group of deck transformations. Then all L^2 -Betti numbers $b_p^{(2)}(\overline{M}; \mathcal{N}(G))$ are integers. **43**

For the precise definition and more information about L^2 -Betti numbers and the group von Neumann algebra $\mathcal{N}(G)$ we refer for instance to [202, 205].

This more geometric formulation of the Atiyah Conjecture is in fact implied by the following more operator theoretic version. (The two would be equivalent if one would work with rational instead of complex coefficients below.)

Conjecture 44: (*Strong Atiyah-Conjecture for Torsion Free Groups.*) Let G be a torsion free group. Then for all (m, n) -matrices A over $\mathbb{C}G$ the von Neumann dimension of the kernel of the induced G -equivariant bounded operator

$$r_A^{(2)}: \ell^2(G)^m \rightarrow \ell^2(G)^n$$

is an integer.

The Strong Atiyah-Conjecture for Torsion Free Groups implies both the Atiyah-Conjecture for Torsion Free Groups 43 [202, lemma 10.5 on page 371] and the Zero-Divisor-Conjecture 42 for $R = \mathbb{C}$ [202, lemma 10.15 on page 376].

Conjecture 45: (*Embedding Conjecture.*) Let G be a torsion free group. Then $\mathbb{C}G$ admits an embedding into a skewfield. **45**

Obviously the Embedding Conjecture implies the Zero-Divisor-Conjecture 42 for $R = \mathbb{C}$. If G is a torsion free amenable group, then the Strong Atiyah-Conjecture for Torsion Free Groups 44 and the Zero-Divisor-Conjecture 42 for $R = \mathbb{C}$ are equivalent [202, lemma 10.16 on page 376]. For more information about the Atiyah Conjecture we refer for instance to [192], [202, chapt. 10] and [261].

Finally we would like to mention the Unit Conjecture.

- 46** **Conjecture 46:** (*Unit-Conjecture.*) Let R be an integral domain and G be a torsion free group. Then every unit in RG is trivial, i.e. of the form $r \cdot g$ for some unit $r \in R^{\text{inv}}$ and $g \in G$.

The Unit Conjecture 46 implies the Zero-Divisor-Conjecture 42. For a proof of this fact and for more information we refer to [187, proposition 6.21 on page 95].

L^2 -Rho-Invariants and L^2 -Signatures

Let M be a closed connected orientable Riemannian manifold. Denote by $\eta(M) \in \mathbb{R}$ the *eta-invariant* of M and by $\eta^{(2)}(\tilde{M}) \in \mathbb{R}$ the L^2 -*eta-invariant* of the $\pi_1(M)$ -covering given by the universal covering $\tilde{M} \rightarrow M$. Let $\varrho^{(2)}(M) \in \mathbb{R}$ be the L^2 -*rho-invariant* which is defined to be the difference $\eta^{(2)}(\tilde{M}) - \eta(M)$. These invariants were studied by Cheeger and Gromov [64, 65]. They show that $\varrho^{(2)}(M)$ depends only on the diffeomorphism type of M and is in contrast to $\eta(M)$ and $\eta^{(2)}(\tilde{M})$ independent of the choice of Riemannian metric on M . The following conjecture is taken from Mathai [214].

- 47** **Conjecture 47:** (*Homotopy Invariance of the L^2 -Rho-Invariant for Torsionfree Groups.*) If $\pi_1(M)$ is torsionfree, then $\varrho^{(2)}(M)$ is a homotopy invariant.

Chang–Weinberger [62] assign to a closed connected oriented $(4k-1)$ -dimensional manifold M a Hirzebruch-type invariant $\tau^{(2)}(M) \in \mathbb{R}$ as follows. By a result of Hausmann [145] there is a closed connected oriented $4k$ -dimensional manifold W with $M = \partial W$ such that the inclusion $\partial W \rightarrow W$ induces an injection on the fundamental groups. Define $\tau^{(2)}(M)$ as the difference $\text{sign}^{(2)}(\tilde{W}) - \text{sign}(W)$ of the L^2 -signature of the $\pi_1(W)$ -covering given by the universal covering $\tilde{W} \rightarrow W$ and the signature of W . This is indeed independent of the choice of W . It is reasonable to believe that $\varrho^{(2)}(M) = \tau^{(2)}(M)$ is always true. Chang–Weinberger [62] use $\tau^{(2)}$ to prove that if $\pi_1(M)$ is not torsionfree there are infinitely many diffeomorphically distinct manifolds of dimension $4k+3$ with $k \geq 1$, which are tangentially simple homotopy equivalent to M .

- 48** **Theorem 48:** (*Homotopy Invariance of $\tau^{(2)}(M)$ and $\varrho^{(2)}(M)$.*) Let M be a closed connected oriented $(4k-1)$ -dimensional manifold M such that $G = \pi_1(M)$ is torsionfree.

- (i) If the assembly map $K_0(BG) \rightarrow K_0(C_{\max}^*(G))$ for the maximal group C^* -algebra is surjective (see Sect. 2.5.1), then $\varrho^{(2)}(M)$ is a homotopy invariant.
- (ii) Suppose that the Farrell–Jones Conjecture for L -theory 19 is rationally true for $R = \mathbb{Z}$, i.e. the rationalized assembly map

$$H_n(BG; \mathbb{L}^{(-\infty)}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L_n^{(-\infty)}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism for $n \in \mathbb{Z}$. Then $\tau^{(2)}(M)$ is a homotopy invariant. If furthermore G is residually finite, then $\varrho^{(2)}(M)$ is a homotopy invariant.

Proof

- (i) This is proved by Keswani [174, 175].
- (ii) This is proved by Chang [61] and Chang–Weinberger [62] using [210].

Remark 49 Let X be a $4n$ -dimensional Poincaré space over \mathbb{Q} . Let $\bar{X} \rightarrow X$ be a normal covering with *torsion-free* covering group G . Suppose that the assembly map $K_0(BG) \rightarrow K_0(C_{\max}^*(G))$ for the maximal group C^* -algebra is surjective (see Sect. 2.5.1) or suppose that the rationalized assembly map for L -theory

$$H_{4n}(BG; \mathbf{L}^{(-\infty)}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L_{4n}^{(-\infty)}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. Then the following L^2 -signature theorem is proved in Lück–Schick [211, theorem 2.3]

$$\text{sign}^{(2)}(\bar{X}) = \text{sign}(X) .$$

If one drops the condition that G is torsionfree this equality becomes false. Namely, Wall has constructed a finite Poincaré space X with a finite G covering $\bar{X} \rightarrow X$ for which $\text{sign}(\bar{X}) \neq |G| \cdot \text{sign}(X)$ holds (see [258, example 22.28], [319, corollary 5.4.1]).

Remark 50 Cochran–Orr–Teichner give in [69] new obstructions for a knot to be slice which are sharper than the Casson–Gordon invariants. They use L^2 -signatures and the Baum–Connes Conjecture 59. We also refer to the survey article [68] about non-commutative geometry and knot theory.

Applications V

Novikov Conjectures

The Baum–Connes and Farrell–Jones Conjectures discussed so far imply obviously that for torsion free groups the rationalized assembly maps

$$\begin{aligned} H_*(BG; \mathbf{K}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} &\rightarrow K_*(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \\ H_*(BG; \mathbf{L}^{(-\infty)}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} &\rightarrow L_*^{(-\infty)}(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \\ K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} &\rightarrow K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q} \end{aligned}$$

are injective. For reasons that will be explained below these “rational injectivity conjectures” are known as “Novikov Conjectures”. In fact one expects these injectivity results also when the groups contain torsion. So there are the following conjectures.

51 Conjecture 51: (*K- and L-theoretic Novikov Conjectures.*) For every group G the assembly maps

$$H_*(BG; \mathbf{K}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$H_*(BG; L^p(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L^p_*(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

are injective.

Observe that, since the $\mathbb{Z}/2$ -Tate cohomology groups vanish rationally, there is no difference between the various decorations in L -theory because of the Rothenberg sequence. We have chosen the p -decoration above.

The Original Novikov Conjecture

We now explain the Novikov Conjecture in its original formulation.

Let G be a (not necessarily torsion free) group and $u: M \rightarrow BG$ be a map from a closed oriented smooth manifold M to BG . Let $\mathcal{L}(M) \in \prod_{k \geq 0} H^k(M; \mathbb{Q})$ be the L -class of M , which is a certain polynomial in the Pontrjagin classes and hence depends a priori on the tangent bundle and hence on the differentiable structure of M . For $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$ define the *higher signature of M associated to x and u* to be

$$\text{sign}_x(M, u) := \langle \mathcal{L}(M) \cup u^*x, [M] \rangle \in \mathbb{Q}. \tag{2.9}$$

The Hirzebruch signature formula says that for $x = 1$ the signature $\text{sign}_1(M, u)$ coincides with the ordinary signature $\text{sign}(M)$ of M , if $\dim(M) = 4n$, and is zero, if $\dim(M)$ is not divisible by four. Recall that for $\dim(M) = 4n$ the *signature* $\text{sign}(M)$ of M is the signature of the non-degenerate bilinear symmetric pairing on the middle cohomology $H^{2n}(M; \mathbb{R})$ given by the intersection pairing $(a, b) \mapsto \langle a \cup b, [M] \rangle$. Obviously $\text{sign}(M)$ depends only on the oriented homotopy type of M . We say that sign_x for $x \in H^*(BG; \mathbb{Q})$ is *homotopy invariant* if for two closed oriented smooth manifolds M and N with reference maps $u: M \rightarrow BG$ and $v: N \rightarrow BG$ we have

$$\text{sign}_x(M, u) = \text{sign}_x(N, v)$$

if there is an orientation preserving homotopy equivalence $f: M \rightarrow N$ such that $v \circ f$ and u are homotopic.

Conjecture 52: (*Novikov Conjecture.*) Let G be a group. Then sign_x is homotopy invariant for all $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$. 52

By Hirzebruch’s signature formula the Novikov Conjecture 52 is true for $x = 1$.

Relations between the Novikov Conjectures

Using surgery theory one can show [260, proposition 6.6 on page 300] the following.

Proposition 53 For a group G the original Novikov Conjecture 52 is equivalent to the L -theoretic Novikov Conjecture, i.e. the injectivity of the assembly map 53

$$H_*(BG; L^p(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L_*^p(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} .$$

In particular for torsion free groups the L -theoretic Farrell–Jones Conjecture 19 implies the Novikov Conjecture 52. Later in Proposition 95 we will prove in particular the following statement.

Proposition 54 The Novikov Conjecture for topological K -theory, i.e. the injectivity of the assembly map 54

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the L -theoretic Novikov Conjecture and hence the original Novikov Conjecture.

For more information about the Novikov Conjectures we refer for instance to [38, 52, 55, 81, 120, 127, 179, 258] and [269].

The Zero-in-the-Spectrum Conjecture

The following Conjecture is due to Gromov [136, page 120].

Conjecture 55: (*Zero-in-the-spectrum Conjecture.*) Suppose that \tilde{M} is the universal covering of an aspherical closed Riemannian manifold M (equipped with the lifted Riemannian metric). Then zero is in the spectrum of the minimal closure 55

$$(\Delta_p)_{\min} : L^2\Omega^p(\tilde{M}) \supset \text{dom}(\Delta_p)_{\min} \rightarrow L^2\Omega^p(\tilde{M}) ,$$

for some $p \in \{0, 1, \dots, \dim M\}$, where Δ_p denotes the Laplacian acting on smooth p -forms on \tilde{M} .

56 **Proposition 56** Suppose that M is an aspherical closed Riemannian manifold with fundamental group G , then the injectivity of the assembly map

$$K_*(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the Zero-in-the-spectrum Conjecture for \tilde{M} .

Proof We give a sketch of the proof. More details can be found in [195, corollary 4]. We only explain that the assumption that in every dimension zero is not in the spectrum of the Laplacian on \tilde{M} , yields a contradiction in the case that $n = \dim(M)$ is even. Namely, this assumption implies that the $C_r^*(G)$ -valued index of the signature operator twisted with the flat bundle $\tilde{M} \times_G C_r^*(G) \rightarrow M$ in $K_0(C_r^*(G))$ is zero, where $G = \pi_1(M)$. This index is the image of the class $[S]$ defined by the signature operator in $K_0(BG)$ under the assembly map $K_0(BG) \rightarrow K_0(C_r^*(G))$. Since by assumption the assembly map is rationally injective, this implies $[S] = 0$ in $K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q}$. Notice that M is aspherical by assumption and hence $M = BG$. The homological Chern character defines an isomorphism

$$K_0(BG) \otimes_{\mathbb{Z}} \mathbb{Q} = K_0(M) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \geq 0} H_{2p}(M; \mathbb{Q}),$$

which sends $[S]$ to the Poincaré dual $\mathcal{L}(M) \cap [M]$ of the Hirzebruch L -class $\mathcal{L}(M) \in \bigoplus_{p \geq 0} H^{2p}(M; \mathbb{Q})$. This implies that $\mathcal{L}(M) \cap [M] = 0$ and hence $\mathcal{L}(M) = 0$. This contradicts the fact that the component of $\mathcal{L}(M)$ in $H^0(M; \mathbb{Q})$ is 1.

More information about the Zero-in-the-spectrum Conjecture 55 can be found for instance in [195] and [202, sect. 12].

2.3 The Conjectures in the General Case

In this chapter we will formulate the Baum–Connes and Farrell–Jones Conjectures. We try to emphasize the unifying principle that underlies these conjectures. The point of view taken in this chapter is that all three conjectures are conjectures about specific equivariant homology theories. Some of the technical details concerning the actual construction of these homology theories are deferred to Chapt. 2.7.

2.3.1 Formulation of the Conjectures

Suppose we are given

- A discrete group G ;
- A family \mathcal{F} of subgroups of G , i.e. a set of subgroups which is closed under conjugation with elements of G and under taking finite intersections;
- A G -homology theory $\mathcal{H}_*^G(-)$.

Then one can formulate the following Meta-Conjecture.

Meta-Conjecture 57 The assembly map

$$A_{\mathcal{F}} : \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \rightarrow \mathcal{H}_n^G(\text{pt})$$

which is the map induced by the projection $E_{\mathcal{F}}(G) \rightarrow \text{pt}$, is an isomorphism for $n \in \mathbb{Z}$.

Here $E_{\mathcal{F}}(G)$ is the classifying space of the family \mathcal{F} , a certain G -space which specializes to the universal free G -space EG if the family contains only the trivial subgroup. A G -homology theory is the “obvious” G -equivariant generalization of the concept of a homology theory to a suitable category of G -spaces, in particular it is a functor on such spaces and the map $A_{\mathcal{F}}$ is simply the map induced by the projection $E_{\mathcal{F}}(G) \rightarrow \text{pt}$. We devote the Sects. 2.3.1 to 2.3.1 below to a discussion of G -homology theories, classifying spaces for families of subgroups and related things. The reader who never encountered these concepts should maybe first take a look at these subsections.

Of course the conjecture above is not true for arbitrary G , \mathcal{F} and $\mathcal{H}_*^G(-)$, but the Farrell–Jones and Baum–Connes Conjectures state that for specific G -homology theories there is a natural choice of a family $\mathcal{F} = \mathcal{F}(G)$ of subgroups for every group G such that $A_{\mathcal{F}(G)}$ becomes an isomorphism for all groups G .

Let R be a ring (with involution). In Proposition 156 we will describe the construction of G -homology theories which will be denoted

$$H_n^G(-; \mathbf{K}_R), \quad H_n^G(-; \mathbf{L}_R^{(-\infty)}) \quad \text{and} \quad H_n^G(-; \mathbf{K}^{\text{top}}).$$

If G is the trivial group, these homology theories specialize to the (non-equivariant) homology theories with similar names that appeared in Chapt. 2.2, namely to

$$H_n(-; \mathbf{K}(R)), \quad H_n(-; \mathbf{L}^{(-\infty)}(R)) \quad \text{and} \quad K_n(-).$$

Another main feature of these G -homology theories is that evaluated on the one point space pt (considered as a trivial G -space) we obtain the K - and L -theory of the group ring RG , respectively the topological K -theory of the reduced C^* -algebra (see Proposition 156 and Theorem 158 (iii))

$$\begin{aligned} K_n(RG) &\cong H_n^G(\text{pt}; \mathbf{K}_R), \\ L_n^{(-\infty)}(RG) &\cong H_n^G(\text{pt}; \mathbf{L}_R^{(-\infty)}) \quad \text{and} \\ K_n(C_r^*(G)) &\cong H_n^G(\text{pt}; \mathbf{K}^{\text{top}}). \end{aligned}$$

We are now prepared to formulate the conjectures around which this article is centered. Let \mathcal{FIN} be the family of finite subgroups and let $\mathcal{V}\mathcal{C}\mathcal{Y}$ be the family of virtually cyclic subgroups.

58 **Conjecture 58:** (*Farrell–Jones Conjecture for K - and L -theory.*) Let R be a ring (with involution) and let G be a group. Then for all $n \in \mathbb{Z}$ the maps

$$\begin{aligned} A_{\text{vcy}}: H_n^G(E_{\text{vcy}}(G); \mathbf{K}_R) &\rightarrow H_n^G(\text{pt}; \mathbf{K}_R) \cong K_n(RG); \\ A_{\text{vcy}}: H_n^G(E_{\text{vcy}}(G); \mathbf{L}_R^{(-\infty)}) &\rightarrow H_n^G(\text{pt}; \mathbf{L}_R^{(-\infty)}) \cong L_n^{(-\infty)}(RG), \end{aligned}$$

which are induced by the projection $E_{\text{vcy}}(G) \rightarrow \text{pt}$, are isomorphisms.

The conjecture for the topological K -theory of C^* -algebras is known as the Baum–Connes Conjecture and reads as follows.

59 **Conjecture 59:** (*Baum–Connes Conjecture.*) Let G be a group. Then for all $n \in \mathbb{Z}$ the map

$$A_{\mathcal{F}\mathcal{I}\mathcal{N}}: H_n^G(E_{\mathcal{F}\mathcal{I}\mathcal{N}}(G); \mathbf{K}^{\text{top}}) \rightarrow H_n^G(\text{pt}; \mathbf{K}^{\text{top}}) \cong K_n(C_r^*(G))$$

induced by the projection $E_{\mathcal{F}\mathcal{I}\mathcal{N}}(G) \rightarrow \text{pt}$ is an isomorphism.

We will explain the analytic assembly map $\text{ind}_G: K_n^G(X) \rightarrow K_n(C_r^*(G))$, which can be identified with the assembly map appearing in the Baum–Connes Conjecture 59 in Sect. 2.8.2.

60 **Remark 60** Of course the conjectures really come to life only if the abstract point of view taken in this chapter is connected up with more concrete descriptions of the assembly maps. We have already discussed a surgery theoretic description in Theorem 28 and an interpretation in terms index theory in Sect. 2.2.8. More information about alternative interpretations of assembly maps can be found in Sects. 2.8.2 and 2.8.8. These concrete interpretations of the assembly maps lead to applications. We already discussed many such applications in Chapt. 2.2 and encourage the reader to go ahead and browse through Chapt. 2.4 in order to get further ideas about these more concrete aspects.

61 **Remark 61:** (*Relation to the “classical” assembly maps.*) The value of an equivariant homology theory $\mathcal{H}_*^G(-)$ on the universal free G -space $EG = E_{\{\}}(G)$ (a free G -CW-complex whose quotient EG/G is a model for BG) can be identified with the corresponding non-equivariant homology theory evaluated on BG , if we assume that \mathcal{H}_*^G is the special value of an equivariant homology theory $\mathcal{H}_*^?$ at $? = G$. This means that there exists an induction structure (a mild condition satisfied in our examples, compare Sect. 2.7.1), which yields an identification

$$\mathcal{H}_n^G(EG) \cong \mathcal{H}_n^{\{\}}(BG) = \mathcal{H}_n(BG).$$

Using these identifications the “classical” assembly maps, which appeared in Chapt. 2.2 in the versions of the Farrell–Jones and Baum–Connes Conjectures for torsion free groups (see Conjectures 11, 19 and 31),

$$\begin{aligned} H_n(BG; \mathbf{K}(R)) &\cong H_n^G(EG; \mathbf{K}_R) \rightarrow H_n^G(\text{pt}; \mathbf{K}_R) \cong K_n(RG) ; \\ H_n(BG; \mathbf{L}^{(-\infty)}(R)) &\cong H_n^G(EG; \mathbf{L}_R^{(-\infty)}) \rightarrow H_n^G(\text{pt}; \mathbf{L}_R^{(-\infty)}) \cong L_n^{(-\infty)}(RG) ; \\ \text{and } K_n(BG) &\cong H_n^G(EG; \mathbf{K}^{\text{top}}) \rightarrow H_n^G(\text{pt}; \mathbf{K}^{\text{top}}) \cong K_n(C_r^*(G)) , \end{aligned}$$

correspond to the assembly maps for the family $\mathcal{F} = \{\}$ consisting only of the trivial group and are simply the maps induced by the projection $EG \rightarrow \text{pt}$.

Remark 62: (*The choice of the right family.*) As explained above the Farrell–Jones and Baum–Connes Conjectures 58 and 59 can be considered as special cases of the Meta-Conjecture 57. In all three cases we are interested in a computation of the right hand side $\mathcal{H}_n^G(\text{pt})$ of the assembly map, which can be identified with $K_n(RG)$, $L_n^{(-\infty)}(RG)$ or $K_n(C_r^*(G))$. The left hand side $\mathcal{H}_n^G(E_{\mathcal{F}}(G))$ of such an assembly map is much more accessible and the smaller \mathcal{F} is, the easier it is to compute $\mathcal{H}_n^G(E_{\mathcal{F}}(G))$ using homological methods like spectral sequences, Mayer–Vietoris arguments and Chern characters.

In the extreme case where $\mathcal{F} = \mathcal{AL}\mathcal{L}$ is the family of all subgroups the assembly map $A_{\mathcal{AL}\mathcal{L}}: \mathcal{H}_n^G(E_{\mathcal{AL}\mathcal{L}}(G)) \rightarrow \mathcal{H}_n^G(\text{pt})$ is always an isomorphism for the trivial reason that the one point space pt is a model for $E_{\mathcal{AL}\mathcal{L}}(G)$ (compare Sect. 2.3.1) and hence the assembly map is the identity. The goal however is to have an isomorphism for a family which is as small as possible.

We have already seen in Remark 4, Remark 25 and Remark 34 that in all three cases the classical assembly map, which corresponds to the trivial family, is not surjective for finite groups. This forces one to include at least the family \mathcal{FIN} of finite groups. The K - or L -theory of the finite subgroups of the given group G will then enter in a computation of the left hand side of the assembly map similar as the K - and L -theory of the trivial subgroup appeared on the left hand side in the classical case, compare e.g. Remark 14. In the Baum–Connes case the family \mathcal{FIN} seems to suffice. However in the case of algebraic K -theory we saw in Remark 15 that already the simplest torsion free group, the infinite cyclic group, causes problems because of the Nil-terms that appear in the Bass–Heller–Swan formula. The infinite dihedral group is a “minimal counterexample” which shows that the family \mathcal{FIN} is not sufficient in the $L_{\mathbb{Z}}^{(-\infty)}$ -case. There are non-vanishing UNil-terms, compare Proposition 75 and its proof. Also the version of the L -theoretic Farrell–Jones Conjecture with the decoration $s, h = \langle 1 \rangle$ or $p = \langle 0 \rangle$ instead of $\langle -\infty \rangle$ is definitely false. Counterexamples are given in [123]. Recall that there were no Nil-terms in the topological K -theory context, compare Remark 35.

The choice of the family $\mathcal{VC}\mathcal{Y}$ of virtually cyclic subgroups in the Farrell–Jones conjectures pushes all the Nil-problems appearing in algebraic K - and L -theory

into the source of the assembly map so that they do not occur if one tries to prove the Farrell–Jones Conjecture 58. Of course they do come up again when one wants to compute the source of the assembly map.

We now take up the promised detailed discussion of some notions like equivariant homology theories and classifying spaces for families we used above. The reader who is familiar with these concepts may of course skip the following subsections.

G-CW-Complexes

A *G-CW-complex* X is a G -space X together with a filtration $X_{-1} = \emptyset \subseteq X_0 \subseteq X_1 \subseteq \dots \subseteq X$ such that $X = \operatorname{colim}_{n \rightarrow \infty} X_n$ and for each n there is a G -pushout

$$\begin{array}{ccc}
 \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\
 \downarrow & & \downarrow \\
 \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n .
 \end{array}$$

This definition makes also sense for topological groups. The following alternative definition only applies to discrete groups. A *G-CW-complex* is a *CW-complex* with a G -action by cellular maps such that for each open cell e and each $g \in G$ with $ge \cap e \neq \emptyset$ we have $gx = x$ for all $x \in e$. There is an obvious notion of a *G-CW-pair*.

A *G-CW-complex* X is called *finite* if it is built out of finitely many G -cells $G/H_i \times D^n$. This is the case if and only if it is *cocompact*, i.e. the quotient space $G \backslash X$ is compact. More information about *G-CW-complexes* can be found for instance in [197, sects. 1 and 2], [304, sects. II.1 and II.2].

Families of Subgroups

A *family \mathcal{F} of subgroups of G* is a set of subgroups of G closed under conjugation, i.e. $H \in \mathcal{F}, g \in G$ implies $g^{-1}Hg \in \mathcal{F}$, and finite intersections, i.e. $H, K \in \mathcal{F}$ implies $H \cap K \in \mathcal{F}$. Throughout the text we will use the notations

$$\{\}, \mathcal{FCY}, \mathcal{FLN}, \mathcal{CYC}, \mathcal{VCY}_1, \mathcal{VCY} \quad \text{and} \quad \mathcal{ALL}$$

for the families consisting of the trivial, all finite cyclic, all finite, all (possibly infinite) cyclic, all virtually cyclic of the first kind, all virtually cyclic, respectively all subgroups of a given group G . Recall that a group is called *virtually cyclic* if it is finite or contains an infinite cyclic subgroup of finite index. A group is *virtually cyclic of the first kind* if it admits a surjection onto an infinite cyclic group with finite kernel, compare Lemma 71.

Classifying Spaces for Families

Let \mathcal{F} be a family of subgroups of G . A G -CW-complex, all whose isotropy groups belong to \mathcal{F} and whose H -fixed point sets are contractible for all $H \in \mathcal{F}$, is called a *classifying space for the family \mathcal{F}* and will be denoted $E_{\mathcal{F}}(G)$. Such a space is unique up to G -homotopy because it is characterized by the property that for any G -CW-complex X , all whose isotropy groups belong to \mathcal{F} , there is up to G -homotopy precisely one G -map from X to $E_{\mathcal{F}}(G)$. These spaces were introduced by tom Dieck [303], [304, I.6].

A functorial “bar-type” construction is given in [82, sect. 7].

If $\mathcal{F} \subset \mathcal{G}$ are families of subgroups for G , then by the universal property there is up to G -homotopy precisely one G -map $E_{\mathcal{F}}(G) \rightarrow E_{\mathcal{G}}(G)$.

The space $E_{\{1\}}(G)$ is the same as the space EG which is by definition the total space of the universal G -principal bundle $G \rightarrow EG \rightarrow BG$, or, equivalently, the universal covering of BG . A model for $E_{\mathcal{A}\mathcal{L}\mathcal{L}}(G)$ is given by the space $G/G = \text{pt}$ consisting of one point.

The space $E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$ is also known as *the classifying space for proper G -actions* and denoted by $\underline{E}G$ in the literature. Recall that a G -CW-complex X is proper if and only if all its isotropy groups are finite (see for instance [197, theorem 1.23 on page 18]). There are often nice models for $E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$. If G is word hyperbolic in the sense of Gromov, then the Rips-complex is a finite model [216, 217].

If G is a discrete subgroup of a Lie group L with finitely many path components, then for any maximal compact subgroup $K \subseteq L$ the space L/K with its left G -action is a model for $E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$ [2, corollary 4.14]. More information about $E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$ can be found for instance in [28, sect. 2], [180, 199, 206, 207] and [282].

G -Homology Theories

Fix a group G and an associative commutative ring Λ with unit. A *G -homology theory \mathcal{H}_*^G with values in Λ -modules* is a collection of covariant functors \mathcal{H}_n^G from the category of G -CW-pairs to the category of Λ -modules indexed by $n \in \mathbb{Z}$ together with natural transformations

$$\partial_n^G(X, A): \mathcal{H}_n^G(X, A) \rightarrow \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A, \emptyset)$$

for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

- (i) G -homotopy invariance
 If f_0 and f_1 are G -homotopic maps $(X, A) \rightarrow (Y, B)$ of G -CW-pairs, then $\mathcal{H}_n^G(f_0) = \mathcal{H}_n^G(f_1)$ for $n \in \mathbb{Z}$.

- (ii) Long exact sequence of a pair
 Given a pair (X, A) of G -CW-complexes, there is a long exact sequence

$$\begin{aligned} \dots \xrightarrow{\mathcal{H}_{n+1}^G(j)} \mathcal{H}_{n+1}^G(X, A) \xrightarrow{\partial_{n+1}^G} \mathcal{H}_n^G(A) \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X) \\ \xrightarrow{\mathcal{H}_n^G(j)} \mathcal{H}_n^G(X, A) \xrightarrow{\partial_n^G} \mathcal{H}_{n-1}^G(A) \xrightarrow{\mathcal{H}_{n-1}^G(i)} \dots, \end{aligned}$$

where $i: A \rightarrow X$ and $j: X \rightarrow (X, A)$ are the inclusions.

(iii) Excision

Let (X, A) be a G -CW-pair and let $f: A \rightarrow B$ be a cellular G -map of G -CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a G -CW-pair. Then the canonical map $(F, f): (X, A) \rightarrow (X \cup_f B, B)$ induces for each $n \in \mathbb{Z}$ an isomorphism

$$\mathcal{H}_n^G(F, f): \mathcal{H}_n^G(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(X \cup_f B, B) .$$

(vi) Disjoint union axiom

Let $\{X_i \mid i \in I\}$ be a family of G -CW-complexes. Denote by $j_i: X_i \rightarrow \coprod_{i \in I} X_i$ the canonical inclusion. Then the map

$$\bigoplus_{i \in I} \mathcal{H}_n^G(j_i): \bigoplus_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G\left(\coprod_{i \in I} X_i\right)$$

is bijective for each $n \in \mathbb{Z}$.

Of course a G -homology theory for the trivial group $G = \{1\}$ is a homology theory (satisfying the disjoint union axiom) in the classical non-equivariant sense.

The disjoint union axiom ensures that we can pass from finite G -CW-complexes to arbitrary ones using the following lemma.

63

Lemma 63 Let \mathcal{H}_*^G be a G -homology theory. Let X be a G -CW-complex and $\{X_i \mid i \in I\}$ be a directed system of G -CW-subcomplexes directed by inclusion such that $X = \cup_{i \in I} X_i$. Then for all $n \in \mathbb{Z}$ the natural map

$$\operatorname{colim}_{i \in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

is bijective.

Proof Compare for example with [301, proposition 7.53 on page 121], where the non-equivariant case for $I = \mathbb{N}$ is treated.

Example 64. (*Bredon Homology.*) The most basic G -homology theory is *Bredon homology*. The orbit category $\operatorname{Or}(G)$ has as objects the homogeneous spaces G/H and as morphisms G -maps. Let X be a G -CW-complex. It defines a contravariant functor from the orbit category $\operatorname{Or}(G)$ to the category of CW-complexes by sending G/H to $\operatorname{map}_G(G/H, X) = X^H$. Composing it with the functor cellular chain complex yields a contravariant functor

$$C_*^c(X): \operatorname{Or}(G) \rightarrow \mathbb{Z}\text{-CHCOM}$$

into the category of \mathbb{Z} -chain complexes. Let Λ be a commutative ring and let

$$M: \operatorname{Or}(G) \rightarrow \Lambda\text{-MODULES}$$

be a covariant functor. Then one can form the tensor product over the orbit category (see for instance [197, 9.12 on page 166]) and obtains the Λ -chain complex $C_*^c(X) \otimes_{\mathbb{Z}\text{Or}(G)} M$. Its homology is the Bredon homology of X with coefficients in M

$$H_*^G(X; M) = H_*(C_*^c(X) \otimes_{\mathbb{Z}\text{Or}(G)} M) .$$

Thus we get a G -homology theory H_*^G with values in Λ -modules. For a trivial group G this reduces to the cellular homology of X with coefficients in the Λ -module M .

More information about equivariant homology theories will be given in Sect. 2.7.1.

Varying the Family of Subgroups

Suppose we are given a family of subgroups \mathcal{F}' and a subfamily $\mathcal{F} \subset \mathcal{F}'$. Since all isotropy groups of $E_{\mathcal{F}}(G)$ lie in \mathcal{F}' we know from the universal property of $E_{\mathcal{F}'}(G)$ (compare Sect. 2.3.1) that there is a G -map $E_{\mathcal{F}}(G) \rightarrow E_{\mathcal{F}'}(G)$ which is unique up to G -homotopy. For every G -homology theory \mathcal{H}_*^G we hence obtain a *relative assembly map*

$$A_{\mathcal{F} \rightarrow \mathcal{F}'}: \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \rightarrow \mathcal{H}_n^G(E_{\mathcal{F}'}(G)) .$$

If $\mathcal{F}' = \mathcal{A}\mathcal{L}\mathcal{L}$, then $E_{\mathcal{F}'}(G) = \text{pt}$ and $A_{\mathcal{F} \rightarrow \mathcal{F}'}$ specializes to the assembly map $A_{\mathcal{F}}$ we discussed in the previous section. If we now gradually increase the family, we obtain qa factorization of the classical assembly $A = A_{\{\} \rightarrow \mathcal{A}\mathcal{L}\mathcal{L}}$ into several relative assembly maps. We obtain for example from the inclusions

$$\{\} \subset \mathcal{F}\mathcal{C}\mathcal{Y} \subset \mathcal{F}\mathcal{I}\mathcal{N} \subset \mathcal{V}\mathcal{C}\mathcal{Y} \subset \mathcal{A}\mathcal{L}\mathcal{L}$$

for every G -homology theory $\mathcal{H}_n^G(-)$ the following commutative diagram.

$$\begin{array}{ccccc}
 \mathcal{H}_n^G(EG) & \xrightarrow{A} & \mathcal{H}_n^G(\text{pt}) & & \\
 \downarrow & & \nearrow A_{\mathcal{F}\mathcal{I}\mathcal{N}} & & \uparrow A_{\mathcal{V}\mathcal{C}\mathcal{Y}} \\
 \mathcal{H}_n^G(E_{\mathcal{F}\mathcal{C}\mathcal{Y}}(G)) & \longrightarrow & \mathcal{H}_n^G(E_{\mathcal{F}\mathcal{I}\mathcal{N}}(G)) & \longrightarrow & \mathcal{H}_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G)) .
 \end{array} \tag{2.10}$$

Here A is the “classical” assembly map and $A_{\mathcal{F}\mathcal{I}\mathcal{N}}$ and $A_{\mathcal{V}\mathcal{C}\mathcal{Y}}$ are the assembly maps that for specific G -homology theories appear in the Baum–Connes and Farrell–Jones Conjectures.

Such a factorization is extremely useful because one can study the relative assembly map $A_{\mathcal{F} \rightarrow \mathcal{F}'}$ in terms of absolute assembly maps corresponding to groups in the bigger family. For example the relative assembly map

$$A_{\mathcal{F}\mathcal{I}\mathcal{N} \rightarrow \mathcal{V}\mathcal{C}\mathcal{Y}}: \mathcal{H}_n^G(E_{\mathcal{F}\mathcal{I}\mathcal{N}}(G)) \rightarrow \mathcal{H}_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G))$$

is an isomorphism if for all virtually cyclic subgroups V of G the assembly map

$$A_{\mathcal{F}, \mathcal{L}, N} = A_{\mathcal{F}, \mathcal{L}, N \rightarrow \mathcal{A}, \mathcal{L}, \mathcal{L}} : \mathcal{H}_n^V(E_{\mathcal{F}, \mathcal{L}, N}(V)) \rightarrow \mathcal{H}_n^V(\text{pt})$$

is an isomorphism. Of course here we need to assume that the G -homology theory \mathcal{H}_*^G and the V -homology theory \mathcal{H}_*^V are somehow related. In fact all the G -homology theories \mathcal{H}_*^G we care about are defined simultaneously for all groups G and for varying G these G -homology theories are related via a so called “induction structure”. Induction structures will be discussed in detail in Sect. 2.7.1.

For a family \mathcal{F} of subgroups of G and a subgroup $H \subset G$ we define a family of subgroups of H

$$\mathcal{F} \cap H = \{K \cap H \mid K \in \mathcal{F}\}.$$

The general statement about relative assembly maps reads now as follows.

65

Theorem 65: (*Transitivity Principle.*) Let $\mathcal{H}_*^{\rho^2}(-)$ be an equivariant homology theory in the sense of Sect. 2.7.1. Suppose $\mathcal{F} \subset \mathcal{F}'$ are two families of subgroups of G . Suppose that $K \cap H \in \mathcal{F}$ for each $K \in \mathcal{F}$ and $H \in \mathcal{F}'$ (this is automatic if \mathcal{F} is closed under taking subgroups). Let N be an integer. If for every $H \in \mathcal{F}'$ and every $n \leq N$ the assembly map

$$A_{\mathcal{F} \cap H \rightarrow \mathcal{A}, \mathcal{L}, \mathcal{L}} : \mathcal{H}_n^H(E_{\mathcal{F} \cap H}(H)) \rightarrow \mathcal{H}_n^H(\text{pt})$$

is an isomorphism, then for every $n \leq N$ the relative assembly map

$$A_{\mathcal{F} \rightarrow \mathcal{F}'} : \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \rightarrow \mathcal{H}_n^G(E_{\mathcal{F}'}(G))$$

is an isomorphism.

Proof If we equip $E_{\mathcal{F}}(G) \times E_{\mathcal{F}'}(G)$ with the diagonal G -action, it is a model for $E_{\mathcal{F}}(G)$. Now apply Lemma 153 in the special case $Z = E_{\mathcal{F}'}(G)$.

This principle can be used in many ways. For example we will derive from it that the general versions of the Baum–Connes and Farrell–Jones Conjectures specialize to the conjectures we discussed in Chapt. 2.2 in the case where the group is torsion free. If we are willing to make compromises, e.g. to invert 2, to rationalize the theories or to restrict ourselves to small dimensions or special classes of groups, then it is often possible to get away with a smaller family, i.e. to conclude from the Baum–Connes or Farrell–Jones Conjectures that an assembly map with respect to a family smaller than the family of finite or virtually cyclic subgroups is an isomorphism. The left hand side becomes more computable and this leads to new corollaries of the Baum–Connes and Farrell–Jones Conjectures.

The General Versions Specialize to the Torsion Free Versions

If G is a torsion free group, then the family \mathcal{FLN} obviously coincides with the trivial family $\{1\}$. Since a nontrivial torsion free virtually cyclic group is infinite cyclic we also know that the family \mathcal{VCY} reduces to the family of all cyclic subgroups, denoted \mathcal{CYC} .

Proposition 66 Let G be a torsion free group.

66

(i) If R is a regular ring, then the relative assembly map

$$A_{\{1\} \rightarrow \mathcal{CYC}} : H_n^G(E_{\{1\}}(G); \mathbf{K}_R) \rightarrow H_n^G(E_{\mathcal{CYC}}(G); \mathbf{K}_R)$$

is an isomorphism.

(ii) For every ring R the relative assembly map

$$A_{\{1\} \rightarrow \mathcal{CYC}} : H_n^G(E_{\{1\}}(G); \mathbf{L}_R^{(-\infty)}) \rightarrow H_n^G(E_{\mathcal{CYC}}(G); \mathbf{L}_R^{(-\infty)})$$

is an isomorphism.

Proof Because of the Transitivity Principle 65 it suffices in both cases to prove that the classical assembly map $A = A_{\{1\} \rightarrow \mathcal{ALL}}$ is an isomorphism in the case where G is an infinite cyclic group. For regular rings in the K -theory case and with the $-\infty$ -decoration in the L -theory case this is true as we discussed in Remark 15 respectively Remark 26.

As an immediate consequence we obtain.

Corollary 67

67

- (i) For a torsion free group the Baum–Connes Conjecture 59 is equivalent to its “torsion free version” Conjecture 31.
- (ii) For a torsion free group the Farrell–Jones Conjecture 58 for algebraic K - is equivalent to the “torsion free version” Conjecture 11, provided R is regular.
- (iii) For a torsion free group the Farrell–Jones Conjecture 58 for algebraic L -theory is equivalent to the “torsion free version” Conjecture 19.

The Baum–Connes Conjecture and the Family \mathcal{VCY}

Replacing the family \mathcal{FLN} of finite subgroups by the family \mathcal{VCY} of virtually cyclic subgroups would not make any difference in the Baum–Connes Conjecture 59. The Transitivity Principle 65 and the fact that the Baum–Connes Conjecture 59 is known for virtually cyclic groups implies the following.

68 **Proposition 68** For every group G and every $n \in \mathbb{Z}$ the relative assembly map for topological K -theory

$$A_{\mathcal{F}\mathcal{L}\mathcal{N} \rightarrow \mathcal{V}\mathcal{C}\mathcal{Y}}: H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}^{\text{top}}) \rightarrow H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G); \mathbf{K}^{\text{top}})$$

is an isomorphism.

The Baum–Connes Conjecture and the Family $\mathcal{F}\mathcal{C}\mathcal{Y}$

The following result is proven in [215].

69 **Proposition 69** For every group G and every $n \in \mathbb{Z}$ the relative assembly map for topological K -theory

$$A_{\mathcal{F}\mathcal{C}\mathcal{Y} \rightarrow \mathcal{F}\mathcal{L}\mathcal{N}}: H_n^G(E_{\mathcal{F}\mathcal{C}\mathcal{Y}}(G); \mathbf{K}^{\text{top}}) \rightarrow H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}^{\text{top}})$$

is an isomorphism.

In particular the Baum–Connes Conjecture predicts that the $\mathcal{F}\mathcal{C}\mathcal{Y}$ -assembly map

$$A_{\mathcal{F}\mathcal{C}\mathcal{Y}}: H_n^G(E_{\mathcal{F}\mathcal{C}\mathcal{Y}}(G); \mathbf{K}^{\text{top}}) \rightarrow K_n(C_r^*(G))$$

is always an isomorphism.

Algebraic K -Theory for Special Coefficient Rings

In the algebraic K -theory case we can reduce to the family of finite subgroups if we assume special coefficient rings.

70 **Proposition 70** Suppose R is a regular ring in which the orders of all finite subgroups of G are invertible. Then for every $n \in \mathbb{Z}$ the relative assembly map for algebraic K -theory

$$A_{\mathcal{F}\mathcal{L}\mathcal{N} \rightarrow \mathcal{V}\mathcal{C}\mathcal{Y}}: H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}_R) \rightarrow H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G); \mathbf{K}_R)$$

is an isomorphism. In particular if R is a regular ring which is a \mathbb{Q} -algebra (for example a field of characteristic 0) the above applies to all groups G .

Proof We first show that RH is regular for a finite group H . Since R is Noetherian and H is finite, RH is Noetherian. It remains to show that every RH -module M has a finite dimensional projective resolution. By assumption M considered as an R -module has a finite dimensional projective resolution. If one applies $RH \otimes_R$ – this

yields a finite dimensional RH -resolution of $RH \otimes_R \text{res } M$. Since $|H|$ is invertible, the RH -module M is a direct summand of $RH \otimes_R \text{res } M$ and hence has a finite dimensional projective resolution.

Because of the Transitivity Principle 65 we need to prove that the \mathcal{FLN} -assembly map $A_{\mathcal{FLN}}$ is an isomorphism for virtually cyclic groups V . Because of Lemma 71 we can assume that either $V \cong H \rtimes \mathbb{Z}$ or $V \cong K_1 *_H K_2$ with finite groups H , K_1 and K_2 . From [313] we obtain in both cases long exact sequences involving the algebraic K -theory of the constituents, the algebraic K -theory of V and also additional Nil-terms. However, in both cases the Nil-terms vanish if RH is a regular ring (compare theorem 4 on page 138 and the remark on page 216 in [313]). Thus we get long exact sequences

$$\dots \rightarrow K_n(RH) \rightarrow K_n(RH) \rightarrow K_n(RV) \rightarrow K_{n-1}(RH) \rightarrow K_{n-1}(RH) \rightarrow \dots$$

and

$$\begin{aligned} \dots \rightarrow K_n(RH) &\rightarrow K_n(RK_1) \oplus K_n(RK_2) \rightarrow K_n(RV) \\ &\rightarrow K_{n-1}(RH) \rightarrow K_{n-1}(RK_1) \oplus K_{n-1}(RK_2) \rightarrow \dots \end{aligned}$$

One obtains analogous exact sequences for the sources of the various assembly maps from the fact that the sources are equivariant homology theories and one can find specific models for $E_{\mathcal{FLN}}(V)$. These sequences are compatible with the assembly maps. The assembly maps for the finite groups H, K_1 and K_2 are bijective. Now a Five-Lemma argument shows that also the one for V is bijective.

In particular for regular coefficient rings R which are \mathbb{Q} -algebras the K -theoretic Farrell–Jones Conjecture specializes to the conjecture that the assembly map

$$A_{\mathcal{FLN}}: H_n^G(E_{\mathcal{FLN}}(G); \mathbf{K}_R) \rightarrow H_n^G(\text{pt}; \mathbf{K}_R) \cong K_n(RG)$$

is an isomorphism.

In the proof above we used the following important fact about virtually cyclic groups.

Lemma 71 If G is an infinite virtually cyclic group then we have the following dichotomy. 71

- (I) Either G admits a surjection with finite kernel onto the infinite cyclic group \mathbb{Z} , or
- (II) G admits a surjection with finite kernel onto the infinite dihedral group $\mathbb{Z}/2 * \mathbb{Z}/2$.

Proof This is not difficult and proven as lemma 2.5 in [113].

Splitting off Nil-Terms and Rationalized Algebraic K -Theory

Recall that the Nil-terms, which prohibit the classical assembly map from being an isomorphism, are direct summands of the K -theory of the infinite cyclic group (see Remark 15). Something similar remains true in general [16].

72 Proposition 72

(i) For every group G , every ring R and every $n \in \mathbb{Z}$ the relative assembly map

$$A_{\mathcal{F}\mathcal{L}\mathcal{N} \rightarrow \mathcal{V}\mathcal{C}\mathcal{Y}} : H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}_R) \rightarrow H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G); \mathbf{K}_R)$$

is split-injective.

(ii) Suppose R is such that $K_{-i}(RV) = 0$ for all virtually cyclic subgroups V of G and for sufficiently large i (for example $R = \mathbb{Z}$ will do, compare Proposition 78). Then the relative assembly map

$$A_{\mathcal{F}\mathcal{L}\mathcal{N} \rightarrow \mathcal{V}\mathcal{C}\mathcal{Y}} : H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{L}_R^{(-\infty)}) \rightarrow H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G); \mathbf{L}_R^{(-\infty)})$$

is split-injective.

Combined with the Farrell–Jones Conjectures we obtain that the homology group $H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}_R)$ is a direct summand in $K_n(RG)$. It is much better understood (compare Chapt. 2.9) than the remaining summand which is isomorphic to $H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G), E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}_R)$. This remaining summand is the one which plays the role of the Nil-terms for a general group. It is known that for $R = \mathbb{Z}$ the negative dimensional Nil-groups which are responsible for virtually cyclic groups vanish [113]. They vanish rationally, in dimension 0 by [76] and in higher dimensions by [182]. For more information see also [75]. Analogously to the proof of Proposition 70 we obtain the following proposition.

73 Proposition 73 We have

$$H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G), E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}_{\mathbb{Z}}) = 0 \quad \text{for } n < 0 \quad \text{and}$$

$$H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G), E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} = 0 \quad \text{for all } n \in \mathbb{Z}.$$

In particular the Farrell–Jones Conjecture for the algebraic K -theory of the integral group ring predicts that the map

$$A_{\mathcal{F}\mathcal{L}\mathcal{N}} : H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is always an isomorphism.

Inverting 2 in L -Theory

Proposition 74 For every group G , every ring R with involution, every decoration j and all $n \in \mathbb{Z}$ the relative assembly map

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$$A_{\mathcal{F}\mathcal{L}\mathcal{N} \rightarrow \mathcal{V}\mathcal{C}\mathcal{Y}} : H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{L}_R^{(j)}) \left[\frac{1}{2} \right] \rightarrow H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G); \mathbf{L}_R^{(j)}) \left[\frac{1}{2} \right]$$

is an isomorphism.

Proof According to the Transitivity Principle it suffices to prove the claim for a virtually cyclic group. Now argue analogously to the proof of Proposition 70 using the exact sequences in [48] and the fact that the UNil-terms appearing there vanish after inverting two [48]. Also recall from Remark 22 that after inverting 2 there are no differences between the decorations.

In particular the L -theoretic Farrell–Jones Conjecture implies that for every decoration j the assembly map

$$A_{\mathcal{F}\mathcal{L}\mathcal{N}} : H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{L}_R^{(j)}) \left[\frac{1}{2} \right] \rightarrow L_n^{(j)}(RG) \left[\frac{1}{2} \right]$$

is an isomorphism.

L -Theory and Virtually Cyclic Subgroups of the First Kind

Recall that a group is virtually cyclic of the first kind if it admits a surjection with finite kernel onto the infinite cyclic group. The family of these groups is denoted $\mathcal{V}\mathcal{C}\mathcal{Y}_I$.

Proposition 75 For all groups G , all rings R and all $n \in \mathbb{Z}$ the relative assembly map

75

$$A_{\mathcal{F}\mathcal{L}\mathcal{N} \rightarrow \mathcal{V}\mathcal{C}\mathcal{Y}_I} : H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{L}_R^{(-\infty)}) \rightarrow H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}_I}(G); \mathbf{L}_R^{(-\infty)})$$

is an isomorphism.

Proof The point is that there are no UNil-terms for infinite virtually cyclic groups of the first kind. This follows essentially from [254] and [255] as carried out in [204].

Rationally \mathcal{FLN} Reduces to \mathcal{FCY}

We will see later (compare Theorem 172, 173 and 178) that in all three cases, topological K -theory, algebraic K -theory and L -theory, the rationalized left hand side of the \mathcal{FLN} -assembly map can be computed very explicitly using the equivariant Chern-Character. As a by-product these computations yield that after rationalizing the family \mathcal{FLN} can be reduced to the family \mathcal{FCY} of finite cyclic groups and that the rationalized relative assembly maps $A_{\{\}} \rightarrow \mathcal{FCY}$ are injective.

76 Proposition 76 For every ring R , every group G and all $n \in \mathbb{Z}$ the relative assembly maps

$$A_{\mathcal{FCY} \rightarrow \mathcal{FLN}} : H_n^G(E_{\mathcal{FCY}}(G); \mathbf{K}_R) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H_n^G(E_{\mathcal{FLN}}(G); \mathbf{K}_R) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$A_{\mathcal{FCY} \rightarrow \mathcal{FLN}} : H_n^G(E_{\mathcal{FCY}}(G); \mathbf{L}_R^{(-\infty)}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H_n^G(E_{\mathcal{FLN}}(G); \mathbf{L}_R^{(-\infty)}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$A_{\mathcal{FCY} \rightarrow \mathcal{FLN}} : H_n^G(E_{\mathcal{FCY}}(G); \mathbf{K}^{\text{top}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H_n^G(E_{\mathcal{FLN}}(G); \mathbf{K}^{\text{top}}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

are isomorphisms and the corresponding relative assembly maps $A_{\{\}} \rightarrow \mathcal{FCY}$ are all rationally injective.

Recall that the statement for topological K -theory is even known integrally, compare Proposition 69. Combining the above with Proposition 73 and Proposition 74 we see that the Farrell-Jones Conjecture predicts in particular that the \mathcal{FCY} -assembly maps

$$A_{\mathcal{FCY}} : H_n^G(E_{\mathcal{FCY}}(G); \mathbf{L}_R^{(-\infty)}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L_n^{(-\infty)}(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$A_{\mathcal{FCY}} : H_n^G(E_{\mathcal{FCY}}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

are always isomorphisms.

2.4 More Applications

2.4.1 Applications VI

Low Dimensional Algebraic K -Theory

As opposed to topological K -theory and L -theory, which are periodic, the algebraic K -theory groups of coefficient rings such as \mathbb{Z} , \mathbb{Q} or \mathbb{C} are known to be bounded below. Using the spectral sequences for the left hand side of an assembly map that will be discussed in Sect. 2.9.4, this leads to vanishing results in negative dimensions and a concrete description of the groups in the first non-vanishing dimension.

The following conjecture is a consequence of the K -theoretic Farrell-Jones Conjecture in the case $R = \mathbb{Z}$. Note that by the results discussed in Proposition 72

we know that in negative dimensions we can reduce to the family of finite subgroups. Explanations about the colimit that appears follow below.

Conjecture 77: (The Farrell–Jones Conjecture for $K_n(\mathbb{Z}G)$ for $n \leq -1$.) For every group G we have

77

$$K_{-n}(\mathbb{Z}G) = 0 \quad \text{for } n \geq 2,$$

and the map

$$\text{colim}_{H \in \text{Sub}_{\mathcal{F}, \mathcal{N}}(G)} K_{-1}(\mathbb{Z}H) \xrightarrow{\cong} K_{-1}(\mathbb{Z}G)$$

is an isomorphism.

We can consider a family \mathcal{F} of subgroups of G as a category $\text{Sub}_{\mathcal{F}}(G)$ as follows. The objects are the subgroups H with $H \in \mathcal{F}$. For $H, K \in \mathcal{F}$ let $\text{conhom}_G(H, K)$ be the set of all group homomorphisms $f: H \rightarrow K$, for which there exists a group element $g \in G$ such that f is given by conjugation with g . The group of inner automorphisms $\text{inn}(K)$ consists of those automorphisms $K \rightarrow K$, which are given by conjugation with an element $k \in K$. It acts on $\text{conhom}(H, K)$ from the left by composition. Define the set of morphisms in $\text{Sub}_{\mathcal{F}}(G)$ from H to K to be $\text{inn}(K) \backslash \text{conhom}(H, K)$. Composition of group homomorphisms defines the composition of morphisms in $\text{Sub}_{\mathcal{F}}(G)$. We mention that $\text{Sub}_{\mathcal{F}}(G)$ is a quotient category of the orbit category $\text{Or}_{\mathcal{F}}(G)$ which we will introduce in Sect. 2.7.4. Note that there is a morphism from H to K only if H is conjugate to a subgroup of K . Clearly $K_n(R(-))$ yields a functor from $\text{Sub}_{\mathcal{F}}(G)$ to abelian groups since inner automorphisms on a group G induce the identity on $K_n(RG)$. Using the inclusions into G , one obtains a map

$$\text{colim}_{H \in \text{Sub}_{\mathcal{F}}(G)} K_n(RH) \rightarrow K_n(RG).$$

The colimit can be interpreted as the 0-th Bredon homology group

$$H_0^G(E_{\mathcal{F}}(G); K_n(R(?)))$$

(compare Example 64) and the map is the edge homomorphism in the equivariant Atiyah–Hirzebruch spectral sequence discussed in Sect. 2.9.4. In Conjecture 77 we consider the first non-vanishing entry in the lower left hand corner of the E_2 -term because of the following vanishing result [113, theorem 2.1] which generalizes vanishing results for finite groups from [57].

Proposition 78 If V is a virtually cyclic group, then $K_{-n}(\mathbb{Z}V) = 0$ for $n \geq 2$.

78

If our coefficient ring R is a regular ring in which the orders of all finite subgroups of G are invertible, then we know already from Sect. 2.3.2 that we can reduce to the family of finite subgroups. In the proof of Proposition 70 we have seen that then RH is again regular if $H \subset G$ is finite. Since negative K -groups vanish for

regular rings [268, 5.3.30 on page 295], the following is implied by the Farrell–Jones Conjecture 58.

79 **Conjecture 79:** (*Farrell–Jones Conjecture for $K_0(\mathbb{Q}G)$.*) Suppose R is a regular ring in which the orders of all finite subgroups of G are invertible (for example a field of characteristic 0), then

$$K_{-n}(RG) = 0 \quad \text{for } n \geq 1$$

and the map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FLN}}(G)} K_0(RH) \xrightarrow{\cong} K_0(RG)$$

is an isomorphism.

The conjecture above holds if G is virtually poly-cyclic. Surjectivity is proven in [227] (see also [67] and chapt. 8 in [235]), injectivity in [271]. We will show in Lemma 87 (i) that the map appearing in the conjecture is always rationally injective for $R = \mathbb{C}$.

The conjectures above describe the first non-vanishing term in the equivariant Atiyah–Hirzebruch spectral sequence. Already the next step is much harder to analyze in general because there are potentially non-vanishing differentials. We know however that after rationalizing the equivariant Atiyah–Hirzebruch spectral sequence for the left hand side of the \mathcal{FLN} -assembly map collapses. As a consequence we obtain that the following conjecture follows from the K -theoretic Farrell–Jones Conjecture 58.

80 **Conjecture 80** For every group G , every ring R and every $n \in \mathbb{Z}$ the map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FLN}}(G)} K_n(RH) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective.

Note that for $K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$ the conjecture above is always true but not very interesting, because for a finite group H it is known that $\tilde{K}_0(\mathbb{Z}H) \otimes_{\mathbb{Z}} \mathbb{Q} = 0$, compare [298, proposition 9.1], and hence the left hand side reduces to $K_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$. However, the full answer for $K_0(\mathbb{Z}G)$ should involve the negative K -groups, compare Example 176.

Analogously to Conjecture 80 the following can be derived from the K -theoretic Farrell–Jones Conjecture 58, compare [208].

81 **Conjecture 81** The map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FLN}}(G)} \operatorname{Wh}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \operatorname{Wh}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is always injective.

In general one does not expect this map to be an isomorphism. There should be additional contributions coming from negative K -groups. Conjecture 81 is true for groups satisfying a mild homological finiteness condition, see Theorem 145.

Remark 82: (*The Conjectures as Generalized Induction Theorems.*) The above discussion shows that one may think of the Farrell–Jones Conjectures 58 and the Baum–Connes Conjecture 59 as “generalized induction theorems”. The prototype of an induction theorem is Artin’s theorem about the complex representation ring $R_{\mathbb{C}}(G)$ of a finite group G . Let us recall Artin’s theorem.

82

For finite groups H the complex representation ring $R_{\mathbb{C}}(H)$ coincides with $K_0(\mathbb{C}H)$. Artin’s Theorem [283, theorem 17 in 9.2 on page 70] implies that the obvious induction homomorphism

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\text{cyc}}(G)} R_{\mathbb{C}}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} R_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. Note that this is a very special case of Theorem 172 or 173, compare Remark 177.

Artin’s theorem says that rationally one can compute $R_{\mathbb{C}}(G)$ if one knows all the values $R_{\mathbb{C}}(C)$ (including all maps coming from induction with group homomorphisms induced by conjugation with elements in G) for all cyclic subgroups $C \subseteq G$. The idea behind the Farrell–Jones Conjectures 58 and the Baum–Connes Conjecture 59 is analogous. We want to compute the functors $K_n(RG)$, $L_n(RG)$ and $K_n(C_r^*(G))$ from their values (including their functorial properties under induction) on elements of the family \mathcal{FIN} or \mathcal{VCY} .

The situation in the Farrell Jones and Baum–Connes Conjectures is more complicated than in Artin’s Theorem, since we have already seen in Remarks 15, 26 and 35 that a computation of $K_n(RG)$, $L_n^{(-\infty)}(RG)$ and $K_n(C_r^*(G))$ does involve also the values $K_p(RH)$, $L_p^{(-\infty)}(RH)$ and $K_p(C_r^*(H))$ for $p \leq n$. A degree mixing occurs.

G-Theory

Instead of considering finitely generated projective modules one may apply the standard K -theory machinery to the category of finitely generated modules. This leads to the definition of the groups $G_n(R)$ for $n \geq 0$. For instance $G_0(R)$ is the abelian group whose generators are isomorphism classes $[M]$ of finitely generated R -modules and whose relations are given by $[M_0] - [M_1] + [M_2]$ for any exact sequence $0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0$ of finitely generated modules. One may ask whether versions of the Farrell–Jones Conjectures for G -theory instead of K -theory might be true. The answer is negative as the following discussion explains.

For a finite group H the ring $\mathbb{C}H$ is semisimple. Hence any finitely generated $\mathbb{C}H$ -module is automatically projective and $K_0(\mathbb{C}H) = G_0(\mathbb{C}H)$. Recall that a group G is called *virtually poly-cyclic* if there exists a subgroup of finite index $H \subseteq G$ together with a filtration $\{1\} = H_0 \subseteq H_1 \subseteq H_2 \subseteq \dots \subseteq H_r = H$ such that

H_{i-1} is normal in H_i and the quotient H_i/H_{i-1} is cyclic. More generally for all $n \in \mathbb{Z}$ the forgetful map

$$f: K_n(\mathbb{C}G) \rightarrow G_n(\mathbb{C}G)$$

is an isomorphism if G is virtually poly-cyclic, since then $\mathbb{C}G$ is regular [273, theorem 8.2.2 and theorem 8.2.20] and the forgetful map f is an isomorphism for regular rings, compare [268, corollary 53.26 on page 293]. In particular this applies to virtually cyclic groups and so the left hand side of the Farrell–Jones assembly map does not see the difference between K - and G -theory if we work with complex coefficients. We obtain a commutative diagram

$$\begin{array}{ccc} \operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{F}, \mathcal{N}}(G)} K_0(\mathbb{C}H) & \longrightarrow & K_0(\mathbb{C}G) \\ \cong \downarrow & & \downarrow f \\ \operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{F}, \mathcal{N}}(G)} G_0(\mathbb{C}H) & \longrightarrow & G_0(\mathbb{C}G) \end{array} \quad (2.11)$$

where, as indicated, the left hand vertical map is an isomorphism. Conjecture 79, which is implied by the Farrell–Jones Conjecture, says that the upper horizontal arrow is an isomorphism. A G -theoretic analogue of the Farrell–Jones Conjecture would say that the lower horizontal map is an isomorphism. There are however cases where the upper horizontal arrow is known to be an isomorphism, but the forgetful map f on the right is not injective or not surjective:

If G contains a non-abelian free subgroup, then the class $[\mathbb{C}G] \in G_0(\mathbb{C}G)$ vanishes [202, theorem 9.66 on page 364] and hence the map $f: K_0(\mathbb{C}G) \rightarrow G_0(\mathbb{C}G)$ has an infinite kernel ($[\mathbb{C}G]$ generates an infinite cyclic subgroup in $K_0(\mathbb{C}G)$). The Farrell–Jones Conjecture for $K_0(\mathbb{C}G)$ is known for non-abelian free groups.

The Farrell–Jones Conjecture is also known for $A = \bigoplus_{n \in \mathbb{Z}} \mathbb{Z}/2$ and hence $K_0(\mathbb{C}A)$ is countable, whereas $G_0(\mathbb{C}A)$ is not countable [202, example 10.13 on page 375]. Hence the map f cannot be surjective.

At the time of writing we do not know a counterexample to the statement that for an amenable group G , for which there is an upper bound on the orders of its finite subgroups, the forgetful map $f: K_0(\mathbb{C}G) \rightarrow G_0(\mathbb{C}G)$ is an isomorphism. We do not know a counterexample to the statement that for a group G , which is not amenable, $G_0(\mathbb{C}G) = \{0\}$. We also do not know whether $G_0(\mathbb{C}G) = \{0\}$ is true for $G = \mathbb{Z} * \mathbb{Z}$.

For more information about $G_0(\mathbb{C}G)$ we refer for instance to [202, sect. 9.5.3].

Bass Conjectures

Complex representations of a finite group can be studied using characters. We now want to define the Hattori–Stallings rank of a finitely generated projective $\mathbb{C}G$ -module which should be seen as a generalization of characters to infinite groups.

Let $\text{con}(G)$ be the set of conjugacy classes (g) of elements $g \in G$. Denote by $\text{con}(G)_f$ the subset of $\text{con}(G)$ consisting of those conjugacy classes (g) for which each representative g has finite order. Let $\text{class}_0(G)$ and $\text{class}_0(G)_f$ be the \mathbb{C} -vector space with the set $\text{con}(G)$ and $\text{con}(G)_f$ as basis. This is the same as the \mathbb{C} -vector space of \mathbb{C} -valued functions on $\text{con}(G)$ and $\text{con}(G)_f$ with finite support. Define the *universal \mathbb{C} -trace* as

$$\text{tr}_{\mathbb{C}G}^u : \mathbb{C}G \rightarrow \text{class}_0(G), \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \lambda_g \cdot (g) . \tag{2.12}$$

It extends to a function $\text{tr}_{\mathbb{C}G}^u : M_n(\mathbb{C}G) \rightarrow \text{class}_0(G)$ on (n, n) -matrices over $\mathbb{C}G$ by taking the sum of the traces of the diagonal entries. Let P be a finitely generated projective $\mathbb{C}G$ -module. Choose a matrix $A \in M_n(\mathbb{C}G)$ such that $A^2 = A$ and the image of the $\mathbb{C}G$ -map $r_A : \mathbb{C}G^n \rightarrow \mathbb{C}G^n$ given by right multiplication with A is $\mathbb{C}G$ -isomorphic to P . Define the *Hattori–Stallings rank* of P as

$$\text{HS}_{\mathbb{C}G}(P) = \text{tr}_{\mathbb{C}G}^u(A) \in \text{class}_0(G) . \tag{2.13}$$

The Hattori–Stallings rank depends only on the isomorphism class of the $\mathbb{C}G$ -module P and induces a homomorphism $\text{HS}_{\mathbb{C}G} : K_0(\mathbb{C}G) \rightarrow \text{class}_0(G)$.

Conjecture 83: (*Strong Bass Conjecture for $K_0(\mathbb{C}G)$.*)

83

The \mathbb{C} -vector space spanned by the image of the map

$$\text{HS}_{\mathbb{C}G} : K_0(\mathbb{C}G) \rightarrow \text{class}_0(G)$$

is $\text{class}_0(G)_f$.

This conjecture is implied by the surjectivity of the map

$$\text{colim}_{H \in \text{Sub}_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)} K_0(CH) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} , \tag{2.14}$$

(compare Conjecture 79) and hence by the K -theoretic Farrell–Jones Conjecture for $K_0(\mathbb{C}G)$. We will see below that the surjectivity of the map (2.14) also implies that the map $K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \text{class}_0(G)$, which is induced by the Hattori–Stallings rank, is injective. Hence we do expect that the Hattori–Stallings rank induces an isomorphism

$$K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \text{class}_0(G)_f .$$

There are also versions of the Bass conjecture for other coefficients than \mathbb{C} . It follows from results of Linnell [191, theorem 4.1 on page 96] that the following version is implied by the Strong Bass Conjecture for $K_0(\mathbb{C}G)$.

Conjecture 84: (*The Strong Bass Conjecture for $K_0(\mathbb{Z}G)$.*) The image of the composition

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$$K_0(\mathbb{Z}G) \rightarrow K_0(\mathbb{C}G) \xrightarrow{\text{HS}_{\mathbb{C}G}} \text{class}_0(G)$$

is contained in the \mathbb{C} -vector space of those functions $f: \text{con}(G) \rightarrow \mathbb{C}$ which vanish for $(g) \in \text{con}(g)$ with $g \neq 1$.

The conjecture says that for every finitely generated projective $\mathbb{Z}G$ -module P the Hattori–Stallings rank of $\mathbb{C}G \otimes_{\mathbb{Z}G} P$ looks like the Hattori–Stallings rank of a free $\mathbb{C}G$ -module. A natural explanation for this behaviour is the following conjecture which clearly implies the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$.

85 Conjecture 85: (*Rational $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ -Conjecture.*) For every group G the map

$$\tilde{K}_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \tilde{K}_0(\mathbb{Q}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

induced by the change of coefficients is trivial.

Finally we mention the following variant of the Bass Conjecture.

86 Conjecture 86: (*The Weak Bass Conjecture.*) Let P be a finitely generated projective $\mathbb{Z}G$ -module. The value of the Hattori–Stallings rank of $\mathbb{C}G \otimes_{\mathbb{Z}G} P$ at the conjugacy class of the identity element is given by

$$\text{HS}_{\mathbb{C}G}(\mathbb{C}G \otimes_{\mathbb{Z}G} P)((1)) = \dim_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P).$$

Here \mathbb{Z} is considered as a $\mathbb{Z}G$ -module via the augmentation.

The K -theoretic Farrell–Jones Conjecture implies all four conjectures above. More precisely we have the following proposition.

87 Proposition 87

(i) The map

$$\text{colim}_{H \in \text{Sub}_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)} K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is always injective. If the map is also surjective (compare Conjecture 79) then the Hattori–Stallings rank induces an isomorphism

$$K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \text{class}_0(G)_f$$

and in particular the Strong Bass Conjecture for $K_0(\mathbb{C}G)$ and hence also the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$ hold.

(ii) The surjectivity of the map

$$A_{\text{vCy}}: H_0^G(\text{Evcy}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

implies the Rational $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ Conjecture and hence also the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$.

(iii) The Strong Bass Conjecture for $K_0(\mathbb{C}G)$ implies the Strong Bass Conjecture for $K_0(\mathbb{Z}G)$. The Strong Bass Conjecture for $K_0(\mathbb{Z}G)$ implies the Weak Bass Conjecture.

Proof (i) follows from the following commutative diagram, compare [198, lemma 2.15 on page 220].

$$\begin{array}{ccccc}
 \operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}}(G)} K_0(\mathbb{C}H) \otimes_{\mathbb{Z}} \mathbb{C} & \longrightarrow & & \longrightarrow & K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\
 \cong \downarrow & & & & \downarrow \\
 \operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}}(G)} \operatorname{class}_0(H) & \xrightarrow{\cong} & \operatorname{class}_0(G)_f & \xrightarrow{i} & \operatorname{class}_0(G) .
 \end{array}$$

Here the vertical maps are induced by the Hattori–Stallings rank, the map i is the natural inclusion and in particular injective and we have the indicated isomorphisms.

(ii) According to Proposition 73 the surjectivity of the map $A_{V\mathcal{C}\mathcal{Y}}$ appearing in 87 implies the surjectivity of the corresponding assembly map $A_{\mathcal{FIN}}$ (rationalized and with \mathbb{Z} as coefficient ring) for the family of finite subgroups. The map $A_{\mathcal{FIN}}$ is natural with respect to the change of the coefficient ring from \mathbb{Z} to \mathbb{Q} . By Theorem 173 we know that for every coefficient ring R there is an isomorphism from

$$\bigoplus_{p,q,p+q=0} \bigoplus_{(C) \in (\mathcal{F}\mathcal{C}\mathcal{Y})} H_p(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \Theta_C \cdot K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q}$$

to the 0-dimensional part of the left hand side of the rationalized \mathcal{FIN} -assembly map $A_{\mathcal{FIN}}$. The isomorphism is natural with respect to a change of coefficient rings. To see that the Rational $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ Conjecture follows, it hence suffices to show that the summand corresponding to $C = \{1\}$ and $p = q = 0$ is the only one where the map induced from $\mathbb{Z} \rightarrow \mathbb{Q}$ is possibly non-trivial. But $K_q(\mathbb{Q}C) = 0$ if $q < 0$, because $\mathbb{Q}C$ is semisimple and hence regular, and for a finite cyclic group $C \neq \{1\}$ we have by [198, lemma 7.4]

$$\Theta_C \cdot K_0(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q} = \operatorname{coker} \left(\bigoplus_{D \subseteq C} K_0(\mathbb{Z}D) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q} \right) = 0 ,$$

since by a result of Swan $K_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(\mathbb{Z}H) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an isomorphism for a finite group H , see [298, proposition 9.1].

(iii) As already mentioned the first statement follows from [191, theorem 4.1 on page 96]. The second statement follows from the formula

$$\sum_{(g) \in \operatorname{con}(G)} \operatorname{HS}_{\mathbb{C}G}(\mathbb{C} \otimes_{\mathbb{Z}} P)(g) = \dim_{\mathbb{Z}}(\mathbb{Z} \otimes_{\mathbb{Z}G} P) .$$

The next result is due to Berrick, Chatterji and Mislin [36, theorem 5.2]. The Bost Conjecture is a variant of the Baum–Connes Conjecture and is explained in Sect. 2.5.1.

88 **Theorem 88** If the assembly map appearing in the Bost Conjecture 104 is rationally surjective, then the Strong Bass Conjecture for $K_0(\mathbb{C}G)$ (compare 83) is true.

We now discuss some further questions and facts that seem to be relevant in the context of the Bass Conjectures.

89 **Remark 89:** (*Integral $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ -Conjecture.*) We do not know a counterexample to the Integral $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ Conjecture, i.e. to the statement that the map

$$\tilde{K}_0(\mathbb{Z}G) \rightarrow \tilde{K}_0(\mathbb{Q}G)$$

itself is trivial. But we also do not know a proof which shows that the K -theoretic Farrell–Jones Conjecture implies this integral version. Note that the Integral $\tilde{K}_0(\mathbb{Z}G)$ -to- $\tilde{K}_0(\mathbb{Q}G)$ Conjecture would imply that the following diagram commutes.

$$\begin{array}{ccc} K_0(\mathbb{Z}G) & \longrightarrow & K_0(\mathbb{Q}G) \\ p_* \downarrow & \text{dim}_{\mathbb{Z}} & \uparrow i \\ K_0(\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \end{array}$$

Here p_* is induced by the projection $G \rightarrow \{1\}$ and i sends $1 \in \mathbb{Z}$ to the class of $\mathbb{Q}G$.

90 **Remark 90:** (*The passage from $\tilde{K}_0(\mathbb{Z}G)$ to $\tilde{K}_0(\mathcal{N}(G))$.*) Let $\mathcal{N}(G)$ denote the group von Neumann algebra of G . It is known that for every group G the composition

$$\tilde{K}_0(\mathbb{Z}G) \rightarrow \tilde{K}_0(\mathbb{Q}G) \rightarrow \tilde{K}_0(\mathbb{C}G) \rightarrow \tilde{K}_0(C_r^*(G)) \rightarrow \tilde{K}_0(\mathcal{N}(G))$$

is the zero-map (see for instance [202, theorem 9.62 on page 362]). Since the group von Neumann algebra $\mathcal{N}(G)$ is not functorial under arbitrary group homomorphisms such as $G \rightarrow \{1\}$, this does *not* imply that the diagram

$$\begin{array}{ccc} K_0(\mathbb{Z}G) & \longrightarrow & K_0(\mathcal{N}(G)) \\ p_* \downarrow & \text{dim}_{\mathbb{Z}} & \uparrow i \\ K_0(\mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \end{array}$$

commutes. However, commutativity would follow from the Weak Bass Conjecture 86. For a discussion of these questions see [93].

More information and further references about the Bass Conjecture can be found for instance in [24], [36, sect. 7], [42, 92, 93, 118, 191] [202, sect. 9.5.2], and [225, pp. 66ff].

Applications VII

2.4.2

Novikov Conjectures

In Sect. 2.2.9 we discussed the Novikov Conjectures. Recall that one possible reformulation of the original Novikov Conjecture says that for every group G the rationalized classical assembly map in L -theory

$$A: H_n(BG; \mathbf{L}^p(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow L_n^p(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective. Since A can be identified with $A_{\{\emptyset\} \rightarrow \mathcal{AL}\mathcal{L}}$ and we know from Sect. 2.3.2 that the relative assembly map

$$A_{\{\emptyset\} \rightarrow \mathcal{F}\mathcal{L}\mathcal{N}}: H_n^G(E_{\{\emptyset\}}(G); \mathbf{L}_{\mathbb{Z}}^p) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{L}_{\mathbb{Z}}^p) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective we obtain the following proposition.

Proposition 91 The rational injectivity of the assembly map appearing in L -theoretic Farrell–Jones Conjecture (Conjecture 58) implies the L -theoretic Novikov Conjecture (Conjecture 51) and hence the original Novikov Conjecture 52.

91

Similarly the Baum–Connes Conjecture 59 implies the injectivity of the rationalized classical assembly map

$$A: H_n(BG; \mathbf{K}^{\text{top}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

In the next subsection we discuss how one can relate assembly maps for topological K -theory with L -theoretic assembly maps. The results imply in particular the following proposition.

Proposition 92 The rational injectivity of the assembly map appearing in the Baum–Connes Conjecture (Conjecture 59) implies the Novikov Conjecture (Conjecture 52).

92

Finally we would like to mention that by combining the results about the rationalization of $A_{\{\emptyset\} \rightarrow \mathcal{F}\mathcal{L}\mathcal{N}}$ from Sect. 2.3.2 with the splitting result about $A_{\mathcal{F}\mathcal{L}\mathcal{N} \rightarrow \mathcal{V}\mathcal{C}\mathcal{Y}}$ from Proposition 72 we obtain the following result

93 **Proposition 93** The rational injectivity of the assembly map appearing in the Farrell–Jones Conjecture for algebraic K -theory (Conjecture 58) implies the K -theoretic Novikov Conjecture, i.e. the injectivity of

$$A : H_n(BG; \mathbf{K}(R)) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q} .$$

94 **Remark 94:** (*Integral Injectivity Fails.*) In general the classical assembly maps $A = A_{\{\}}^j$ themselves, i.e. without rationalizing, are not injective. For example one can use the Atiyah–Hirzebruch spectral sequence to see that for $G = \mathbb{Z}/5$

$$H_1(BG; \mathbf{K}^{\text{top}}) \quad \text{and} \quad H_1(BG; \mathbf{L}^{(-\infty)}(\mathbb{Z}))$$

contain 5-torsion, whereas for every finite group G the topological K -theory of $\mathbb{C}G$ is torsionfree and the torsion in the L -theory of $\mathbb{Z}G$ is always 2-torsion, compare Proposition 169 (i) and Proposition 171 (i).

Relating Topological K -Theory and L -Theory

For every real C^* -algebra A there is an isomorphism $L_n^p(A)[1/2] \xrightarrow{\cong} K_n(A)[1/2]$ [269]. This can be used to compare L -theory to topological K -theory and leads to the following result.

95 **Proposition 95** Let $\mathcal{F} \subseteq \mathcal{FIN}$ be a family of finite subgroups of G . If the topological K -theory assembly map

$$A_{\mathcal{F}} : H_n^G(E_{\mathcal{F}}(G); \mathbf{K}^{\text{top}}) \left[\frac{1}{2} \right] \rightarrow K_n(C_r^*(G)) \left[\frac{1}{2} \right]$$

is injective, then for an arbitrary decoration j also the map

$$A_{\mathcal{F}} : H_n^G(E_{\mathcal{F}}(G); \mathbf{L}_{\mathbb{Z}}^{(j)}) \left[\frac{1}{2} \right] \rightarrow L_n^{(j)}(\mathbb{Z}G) \left[\frac{1}{2} \right]$$

is injective.

Proof First recall from Remark 22 that after inverting 2 there is no difference between the different decorations and we can hence work with the p -decoration. One can construct for any subfamily $\mathcal{F} \subseteq \mathcal{FIN}$ the following commutative diagram [200, sect. 7.5]

$$\begin{array}{ccc}
 H_n^G(E_{\mathcal{F}}(G); \mathbf{L}_{\mathbb{Z}}^p[1/2]) & \xrightarrow{A_{\mathcal{F}}^1} & L_n^p(\mathbb{Z}G)[1/2] \\
 i_1 \downarrow \cong & & j_1 \downarrow \cong \\
 H_n^G(E_{\mathcal{F}}(G); \mathbf{L}_{\mathbb{Q}}^p[1/2]) & \xrightarrow{A_{\mathcal{F}}^2} & L_n^p(\mathbb{Q}G)[1/2] \\
 i_2 \downarrow \cong & & j_2 \downarrow \\
 H_n^G(E_{\mathcal{F}}(G); \mathbf{L}_{\mathbb{R}}^p[1/2]) & \xrightarrow{A_{\mathcal{F}}^3} & L_n^p(\mathbb{R}G)[1/2] \\
 i_3 \downarrow \cong & & j_3 \downarrow \\
 H_n^G(E_{\mathcal{F}}(G); \mathbf{L}_{C_r^*(?;\mathbb{R})}^p[1/2]) & \xrightarrow{A_{\mathcal{F}}^4} & L_n^p(C_r^*(G; \mathbb{R}))[1/2] \\
 i_4 \downarrow \cong & & j_4 \downarrow \cong \\
 H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{R}}^{\text{top}}[1/2]) & \xrightarrow{A_{\mathcal{F}}^5} & K_n(C_r^*(G; \mathbb{R}))[1/2] \\
 i_5 \downarrow & & j_5 \downarrow \\
 H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{C}}^{\text{top}}[1/2]) & \xrightarrow{A_{\mathcal{F}}^6} & K_n(C_r^*(G))[1/2]
 \end{array}$$

Here

$$\mathbf{L}_{\mathbb{Z}}^p[1/2], \quad \mathbf{L}_{\mathbb{Q}}^p[1/2], \quad \mathbf{L}_{\mathbb{R}}^p[1/2], \quad \mathbf{L}_{C_r^*(?;\mathbb{R})}^p[1/2], \\
 \mathbf{K}_{\mathbb{R}}^{\text{top}}[1/2] \quad \text{and} \quad \mathbf{K}_{\mathbb{C}}^{\text{top}}[1/2]$$

are covariant $\text{Or}(G)$ -spectra (compare Sect. 2.7.2 and in particular Proposition 156) such that the n -th homotopy group of their evaluations at G/H are given by

$$L_n^p(\mathbb{Z}H)[1/2], \quad L_n^p(\mathbb{Q}H)[1/2], \quad L_n^p(\mathbb{R}H)[1/2], \quad L_n^p(C_r^*(H; \mathbb{R}))[1/2], \\
 K_n(C_r^*(H; \mathbb{R}))[1/2] \quad \text{respectively} \quad K_n(C_r^*(H))[1/2].$$

All horizontal maps are assembly maps induced by the projection $\text{pr}: E_{\mathcal{F}}(G) \rightarrow \text{pt}$. The maps i_k and j_k for $k = 1, 2, 3$ are induced from a change of rings. The isomorphisms i_4 and j_4 come from the general isomorphism for any real C^* -algebra A

$$L_n^p(A)[1/2] \xrightarrow{\cong} K_n(A)[1/2]$$

and its spectrum version [269, theorem 1.11 on page 350]. The maps i_1, j_1, i_2 are isomorphisms by [256, page 376] and [258, proposition 22.34 on page 252]. The map i_3 is bijective since for a finite group H we have $\mathbb{R}H = C_r^*(H; \mathbb{R})$. The maps i_5

and j_5 are given by extending the scalars from \mathbb{R} to \mathbb{C} by induction. For every real C^* -algebra A the composition

$$K_n(A)[1/2] \rightarrow K_n(A \otimes_{\mathbb{R}} \mathbb{C})[1/2] \rightarrow K_n(M_2(A))[1/2]$$

is an isomorphism and hence j_5 is split injective. An $\text{Or}(G)$ -spectrum version of this argument yields that also i_5 is split injective.

96 Remark 96 One may conjecture that the right vertical maps j_2 and j_3 are isomorphisms and try to prove this directly. Then if we invert 2 everywhere the Baum–Connes Conjecture 59 for the real reduced group C^* -algebra, would be equivalent to the Farrell–Jones Isomorphism Conjecture for $L_*(\mathbb{Z}G)[1/2]$.

2.4.3 Applications VIII

The Modified Trace Conjecture

Denote by Λ^G the subring of \mathbb{Q} which is obtained from \mathbb{Z} by inverting all orders $|H|$ of finite subgroups H of G , i.e.

$$\Lambda^G = \mathbb{Z} [|H|^{-1} \mid H \subset G, |H| < \infty] . \tag{2.15}$$

The following conjecture generalizes Conjecture 37 to the case where the group need no longer be torsionfree. For the standard trace compare (2.7).

97 Conjecture 97: (*Modified Trace Conjecture for a group G .*) Let G be a group. Then the image of the homomorphism induced by the standard trace

$$\text{tr}_{C_r^*(G)} : K_0(C_r^*(G)) \rightarrow \mathbb{R} \tag{2.16}$$

is contained in Λ^G .

The following result is proved in [203, theorem 0.3].

98 Theorem 98 Let G be a group. Then the image of the composition

$$K_0^G(E_{\mathcal{F}, \mathcal{I}, \mathcal{N}}(G)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{A_{\mathcal{F}, \mathcal{I}, \mathcal{N}} \otimes_{\mathbb{Z}} \text{id}} K_0(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\text{tr}_{C_r^*(G)}} \mathbb{R}$$

is Λ^G . Here $A_{\mathcal{F}, \mathcal{I}, \mathcal{N}}$ is the map appearing in the Baum–Connes Conjecture 59. In particular the Baum–Connes Conjecture 59 implies the Modified Trace Conjecture.

The original version of the Trace Conjecture due to Baum and Connes [27, page 21] makes the stronger statement that the image of $\text{tr}_{C_r^*(G)} : K_0(C_r^*(G)) \rightarrow \mathbb{R}$ is the additive subgroup of \mathbb{Q} generated by all numbers $\frac{1}{|H|}$, where $H \subset G$ runs through

all finite subgroups of G . Roy has constructed a counterexample to this version in [274] based on her article [275]. The examples of Roy do *not* contradict the Modified Trace Conjecture 97 or the Baum–Connes Conjecture 59.

The Stable Gromov–Lawson–Rosenberg Conjecture

The Stable Gromov–Lawson–Rosenberg Conjecture is a typical conjecture relating Riemannian geometry to topology. It is concerned with the question when a given manifold admits a metric of positive scalar curvature. To discuss its relation with the Baum–Connes Conjecture we will need the real version of the Baum–Connes Conjecture, compare Sect. 2.5.1.

Let $\Omega_n^{\text{Spin}}(BG)$ be the bordism group of closed Spin-manifolds M of dimension n with a reference map to BG . Let $C_r^*(G; \mathbb{R})$ be the real reduced group C^* -algebra and let $KO_n(C_r^*(G; \mathbb{R})) = K_n(C_r^*(G; \mathbb{R}))$ be its topological K -theory. We use KO instead of K as a reminder that we here use the real reduced group C^* -algebra. Given an element $[u: M \rightarrow BG] \in \Omega_n^{\text{Spin}}(BG)$, we can take the $C_r^*(G; \mathbb{R})$ -valued index of the equivariant Dirac operator associated to the G -covering $\overline{M} \rightarrow M$ determined by u . Thus we get a homomorphism

$$\text{ind}_{C_r^*(G; \mathbb{R})} : \Omega_n^{\text{Spin}}(BG) \rightarrow KO_n(C_r^*(G; \mathbb{R})) . \tag{2.17}$$

A *Bott manifold* is any simply connected closed Spin-manifold B of dimension 8 whose \widehat{A} -genus $\widehat{A}(B)$ is 8. We fix such a choice, the particular choice does not matter for the sequel. Notice that $\text{ind}_{C_r^*(\{1\}; \mathbb{R})}(B) \in KO_8(\mathbb{R}) \cong \mathbb{Z}$ is a generator and the product with this element induces the Bott periodicity isomorphisms $KO_n(C_r^*(G; \mathbb{R})) \xrightarrow{\cong} KO_{n+8}(C_r^*(G; \mathbb{R}))$. In particular

$$\text{ind}_{C_r^*(G; \mathbb{R})}(M) = \text{ind}_{C_r^*(G; \mathbb{R})}(M \times B) , \tag{2.18}$$

if we identify $KO_n(C_r^*(G; \mathbb{R})) = KO_{n+8}(C_r^*(G; \mathbb{R}))$ via Bott periodicity.

Conjecture 99: (*Stable Gromov–Lawson–Rosenberg Conjecture.*) Let M be a closed connected Spin-manifold of dimension $n \geq 5$. Let $u_M: M \rightarrow B\pi_1(M)$ be the classifying map of its universal covering. Then $M \times B^k$ carries for some integer $k \geq 0$ a Riemannian metric with positive scalar curvature if and only if

$$\text{ind}_{C_r^*(\pi_1(M); \mathbb{R})}([M, u_M]) = 0 \quad \in KO_n(C_r^*(\pi_1(M); \mathbb{R})) .$$

If M carries a Riemannian metric with positive scalar curvature, then the index of the Dirac operator must vanish by the Bochner–Lichnerowicz formula [267]. The converse statement that the vanishing of the index implies the existence of a Riemannian metric with positive scalar curvature is the hard part of the conjecture. The following result is due to Stolz. A sketch of the proof can be found in [297, sect. 3], details are announced to appear in a different paper.

Theorem 100 If the assembly map for the real version of the Baum–Connes Conjecture (compare Sect. 2.5.1) is injective for the group G , then the Stable Gromov–Lawson–Rosenberg Conjecture 99 is true for all closed Spin-manifolds of dimension ≥ 5 with $\pi_1(M) \cong G$.

The requirement $\dim(M) \geq 5$ is essential in the Stable Gromov–Lawson–Rosenberg Conjecture, since in dimension four new obstructions, the Seiberg–Witten invariants, occur. The unstable version of this conjecture says that M carries a Riemannian metric with positive scalar curvature if and only if $\text{ind}_{C_r^*(\pi_1(M); \mathbb{R})}([M, u_M]) = 0$. Schick [278] constructs counterexamples to the unstable version using minimal hypersurface methods due to Schoen and Yau (see also [90]). It is not known at the time of writing whether the unstable version is true for finite fundamental groups. Since the Baum–Connes Conjecture 59 is true for finite groups (for the trivial reason that $E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G) = \text{pt}$ for finite groups G), Theorem 100 implies that the Stable Gromov–Lawson Conjecture 99 holds for finite fundamental groups (see also [270]).

The index map appearing in (2.17) can be factorized as a composition

$$\text{ind}_{C_r^*(G; \mathbb{R})} : \Omega_n^{\text{Spin}}(BG) \xrightarrow{D} KO_n(BG) \xrightarrow{A} KO_n(C_r^*(G; \mathbb{R})), \tag{2.19}$$

where D sends $[M, u]$ to the class of the G -equivariant Dirac operator of the G -manifold \overline{M} given by u and $A = A_{\{\}} is the real version of the classical assembly map. The homological Chern character defines an isomorphism$

$$KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \bigoplus_{p \in \mathbb{Z}} H_{n+4p}(BG; \mathbb{Q}).$$

Recall that associated to M there is the \widehat{A} -class

$$\widehat{\mathcal{A}}(M) \in \prod_{p \geq 0} H^p(M; \mathbb{Q}) \tag{2.20}$$

which is a certain polynomial in the Pontrjagin classes. The map D appearing in (2.19) sends the class of $u : M \rightarrow BG$ to $u_*(\widehat{\mathcal{A}}(M) \cap [M])$, i.e. the image of the Poincaré dual of $\widehat{\mathcal{A}}(M)$ under the map induced by u in rational homology. Hence $D([M, u]) = 0$ if and only if $u_*(\widehat{\mathcal{A}}(M) \cap [M])$ vanishes. For $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$ define the *higher \widehat{A} -genus of (M, u) associated to x* to be

$$\widehat{A}_x(M, u) = \langle \widehat{\mathcal{A}}(M) \cup u^*x, [M] \rangle = \langle x, u_*(\widehat{\mathcal{A}}(M) \cap [M]) \rangle \in \mathbb{Q}. \tag{2.21}$$

The vanishing of $\widehat{\mathcal{A}}(M)$ is equivalent to the vanishing of all higher \widehat{A} -genera $\widehat{A}_x(M, u)$. The following conjecture is a weak version of the Stable Gromov–Lawson–Rosenberg Conjecture.

Conjecture 101: (*Homological Gromov–Lawson–Rosenberg Conjecture.*) Let G be a group. Then for any closed Spin-manifold M , which admits a Riemannian metric

with positive scalar curvature, the \widehat{A} -genus $\widehat{A}_x(M, u)$ vanishes for all maps $u: M \rightarrow BG$ and elements $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$.

From the discussion above we obtain the following result.

Proposition 102 If the assembly map

102

$$KO_n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow KO_n(C_r^*(G; \mathbb{R})) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective for all $n \in \mathbb{Z}$, then the Homological Gromov–Lawson–Rosenberg Conjecture holds for G .

Generalizations and Related Conjectures

2.5

Variants of the Baum–Connes Conjecture

2.5.1

The Real Version

There is an obvious real version of the Baum–Connes Conjecture, which predicts that for all $n \in \mathbb{Z}$ and groups G the assembly map

$$A_{\mathcal{F}, \mathcal{L}, \mathcal{N}}^{\mathbb{R}}: H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{R}}^{\text{top}}) \rightarrow KO_n(C_r^*(G; \mathbb{R}))$$

is an isomorphism. Here $H_n^G(-; \mathbf{K}_{\mathbb{R}}^{\text{top}})$ is an equivariant homology theory whose distinctive feature is that $H_n^G(G/H; \mathbf{K}_{\mathbb{R}}^{\text{top}}) \cong KO_n(C_r^*(H; \mathbb{R}))$. Recall that we write $KO_n(-)$ only to remind ourselves that the C^* -algebra we apply it to is a real C^* -algebra, like for example the real reduced group C^* -algebra $C_r^*(G; \mathbb{R})$. The following result appears in [31].

Proposition 103 The Baum–Connes Conjecture 59 implies the real version of the Baum–Connes Conjecture.

103

In the proof of Proposition 95 we have already seen that after inverting 2 the “real assembly map” is a retract of the complex assembly map. In particular with 2-inverted or after rationalizing also injectivity results or surjectivity results about the complex Baum–Connes assembly map yield the corresponding results for the real Baum–Connes assembly map.

The Version for Maximal Group C^* -Algebras

For a group G let $C_{\max}^*(G)$ be its *maximal group C^* -algebra*, compare [242, 7.1.5 on page 229]. The maximal group C^* -algebra has the advantage that every homomorphism of groups $\phi: G \rightarrow H$ induces a homomorphism $C_{\max}^*(G) \rightarrow C_{\max}^*(H)$ of C^* -algebras. This is not true for the reduced group C^* -algebra $C_r^*(G)$. Here

is a counterexample: since $C_r^*(F)$ is a simple algebra if F is a non-abelian free group [245], there is no unital algebra homomorphism $C_r^*(F) \rightarrow C_r^*(\{1\}) = \mathbb{C}$.

One can construct a version of the Baum–Connes assembly map using an equivariant homology theory $H_n^G(-; \mathbf{K}_{\max}^{\text{top}})$ which evaluated on G/H yields the K -theory of $C_{\max}^*(H)$ (use Proposition 156 and a suitable modification of \mathbf{K}^{top} , compare Sect. 2.7.3).

Since on the left hand side of a \mathcal{FIN} -assembly map only the maximal group C^* -algebras for finite groups H matter, and clearly $C_{\max}^*(H) = \mathbb{C}H = C_r^*(H)$ for such H , this left hand side coincides with the left hand side of the usual Baum–Connes Conjecture. There is always a C^* -homomorphism $p: C_{\max}^*(G) \rightarrow C_r^*(G)$ (it is an isomorphism if and only if G is amenable [242, theorem 7.3.9 on page 243]) and hence we obtain the following factorization of the usual Baum–Connes assembly map

$$\begin{array}{ccc}
 & & K_n(C_{\max}^*(G)) \\
 & \nearrow^{A_{\mathcal{FIN}}^{\max}} & \downarrow K_n(p) \\
 H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}^{\text{top}}) & \xrightarrow{A_{\mathcal{FIN}}} & K_n(C_r^*(G)) .
 \end{array} \tag{2.22}$$

It is known that the map $A_{\mathcal{FIN}}^{\max}$ is in general not surjective. The Baum–Connes Conjecture would imply that the map is $A_{\mathcal{FIN}}^{\max}$ is always injective, and that it is surjective if and only if the vertical map $K_n(p)$ is injective.

A countable group G is called *K-amenable* if the map $p: C_{\max}^*(G) \rightarrow C_r^*(G)$ induces a KK -equivalence (compare [78]). This implies in particular that the vertical map $K_n(p)$ is an isomorphism for all $n \in \mathbb{Z}$. Note that for K -amenable groups the Baum–Connes Conjecture holds if and only if the “maximal” version of the assembly map $A_{\mathcal{FIN}}^{\max}$ is an isomorphism for all $n \in \mathbb{Z}$. A-T-menable groups are K -amenable, compare Theorem 120. But $K_0(p)$ is not injective for every infinite group which has property (T) such as for example $SL_n(\mathbb{Z})$ for $n \geq 3$, compare for instance the discussion in [163]. There are groups with property (T) for which the Baum–Connes Conjecture is known (compare Sect. 2.6.1) and hence there are counterexamples to the conjecture that $A_{\mathcal{FIN}}^{\max}$ is an isomorphism.

In Theorem 48 and Remark 49 we discussed applications of the maximal C^* -algebra version of the Baum–Connes Conjecture.

The Bost Conjecture

Some of the strongest results about the Baum–Connes Conjecture are proven using the so called Bost Conjecture (see [186]). The Bost Conjecture is the version of the Baum–Connes Conjecture, where one replaces the reduced group C^* -algebra $C_r^*(G)$ by the Banach algebra $l^1(G)$ of absolutely summable functions on G . Again one can use the spectra approach (compare Sects. 2.7.2 and 2.7.3 and in particular Proposition 156) to produce a variant of equivariant K -homology denoted

$H_n^G(-; \mathbf{K}_1^{\text{top}})$ which this time evaluated on G/H yields $K_n(l^1(H))$, the topological K -theory of the Banach algebra $l^1(H)$.

As explained in the beginning of Chapt. 2.3, we obtain an associated assembly map and we believe that it coincides with the one defined using a Banach-algebra version of KK -theory in [186].

Conjecture 104: (*Bost Conjecture.*) Let G be a countable group. Then the assembly map

$$A_{\mathcal{F}\mathcal{L}\mathcal{N}}^1 : H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}_1^{\text{top}}) \rightarrow K_n(l^1(G))$$

is an isomorphism.

Again the left hand side coincides with the left hand side of the Baum–Connes assembly map because for finite groups H we have $l^1(H) = \mathbb{C}H = C_r^*(H)$. There is always a homomorphism of Banach algebras $q: l^1(G) \rightarrow C_r^*(G)$ and one obtains a factorization of the usual Baum–Connes assembly map

$$\begin{array}{ccc}
 & & K_n(l^1(G)) \\
 & \nearrow A_{\mathcal{F}\mathcal{L}\mathcal{N}}^1 & \downarrow K_n(q) \\
 H_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); \mathbf{K}_1^{\text{top}}) & \xrightarrow{A_{\mathcal{F}\mathcal{L}\mathcal{N}}} & K_n(C_r^*(G)) .
 \end{array}$$

Every group homomorphism $G \rightarrow H$ induces a homomorphism of Banach algebras $l^1(G) \rightarrow l^1(H)$. So similar as in the maximal group C^* -algebra case this approach repairs the lack of functoriality for the reduced group C^* -algebra.

The disadvantage of $l^1(G)$ is however that indices of operators tend to take values in the topological K -theory of the group C^* -algebras, not in $K_n(l^1(G))$. Moreover the representation theory of G is closely related to the group C^* -algebra, whereas the relation to $l^1(G)$ is not well understood.

For more information about the Bost Conjecture 104 see [186, 288].

The Baum–Connes Conjecture with Coefficients

The Baum–Connes Conjecture 59 can be generalized to the Baum–Connes Conjecture with Coefficients. Let A be a separable C^* -algebra with an action of the countable group G . Then there is an assembly map

$$KK_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); A) \rightarrow K_n(A \rtimes G) \tag{2.23}$$

defined in terms of equivariant KK -theory, compare Sects. 2.8.3 and 2.8.4.

Conjecture 105: (*Baum–Connes Conjecture with Coefficients.*) For every separable C^* -algebra A with an action of a countable group G and every $n \in \mathbb{Z}$ the assembly map (2.23) is an isomorphism.

There are counterexamples to the Baum–Connes Conjecture with Coefficients, compare Remark 122. If we take $A = \mathbb{C}$ with the trivial action, the map (2.23) can be identified with the assembly map appearing in the ordinary Baum–Connes Conjecture 59.

106 Remark 106: (*A Spectrum Level Description.*) There is a formulation of the Baum–Connes Conjecture with Coefficients in the framework explained in Sect. 2.7.2. Namely, construct an appropriate covariant functor $\mathbf{K}^{\text{top}}(A \rtimes \mathcal{G}^G(-)) : \text{Or}(G) \rightarrow \text{SPECTRA}$ such that

$$\pi_n(\mathbf{K}^{\text{top}}(A \rtimes \mathcal{G}^G(G/H))) \cong K_n(A \rtimes H)$$

holds for all subgroups $H \subseteq G$ and all $n \in \mathbb{Z}$, and consider the associated G -homology theory $H_*^G(-; \mathbf{K}^{\text{top}}(A \rtimes \mathcal{G}^G(-)))$. Then the map (2.23) can be identified with the map which the projection $\text{pr} : E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G) \rightarrow \text{pt}$ induces for this homology theory.

107 Remark 107: (*Farrell–Jones Conjectures with Coefficients.*) One can also formulate a “Farrell–Jones Conjecture with Coefficients”. (This should not be confused with the Fibered Farrell–Jones Conjecture discussed in Sect. 2.5.2.) Fix a ring S and an action of G on it by isomorphisms of rings. Construct an appropriate covariant functor $\mathbf{K}(S \rtimes \mathcal{G}^G(-)) : \text{Or}(G) \rightarrow \text{SPECTRA}$ such that

$$\pi_n(\mathbf{K}(S \rtimes \mathcal{G}^G(G/H))) \cong K_n(S \rtimes H)$$

holds for all subgroups $H \subseteq G$ and $n \in \mathbb{Z}$, where $S \rtimes H$ is the associated twisted group ring. Now consider the associated G -homology theory $H_*^G(-; \mathbf{K}(S \rtimes \mathcal{G}^G(-)))$. There is an analogous construction for L -theory. A *Farrell–Jones Conjecture with Coefficients* would say that the map induced on these homology theories by the projection $\text{pr} : E_{\text{VCY}}(G) \rightarrow \text{pt}$ is always an isomorphism. We do not know whether there are counterexamples to the Farrell–Jones Conjectures with Coefficients, compare Remark 122.

The Coarse Baum Connes Conjecture

We briefly explain the Coarse Baum–Connes Conjecture, a variant of the Baum–Connes Conjecture, which applies to metric spaces. Its importance lies in the fact that isomorphism results about the Coarse Baum–Connes Conjecture can be used to prove injectivity results about the classical assembly map for topological K -theory. Compare also Sect. 2.8.10.

Let X be a proper (closed balls are compact) metric space and H_X a separable Hilbert space with a faithful nondegenerate $*$ -representation of $C_0(X)$, the algebra of complex valued continuous functions which vanish at infinity. A bounded linear

operator T has a support $\text{supp } T \subset X \times X$, which is defined as the complement of the set of all pairs (x, x') , for which there exist functions ϕ and $\phi' \in C_0(X)$ such that $\phi(x) \neq 0$, $\phi'(x') \neq 0$ and $\phi'T\phi = 0$. The operator T is said to be a finite propagation operator if there exists a constant α such that $d(x, x') \leq \alpha$ for all pairs in the support of T . The operator is said to be *locally compact* if ϕT and $T\phi$ are compact for every $\phi \in C_0(X)$. An operator is called *pseudolocal* if $\phi T \psi$ is a compact operator for all pairs of continuous functions ϕ and ψ with compact and disjoint supports.

The Roe-algebra $C^*(X) = C(X, H_X)$ is the operator-norm closure of the $*$ -algebra of all locally compact finite propagation operators on H_X . The algebra $D^*(X) = D^*(X, H_X)$ is the operator-norm closure of the pseudolocal finite propagation operators. One can show that the topological K -theory of the quotient algebra $D^*(X)/C^*(X)$ coincides up to an index shift with the analytically defined (non-equivariant) K -homology $K_*(X)$, compare Sect. 2.8.1. For a uniformly contractible proper metric space the coarse assembly map $K_n(X) \rightarrow K_n(C^*(X))$ is the boundary map in the long exact sequence associated to the short exact sequence of C^* -algebras

$$0 \rightarrow C^*(X) \rightarrow D^*(X) \rightarrow D^*(X)/C^*(X) \rightarrow 0 .$$

For general metric spaces one first approximates the metric space by spaces with nice local behaviour, compare [263]. For simplicity we only explain the case, where X is a discrete metric space. Let $P_d(X)$ the Rips complex for a fixed distance d , i.e. the simplicial complex with vertex set X , where a simplex is spanned by every collection of points in which every two points are a distance less than d apart. Equip $P_d(X)$ with the spherical metric, compare [335].

A discrete metric space has *bounded geometry* if for each $r > 0$ there exists a $N(r)$ such that for all x the ball of radius r centered at $x \in X$ contains at most $N(r)$ elements.

Conjecture 108: (*Coarse Baum–Connes Conjecture.*) Let X be a proper discrete metric space of bounded geometry. Then for $n = 0, 1$ the coarse assembly map

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$$\text{colim}_d K_n(P_d(X)) \rightarrow \text{colim}_d K_n(C^*(P_d(X))) \cong K_n(C^*(X))$$

is an isomorphism.

The conjecture is false if one drops the bounded geometry hypothesis. A counterexample can be found in [336, sect. 8]. Our interest in the conjecture stems from the following fact, compare [263, chapt. 8].

Proposition 109 Suppose the finitely generated group G admits a classifying space BG of finite type. If G considered as a metric space via a word length metric satisfies the Coarse Baum–Connes Conjecture 108 then the classical assembly map $A: K_*(BG) \rightarrow K_*(C_r^*G)$ which appears in Conjecture 31 is injective.

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The Coarse Baum–Connes Conjecture for a discrete group G (considered as a metric space) can be interpreted as a case of the Baum–Connes Conjecture with Coefficients 105 for the group G with a certain specific choice of coefficients, compare [339].

Further information about the coarse Baum–Connes Conjecture can be found for instance in [151, 152, 154, 263, 334, 335, 337, 340], and [338].

The Baum–Connes Conjecture for Non-Discrete Groups

Throughout this subsection let T be a locally compact second countable topological Hausdorff group. There is a notion of a classifying space for proper T -actions \underline{ET} (see [28, sects. 1 and 2] [304, sect. I.6], [207, sect. 1]) and one can define its equivariant topological K -theory $K_n^T(\underline{ET})$. The definition of a reduced C^* -algebra $C_r^*(T)$ and its topological K -theory $K_n(C_r^*(T))$ makes sense also for T . There is an assembly map defined in terms of equivariant index theory

$$A_{\mathcal{K}}: K_n^T(\underline{ET}) \rightarrow K_n(C_r^*(T)). \tag{2.24}$$

The Baum–Connes Conjecture for T says that this map is bijective for all $n \in \mathbb{Z}$ [28, conjecture 3.15 on page 254].

Now consider the special case where T is a connected Lie group. Let \mathcal{K} be the family of compact subgroups of T . There is a notion of a T -CW-complex and of a classifying space $E_{\mathcal{K}}(T)$ defined as in Sects. 2.3.1 and 2.3.1. The classifying space $E_{\mathcal{K}}(T)$ yields a model for \underline{ET} . Let $K \subset T$ be a maximal compact subgroup. It is unique up to conjugation. The space T/K is contractible and in fact a model for \underline{ET} (see [1, appendix, theorem A.5], [2, corollary 4.14], [207, sect. 1]). One knows (see [28, proposition 4.22], [170])

$$K_n^T(\underline{ET}) = K_n^T(T/K) = \begin{cases} R_{\mathbb{C}}(K) & n = \dim(T/K) \pmod{2}, \\ 0 & n = 1 + \dim(T/K) \pmod{2}, \end{cases}$$

where $R_{\mathbb{C}}(K)$ is the complex representation ring of K .

Next we consider the special case where T is a totally disconnected group. Let $\mathcal{K}\mathcal{O}$ be the family of compact-open subgroups of T . A T -CW-complex and a classifying space $E_{\mathcal{K}\mathcal{O}}(T)$ for T and $\mathcal{K}\mathcal{O}$ are defined similar as in Sect. 2.3.1 and 2.3.1. Then $E_{\mathcal{K}\mathcal{O}}(T)$ is a model for \underline{ET} since any compact subgroup is contained in a compact-open subgroup, and the Baum–Connes Conjecture says that the assembly map yields for $n \in \mathbb{Z}$ an isomorphism

$$A_{\mathcal{K}\mathcal{O}}: K_n^T(E_{\mathcal{K}\mathcal{O}}(T)) \rightarrow K_n(C_r^*(T)). \tag{2.25}$$

For more information see [30].

2.5.2 Variants of the Farrell–Jones Conjecture

Pseudoisotopy Theory

An important variant of the Farrell–Jones Conjecture deals with the pseudoisotopy spectrum functor \mathbb{P} , which we already discussed briefly in Sect. 2.2.4. In fact it is

this variant of the Farrell–Jones Conjecture (and its fibered version which will be explained in the next subsection) for which the strongest results are known at the time of writing.

In Proposition 157 we will explain that every functor $\mathbf{E}: \text{GROUPOIDS} \rightarrow \text{SPECTRA}$, which sends equivalences of groupoids to stable weak equivalences of spectra, yields a corresponding equivariant homology theory $H_n^G(-; \mathbf{E})$. Now whenever we have a functor $\mathbf{F}: \text{SPACES} \rightarrow \text{SPECTRA}$, we can precompose it with the functor “classifying space” which sends a groupoid \mathcal{G} to its classifying space $B\mathcal{G}$. (Here $B\mathcal{G}$ is simply the realization of the nerve of \mathcal{G} considered as a category.) In particular this applies to the pseudoisotopy functor \mathbf{P} . Thus we obtain a homology theory $H_n^G(-; \mathbf{P} \circ B)$ whose essential feature is that

$$H_n^G(G/H; \mathbf{P} \circ B) \cong \pi_n(\mathbf{P}(BH)),$$

i.e. evaluated at G/H one obtains the homotopy groups of the pseudoisotopy spectrum of the classifying space BH of the group H . As the reader may guess there is the following conjecture.

Conjecture 110: (*Farrell–Jones Conjecture for Pseudoisotopies of Aspherical Spaces.*) For every group G and all $n \in \mathbb{Z}$ the assembly map

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$$H_n^G(E_{\text{vcy}}(G); \mathbf{P} \circ B) \rightarrow H_n^G(\text{pt}; \mathbf{P} \circ B) \cong \pi_n(\mathbf{P}(BG))$$

is an isomorphism. Similarly for \mathbf{P}^{diff} , the pseudoisotopy functor which is defined using differentiable pseudoisotopies.

A formulation of a conjecture for spaces which are not necessarily aspherical will be given in the next subsection, see in particular Remark 115.

Remark 111: (*Relating K -theory and Pseudoisotopy Theory.*) We already outlined in Sect. 2.2.4 the relationship between K -theory and pseudoisotopies. The comparison in positive dimensions described there can be extended to all dimensions. Vogell constructs in [309] a version of A -theory using retractive spaces that are bounded over \mathbb{R}^k (compare Sects. 2.2.2 and 2.2.4). This leads to a functor $\mathbf{A}^{-\infty}$ from spaces to non-connective spectra. Compare also [56, 310, 311] and [326]. We define $\mathbf{Wh}_{PL}^{-\infty}$ via the fibration sequence

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$$X_+ \wedge \mathbf{A}^{-\infty}(\text{pt}) \rightarrow \mathbf{A}^{-\infty}(X) \rightarrow \mathbf{Wh}_{PL}^{-\infty}(X),$$

where the first map is the assembly map. The natural equivalence

$$\Omega^2 \mathbf{Wh}_{PL}^{-\infty}(X) \simeq \mathbf{P}(X)$$

seems to be hard to trace down in the literature but should be true. We will assume it in the following discussion.

Precompose the functors above with the classifying space functor B to obtain functors from groupoids to spectra. The pseudoisotopy assembly map which ap-

pears in Conjecture 110 is an isomorphism if and only if the A -theory assembly map

$$H_{n+2}^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G); \mathbf{A}^{-\infty} \circ B) \rightarrow H_{n+2}^G(\text{pt}; \mathbf{A}^{-\infty} \circ B) \cong \pi_{n+2}(\mathbf{A}^{-\infty}(BG))$$

is an isomorphism. This uses a 5-lemma argument and the fact that for a fixed spectrum \mathbf{E} the assembly map

$$H_n^G(E_{\mathcal{F}}(G); B\mathcal{G}^G(-)_+ \wedge \mathbf{E}) \rightarrow H_n^G(\text{pt}; B\mathcal{G}^G(-)_+ \wedge \mathbf{E})$$

is always bijective. There is a linearization map $\mathbf{A}^{-\infty}(X) \rightarrow \mathbf{K}(\mathbb{Z}\Pi(X)_{\oplus})$ (see the next subsection for the notation) which is always 2-connected and a rational equivalence if X is aspherical (recall that \mathbf{K} denotes the non-connective K -theory spectrum). For finer statements about the linearization map, compare also [230].

The above discussion yields in particular the following, compare [111, 1.6.7 on page 261].

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Proposition 112 The rational version of the K -theoretic Farrell–Jones Conjecture 58 is equivalent to the rational version of the Farrell–Jones Conjecture for Pseudoisotopies of Aspherical Spaces 110. If the assembly map in the conjecture for pseudoisotopies is (integrally) an isomorphism for $n \leq -1$, then so is the assembly map in the K -theoretic Farrell–Jones Conjecture for $n \leq 1$.

Fibered Versions

Next we present the more general fibered versions of the Farrell–Jones Conjectures. These fibered versions have better inheritance properties, compare Sect. 2.6.4.

In the previous section we considered functors $\mathbf{F}: \text{SPACES} \rightarrow \text{SPECTRA}$, like \mathbf{P} , \mathbf{P}^{diff} and $\mathbf{A}^{-\infty}$, and the associated equivariant homology theories $H_n^G(-; \mathbf{F} \circ B)$ (compare Proposition 157). Here B denotes the classifying space functor, which sends a groupoid \mathcal{G} to its classifying space $B\mathcal{G}$. In fact all equivariant homology theories we considered so far can be obtained in this fashion for special choices of \mathbf{F} . Namely, let \mathbf{F} be one of the functors

$$\mathbf{K}(R\Pi(-)_{\oplus}), \quad \mathbf{L}^{(-\infty)}(R\Pi(-)_{\oplus}) \quad \text{or} \quad \mathbf{K}^{\text{top}}(C_r^*\Pi(-)_{\oplus}),$$

where $\Pi(X)$ denotes the fundamental groupoid of a space, $R\mathcal{G}_{\oplus}$ respectively $C_r^*\mathcal{G}_{\oplus}$ is the R -linear respectively the C^* -category associated to a groupoid \mathcal{G} and \mathbf{K} , $\mathbf{L}^{(-\infty)}$ and \mathbf{K}^{top} are suitable functors which send additive respectively C^* -categories to spectra, compare the proof of Theorem 158. There is a natural equivalence $\mathcal{G} \rightarrow \Pi B\mathcal{G}$. Hence, if we precompose the functors above with the classifying space functor B we obtain functors which are equivalent to the functors we have so far been calling

$$\mathbf{K}_R, \quad \mathbf{L}_R^{(-\infty)} \quad \text{and} \quad \mathbf{K}^{\text{top}},$$

compare Theorem 158. Note that in contrast to these three cases the pseudoisotopy functor \mathbf{P} depends on more than just the fundamental groupoid. However Conjecture 110 above only deals with aspherical spaces.

Given a G -CW-complex Z and a functor \mathbf{F} from spaces to spectra we obtain a functor $X \mapsto \mathbf{F}(Z \times_G X)$ which digests G -CW-complexes. In particular we can restrict it to the orbit category to obtain a functor

$$\mathbf{F}(Z \times_G -): \text{Or}(G) \rightarrow \text{SPECTRA} .$$

According to Proposition 156 we obtain a corresponding G -homology theory

$$H_n^G(-; \mathbf{F}(Z \times_G -))$$

and associated assembly maps. Note that restricted to the orbit category the functor $EG \times_G -$ is equivalent to the classifying space functor B and so $H_n^G(-; \mathbf{F} \circ B)$ can be considered as a special case of this construction.

Conjecture 113: (*Fibered Farrell–Jones Conjectures.*) Let R be a ring (with involution). Let $\mathbf{F}: \text{SPACES} \rightarrow \text{SPECTRA}$ be one of the functors

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$$\mathbf{K}(R\Pi(-)_\oplus), \quad \mathbf{L}^{(-\infty)}(R\Pi(-)_\oplus), \quad \mathbf{P}(-), \quad \mathbf{P}^{\text{diff}}(-) \quad \text{or} \quad \mathbf{A}^{-\infty}(-) .$$

Then for every free G -CW-complex Z and all $n \in \mathbb{Z}$ the associated assembly map

$$H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G); \mathbf{F}(Z \times_G -)) \rightarrow H_n^G(\text{pt}; \mathbf{F}(Z \times_G -)) \cong \pi_n(\mathbf{F}(Z/G))$$

is an isomorphism.

Remark 114: (*A Fibered Baum–Connes Conjecture.*) With the family \mathcal{FIN} instead of $\mathcal{V}\mathcal{C}\mathcal{Y}$ and the functor $\mathbf{F} = \mathbf{K}^{\text{top}}(C_r^*\Pi(-)_\oplus)$ one obtains a *Fibered Baum–Connes Conjecture*.

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Remark 115: (*The Special Case $Z = \tilde{X}$.*) Suppose $Z = \tilde{X}$ is the universal covering of a space X equipped with the action of its fundamental group $G = \pi_1(X)$. Then in the algebraic K - and L -theory case the conjecture above specializes to the “ordinary” Farrell–Jones Conjecture 58. In the pseudoisotopy and A -theory case one obtains a formulation of an (unfibered) conjecture about $\pi_n(\mathbf{P}(X))$ or $\pi_n(\mathbf{A}^{-\infty}(X))$ for spaces X which are not necessarily aspherical.

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Remark 116: (*Relation to the Original Formulation.*) In [111] Farrell and Jones formulate a fibered version of their conjectures for every (Serre) fibration $Y \rightarrow X$ over a connected CW-complex X . In our set-up this corresponds to choosing Z to be the total space of the fibration obtained from $Y \rightarrow X$ by pulling back along the

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universal covering projection $\tilde{X} \rightarrow X$. This space is a free G -space for $G = \pi_1(X)$. Note that an arbitrary free G -CW-complex Z can always be obtained in this fashion from a map $Z/G \rightarrow BG$, compare [111, corollary 2.2.1 on page 264].

117 Remark 117: (*Relating K -theory and Pseudoisotopy Theory in the Fibered Case.*) The linearization map $\pi_n(\mathbf{A}^{-\infty}(X)) \rightarrow K_n(\mathbb{Z}\Pi(X))$ is always 2-connected, but for spaces which are not aspherical it need not be a rational equivalence. Hence the comparison results discussed in Remark 111 apply for the fibered versions only in dimensions $n \leq 1$.

The Isomorphism Conjecture for NK -Groups

In Remark 15 we defined the groups $NK_n(R)$ for a ring R . They are the simplest kind of Nil-groups responsible for the infinite cyclic group. Since the functor \mathbf{K}_R is natural with respect to ring homomorphism we can define \mathbf{NK}_R as the (objectwise) homotopy cofiber of $\mathbf{K}_R \rightarrow \mathbf{K}_{R[t]}$. There is an associated assembly map.

118 Conjecture 118: (*Isomorphism Conjecture for NK -groups.*) The assembly map

$$H_n^G(E_{\text{Vey}}(G); \mathbf{NK}_R) \rightarrow H_n^G(\text{pt}; \mathbf{NK}_R) \cong NK_n(RG)$$

is always an isomorphism.

There is a weak equivalence $\mathbf{K}_{R[t]} \simeq \mathbf{K}_R \vee \mathbf{NK}_R$ of functors from GROUPOIDS to SPECTRA. This implies for a fixed family \mathcal{F} of subgroups of G and $n \in \mathbb{Z}$ that whenever two of the three assembly maps

$$A_{\mathcal{F}} : H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_{R[t]}) \rightarrow K_n(R[t][G]),$$

$$A_{\mathcal{F}} : H_n^G(E_{\mathcal{F}}(G); \mathbf{K}_R) \rightarrow K_n(R[G]),$$

$$A_{\mathcal{F}} : H_n^G(E_{\mathcal{F}}(G); \mathbf{NK}_R) \rightarrow NK_n(RG)$$

are bijective, then so is the third (compare [19, sect. 7]). Similarly one can define a functor \mathbf{E}_R from the category GROUPOIDS to SPECTRA and weak equivalences

$$\mathbf{K}_{R[t,t^{-1}]} \rightarrow \mathbf{E}_R \leftarrow \mathbf{K}_R \vee \Sigma \mathbf{K}_R \vee \mathbf{NK}_R \vee \mathbf{NK}_R,$$

which on homotopy groups corresponds to the Bass–Heller–Swan decomposition (see Remark 15). One obtains a two-out-of-three statement as above with the $\mathbf{K}_{R[t]}$ -assembly map replaced by the $\mathbf{K}_{R[t,t^{-1}]}$ -assembly map.

Algebraic K -Theory of the Hecke Algebra

In Sect. 2.5.1 we mentioned the classifying space $E_{\mathcal{K}\mathcal{O}}(G)$ for the family of compact-open subgroups and the Baum–Connes Conjecture for a totally disconnected

group T . There is an analogous conjecture dealing with the algebraic K -theory of the Hecke algebra.

Let $\mathcal{H}(T)$ denote the Hecke algebra of T which consists of locally constant functions $G \rightarrow \mathbb{C}$ with compact support and inherits its multiplicative structure from the convolution product. The Hecke algebra $\mathcal{H}(T)$ plays the same role for T as the complex group ring $\mathbb{C}G$ for a discrete group G and reduces to this notion if T happens to be discrete. There is a T -homology theory \mathcal{H}_*^T with the property that for any open and closed subgroup $H \subseteq T$ and all $n \in \mathbb{Z}$ we have $\mathcal{H}_n^T(T|H) = K_n(\mathcal{H}(H))$, where $K_n(\mathcal{H}(H))$ is the algebraic K -group of the Hecke algebra $\mathcal{H}(H)$.

Conjecture 119: (*Isomorphism Conjecture for the Hecke-Algebra.*) For a totally disconnected group T the assembly map

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$$A_{\mathcal{K}\mathcal{O}}: \mathcal{H}_n^T(E_{\mathcal{K}\mathcal{O}}(T)) \rightarrow \mathcal{H}^T(\text{pt}) = K_n(\mathcal{H}(T)) \tag{2.26}$$

induced by the projection $\text{pr}: E_{\mathcal{K}\mathcal{O}}(T) \rightarrow \text{pt}$ is an isomorphism for all $n \in \mathbb{Z}$.

In the case $n = 0$ this reduces to the statement that

$$\text{colim}_{T|H \in \text{Or}_{\mathcal{K}\mathcal{O}}(T)} K_0(\mathcal{H}(H)) \rightarrow K_0(\mathcal{H}(T)) . \tag{2.27}$$

is an isomorphism. For $n \leq -1$ one obtains the statement that $K_n(\mathcal{H}(G)) = 0$. The group $K_0(\mathcal{H}(T))$ has an interpretation in terms of the smooth representations of T . The G -homology theory can be constructed using an appropriate functor $E: \text{Or}_{\mathcal{K}\mathcal{O}}(T) \rightarrow \text{SPECTRA}$ and the recipe explained in Sect. 2.7.2. The desired functor E is given in [276].

Status of the Conjectures

2.6

In this section, we give the status, at the time of writing, of some of the conjectures mentioned earlier. We begin with the Baum–Connes Conjecture.

Status of the Baum–Connes-Conjecture

2.6.1

The Baum–Connes Conjecture with Coefficients

We begin with the Baum–Connes Conjecture with Coefficients 105. It has better inheritance properties than the Baum–Connes Conjecture 59 itself and contains it as a special case.

Theorem 120: (*Baum–Connes Conjecture with Coefficients and a - T -menable Groups.*) The discrete group G satisfies the Baum–Connes Conjecture with Coefficients 105 and is K -amenable provided that G is a - T -menable.

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This theorem is proved in Higson–Kasparov [149, theorem 1.1], where more generally second countable locally compact topological groups are treated (see also [164]).

A group G is *a-T-menable*, or, equivalently, has the *Haagerup property* if G admits a metrically proper isometric action on some affine Hilbert space. *Metrically proper* means that for any bounded subset B the set $\{g \in G \mid gB \cap B \neq \emptyset\}$ is finite. An extensive treatment of such groups is presented in [66]. Any a-T-menable group is countable. The class of a-T-menable groups is closed under taking subgroups, under extensions with finite quotients and under finite products. It is not closed under semi-direct products. Examples of a-T-menable groups are countable amenable groups, countable free groups, discrete subgroups of $SO(n, 1)$ and $SU(n, 1)$, Coxeter groups, countable groups acting properly on trees, products of trees, or simply connected $CAT(0)$ cubical complexes. A group G has Kazhdan's *property (T)* if, whenever it acts isometrically on some affine Hilbert space, it has a fixed point. An infinite a-T-menable group does not have property (T). Since $SL(n, \mathbb{Z})$ for $n \geq 3$ has property (T), it cannot be a-T-menable.

Using the Higson–Kasparov result Theorem 120 and known inheritance properties of the Baum–Connes Conjecture with Coefficients (compare Sect. 2.6.4 and [233, 234]) Mislin describes an even larger class of groups for which the conjecture is known [225, theorem 5.23].

121

Theorem 121: (*The Baum–Connes Conjecture with Coefficients and the Class of Groups $LH\mathcal{E}\mathcal{T}\mathcal{H}$.*) The discrete group G satisfies the Baum–Connes Conjecture with Coefficients 105 provided that G belongs to the class $LH\mathcal{E}\mathcal{T}\mathcal{H}$.

The class $LH\mathcal{E}\mathcal{T}\mathcal{H}$ is defined as follows. Let $H\mathcal{T}\mathcal{H}$ be the smallest class of groups which contains all a-T-menable groups and contains a group G if there is a 1-dimensional contractible G -CW-complex whose stabilizers belong already to $H\mathcal{T}\mathcal{H}$. Let $H\mathcal{E}\mathcal{T}\mathcal{H}$ be the smallest class of groups containing $H\mathcal{T}\mathcal{H}$ and containing a group G if either G is countable and admits a surjective map $p: G \rightarrow Q$ with Q and $p^{-1}(F)$ in $H\mathcal{E}\mathcal{T}\mathcal{H}$ for every finite subgroup $F \subseteq Q$ or if G admits a 1-dimensional contractible G -CW-complex whose stabilizers belong already to $H\mathcal{E}\mathcal{T}\mathcal{H}$. Let $LH\mathcal{E}\mathcal{T}\mathcal{H}$ be the class of groups G whose finitely generated subgroups belong to $H\mathcal{E}\mathcal{T}\mathcal{H}$.

The class $LH\mathcal{E}\mathcal{T}\mathcal{H}$ is closed under passing to subgroups, under extensions with torsion free quotients and under finite products. It contains in particular one-relator groups and Haken 3-manifold groups (and hence all knot groups). All these facts of the class $LH\mathcal{E}\mathcal{T}\mathcal{H}$ and more information can be found in [225].

Vincent Lafforgue has an unpublished proof of the Baum–Connes Conjecture with Coefficients 105 for word-hyperbolic groups.

122

Remark 122 There are counterexamples to the Baum–Connes Conjecture with (commutative) Coefficients 105 as soon as the existence of finitely generated groups containing arbitrary large expanders in their Cayley graph is shown [150, sect. 7]. The existence of such groups has been claimed by Gromov [138, 139]. Details of

the construction are described by Ghys in [134]. At the time of writing no counterexample to the Baum–Connes Conjecture 59 (without coefficients) is known to the authors.

The Baum–Connes Conjecture

Next we deal with the Baum–Connes Conjecture 59 itself. Recall that all groups which satisfy the Baum–Connes Conjecture with Coefficients 105 do in particular satisfy the Baum–Connes Conjecture 59.

Theorem 123: (*Status of the Baum–Connes Conjecture.*) A group G satisfies the Baum–Connes Conjecture 59 if it satisfies one of the following conditions.

123

- (i) It is a discrete subgroup of a connected Lie groups L , whose Levi–Malcev decomposition $L = RS$ into the radical R and semisimple part S is such that S is locally of the form

$$S = K \times SO(n_1, 1) \times \dots \times SO(n_k, 1) \times SU(m_1, 1) \times \dots \times SU(m_l, 1)$$

for a compact group K .

- (ii) The group G has property (RD) and admits a proper isometric action on a strongly bolic weakly geodesic uniformly locally finite metric space.
- (iii) G is a subgroup of a word hyperbolic group.
- (vi) G is a discrete subgroup of $Sp(n, 1)$.

Proof The proof under condition 123 is due to Julg–Kasparov [166]. The proof under condition 123 is due to Lafforgue [183] (see also [288]). Word hyperbolic groups have property (RD) [84]. Any subgroup of a word hyperbolic group satisfies the conditions appearing in the result of Lafforgue and hence satisfies the Baum–Connes Conjecture 59 [222, theorem 20]. The proof under condition 123 is due to Julg [165].

Lafforgue’s result about groups satisfying condition 123 yielded the first examples of infinite groups which have Kazhdan’s property (T) and satisfy the Baum–Connes Conjecture 59. Here are some explanations about condition 123.

A *length function* on G is a function $L: G \rightarrow \mathbb{R}_{\geq 0}$ such that $L(1) = 0$, $L(g) = L(g^{-1})$ for $g \in G$ and $L(g_1g_2) \leq L(g_1) + L(g_2)$ for $g_1, g_2 \in G$ holds. The word length metric L_S associated to a finite set S of generators is an example. A length function L on G has *property (RD)* (“rapid decay”) if there exist $C, s > 0$ such that for any $u = \sum_{g \in G} \lambda_g \cdot g \in \mathbb{C}G$ we have

$$\|\varrho_G(u)\|_{\infty} \leq C \cdot \left(\sum_{g \in G} |\lambda_g|^2 \cdot (1 + L(g))^{2s} \right)^{1/2},$$

where $\|\varrho_G(u)\|_\infty$ is the operator norm of the bounded G -equivariant operator $l^2(G) \rightarrow l^2(G)$ coming from right multiplication with u . A group G has *property (RD)* if there is a length function which has property (RD). More information about property (RD) can be found for instance in [63, 184] and [307, chapt. 8]. Bolicity generalizes Gromov's notion of hyperbolicity for metric spaces. We refer to [169] for a precise definition.

124 Remark 124 We do not know whether all groups appearing in Theorem 123 satisfy also the Baum–Connes Conjecture with Coefficients 105.

125 Remark 125 It is not known at the time of writing whether the Baum–Connes Conjecture is true for $SL(n, \mathbb{Z})$ for $n \geq 3$.

126 Remark 126: (*The Status for Topological Groups.*) We only dealt with the Baum–Connes Conjecture for discrete groups. We already mentioned that Higson–Kasparov [149] treat second countable locally compact topological groups. The Baum–Connes Conjecture for second countable almost connected groups G has been proven by Chabert–Echterhoff–Nest [60] based on the work of Higson–Kasparov [149] and Lafforgue [186]. The Baum–Connes Conjecture with Coefficients 105 has been proved for the connected Lie groups L appearing in Theorem 123 (i) by [166] and for $Sp(n, 1)$ by Julg [165].

The Injectivity Part of the Baum–Connes Conjecture

In this subsection we deal with injectivity results about the assembly map appearing in the Baum–Connes Conjecture 59. Recall that rational injectivity already implies the Novikov Conjecture 52 (see Proposition 92) and the Homological Stable Gromov–Lawson–Rosenberg Conjecture 101 (see Propositions 102 and 76).

127 Theorem 127: (*Rational Injectivity of the Baum–Connes Assembly Map.*) The assembly map appearing in the Baum–Connes Conjecture 59 is rationally injective if G belongs to one of the classes of groups below.

- (i) Groups acting properly isometrically on complete manifolds with non-positive sectional curvature.
- (ii) Discrete subgroups of Lie groups with finitely many path components.
- (iii) Discrete subgroups of p -adic groups.

Proof The proof of assertions (i) and (ii) is due to Kasparov [171], the one of assertion (iii) to Kasparov–Skandalis [172].

A metric space (X, d) admits a *uniform embedding into Hilbert space* if there exist a separable Hilbert space H , a map $f: X \rightarrow H$ and non-decreasing functions φ_1 and φ_2 from $[0, \infty) \rightarrow \mathbb{R}$ such that $\varphi_1(d(x, y)) \leq \|f(x) - f(y)\| \leq \varphi_2(d(x, y))$ for $x, y \in X$ and $\lim_{r \rightarrow \infty} \varphi_i(r) = \infty$ for $i = 1, 2$. A metric is *proper* if for each $r > 0$ and $x \in X$ the closed ball of radius r centered at x is compact. The question whether a discrete group G equipped with a proper left G -invariant metric d admits a uniform embedding into Hilbert space is independent of the choice of d , since the induced coarse structure does not depend on d [289, page 808]. For more information about groups admitting a uniform embedding into Hilbert space we refer to [87, 140].

The class of finitely generated groups, which embed uniformly into Hilbert space, contains a subclass A , which contains all word hyperbolic groups, finitely generated discrete subgroups of connected Lie groups and finitely generated amenable groups and is closed under semi-direct products [338, definition 2.1, theorem 2.2 and proposition 2.6]. Gromov [138, 139] has announced examples of finitely generated groups which do not admit a uniform embedding into Hilbert space. Details of the construction are described in Ghys [134].

The next theorem is proved by Skandalis–Tu–Yu [289, theorem 6.1] using ideas of Higson [148].

Theorem 128: (*Injectivity of the Baum–Connes Assembly Map.*) Let G be a countable group. Suppose that G admits a G -invariant metric for which G admits a uniform embedding into Hilbert space. Then the assembly map appearing in the Baum–Connes Conjecture with Coefficients 105 is injective.

128

We now discuss conditions which can be used to verify the assumption in Theorem 128.

Remark 129: (*Linear Groups.*) A group G is called *linear* if it is a subgroup of $GL_n(F)$ for some n and some field F . Guentner–Higson–Weinberger [140] show that every countable linear group admits a uniform embedding into Hilbert space and hence Theorem 128 applies.

129

Remark 130: (*Groups Acting Amenably on a Compact Space.*) A continuous action of a discrete group G on a compact space X is called *amenable* if there exists a sequence

130

$$p_n: X \rightarrow M^1(G) = \left\{ f: G \rightarrow [0, 1] \mid \sum_{g \in G} f(g) = 1 \right\}$$

of weak- $*$ -continuous maps such that for each $g \in G$ one has

$$\lim_{n \rightarrow \infty} \sup_{x \in X} \|g * (p_n(x) - p_n(g \cdot x))\|_1 = 0.$$

Note that a group G is amenable if and only if its action on the one-point-space is amenable. More information about this notion can be found for instance in [5, 6].

Higson–Roe [153, theorem 1.1 and proposition 2.3] show that a finitely generated group equipped with its word length metric admits an amenable action on a compact metric space, if and only if it belongs to the class A defined in [338, definition 2.1], and hence admits a uniform embedding into Hilbert space. Hence Theorem 128 implies the result of Higson [148, theorem 1.1] that the assembly map appearing in the Baum–Connes Conjecture with Coefficients 105 is injective if G admits an amenable action on some compact space.

Word hyperbolic groups and the class of groups mentioned in Theorem 127 (ii) fall under the class of groups admitting an amenable action on some compact space [153, sect. 4].

131 **Remark 131** Higson [148, theorem 5.2] shows that the assembly map appearing in the Baum–Connes Conjecture with Coefficients 105 is injective if $\underline{E}G$ admits an appropriate compactification. This is a C^* -version of the result for K - and L -theory due to Carlsson–Pedersen [55], compare Theorem 146.

132 **Remark 132** We do not know whether the groups appearing in Theorem 127 and 128 satisfy the Baum–Connes Conjecture 59.

Next we discuss injectivity results about the classical assembly map for topological K -theory.

The *asymptotic dimension* of a proper metric space X is the infimum over all integers n such that for any $R > 0$ there exists a cover \mathcal{U} of X with the property that the diameter of the members of \mathcal{U} is uniformly bounded and every ball of radius R intersects at most $(n + 1)$ elements of \mathcal{U} (see [137, page 28]).

The next result is due to Yu [337].

133 **Theorem 133:** (*The C^* -Theoretic Novikov Conjecture and Groups of Finite Asymptotic Dimension.*) Let G be a group which possesses a finite model for BG and has finite asymptotic dimension. Then the assembly map in the Baum–Connes Conjecture 31

$$K_n(BG) \rightarrow K_n(C_r^*(G))$$

is injective for all $n \in \mathbb{Z}$.

The Coarse Baum–Connes Conjecture

The coarse Baum–Connes Conjecture was explained in Sect. 2.5.1. Recall the descent principle (Proposition 109): if a countable group can be equipped with a G -

invariant metric such that the resulting metric space satisfies the Coarse Baum–Connes Conjecture, then the classical assembly map for topological K -theory is injective.

Recall that a discrete metric space has bounded geometry if for each $r > 0$ there exists a $N(r)$ such that for all x the ball of radius $N(r)$ centered at $x \in X$ contains at most $N(r)$ elements.

The next result is due to Yu [338, theorem 2.2 and proposition 2.6].

Theorem 134: (*Status of the Coarse Baum–Connes Conjecture.*) The Coarse Baum–Connes Conjecture 108 is true for a discrete metric space X of bounded geometry if X admits a uniform embedding into Hilbert space. In particular a countable group G satisfies the Coarse Baum–Connes Conjecture 108 if G equipped with a proper left G -invariant metric admits a uniform embedding into Hilbert space.

134

Also Yu’s Theorem 133 is proven via a corresponding result about the Coarse Baum–Connes Conjecture.

Status of the Farrell–Jones Conjecture

2.6.2

Next we deal with the Farrell–Jones Conjecture.

The Fibered Farrell–Jones Conjecture

The Fibered Farrell–Jones Conjecture 113 was discussed in Sect. 2.5.2. Recall that it has better inheritance properties (compare Sect. 2.6.4) and contains the ordinary Farrell–Jones Conjecture 58 as a special case.

Theorem 135: (*Status of the Fibered Farrell–Jones Conjecture.*)

135

- (i) Let G be a discrete group which satisfies one of the following conditions.
- (a) There is a Lie group L with finitely many path components and G is a co-compact discrete subgroup of L .
 - (b) The group G is virtually torsionfree and acts properly discontinuously, co-compactly and via isometries on a simply connected complete nonpositively curved Riemannian manifold.

Then

- (1) The version of the Fibered Farrell–Jones Conjecture 113 for the topological and the differentiable pseudoisotopy functor is true for G .
- (2) The version of the Fibered Farrell–Jones Conjecture 113 for K -theory and $R = \mathbb{Z}$ is true for G in the range $n \leq 1$, i.e. the assembly map is bijective for $n \leq 1$.

Moreover we have the following statements.

- (ii) The version of the Fibered Farrell–Jones Conjecture 113 for K -theory and $R = \mathbb{Z}$ is true in the range $n \leq 1$ for braid groups.
- (iii) The L -theoretic version of the Fibered Farrell–Jones Conjecture 113 with $R = \mathbb{Z}$ holds after inverting 2 for elementary amenable groups.

Proof (i) For assertion (1) see [111, theorem 2.1 on page 263], [111, proposition 2.3] and [119, theorem A]. Assertion (2) follows from (1) by Remark 117.

(ii) See [119].

(iii) is proven in [117, theorem 5.2]. For crystallographic groups see also [333].

A surjectivity result about the Fibered Farrell–Jones Conjecture for Pseudoisotopies appears as the last statement in Theorem 139.

The rational comparison result between the K -theory and the pseudoisotopy version (see Proposition 112) does not work in the fibered case, compare Remark 117. However, in order to exploit the good inheritance properties one can first use the pseudoisotopy functor in the fibered set-up, then specialize to the unfibered situation and finally do the rational comparison to K -theory.

136 **Remark 136** The version of the Fibered Farrell–Jones Conjecture 113 for L -theory and $R = \mathbb{Z}$ seems to be true if G satisfies the condition (a) appearing in Theorem 135. Farrell and Jones [111, remark 2.1.3 on page 263] say that they can also prove this version without giving the details.

137 **Remark 137** Let G be a virtually poly-cyclic group. Then it contains a maximal normal finite subgroup N such that the quotient G/N is a discrete cocompact subgroup of a Lie group with finitely many path components [331, theorem 3, remark 4 on page 200]. Hence by Sect. 2.6.4 and Theorem 135 the version of the Fibered Farrell–Jones Conjecture 113 for the topological and the differentiable pseudoisotopy functor, and for K -theory and $R = \mathbb{Z}$ in the range $n \leq 1$, is true for G . Earlier results of this type were treated for example in [100, 105].

The Farrell–Jones Conjecture

Here is a sample of some results one can deduce from Theorem 135.

138 **Theorem 138:** (*The Farrell–Jones Conjecture and Subgroups of Lie groups.*) Suppose H is a subgroup of G , where G is a discrete cocompact subgroup of a Lie group L with finitely many path components. Then

- (i) The version of the Farrell–Jones Conjecture for K -theory and $R = \mathbb{Z}$ is true for H rationally, i.e. the assembly map appearing in Conjecture 58 is an isomorphism after applying $- \otimes_{\mathbb{Z}} \mathbb{Q}$.
- (ii) The version of the Farrell–Jones Conjecture for K -theory and $R = \mathbb{Z}$ is true for H in the range $n \leq 1$, i.e. the assembly map appearing in Conjecture 58 is an isomorphism for $n \leq 1$.

Proof The results follow from Theorem 135, since the Fibered Farrell–Jones Conjecture 113 passes to subgroups [111, theorem A.8 on page 289] (compare Sect. 2.6.4) and implies the Farrell–Jones Conjecture 58.

We now discuss results for torsion free groups. Recall that for $R = \mathbb{Z}$ the K -theoretic Farrell–Jones Conjecture in dimensions ≤ 1 together with the L -theoretic version implies already the Borel Conjecture 27 in dimension ≥ 5 (see Theorem 28).

A complete Riemannian manifold M is called *A-regular* if there exists a sequence of positive real numbers A_0, A_1, A_2, \dots such that $\|\nabla^n K\| \leq A_n$, where $\|\nabla^n K\|$ is the supremum-norm of the n -th covariant derivative of the curvature tensor K . Every locally symmetric space is A-regular since ∇K is identically zero. Obviously every closed Riemannian manifold is A-regular.

Theorem 139: (*Status of the Farrell–Jones Conjecture for Torsionfree Groups.*)

139

Consider the following conditions for the group G .

- (i) $G = \pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature which is A-regular.
- (ii) $G = \pi_1(M)$ for a closed Riemannian manifold M with non-positive sectional curvature.
- (iii) $G = \pi_1(M)$ for a complete Riemannian manifold with negatively pinched sectional curvature.
- (iv) G is a torsion free discrete subgroup of $GL(n, \mathbb{R})$.
- (v) G is a torsion free solvable discrete subgroup of $GL(n, \mathbb{C})$.
- (vi) $G = \pi_1(X)$ for a non-positively curved finite simplicial complex X .
- (vii) G is a strongly poly-free group in the sense of Aravinda–Farrell–Roushon [10, definition 1.1]. The pure braid group satisfies this hypothesis.

Then

- (1) Suppose that G satisfies one of the conditions (i) to (vii). Then the K -theoretic Farrell–Jones Conjecture is true for $R = \mathbb{Z}$ in dimensions $n \leq 1$. In particular Conjecture 3 holds for G .
- (2) Suppose that G satisfies one of the conditions (i), (ii), (iii) or (iv). Then G satisfies the Farrell–Jones Conjecture for Torsion Free Groups and L -theory 19 for $R = \mathbb{Z}$.
- (3) Suppose that G satisfies (ii). Then the Farrell–Jones Conjecture for Pseudoisotopies of Aspherical Spaces 110 holds for G .
- (4) Suppose that G satisfies one of the conditions (i), (iii), or (iv). Then the assembly map appearing in the version of the Fibered Farrell–Jones Conjecture for Pseudoisotopies 113 is surjective, provided that the G -space Z appearing in Conjecture 113 is connected.

Proof Note that (ii) is a special case of (i) because every closed Riemannian manifold is A-regular. If M is a pinched negatively curved complete Riemannian

manifold, then there is another Riemannian metric for which M is negatively curved complete and A-regular. This fact is mentioned in [115, page 216] and attributed there to Abresch [3] and Shi [285]. Hence also (iii) can be considered as a special case of (i). The manifold $M = G \backslash GL(n, \mathbb{R})/O(n)$ is a non-positively curved complete locally symmetric space and hence in particular A-regular. So (iv) is a special case of (i).

Assertion (1) under the assumption (i) is proved by Farrell–Jones in [115, proposition 0.10 and lemma 0.12]. The earlier work [110] treated the case (ii). Under assumption (v) assertion (1) is proven by Farrell–Linnell [117, theorem 1.1]. The result under assumption (vi) is proved by Hu [156], under assumption (vii) it is proved by Aravinda–Farrell–Roushon [10, theorem 1.3].

Assertion (2) under assumption (i) is proven by Farrell–Jones in [115]. The case (ii) was treated earlier in [112].

Assertion (3) is proven by Farrell–Jones in [110] and assertion (4) by Jones in [160].

140

Remark 140 As soon as certain collapsing results (compare [114], [116]) are extended to orbifolds, the results under (4) above would also apply to groups with torsion and in particular to $SL_n(\mathbb{Z})$ for arbitrary n .

The Farrell–Jones Conjecture for Arbitrary Coefficients

The following result due to Bartels–Reich [21] deals with algebraic K -theory for arbitrary coefficient rings R . It extends Bartels–Farrell–Jones–Reich [19].

141

Theorem 141 Suppose that G is the fundamental group of a closed Riemannian manifold with negative sectional curvature. Then the K -theoretic part of the Farrell–Jones Conjecture 58 is true for any ring R , i.e. the assembly map

$$A_{\mathcal{VCY}}: H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) \rightarrow K_n(RG)$$

is an isomorphism for all $n \in \mathbf{Z}$.

Note that the assumption implies that G is torsion free and hence the family \mathcal{VCY} reduces to the family \mathcal{CYC} of cyclic subgroups. Recall that for a regular ring R the theorem above implies that the classical assembly

$$A: H_n(BG; \mathbf{K}(R)) \rightarrow K_n(RG)$$

is an isomorphism, compare Proposition 66 (i).

Injectivity Part of the Farrell–Jones Conjecture

The next result about the classical K -theoretic assembly map is due to Bökstedt–Hsiang–Madsen [38].

Theorem 142: (*Rational Injectivity of the Classical K -Theoretic Assembly Map.*)

Let G be a group such that the integral homology $H_j(BG; \mathbb{Z})$ is finitely generated for each $j \in \mathbb{Z}$. Then the rationalized assembly map

$$A: H_n(BG; \mathbf{K}(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H_n^G(E_{\square}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective for all $n \in \mathbb{Z}$.

Because of the homological Chern character (see Remark 12) we obtain for the groups treated in Theorem 142 an injection

$$\bigoplus_{s+t=n} H_s(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} (K_t(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}) \rightarrow K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{2.28}$$

Next we describe a generalization of Theorem 142 above from the trivial family $\{\square\}$ to the family \mathcal{FIN} of finite subgroups due to Lück–Reich–Rognes–Varisco [208]. Let $\mathbf{K}_{\mathbb{Z}}^{\text{con}}: \text{GROUPOIDS} \rightarrow \text{SPECTRA}$ be the connective version of the functor $\mathbf{K}_{\mathbb{Z}}$ of (2.37). In particular $H_n(G/H; \mathbf{K}_{\mathbb{Z}}^{\text{con}})$ is isomorphic to $K_n(\mathbb{Z}H)$ for $n \geq 0$ and vanishes in negative dimensions. For a prime p we denote by \mathbb{Z}_p the p -adic integers. Let $K_n(R; \mathbb{Z}_p)$ denote the homotopy groups $\pi_n(\mathbf{K}^{\text{con}}(R)_{\widehat{p}})$ of the p -completion of the connective K -theory spectrum of the ring R .

Theorem 143: (*Rational Injectivity of the Farrell–Jones Assembly Map for Connective K -theory.*) Suppose that the group G satisfies the following two conditions:

- (H) For each finite cyclic subgroup $C \subseteq G$ and all $j \geq 0$ the integral homology group $H_j(BZ_G C; \mathbb{Z})$ of the centralizer $Z_G C$ of C in G is finitely generated.
- (K) There exists a prime p such that for each finite cyclic subgroup $C \subseteq G$ and each $j \geq 1$ the map induced by the change of coefficients homomorphism

$$K_j(\mathbb{Z}C; \mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_j(\mathbb{Z}_p C; \mathbb{Z}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is injective.

Then the rationalized assembly map

$$A_{\text{vce}\mathcal{Y}}: H_n^G(E_{\text{vce}\mathcal{Y}}(G); \mathbf{K}_{\mathbb{Z}}^{\text{con}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an injection for all $n \in \mathbb{Z}$.

Remark 144 The methods of Chapter 2.9 apply also to $\mathbf{K}_{\mathbb{Z}}^{\text{con}}$ and yield under assumption (H) and+(K) an injection

$$\bigoplus_{s+t=n, t \geq 0} \bigoplus_{(C) \in (\mathcal{FCE}\mathcal{Y})} H_s(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C \cdot K_t(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_n(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Notice that in the index set for the direct sum appearing in the source we require $t \geq 0$. This reflects the fact that the result deals only with the connective K -

theory spectrum. If one drops the restriction $t \geq 0$ the Farrell–Jones Conjecture 58 predicts that the map is an isomorphism, compare Sect. 2.3.2 and Theorem 173. If we restrict the injection to the direct sum given by $C = 1$, we rediscover the map (2.28) whose injectivity follows already from Theorem 142.

The condition (K) appearing in Theorem 143 is conjectured to be true for all primes p (compare [280, 290] and [291]) but no proof is known. The weaker version of condition (K), where C is the trivial group is also needed in Theorem 142. But that case is known to be true and hence does not appear in its formulation. The special case of condition (K), where $j = 1$ is implied by the Leopoldt Conjecture for abelian fields (compare [229, IX, § 3]), which is known to be true [229, theorem 10.3.16]. This leads to the following result.

145 **Theorem 145:** (*Rational Contribution of Finite Subgroups to $Wh(G)$.*) Let G be a group. Suppose that for each finite cyclic subgroup $C \subseteq G$ and each $j \leq 4$ the integral homology group $H_j(BZ_G C)$ of the centralizer $Z_G C$ of C in G is finitely generated. Then the map

$$\operatorname{colim}_{H \in \operatorname{Sub}_{\mathcal{FIN}}(G)} \operatorname{Wh}(H) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \operatorname{Wh}(G) \otimes_{\mathbb{Z}} \mathbb{Q} .$$

is injective, compare Conjecture 81.

The result above should be compared to the result which is proven using Fuglede–Kadison determinants in [209, sect. 5], [202, theorem 9.38 on page 354]: for every (discrete) group G and every finite normal subgroup $H \subseteq G$ the map $\operatorname{Wh}(H) \otimes_{\mathbb{Z}G} \mathbb{Z} \rightarrow \operatorname{Wh}(G)$ induced by the inclusion $H \rightarrow G$ is rationally injective.

The next result is taken from Rosenthal [271], where the techniques and results of Carlsson–Pedersen [55] are extended from the trivial family $\{1\}$ to the family of finite subgroups \mathcal{FIN} .

146 **Theorem 146** Suppose there exists a model E for the classifying space $E_{\mathcal{FIN}}(G)$ which admits a metrizable compactification \bar{E} to which the group action extends. Suppose \bar{E}^H is contractible and E^H is dense in \bar{E}^H for every finite subgroup $H \subset G$. Suppose compact subsets of E become small near $\bar{E} - E$. Then for every ring R the assembly map

$$A_{\mathcal{FIN}} : H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) \rightarrow K_n(RG)$$

is split injective.

A compact subset $K \subset E$ is said to become *small near $\bar{E} - E$* if for every neighbourhood $U \subset \bar{E}$ of a point $x \in \bar{E} - E$ there exists a neighbourhood $V \subset \bar{E}$ such that $g \in G$ and $gK \cap V \neq \emptyset$ implies $gK \subset U$. Presumably there is an analogous result for $L^{(-\infty)}$ -theory under the assumption that $K_{-n}(RH)$ vanishes for finite subgroups H

of G and n large enough. This would extend the corresponding result for the family $\mathcal{F} = \{\}$ which appears in Carlsson–Pedersen [55].

We finally discuss injectivity results about assembly maps for the trivial family. The following result is due to Ferry–Weinberger [129, corollary 2.3] extending earlier work of Farrell–Hsiang [99].

Theorem 147 Suppose $G = \pi_1(M)$ for a complete Riemannian manifold of non-positive sectional curvature. Then the L -theory assembly map

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$$A: H_n(BG; \mathbf{L}_{\mathbb{Z}}^\varepsilon) \rightarrow L_n^\varepsilon(\mathbb{Z}G)$$

is injective for $\varepsilon = h, s$.

In fact Ferry–Weinberger also prove a corresponding splitting result for the classical A -theory assembly map. In [155] Hu shows that a finite complex of non-positive curvature is a retract of a non-positively curved PL -manifold and concludes split injectivity of the classical L -theoretic assembly map for $R = \mathbb{Z}$.

The next result due to Bartels [17] is the algebraic K - and L -theory analogue of Theorem 133.

Theorem 148: (*The K - and L -Theoretic Novikov Conjecture and Groups of Finite Asymptotic Dimension.*) Let G be a group which admits a finite model for BG . Suppose that G has finite asymptotic dimension. Then

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(i) The assembly maps appearing in the Farrell–Jones Conjecture 11

$$A: H_n(BG; \mathbf{K}(R)) \rightarrow K_n(RG)$$

is injective for all $n \in \mathbb{Z}$.

(ii) If furthermore R carries an involution and $K_{-j}(R)$ vanishes for sufficiently large j , then the assembly maps appearing in the Farrell–Jones Conjecture 19

$$A: H_n(BG; \mathbf{L}^{(-\infty)}(R)) \rightarrow L_n^{(-\infty)}(RG)$$

is injective for all $n \in \mathbb{Z}$.

Further results related to the Farrell–Jones Conjecture 58 can be found for instance in [9, 33].

List of Groups Satisfying the Conjecture

2.6.3

In the following table we list prominent classes of groups and state whether they are known to satisfy the Baum–Connes Conjecture 59 (with coefficients 105) or the Farrell–Jones Conjecture 58 (fibered 113). Some of the classes are redundant. A question mark means that the authors do not know about a corresponding result. The reader should keep in mind that there may exist results of which the authors are not aware.

type of group	Baum–Connes Conjecture 59 (with coefficients 105)	Farrell–Jones Conjecture 58 for K -theory for $R=\mathbb{Z}$ (fibered 113)	Farrell–Jones Conjecture 58 for L -theory for $R=\mathbb{Z}$ (fibered 113)
a-T-menenable groups	true with coefficients (see Theorem 120)	?	injectivity is true after inverting 2 (see Propositions 74 and 95)
amenable groups	true with coefficients (see Theorem 120)	?	injectivity is true after inverting 2 (see Propositions 74 and 95)
elementary amenable groups	true with coefficients (see Theorem 120)	?	true fibered after inverting 2 (see Theorem 135)
virtually poly-cyclic	true with coefficients (see Theorem 120)	true rationally, true fibered in the range $n \leq 1$ (compare Remark 137)	true fibered after inverting 2 (see Theorem 135)
torsion free virtually solvable subgroups of $GL(n, \mathbb{C})$	true with coefficients (see Theorem 120)	true in the range ≤ 1 [117, theorem 1.1]	true fibered after inverting 2 [117, corollary 5.3]
discrete subgroups of Lie groups with finitely many path components	injectivity true (see Theorem 128 and Remark 130)	?	injectivity is true after inverting 2 (see Propositions 74 and 95)
subgroups of groups which are discrete cocompact subgroups of Lie groups with finitely many path components	injectivity true (see Theorem 128 and Remark 130)	true rationally, true fibered in the range $n \leq 1$ (see Theorem 135)	probably true fibered (see Remark 136). Injectivity is true after inverting 2 (see Propositions 74 and 95)
linear groups	injectivity is true (see Theorem 128 and Remark 129)	?	injectivity is true after inverting 2 (see Propositions 74 and 95)
arithmetic groups	injectivity is true (see Theorem 128 and Remark 129)	?	injectivity is true after inverting 2 (see Propositions 74 and 95)
torsion free discrete subgroups of $GL(n, \mathbb{R})$	injectivity is true (see Theorem 128 and Remark 130)	true in the range $n \leq 1$ (see Theorem 139)	true (see Theorem 139)

type of group	Baum–Connes Conjecture 59 (with coefficients 105)	Farrell–Jones Conjecture 58 for K -theory for $R = \mathbb{Z}$ (fibered 113)	Farrell–Jones Conjecture 58 for L -theory for $R = \mathbb{Z}$ (fibered 113)
Groups with finite BG and finite asymptotic dimension	injectivity is true (see Theorem 133)	injectivity is true for arbitrary coefficients R (see Theorem 148)	injectivity is true for regular R as coefficients (see Theorem 148)
G acts properly and isometrically on a complete Riemannian manifold M with non-positive sectional curvature	rational injectivity is true (see Theorem 127)	?	rational injectivity is true (see Propositions 74 and 95)
$\pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature	rationally injective (see Theorem 127)	?	injectivity true (see Theorem 147)
$\pi_1(M)$ for a complete Riemannian manifold M with non-positive sectional curvature which is A-regular	rationally injective (see Theorem 127)	true in the range $n \leq 1$, rationally surjective (see Theorem 139)	true (see Theorem 139)
$\pi_1(M)$ for a complete Riemannian manifold M with pinched negative sectional curvature	rational injectivity is true (see Theorem 128)	true in the range $n \leq 1$, rationally surjective (see Theorem 139)	true (see Theorem 139)
$\pi_1(M)$ for a closed Riemannian manifold M with non-positive sectional curvature	rationally injective (see Theorem 127)	true fibered in the range $n \leq 1$, true rationally (see Theorem 139)	true (see Theorem 139)
$\pi_1(M)$ for a closed Riemannian manifold M with negative sectional curvature	true for all subgroups (see Theorem 123)	true for all coefficients R (see Theorem 141)	true (see Theorem 139)

type of group	Baum–Connes Conjecture 59 (with coefficients 105)	Farrell–Jones Conjecture 58 for K -theory and $R = \mathbb{Z}$ (fibered 113)	Farrell–Jones Conjecture 58 for L -theory for $R = \mathbb{Z}$ (fibered 113)
word hyperbolic groups	true for all subgroups (see Theorem 123). Unpublished proof with coefficients by V. Laforgue	?	injectivity is true after inverting 2 (see Propositions 74 and 95)
one-relator groups	true with coefficients (see Theorem 121)	rational injectivity is true for the fibered version (see [20])	injectivity is true after inverting 2 (see Propositions 74 and 95)
torsion free one-relator groups	true with coefficients (see Theorem 121)	true for R regular [313, theorem 19.4 on page 249 and theorem 19.5 on page 250]	true after inverting 2 [47, corollary 8]
Haken 3-manifold groups (in particular knot groups)	true with coefficients (see Theorem 121)	true in the range $n \leq 1$ for R regular [313, theorem 19.4 on page 249 and theorem 19.5 on page 250]	true after inverting 2 [47, corollary 8]
$SL(n, \mathbb{Z}), n \geq 3$	injectivity is true	compare Remark 140	injectivity is true after inverting 2 (see Propositions 74 and 95)
Artin’s braid group B_n	true with coefficients [225, theorem 5.25], [277]	true fibered in the range $n \leq 1$, true rationally [119]	injectivity is true after inverting 2 (see Propositions 74 and 95)
pure braid group C_n	true with coefficients	true in the range $n \leq 1$ (see Theorem 139)	injectivity is true after inverting 2 (see Propositions 74 and 95)
Thompson’s group F	true with coefficients [94]	?	injectivity is true after inverting 2 (see Propositions 74 and 95)

Remark 149 The authors have no information about the status of these conjectures for mapping class groups of higher genus or the group of outer automorphisms of free groups. Since all of these spaces have finite models for $E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$ Theorem 143 applies in these cases.

Inheritance Properties

In this Subsection we list some inheritance properties of the various conjectures.

Directed Colimits

Let $\{G_i \mid i \in I\}$ be a directed system of groups. Let $G = \operatorname{colim}_{i \in I} G_i$ be the colimit. We do not require that the structure maps are injective. If the Fibered Farrell–Jones Conjecture 113 is true for each G_i , then it is true for G [117, theorem 6.1].

Suppose that $\{G_i \mid i \in I\}$ is a system of subgroups of G directed by inclusion such that $G = \operatorname{colim}_{i \in I} G_i$. If each G_i satisfies the Farrell–Jones Conjecture 58, the Baum–Connes Conjecture 59 or the Baum–Connes Conjecture with Coefficients 105, then the same is true for G [32, theorem 1.1], [225, lemma 5.3]. We do not know a reference in Farrell–Jones case. An argument in that case uses Lemma 63, the fact that $K_n(RG) = \operatorname{colim}_{i \in I} K_n(RG_i)$ and that for suitable models we have $E_{\mathcal{F}}(G) = \bigcup_{i \in I} G \times_{G_i} E_{\mathcal{F} \cap G_i}(G_i)$.

Passing to Subgroups

The Baum–Connes Conjecture with Coefficients 105 and the Fibered Farrell–Jones Conjecture 113 pass to subgroups, i.e. if they hold for G , then also for any subgroup $H \subseteq G$. This claim for the Baum–Connes Conjecture with Coefficients 105 has been stated in [28], a proof can be found for instance in [59, theorem 2.5]. For the Fibered Farrell–Jones Conjecture this is proved in [111, theorem A.8 on page 289] for the special case $R = \mathbb{Z}$, but the proof also works for arbitrary rings R .

It is not known whether the Baum–Connes Conjecture 59 or the Farrell–Jones Conjecture 58 itself passes to subgroups.

Extensions of Groups

Let $p: G \rightarrow K$ be a surjective group homomorphism. Suppose that the Baum–Connes Conjecture with Coefficients 105 or the Fibered Farrell–Jones Conjecture 113 respectively holds for K and for $p^{-1}(H)$ for any subgroup $H \subset K$ which is finite or virtually cyclic respectively. Then the Baum–Connes Conjecture with Coefficients 105 or the Fibered Farrell–Jones Conjecture 113 respectively holds for G . This is proved in [233, theorem 3.1] for the Baum–Connes Conjecture with Coefficients 105, and in [111, proposition 2.2 on page 263] for the Fibered Farrell–Jones Conjecture 113 in the case $R = \mathbb{Z}$. The same proof works for arbitrary coefficient rings.

It is not known whether the corresponding statement holds for the Baum–Connes Conjecture 59 or the Farrell–Jones Conjecture 58 itself.

Let $H \subseteq G$ be a normal subgroup of G . Suppose that H is a -T-menable. Then G satisfies the Baum–Connes Conjecture with Coefficients 105 if and only if G/H does [59, corollary 3.14]. The corresponding statement is not known for the Baum–Connes Conjecture 59.

Products of Groups

The group $G_1 \times G_2$ satisfies the Baum–Connes Conjecture with Coefficients 105 if and only if both G_1 and G_2 do [59, theorem 3.17], [233, corollary 7.12]. The corresponding statement is not known for the Baum–Connes Conjecture 59.

Let $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$ denote the infinite dihedral group. Whenever a version of the Fibered Farrell–Jones Conjecture 113 is known for $G = \mathbb{Z} \times \mathbb{Z}$, $G = \mathbb{Z} \times D_\infty$ and $D_\infty \times D_\infty$, then that version of the Fibered Farrell–Jones Conjecture is true for $G_1 \times G_2$ if and only if it is true for G_1 and G_2 .

Subgroups of Finite Index

It is not known whether the Baum–Connes Conjecture 59, the Baum–Connes Conjecture with Coefficients 105, the Farrell–Jones Conjecture 58 or the Fibered Farrell–Jones Conjecture 113 is true for a group G if it is true for a subgroup $H \subseteq G$ of finite index.

Groups Acting on Trees

Let G be a countable discrete group acting without inversion on a tree T . Then the Baum–Connes Conjecture with Coefficients 105 is true for G if and only if it holds for all stabilizers of the vertices of T . This is proved by Oyono-Oyono [234, theorem 1.1]. This implies that Baum–Connes Conjecture with Coefficients 105 is stable under amalgamated products and HNN-extensions. Actions on trees in the context the Farrell–Jones Conjecture 58 will be treated in [20].

2.7

Equivariant Homology Theories

We already defined the notion of a G -homology theory in Sect. 2.3.1. If G -homology theories for different G are linked via a so called induction structure one obtains the notion of an equivariant homology theory. In this section we give a precise definition and we explain how a functor from the orbit category $\text{Or}(G)$ to the category of spectra leads to a G -homology theory (see Proposition 156) and how more generally a functor from the category of groupoids leads to an equivariant homology theory (see Proposition 157). We then describe the main examples of such spectra valued functors which were already used in order to formulate the Farrell–Jones and the Baum–Connes Conjectures in Chapt. 2.3.

2.7.1

The Definition of an Equivariant Homology Theory

The notion of a G -homology theory \mathcal{H}_*^G with values in Λ -modules for a commutative ring Λ was defined in Sect. 2.3.1. We now recall the axioms of an equivariant homology theory from [201, sect. 1]. We will see in Sect. 2.7.3 that the G -homology theories we used in the formulation of the Baum–Connes and the Farrell–Jones Conjectures in Chapt. 2.3 are in fact the values at G of suitable equivariant homology theories.

Let $\alpha: H \rightarrow G$ be a group homomorphism. Given a H -space X , define the *induction of X with α* to be the G -space $\text{ind}_\alpha X$ which is the quotient of $G \times X$ by the right H -action $(g, x) \cdot h := (g\alpha(h), h^{-1}x)$ for $h \in H$ and $(g, x) \in G \times X$. If $\alpha: H \rightarrow G$ is an inclusion, we also write ind_H^G instead of ind_α .

An *equivariant homology theory $\mathcal{H}_*^?$ with values in Λ -modules* consists of a G -homology theory \mathcal{H}_*^G with values in Λ -modules for each group G together with the following so called *induction structure*: given a group homomorphism $\alpha: H \rightarrow G$ and a H -CW-pair (X, A) such that $\ker(\alpha)$ acts freely on X , there are for each $n \in \mathbb{Z}$ natural isomorphisms

$$\text{ind}_\alpha: \mathcal{H}_n^H(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(\text{ind}_\alpha(X, A))$$

satisfying the following conditions.

- (i) Compatibility with the boundary homomorphisms

$$\partial_n^G \circ \text{ind}_\alpha = \text{ind}_\alpha \circ \partial_n^H.$$

- (ii) Functoriality

Let $\beta: G \rightarrow K$ be another group homomorphism such that $\ker(\beta \circ \alpha)$ acts freely on X . Then we have for $n \in \mathbb{Z}$

$$\text{ind}_{\beta\alpha} = \mathcal{H}_n^K(f_1) \circ \text{ind}_\beta \circ \text{ind}_\alpha: \mathcal{H}_n^H(X, A) \rightarrow \mathcal{H}_n^K(\text{ind}_{\beta\alpha}(X, A)),$$

where $f_1: \text{ind}_\beta \text{ind}_\alpha(X, A) \xrightarrow{\cong} \text{ind}_{\beta\alpha}(X, A)$, $(k, g, x) \mapsto (k\beta(g), x)$ is the natural K -homeomorphism.

- (iii) Compatibility with conjugation

For $n \in \mathbb{Z}$, $g \in G$ and a G -CW-pair (X, A) the homomorphism

$$\text{ind}_{c(g): G \rightarrow G}: \mathcal{H}_n^G(X, A) \rightarrow \mathcal{H}_n^G(\text{ind}_{c(g): G \rightarrow G}(X, A))$$

agrees with $\mathcal{H}_n^G(f_2)$, where the G -homeomorphism

$$f_2: (X, A) \rightarrow \text{ind}_{c(g): G \rightarrow G}(X, A)$$

sends x to $(1, g^{-1}x)$ and $c(g): G \rightarrow G$ sends g' to $gg'g^{-1}$.

This induction structure links the various homology theories for different groups G .

If the G -homology theory \mathcal{H}_*^G is defined or considered only for proper G -CW-pairs (X, A) , we call it a *proper G -homology theory \mathcal{H}_*^G with values in Λ -modules*.

Example 150. Let \mathcal{K}_* be a homology theory for (non-equivariant) CW-pairs with values in Λ -modules. Examples are singular homology, oriented bordism theory or topological K -homology. Then we obtain two equivariant homology theories with values in Λ -modules, whose underlying G -homology theories for a group G are given by the following constructions

$$\mathcal{H}_n^G(X, A) = \mathcal{K}_n(G \setminus X, G \setminus A);$$

$$\mathcal{H}_n^G(X, A) = \mathcal{K}_n(EG \times_G (X, A)).$$

Example 151. Given a proper G -CW-pair (X, A) , one can define the G -bordism group $\Omega_n^G(X, A)$ as the abelian group of G -bordism classes of maps $f: (M, \partial M) \rightarrow (X, A)$ whose sources are oriented smooth manifolds with cocompact orientation preserving proper smooth G -actions. The definition is analogous to the one in the non-equivariant case. This is also true for the proof that this defines a proper G -homology theory. There is an obvious induction structure coming from induction of equivariant spaces. Thus we obtain an equivariant proper homology theory $\Omega_*^?$.

Example 152. Let Λ be a commutative ring and let

$$M: \text{GROUPOIDS} \rightarrow \Lambda\text{-MODULES}$$

be a contravariant functor. For a group G we obtain a covariant functor

$$M^G: \text{Or}(G) \rightarrow \Lambda\text{-MODULES}$$

by its composition with the transport groupoid functor \mathcal{G}^G defined in (2.30). Let $H_*^G(-; M)$ be the G -homology theory given by the Bredon homology with coefficients in M^G as defined in Example 64. There is an induction structure such that the collection of the $H_*^G(-; M)$ defines an equivariant homology theory $H_*^?(-; M)$. This can be interpreted as the special case of Proposition 157, where the covariant functor $\text{GROUPOIDS} \rightarrow \Omega\text{-SPECTRA}$ is the composition of M with the functor sending a Λ -module to the associated Eilenberg–MacLane spectrum. But there is also a purely algebraic construction.

The next lemma was used in the proof of the Transitivity Principle 65.

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Lemma 153 Let $\mathcal{H}_*^?$ be an equivariant homology theory with values in Λ -modules. Let G be a group and let \mathcal{F} a family of subgroups of G . Let Z be a G -CW-complex. Consider $N \in \mathbb{Z} \cup \{\infty\}$. For $H \subseteq G$ let $\mathcal{F} \cap H$ be the family of subgroups of H given by $\{K \cap H \mid K \in \mathcal{F}\}$. Suppose for each $H \subset G$, which occurs as isotropy group in Z , that the map induced by the projection $\text{pr}: E_{\mathcal{F} \cap H}(H) \rightarrow \text{pt}$

$$\mathcal{H}_n^H(\text{pr}): \mathcal{H}_n^H(E_{\mathcal{F} \cap H}(H)) \rightarrow \mathcal{H}_n^H(\text{pt})$$

is bijective for all $n \in \mathbb{Z}, n \leq N$.

Then the map induced by the projection $\text{pr}_2: E_{\mathcal{F}}(G) \times Z \rightarrow Z$

$$\mathcal{H}_n^G(\text{pr}_2): \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z) \rightarrow \mathcal{H}_n^G(Z)$$

is bijective for $n \in \mathbb{Z}, n \leq N$.

Proof We first prove the claim for finite-dimensional G -CW-complexes by induction over $d = \dim(Z)$. The induction beginning $\dim(Z) = -1$, i.e. $Z = \emptyset$, is trivial.

In the induction step from $(d - 1)$ to d we choose a G -pushout

$$\begin{array}{ccc} \coprod_{i \in I_d} G/H_i \times S^{d-1} & \longrightarrow & Z_{d-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_d} G/H_i \times D^d & \longrightarrow & Z_d . \end{array}$$

If we cross it with $E_{\mathcal{F}}(G)$, we obtain another G -pushout of G -CW-complexes. The various projections induce a map from the Mayer–Vietoris sequence of the latter G -pushout to the Mayer–Vietoris sequence of the first G -pushout. By the Five-Lemma it suffices to prove that the following maps

$$\begin{aligned} \mathcal{H}_n^G(\text{pr}_2): \mathcal{H}_n^G\left(E_{\mathcal{F}}(G) \times \coprod_{i \in I_d} G/H_i \times S^{d-1}\right) &\rightarrow \mathcal{H}_n^G\left(\coprod_{i \in I_d} G/H_i \times S^{d-1}\right); \\ \mathcal{H}_n^G(\text{pr}_2): \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times Z_{d-1}) &\rightarrow \mathcal{H}_n^G(Z_{d-1}); \\ \mathcal{H}_n^G(\text{pr}_2): \mathcal{H}_n^G\left(E_{\mathcal{F}}(G) \times \coprod_{i \in I_d} G/H_i \times D^d\right) &\rightarrow \mathcal{H}_n^G\left(\coprod_{i \in I_d} G/H_i \times D^d\right) \end{aligned}$$

are bijective for $n \in \mathbb{Z}, n \leq N$. This follows from the induction hypothesis for the first two maps. Because of the disjoint union axiom and G -homotopy invariance of $\mathcal{H}_*^?$ the claim follows for the third map if we can show for any $H \subseteq G$ which occurs as isotropy group in Z that the map

$$\mathcal{H}_n^G(\text{pr}_2): \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times G/H) \rightarrow \mathcal{H}_n^G(G/H) \tag{2.29}$$

is bijective for $n \in \mathbb{Z}, n \leq N$. The G -map

$$G \times_H \text{res}_G^H E_{\mathcal{F}}(G) \rightarrow G/H \times E_{\mathcal{F}}(G) \quad (g, x) \mapsto (gH, gx)$$

is a G -homeomorphism where res_G^H denotes the restriction of the G -action to an H -action. Obviously $\text{res}_G^H E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F} \cap H}(H)$. We conclude from the induction structure that the map (2.29) can be identified with the map

$$\mathcal{H}_n^G(\text{pr}): \mathcal{H}_n^H(E_{\mathcal{F} \cap H}(H)) \rightarrow \mathcal{H}_n^H(\text{pt}),$$

which is bijective for all $n \in \mathbb{Z}, n \leq N$ by assumption. This finishes the proof in the case that Z is finite-dimensional. The general case follows by a colimit argument using Lemma 63.

Constructing Equivariant Homology Theories

Recall that a (non-equivariant) spectrum yields an associated (non-equivariant) homology theory. In this section we explain how a spectrum over the orbit category of a group G defines a G -homology theory. We would like to stress that our approach using spectra over the orbit category should be distinguished from

approaches to equivariant homology theories using spectra with G -action or the more complicated notion of G -equivariant spectra in the sense of [190], see for example [53] for a survey. The latter approach leads to a much richer structure but only works for compact Lie groups.

We briefly fix some conventions concerning spectra. We always work in the very convenient category SPACES of compactly generated spaces (see [295], [330, I.4]). In that category the adjunction homeomorphism $\text{map}(X \times Y, Z) \xrightarrow{\cong} \text{map}(X, \text{map}(Y, Z))$ holds without any further assumptions such as local compactness and the product of two CW-complexes is again a CW-complex. Let SPACES⁺ be the category of pointed compactly generated spaces. Here the objects are (compactly generated) spaces X with base points for which the inclusion of the base point is a cofibration. Morphisms are pointed maps. If X is a space, denote by X_+ the pointed space obtained from X by adding a disjoint base point. For two pointed spaces $X = (X, x)$ and $Y = (Y, y)$ define their smash product as the pointed space

$$X \wedge Y = X \times Y / (\{x\} \times Y \cup X \times \{y\}) ,$$

and the reduced cone as

$$\text{cone}(X) = X \times [0, 1] / (X \times \{1\} \cup \{x\} \times [0, 1]) .$$

A spectrum $\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$ is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called structure maps $\sigma(n): E(n) \wedge S^1 \rightarrow E(n+1)$. A map of spectra $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{E}'$ is a sequence of maps $f(n): E(n) \rightarrow E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e. we have $f(n+1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \text{id}_{S^1})$ for all $n \in \mathbb{Z}$. Maps of spectra are sometimes called functions in the literature, they should not be confused with the notion of a map of spectra in the stable category (see [4, III.2.]). The category of spectra and maps will be denoted SPECTRA. Recall that the homotopy groups of a spectrum are defined by

$$\pi_i(\mathbf{E}) = \text{colim}_{k \rightarrow \infty} \pi_{i+k}(E(k)) ,$$

where the system $\pi_{i+k}(E(k))$ is given by the composition

$$\pi_{i+k}(E(k)) \xrightarrow{S} \pi_{i+k+1}(E(k) \wedge S^1) \xrightarrow{\sigma(k)*} \pi_{i+k+1}(E(k+1))$$

of the suspension homomorphism S and the homomorphism induced by the structure map. A weak equivalence of spectra is a map $\mathbf{f}: \mathbf{E} \rightarrow \mathbf{F}$ of spectra inducing an isomorphism on all homotopy groups.

Given a spectrum \mathbf{E} and a pointed space X , we can define their smash product $X \wedge \mathbf{E}$ by $(X \wedge \mathbf{E})(n) := X \wedge E(n)$ with the obvious structure maps. It is a classical result that a spectrum \mathbf{E} defines a homology theory by setting

$$H_n(X, A; \mathbf{E}) = \pi_n((X_+ \cup_{A_+} \text{cone}(A_+)) \wedge \mathbf{E}) .$$

We want to extend this to G -homology theories. This requires the consideration of spaces and spectra over the orbit category. Our presentation follows [82], where more details can be found.

In the sequel \mathcal{C} is a small category. Our main example is the orbit category $\text{Or}(G)$, whose objects are homogeneous G -spaces G/H and whose morphisms are G -maps.

Definition 154 A covariant (contravariant) \mathcal{C} -space X is a covariant (contravariant) functor

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$$X: \mathcal{C} \rightarrow \text{SPACES} .$$

A map between \mathcal{C} -spaces is a natural transformation of such functors. Analogously a pointed \mathcal{C} -space is a functor from \mathcal{C} to SPACES^+ and a \mathcal{C} -spectrum a functor to SPECTRA .

Example 155. Let Y be a left G -space. Define the associated contravariant $\text{Or}(G)$ -space $\text{map}_G(-, Y)$ by

$$\text{map}_G(-, Y): \text{Or}(G) \rightarrow \text{SPACES}, \quad G/H \mapsto \text{map}_G(G/H, Y) = Y^H .$$

If Y is pointed then $\text{map}_G(-, Y)$ takes values in pointed spaces.

Let X be a contravariant and Y be a covariant \mathcal{C} -space. Define their *balanced product* to be the space

$$X \times_{\mathcal{C}} Y := \coprod_{c \in \text{ob}(\mathcal{C})} X(c) \times Y(c) / \sim ,$$

where \sim is the equivalence relation generated by $(x\phi, y) \sim (x, \phi y)$ for all morphisms $\phi: c \rightarrow d$ in \mathcal{C} and points $x \in X(d)$ and $y \in Y(c)$. Here $x\phi$ stands for $X(\phi)(x)$ and ϕy for $Y(\phi)(y)$. If X and Y are pointed, then one defines analogously their *balanced smash product* to be the pointed space

$$X \wedge_{\mathcal{C}} Y = \bigvee_{c \in \text{ob}(\mathcal{C})} X(c) \wedge Y(c) / \sim .$$

In [82] the notation $X \otimes_{\mathcal{C}} Y$ was used for this space. Doing the same construction level-wise one defines the *balanced smash product* $X \wedge_{\mathcal{C}} \mathbf{E}$ of a contravariant pointed \mathcal{C} -space and a covariant \mathcal{C} -spectrum \mathbf{E} .

The proof of the next result is analogous to the non-equivariant case. Details can be found in [82, lemma 4.4], where also cohomology theories are treated.

Proposition 156: (*Constructing G -Homology Theories.*) Let \mathbf{E} be a covariant $\text{Or}(G)$ -spectrum. It defines a G -homology theory $H_*^G(-; \mathbf{E})$ by

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$$H_n^G(X, A; \mathbf{E}) = \pi_n(\text{map}_G(-, (X_+ \cup_{A_+} \text{cone}(A_+))) \wedge_{\text{Or}(G)} \mathbf{E}) .$$

In particular we have

$$H_n^G(G/H; \mathbf{E}) = \pi_n(\mathbf{E}(G/H)) .$$

Recall that we seek an equivariant homology theory and not only a G -homology theory. If the $\text{Or}(G)$ -spectrum in Proposition 156 is obtained from a GROUPOIDS-spectrum in a way we will now describe, then automatically we obtain the desired induction structure.

Let GROUPOIDS be the category of small groupoids with covariant functors as morphisms. Recall that a groupoid is a category in which all morphisms are isomorphisms. A covariant functor $f: \mathcal{G}_0 \rightarrow \mathcal{G}_1$ of groupoids is called *injective*, if for any two objects x, y in \mathcal{G}_0 the induced map $\text{mor}_{\mathcal{G}_0}(x, y) \rightarrow \text{mor}_{\mathcal{G}_1}(f(x), f(y))$ is injective. Let $\text{GROUPOIDS}^{\text{inj}}$ be the subcategory of GROUPOIDS with the same objects and injective functors as morphisms. For a G -set S we denote by $\mathcal{G}^G(S)$ its associated *transport groupoid*. Its objects are the elements of S . The set of morphisms from s_0 to s_1 consists of those elements $g \in G$ which satisfy $gs_0 = s_1$. Composition in $\mathcal{G}^G(S)$ comes from the multiplication in G . Thus we obtain for a group G a covariant functor

$$\mathcal{G}^G: \text{Or}(G) \rightarrow \text{GROUPOIDS}^{\text{inj}}, \quad G/H \mapsto \mathcal{G}^G(G/H). \quad (2.30)$$

A functor of small categories $F: \mathcal{C} \rightarrow \mathcal{D}$ is called an *equivalence* if there exists a functor $G: \mathcal{D} \rightarrow \mathcal{C}$ such that both $F \circ G$ and $G \circ F$ are naturally equivalent to the identity functor. This is equivalent to the condition that F induces a bijection on the set of isomorphisms classes of objects and for any objects $x, y \in \mathcal{C}$ the map $\text{mor}_{\mathcal{C}}(x, y) \rightarrow \text{mor}_{\mathcal{D}}(F(x), F(y))$ induced by F is bijective.

157 Proposition 157: (*Constructing Equivariant Homology Theories.*) Consider a covariant GROUPOIDS^{inj}-spectrum

$$\mathbf{E}: \text{GROUPOIDS}^{\text{inj}} \rightarrow \text{SPECTRA}.$$

Suppose that \mathbf{E} respects equivalences, i.e. it sends an equivalence of groupoids to a weak equivalence of spectra. Then \mathbf{E} defines an equivariant homology theory $H_*^2(-; \mathbf{E})$, whose underlying G -homology theory for a group G is the G -homology theory associated to the covariant $\text{Or}(G)$ -spectrum $\mathbf{E} \circ \mathcal{G}^G: \text{Or}(G) \rightarrow \text{SPECTRA}$ in the previous Proposition 156, i.e.

$$H_*^G(X, A; \mathbf{E}) = H_*^G(X, A; \mathbf{E} \circ \mathcal{G}^G).$$

In particular we have

$$H_n^G(G/H; \mathbf{E}) \cong H_n^H(\text{pt}; \mathbf{E}) \cong \pi_n(\mathbf{E}(I(H))),$$

where $I(H)$ denotes H considered as a groupoid with one object. The whole construction is natural in \mathbf{E} .

Proof We have to specify the induction structure for a homomorphism $\alpha: H \rightarrow G$. We only sketch the construction in the special case where α is injective and $A = \emptyset$. The details of the full proof can be found in [276, theorem 2.10 on page 21].

The functor induced by α on the orbit categories is denoted in the same way

$$\alpha: \text{Or}(H) \rightarrow \text{Or}(G), \quad H/L \mapsto \text{ind}_\alpha(H/L) = G/\alpha(L).$$

There is an obvious natural equivalence of functors $\text{Or}(H) \rightarrow \text{GROUPOIDS}^{\text{inj}}$

$$T: \mathcal{G}^H \rightarrow \mathcal{G}^G \circ \alpha.$$

Its evaluation at H/L is the equivalence of groupoids $\mathcal{G}^H(H/L) \rightarrow \mathcal{G}^G(G/\alpha(L))$ which sends an object hL to the object $\alpha(h)\alpha(L)$ and a morphism given by $h \in H$ to the morphism $\alpha(h) \in G$. The desired isomorphism

$$\text{ind}_\alpha: H_n^H(X; \mathbf{E} \circ \mathcal{G}^H) \rightarrow H_n^G(\text{ind}_\alpha X; \mathbf{E} \circ \mathcal{G}^G)$$

is induced by the following map of spectra

$$\begin{aligned} \text{map}_H(-, X_+) \wedge_{\text{Or}(H)} \mathbf{E} \circ \mathcal{G}^H &\xrightarrow{\text{id} \wedge \mathbf{E}(T)} \text{map}_H(-, X_+) \wedge_{\text{Or}(H)} \mathbf{E} \circ \mathcal{G}^G \circ \alpha \\ &\xleftarrow{\sim} (\alpha_* \text{map}_H(-, X_+)) \wedge_{\text{Or}(G)} \mathbf{E} \circ \mathcal{G}^G \xleftarrow{\sim} \text{map}_G(-, \text{ind}_\alpha X_+) \wedge_{\text{Or}(G)} \mathbf{E} \circ \mathcal{G}^G. \end{aligned}$$

Here $\alpha_* \text{map}_H(-, X_+)$ is the pointed $\text{Or}(G)$ -space which is obtained from the pointed $\text{Or}(H)$ -space $\text{map}_H(-, X_+)$ by induction, i.e. by taking the balanced product over $\text{Or}(H)$ with the $\text{Or}(H)$ - $\text{Or}(G)$ bimodule $\text{mor}_{\text{Or}(G)}(??, \alpha(?))$ [82, definition 1.8]. Notice that $\mathbf{E} \circ \mathcal{G}^G \circ \alpha$ is the same as the restriction of the $\text{Or}(G)$ -spectrum $\mathbf{E} \circ \mathcal{G}^G$ along α which is often denoted by $\alpha^*(\mathbf{E} \circ \mathcal{G}^G)$ in the literature [82, definition 1.8]. The second map is given by the adjunction homeomorphism of induction α_* and restriction α^* (see [82, lemma 1.9]). The third map is the homeomorphism of $\text{Or}(G)$ -spaces which is the adjoint of the obvious map of $\text{Or}(H)$ -spaces $\text{map}_H(-, X_+) \rightarrow \alpha^* \text{map}_G(-, \text{ind}_\alpha X_+)$ whose evaluation at H/L is given by ind_α .

K - and L -theory Spectra over Groupoids

2.7.3

Let RINGS be the category of associative rings with unit. An *involution* on a R is a map $R \rightarrow R$, $r \mapsto \bar{r}$ satisfying $\overline{\bar{1}} = 1$, $\overline{\bar{x} + y} = \bar{x} + \bar{y}$ and $\overline{\bar{x} \cdot y} = \bar{y} \cdot \bar{x}$ for all $x, y \in R$. Let RINGS^{inv} be the category of rings with involution. Let C^* -ALGEBRAS be the category of C^* -algebras. There are classical functors for $j \in -\infty \amalg \{j \in \mathbb{Z} \mid j \leq 2\}$

$$\mathbf{K}: \text{RINGS} \rightarrow \text{SPECTRA}; \tag{2.31}$$

$$\mathbf{L}^{(j)}: \text{RINGS}^{\text{inv}} \rightarrow \text{SPECTRA}; \tag{2.32}$$

$$\mathbf{K}^{\text{top}}: C^*\text{-ALGEBRAS} \rightarrow \text{SPECTRA}. \tag{2.33}$$

The construction of such a non-connective algebraic K -theory functor goes back to Gersten [133] and Wagoner [312]. The spectrum for quadratic algebraic L -theory is constructed by Ranicki in [258]. In a more geometric formulation it goes back to Quinn [250]. In the topological K -theory case a construction using Bott periodicity for C^* -algebras can easily be derived from the Kuiper–Mingo Theorem (see [281,

sect. 2.2]). The homotopy groups of these spectra give the algebraic K -groups of Quillen (in high dimensions) and of Bass (in negative dimensions), the decorated quadratic L -theory groups, and the topological K -groups of C^* -algebras.

We emphasize again that in all three cases we need the non-connective versions of the spectra, i.e. the homotopy groups in negative dimensions are non-trivial in general. For example the version of the Farrell–Jones Conjecture where one uses connective K -theory spectra is definitely false in general, compare Remark 15.

Now let us fix a coefficient ring R (with involution). Then sending a group G to the group ring RG yields functors $R(-): \text{GROUPS} \rightarrow \text{RINGS}$, respectively $R(-): \text{GROUPS} \rightarrow \text{RINGS}^{\text{inv}}$, where GROUPS denotes the category of groups. Let $\text{GROUPS}^{\text{inj}}$ be the category of groups with injective group homomorphisms as morphisms. Taking the reduced group C^* -algebra defines a functor $C_r^*: \text{GROUPS}^{\text{inj}} \rightarrow C^*\text{-ALGEBRAS}$. The composition of these functors with the functors (2.31), (2.32) and (2.33) above yields functors

$$\mathbf{KR}(-): \text{GROUPS} \rightarrow \text{SPECTRA} ; \tag{2.34}$$

$$\mathbf{L}^{(j)}R(-): \text{GROUPS} \rightarrow \text{SPECTRA} ; \tag{2.35}$$

$$\mathbf{K}^{\text{top}}C_r^*(-): \text{GROUPS}^{\text{inj}} \rightarrow \text{SPECTRA} . \tag{2.36}$$

They satisfy

$$\pi_n(\mathbf{KR}(G)) = K_n(RG) ;$$

$$\pi_n(\mathbf{L}^{(j)}R(G)) = L_n^{(j)}(RG) ;$$

$$\pi_n(\mathbf{K}^{\text{top}}C_r^*(G)) = K_n(C_r^*(G)) ,$$

for all groups G and $n \in \mathbb{Z}$. The next result essentially says that these functors can be extended to groupoids.

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Theorem 158: (*K - and L -theory Spectra over Groupoids.*) Let R be a ring (with involution). There exist covariant functors

$$\mathbf{K}_R: \text{GROUPOIDS} \rightarrow \text{SPECTRA} ; \tag{2.37}$$

$$\mathbf{L}_R^{(j)}: \text{GROUPOIDS} \rightarrow \text{SPECTRA} ; \tag{2.38}$$

$$\mathbf{K}^{\text{top}}: \text{GROUPOIDS}^{\text{inj}} \rightarrow \text{SPECTRA} \tag{2.39}$$

with the following properties:

- (i) If $F: \mathcal{G}_0 \rightarrow \mathcal{G}_1$ is an equivalence of (small) groupoids, then the induced maps $\mathbf{K}_R(F)$, $\mathbf{L}_R^{(j)}(F)$ and $\mathbf{K}^{\text{top}}(F)$ are weak equivalences of spectra.
- (ii) Let $I: \text{GROUPS} \rightarrow \text{GROUPOIDS}$ be the functor sending G to G considered as a groupoid, i.e. to $\mathcal{G}^G(G/G)$. This functor restricts to a functor $\text{GROUPS}^{\text{inj}} \rightarrow \text{GROUPOIDS}^{\text{inj}}$.

There are natural transformations from $\mathbf{KR}(-)$ to $\mathbf{K}_R \circ I$, from $\mathbf{L}^{(j)}R(-)$ to $\mathbf{L}_R^{(j)} \circ I$ and from $\mathbf{KC}_r^*(-)$ to $\mathbf{K}^{\text{top}} \circ I$ such that the evaluation of each of these natural transformations at a given group is an equivalence of spectra.

(iii) For every group G and all $n \in \mathbb{Z}$ we have

$$\begin{aligned} \pi_n(\mathbf{K}_R \circ I(G)) &\cong K_n(RG) ; \\ \pi_n(\mathbf{L}_R^{(j)} \circ I^{\text{inv}}(G)) &\cong L_n^{(j)}(RG) ; \\ \pi_n(\mathbf{K}^{\text{top}} \circ I(G)) &\cong K_n(C_r^*(G)) . \end{aligned}$$

Proof We only sketch the strategy of the proof. More details can be found in [82, sect. 2].

Let \mathcal{G} be a groupoid. Similar to the group ring RG one can define an R -linear category $R\mathcal{G}$ by taking the free R -modules over the morphism sets of \mathcal{G} . Composition of morphisms is extended R -linearly. By formally adding finite direct sums one obtains an additive category $R\mathcal{G}_{\oplus}$. Pedersen–Weibel [237] (compare also [51]) define a non-connective algebraic K -theory functor which digests additive categories and can hence be applied to $R\mathcal{G}_{\oplus}$. For the comparison result one uses that for every ring R (in particular for RG) the Pedersen–Weibel functor applied to R_{\oplus} (a small model for the category of finitely generated free R -modules) yields the non-connective K -theory of the ring R and that it sends equivalences of additive categories to equivalences of spectra. In the L -theory case $R\mathcal{G}_{\oplus}$ inherits an involution and one applies the construction of [258, example 13.6 on page 139] to obtain the $L^{(1)} = L^h$ -version. The versions for $j \leq 1$ can be obtained by a construction which is analogous to the Pedersen–Weibel construction for K -theory, compare [55, sect. 4]. In the C^* -case one obtains from \mathcal{G} a C^* -category $C_r^*(\mathcal{G})$ and assigns to it its topological K -theory spectrum. There is a construction of the topological K -theory spectrum of a C^* -category in [82, sect. 2]. However, the construction given there depends on two statements, which appeared in [130, proposition 1 and proposition 3], and those statements are incorrect, as already pointed out by Thomason in [302]. The construction in [82, sect. 2] can easily be fixed but instead we recommend the reader to look at the more recent construction of Joachim [159].

Assembly Maps in Terms of Homotopy Colimits

2.7.4

In this section we describe a homotopy-theoretic formulation of the Baum–Connes and Farrell–Jones Conjectures. For the classical assembly maps which in our set-up correspond to the trivial family such formulations were described in [328].

For a group G and a family \mathcal{F} of subgroups we denote by $\text{Or}_{\mathcal{F}}(G)$ the *restricted orbit category*. Its objects are homogeneous spaces G/H with $H \in \mathcal{F}$. Morphisms are G -maps. If $\mathcal{F} = \mathcal{ALL}$ we get back the (full) orbit category, i.e. $\text{Or}(G) = \text{Or}_{\mathcal{ALL}}(G)$.

159 **Meta-Conjecture 159:** (*Homotopy-Theoretic Isomorphism Conjecture.*) Let G be a group and \mathcal{F} a family of subgroups. Let $\mathbf{E}: \text{Or}(G) \rightarrow \text{SPECTRA}$ be a covariant functor. Then

$$A_{\mathcal{F}}: \text{hocolim}_{\text{Or}_{\mathcal{F}}(G)} \mathbf{E}|_{\text{Or}_{\mathcal{F}}(G)} \rightarrow \text{hocolim}_{\text{Or}(G)} \mathbf{E} \simeq \mathbf{E}(G/G)$$

is a weak equivalence of spectra.

Here hocolim is the homotopy colimit of a covariant functor to spectra, which is itself a spectrum. The map $A_{\mathcal{F}}$ is induced by the obvious functor $\text{Or}_{\mathcal{F}}(G) \rightarrow \text{Or}(G)$. The equivalence $\text{hocolim}_{\text{Or}(G)} \mathbf{E} \simeq \mathbf{E}(G/G)$ comes from the fact that G/G is a final object in $\text{Or}(G)$. For information about homotopy-colimits we refer to [40], [82, sect. 3] and [88].

160 **Remark 160** If we consider the map on homotopy groups that is induced by the map $A_{\mathcal{F}}$ which appears in the Homotopy-Theoretic Isomorphism Conjecture above, then we obtain precisely the map with the same name in Meta-Conjecture 57 for the homology theory $H_*^G(-; \mathbf{E})$ associated with \mathbf{E} in Proposition 156, compare [82, sect. 5]. In particular the Baum–Connes Conjecture 59 and the Farrell–Jones Conjecture 58 can be seen as special cases of Meta-Conjecture 159.

161 **Remark 161:** (*Universal Property of the Homotopy-Theoretic Assembly Map.*) The Homotopy-Theoretic Isomorphism Conjecture 159 is in some sense the most conceptual formulation of an Isomorphism Conjecture because it has a universal property as the universal approximation from the left by a (weakly) excisive \mathcal{F} -homotopy invariant functor. This is explained in detail in [82, sect. 6]. This universal property is important if one wants to identify different models for the assembly map, compare e.g. [19, sect. 6] and [142].

2.7.5 Naturality under Induction

Consider a covariant functor $\mathbf{E}: \text{GROUPOIDS} \rightarrow \text{SPECTRA}$ which respects equivalences. Let $H_*^G(-; \mathbf{E})$ be the associated equivariant homology theory (see Proposition 157). Then for a group homomorphism $\alpha: H \rightarrow G$ and H -CW-pair (X, A) we obtain a homomorphism

$$\text{ind}_{\alpha}: H_n^H(X, A; \mathbf{E}) \rightarrow H_n^G(\text{ind}_{\alpha}(X, A); \mathbf{E})$$

which is natural in (X, A) . Note that we did not assume that $\ker(\alpha)$ acts freely on X . In fact the construction sketched in the proof of Proposition 157 still works even though ind_{α} may not be an isomorphism as it is the case if $\ker(\alpha)$ acts freely. We still have functoriality as described in 2.7.1 towards the beginning of Sect. 2.7.1.

Now suppose that \mathcal{H} and \mathcal{G} are families of subgroups for H and G such that $\alpha(K) \in \mathcal{G}$ holds for all $K \in \mathcal{H}$. Then we obtain a G -map $f: \text{ind}_\alpha E_{\mathcal{H}}(H) \rightarrow E_{\mathcal{G}}(G)$ from the universal property of $E_{\mathcal{G}}(G)$. Let $p: \text{ind}_\alpha \text{pt} = G/\alpha(H) \rightarrow \text{pt}$ be the projection. Let $I: \text{GROUPS} \rightarrow \text{GROUPOIDS}$ be the functor sending G to $\mathcal{G}^G(G/G)$. Then the following diagram, where the horizontal arrows are induced from the projections to the one point space, commutes for all $n \in \mathbb{Z}$.

$$\begin{array}{ccc}
 H_n^H(E_{\mathcal{H}}(H); \mathbf{E}) & \xrightarrow{A_{\mathcal{H}}} & H_n^H(\text{pt}; \mathbf{E}) = \pi_n(\mathbf{E}(I(H))) \\
 H_n^G(f) \circ \text{ind}_\alpha \downarrow & & \downarrow H_n^G(p) \circ \text{ind}_\alpha = \pi_n(\mathbf{E}(I(\alpha))) \\
 H_n^G(E_{\mathcal{G}}(G); \mathbf{E}) & \xrightarrow{A_{\mathcal{G}}} & H_n^G(\text{pt}; \mathbf{E}) = \pi_n(\mathbf{E}(I(G))) .
 \end{array}$$

If we take the special case $\mathbf{E} = \mathbf{K}_R$ and $\mathcal{H} = \mathcal{G} = \mathcal{VCY}$, we get the following commutative diagram, where the horizontal maps are the assembly maps appearing in the Farrell–Jones Conjecture 58 and α_* is the change of rings homomorphism (induction) associated to α .

$$\begin{array}{ccc}
 H_n^H(E_{\mathcal{VCY}}(H); \mathbf{K}_R) & \xrightarrow{A_{\mathcal{VCY}}} & K_n(RH) \\
 H_n^G(f) \circ \text{ind}_\alpha \downarrow & & \downarrow \alpha_* \\
 H_n^G(E_{\mathcal{VCY}}(G); \mathbf{K}_R) & \xrightarrow{A_{\mathcal{VCY}}} & K_n(RG) .
 \end{array}$$

We see that we can define a kind of induction homomorphism on the source of the assembly maps which is compatible with the induction structure given on their target. We get analogous diagrams for the L -theoretic version of the Farrell–Jones–Isomorphism Conjecture 58, for the Bost Conjecture 104 and for the Baum–Connes Conjecture for maximal group C^* -algebras (see (2.22) in Sect. 2.5.1).

Remark 162 The situation for the Baum–Connes Conjecture 59 itself, where one has to work with reduced C^* -algebras, is more complicated. Recall that not every group homomorphism $\alpha: H \rightarrow G$ induces a homomorphisms of C^* -algebras $C_r^*(H) \rightarrow C_r^*(G)$. (It does if $\ker(\alpha)$ is finite.) But it turns out that the source $H_n^H(E_{\mathcal{FIN}}(H); \mathbf{K}^{\text{top}})$ always admits such a homomorphism. The point is that the isotropy groups of $E_{\mathcal{FIN}}(H)$ are all finite and the spectra-valued functor \mathbf{K}^{top} extends from $\text{GROUPOIDS}^{\text{inj}}$ to the category $\text{GROUPOIDS}^{\text{fin ker}}$, which has small groupoids as objects but as morphisms only those functors $f: \mathcal{G}_0 \rightarrow \mathcal{G}_1$ with finite kernels (in the sense that for each object $x \in \mathcal{G}_0$ the group homomorphism $\text{aut}_{\mathcal{G}_0}(x) \rightarrow \text{aut}_{\mathcal{G}_1}(f(x))$ has finite kernel). This is enough to get for any group homomorphism $\alpha: H \rightarrow G$ an induced map $\text{ind}_\alpha: H_n^H(X, A; \mathbf{K}^{\text{top}}) \rightarrow$

$H_n^G(\text{ind}_\alpha(X, A); \mathbf{K}^{\text{top}})$ provided that X is proper. Hence one can define an induction homomorphism for the source of the assembly map as above.

In particular the Baum–Connes Conjecture 59 predicts that for any group homomorphism $\alpha: H \rightarrow G$ there is an induced induction homomorphism $\alpha_*: K_n(C_r^*(H)) \rightarrow K_n(C_r^*(G))$ on the targets of the assembly maps although there is no induced homomorphism of C^* -algebras $C_r^*(H) \rightarrow C_r^*(G)$ in general.

2.8 Methods of Proof

In Chapt. 2.3, we formulated the Baum–Connes Conjecture 58 and the Farrell–Jones Conjecture 59 in abstract homological terms. We have seen that this formulation was very useful in order to understand formal properties of assembly maps. But in order to actually prove cases of the conjectures one needs to interpret the assembly maps in a way that is more directly related to geometry or analysis. In this chapter we wish to explain such approaches to the assembly maps. We briefly survey some of the methods of proof that are used to attack the Baum–Connes Conjecture 59 and the Farrell–Jones Conjecture 58.

2.8.1 Analytic Equivariant K -Homology

Recall that the covariant functor $\mathbf{K}^{\text{top}}: \text{GROUPOIDS}^{\text{inj}} \rightarrow \text{SPECTRA}$ introduced in (2.39) defines an equivariant homology theory $H_*^?(-; \mathbf{K}^{\text{top}})$ in the sense of Sect. 2.7.1 such that

$$H_n^G(G/H; \mathbf{K}^{\text{top}}) = H_n^H(\text{pt}; \mathbf{K}^{\text{top}}) = \begin{cases} R(H) & \text{for even } n; \\ 0 & \text{for odd } n, \end{cases}$$

holds for all groups G and subgroups $H \subseteq G$ (see Proposition 157). Next we want to give for a proper G -CW-complex X an analytic definition of $H_n^G(X; \mathbf{K}^{\text{top}})$.

Consider a locally compact proper G -space X . Recall that a G -space X is called *proper* if for each pair of points x and y in X there are open neighborhoods V_x of x and W_y of y in X such that the subset $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$ of G is finite. A G -CW-complex X is proper if and only if all its isotropy groups are finite [197, theorem 1.23]. Let $C_0(X)$ be the C^* -algebra of continuous functions $f: X \rightarrow \mathbb{C}$ which vanish at infinity. The C^* -norm is the supremum norm. A *generalized elliptic G -operator* is a triple (U, ϱ, F) , which consists of a unitary representation $U: G \rightarrow \mathcal{B}(H)$ of G on a Hilbert space H , a $*$ -representation $\varrho: C_0(X) \rightarrow \mathcal{B}(H)$ such that $\varrho(f \circ l_{g^{-1}}) = U(g) \circ \varrho(f) \circ U(g)^{-1}$ holds for $g \in G$, and a bounded selfadjoint G -operator $F: H \rightarrow H$ such that the operators $\varrho(f)(F^2 - 1)$ and $[\varrho(f), F]$ are compact for all $f \in C_0(X)$. Here $\mathcal{B}(H)$ is the C^* -algebra of bounded operators $H \rightarrow H$, $l_{g^{-1}}: H \rightarrow H$ is given by multiplication with g^{-1} , and $[\varrho(f), F] = \varrho(f) \circ F - F \circ \varrho(f)$.

We also call such a triple (U, φ, F) an *odd cycle*. If we additionally assume that H comes with a $\mathbb{Z}/2$ -grading such that φ preserves the grading if we equip $C_0(X)$ with the trivial grading, and F reverses it, then we call (U, φ, F) an *even cycle*. This means that we have an orthogonal decomposition $H = H_0 \oplus H_1$ such that U, φ and F look like

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix} \quad \varphi = \begin{pmatrix} \varphi_0 & 0 \\ 0 & \varphi_1 \end{pmatrix} \quad F = \begin{pmatrix} 0 & P^* \\ P & 0 \end{pmatrix}.$$

An important example of an even cocycle is described in Sect. 2.8.5. A cycle (U, φ, f) is called *degenerate*, if for each $f \in C_0(X)$ we have $[\varphi(f), F] = \varphi(f)(F^2 - 1) = 0$. Two cycles (U_0, φ_0, F_0) and (U_1, φ_1, F_1) of the same parity are called *homotopic*, if $U_0 = U_1, \varphi_0 = \varphi_1$ and there exists a norm continuous path $F_t, t \in [0, 1]$ in $\mathcal{B}(H)$ such that for each $t \in [0, 1]$ the triple (U_0, φ_0, F_t) is again a cycle of the same parity. Two cycles (U_0, φ_0, F_0) and (U_1, φ_1, F_1) are called *equivalent*, if they become homotopic after taking the direct sum with degenerate cycles of the same parity. Let $K_n^G(C_0(X))$ for even n be the set of equivalence classes of even cycles and $K_n^G(C_0(X))$ for odd n be the set of equivalence classes of odd cycles. These become abelian groups by the direct sum. The neutral element is represented by any degenerate cycle. The inverse of an even cycle is represented by the cycle obtained by reversing the grading of H . The inverse of an odd cycle (U, φ, F) is represented by $(U, \varphi, -F)$.

A proper G -map $f: X \rightarrow Y$ induces a map of C^* -algebras $C_0(f): C_0(Y) \rightarrow C_0(X)$ by composition and thus in the obvious way a homomorphism of abelian groups $K_0^G(f): K_0^G(C_0(X)) \rightarrow K_0^G(C_0(Y))$. It depends only on the proper G -homotopy class of f . One can show that this construction defines a G -homology theory on the category of finite proper G -CW-complexes. It extends to a G -homology theory K_*^G for all proper G -CW-complexes by

$$K_n^G(X) = \operatorname{colim}_{Y \in I(X)} K_n^G(C_0(Y)), \tag{2.40}$$

where $I(X)$ is the set of proper finite G -CW-subcomplexes $Y \subseteq X$ directed by inclusion. This definition is forced upon us by Lemma 63. The groups $K_n^G(X)$ and $K_n^G(C_0(X))$ agree for finite proper G -CW-complexes, in general they are different.

The cycles were introduced by Atiyah [11]. The equivalence relation, the group structure and the homological properties of $K_n^G(X)$ were established by Kasparov [168]. More information about analytic K -homology can be found in Higson–Roe [154].

The Analytic Assembly Map

For every G -CW-complex X the projection $\operatorname{pr}: X \rightarrow \operatorname{pt}$ induces a map

$$H_n^G(X; \mathbf{K}^{\operatorname{top}}) \rightarrow H_n^G(\operatorname{pt}; \mathbf{K}^{\operatorname{top}}) = K_n(C_r^*(G)). \tag{2.41}$$

In the case where X is the proper G -space $E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G)$ we obtain the assembly map appearing in the Baum–Connes Conjecture 59. We explain its analytic analogue

$$\text{ind}_G: K_n^G(X) \rightarrow K_n(C_r^*(G)). \tag{2.42}$$

Note that we need to assume that X is a proper G -space since $K_n^G(X)$ was only defined for such spaces. It suffices to define the map for a finite proper G -CW-complex X . In this case it assigns to the class in $K_n^G(X) = K_n^G(C_0(X))$ represented by a cycle (U, ϱ, F) its G -index in $K_n(C_r^*(G))$ in the sense of Mishencko–Fomenko [223]. At least in the simple case, where G is finite, we can give its precise definition. The odd K -groups vanish in this case and $K_0(C_r^*(G))$ reduces to the complex representation ring $R(G)$. If we write F in matrix form as in (2.8.1) then $P: H \rightarrow H$ is a G -equivariant Fredholm operator. Hence its kernel and cokernel are G -representations and the G -index of F is defined as $[\ker(P)] - [\text{coker}(P)] \in R(G)$. In the case of an infinite group the kernel and cokernel are a priori not finitely generated projective modules over $C_r^*(G)$, but they are after a certain perturbation. Moreover the choice of the perturbation does not affect $[\ker(P)] - [\text{coker}(P)] \in K_0(C_r^*(G))$.

The identification of the two assembly maps (2.41) and (2.42) has been carried out in Hambleton–Pedersen [142] using the universal characterization of the assembly map explained in [82, sect. 6]. In particular for a proper G -CW-complex X we have an identification $H_n^G(X; \mathbf{K}^{\text{top}}) \cong K_n^G(X)$. Notice that $H_n^G(X; \mathbf{K}^{\text{top}})$ is defined for all G -CW-complexes, whereas $K_n^G(X)$ has only been introduced for proper G -CW-complexes.

Thus the Baum–Connes Conjecture 59 gives an index-theoretic interpretations of elements in $K_0(C_r^*(G))$ as generalized elliptic operators or cycles (U, ϱ, F) . We have explained already in Sect. 2.2.8 an application of this interpretation to the Trace Conjecture for Torsionfree Groups 37 and in Sect. 2.4.3 to the Stable Gromov–Lawson–Rosenberg Conjecture 99.

2.8.3 Equivariant KK -Theory

Kasparov [170] developed *equivariant KK -theory*, which we will briefly explain next. It is one of the basic tools in the proofs of theorems about the Baum–Connes Conjecture 59.

A G - C^* -algebra A is a C^* -algebra with a G -action by $*$ -automorphisms. To any pair of separable G - C^* -algebras (A, B) Kasparov assigns abelian groups $KK_n^G(A, B)$. If G is trivial, we write briefly $KK_n(A, B)$. We do not give the rather complicated definition but state the main properties.

If we equip \mathbb{C} with the trivial G -action, then $KK_n^G(C_0(X), \mathbb{C})$ reduces to the abelian group $K_n^G(C_0(X))$ introduced in Sect. 2.8.1. The topological K -theory $K_n(A)$ of a C^* -algebra coincides with $KK_n(\mathbb{C}, A)$. The equivariant KK -groups are covariant in the second and contravariant in the first variable under homomorphism of C^* -algebras. One of the main features is the bilinear Kasparov product

$$KK_i^G(A, B) \times KK_j^G(B, C) \rightarrow KK_{i+j}(A, C), \quad (\alpha, \beta) \mapsto \alpha \otimes_B \beta. \tag{2.43}$$

It is associative and natural. A homomorphism $\alpha: A \rightarrow B$ defines an element in $KK_0(A, B)$. There are natural *descent homomorphisms*

$$j_G: KK_n^G(A, B) \rightarrow KK_n(A \rtimes_r G, B \rtimes_r G), \tag{2.44}$$

where $A \rtimes_r G$ and $B \rtimes_r G$ denote the reduced crossed product C^* -algebras.

The Dirac-Dual Dirac Method

2.8.4

A G - C^* -algebra A is called *proper* if there exists a locally compact proper G -space X and a G -homomorphism $\sigma: C_0(X) \rightarrow \mathcal{B}(A)$, $f \mapsto \sigma_f$ satisfying $\sigma_f(ab) = a\sigma_f(b) = \sigma_f(a)b$ for $f \in C_0(X)$, $a, b \in A$ and for every net $\{f_i \mid i \in I\}$, which converges to 1 uniformly on compact subsets of X , we have $\lim_{i \in I} \|\sigma_{f_i}(a) - a\| = 0$ for all $a \in A$. A locally compact G -space X is proper if and only if $C_0(X)$ is proper as a G - C^* -algebra.

Given a proper G - CW -complex X and a G - C^* -algebra A , we put

$$KK_n^G(X; A) = \operatorname{colim}_{Y \in I(X)} KK_n^G(C_0(Y), A), \tag{2.45}$$

where $I(Y)$ is the set of proper finite G - CW -subcomplexes $Y \subseteq X$ directed by inclusion. We have $KK_n^G(X; \mathbb{C}) = K_n^G(X)$. There is an analytic index map

$$\operatorname{ind}_G^A: KK_n^G(X; A) \rightarrow K_n(A \rtimes_r G), \tag{2.46}$$

which can be identified with the assembly map appearing in the Baum–Connes Conjecture with Coefficients 105. The following result is proved in Tu [305] extending results of Kasparov–Skandalis [169, 172].

Theorem 163 The Baum–Connes Conjecture with coefficients 105 holds for a proper G - C^* -algebra A , i.e. $\operatorname{ind}_G^A: KK_n^G(E_{\mathcal{F}, L, \mathcal{N}}(G); A) \rightarrow K_n(A \rtimes G)$ is bijective.

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Now we are ready to state the *Dirac-dual Dirac method* which is the key strategy in many of the proofs of the Baum–Connes Conjecture 59 or the Baum–Connes Conjecture with coefficients 105.

Theorem 164: (*Dirac-Dual Dirac Method.*) Let G be a countable (discrete) group. Suppose that there exist a proper G - C^* -algebra A , elements $\alpha \in KK_i^G(A, \mathbb{C})$, called the *Dirac element*, and $\beta \in KK_i^G(\mathbb{C}, A)$, called the *dual Dirac element*, satisfying

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$$\beta \otimes_A \alpha = 1 \in KK_0^G(\mathbb{C}, \mathbb{C}).$$

Then the Baum–Connes Conjecture 59 is true, or, equivalently, the analytic index map

$$\operatorname{ind}_G: K_n^G(X) \rightarrow K_n(C_r^*(G))$$

of 2.42 is bijective.

Proof The index map ind_G is a retract of the bijective index map ind_G^A from Theorem 163. This follows from the following commutative diagram

$$\begin{array}{ccccc}
 K_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G)) & \xrightarrow{-\otimes_C \beta} & KK_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G); A) & \xrightarrow{-\otimes_A \alpha} & K_n^G(E_{\mathcal{F}\mathcal{L}\mathcal{N}}(G)) \\
 \downarrow \text{ind}_G & & \downarrow \text{ind}_G^A & & \downarrow \text{ind}_G \\
 K_n(C_r^*(G)) & \xrightarrow{-\otimes_{C_r^*(G)} j_G(\beta)} & K_n(A \rtimes_r G) & \xrightarrow{-\otimes_{A \rtimes_r G} j_G(\alpha)} & K_n(C_r^*(G))
 \end{array}$$

and the fact that the composition of both the top upper horizontal arrows and lower upper horizontal arrows are bijective.

2.8.5 An Example of a Dirac Element

In order to give a glimpse of the basic ideas from operator theory we briefly describe how to define the Dirac element α in the case where G acts by isometries on a complete Riemannian manifold M . Let $T_{\mathbb{C}}M$ be the complexified tangent bundle and let $\text{Cliff}(T_{\mathbb{C}}M)$ be the associated Clifford bundle. Let A be the proper G - C^* -algebra given by the sections of $\text{Cliff}(T_{\mathbb{C}}M)$ which vanish at infinity. Let H be the Hilbert space $L^2(\wedge T_{\mathbb{C}}^*M)$ of L^2 -integrable differential forms on $T_{\mathbb{C}}M$ with the obvious $\mathbb{Z}/2$ -grading coming from even and odd forms. Let U be the obvious G -representation on H coming from the G -action on M . For a 1-form ω on M and $u \in H$ define a $*$ -homomorphism $\varrho: A \rightarrow \mathcal{B}(H)$ by

$$\varrho_{\omega}(u) := \omega \wedge u + i_{\omega}(u).$$

Now $D = (d + d^*)$ is a symmetric densely defined operator $H \rightarrow H$ and defines a bounded selfadjoint operator $F: H \rightarrow H$ by putting $F = \frac{D}{\sqrt{1+D^2}}$. Then (U, ϱ, F) is an even cocycle and defines an element $\alpha \in K_0^G(M) = KK_0^G(C_0(M), \mathbb{C})$. More details of this construction and the construction of the dual Dirac element β under the assumption that M has non-positive curvature and is simply connected, can be found for instance in [307, chapt. 9].

2.8.6 Banach KK -Theory

Skandalis showed that the Dirac-dual Dirac method cannot work for all groups [287] as long as one works with KK -theory in the unitary setting. The problem is that for a group with property (T) the trivial and the regular unitary representation cannot be connected by a continuous path in the space of unitary representations, compare also the discussion in [163]. This problem can be circumvented if one drops the condition unitary and works with a variant of KK -theory for Banach algebras as worked out by Lafforgue [183, 185, 186].

Controlled Topology and Algebra

To a topological problem one can often associate a notion of “size”. We describe a prototypical example. Let M be a Riemannian manifold. Recall that an h -cobordism W over $M = \partial^- W$ admits retractions $r^\pm: W \times I \rightarrow W$, $(x, t) \mapsto r_t^\pm(x, t)$ which retract W to $\partial^\pm W$, i.e. which satisfy $r_0^\pm = \text{id}_W$ and $r_1^\pm(W) \subset \partial^\pm W$. Given $\varepsilon > 0$ we say that W is ε -controlled if the retractions can be chosen in such a way that for every $x \in W$ the paths (called tracks of the h -cobordism) $p_x^\pm: I \rightarrow M$, $t \mapsto r_t^\pm \circ r_1^\pm(x)$ both lie within an ε -neighbourhood of their starting point. The usefulness of this concept is illustrated by the following theorem [124].

Theorem 165 Let M be a compact Riemannian manifold of dimension ≥ 5 . Then there exists an $\varepsilon = \varepsilon_M > 0$, such that every ε -controlled h -cobordism over M is trivial.

If one studies the s -Cobordism Theorem 5 and its proof one is naturally lead to algebraic analogues of the notions above. A (geometric) R -module over the space X is by definition a family $M = (M_x)_{x \in X}$ of free R -modules indexed by points of X with the property that for every compact subset $K \subset X$ the module $\bigoplus_{x \in K} M_x$ is a finitely generated R -module. A morphism ϕ from M to N is an R -linear map $\phi = (\phi_{y,x}): \bigoplus_{x \in X} M_x \rightarrow \bigoplus_{y \in X} N_y$. Instead of specifying fundamental group data by paths (analogues of the tracks of the h -cobordism) one can work with modules and morphisms over the universal covering \tilde{X} , which are invariant under the operation of the fundamental group $G = \pi_1(X)$ via deck transformations, i.e. we require that $M_{gx} = M_x$ and $\phi_{gy,gx} = \phi_{y,x}$. Such modules and morphisms form an additive category which we denote by $\mathcal{C}^G(\tilde{X}; R)$. In particular one can apply to it the non-connective K -theory functor \mathbf{K} (compare [237]). In the case where X is compact the category is equivalent to the category of finitely generated free RG -modules and hence $\pi_* \mathbf{K} \mathcal{C}^G(\tilde{X}; R) \cong K_*(RG)$. Now suppose \tilde{X} is equipped with a G -invariant metric, then we will say that a morphism $\phi = (\phi_{y,x})$ is ε -controlled if $\phi_{y,x} = 0$, whenever x and y are further than ε apart. (Note that ε -controlled morphisms do not form a category because the composition of two such morphisms will in general be 2ε -controlled.)

Theorem 165 has the following algebraic analogue [251] (see also sect. 4 in [240]).

Theorem 166 Let M be a compact Riemannian manifold with fundamental group G . There exists an $\varepsilon = \varepsilon_M > 0$ with the following property. The K_1 -class of every G -invariant automorphism of modules over \tilde{M} which together with its inverse is ε -controlled lies in the image of the classical assembly map

$$H_1(BG; \mathbf{K}R) \rightarrow K_1(RG) \cong K_1(\mathcal{C}^G(\tilde{M}; R)) .$$

To understand the relation to Theorem 165 note that for $R = \mathbb{Z}$ such an ε -controlled automorphism represents the trivial element in the Whitehead group which is in bijection with the h -cobordisms over M , compare Theorem 5.

There are many variants to the simple concept of “metric ε -control” we used above. In particular it is very useful to not measure size directly in M but instead use a map $p: M \rightarrow X$ to measure size in some auxiliary space X . (For example we have seen in Sects. 2.2.2 and 2.2.4 that “bounded” control over \mathbb{R}^k may be used in order to define or describe negative K -groups.)

Before we proceed we would like to mention that there are analogous control-notions for pseudoisotopies and homotopy equivalences. The tracks of a pseudoisotopy $f: M \times I \rightarrow M \times I$ are defined as the paths in M which are given by the composition

$$p_x: I = \{x\} \times I \subset M \times I \xrightarrow{f} M \times I \xrightarrow{p} M$$

for each $x \in M$, where the last map is the projection onto the M -factor. Suppose $f: N \rightarrow M$ is a homotopy equivalence, $g: M \rightarrow N$ its inverse and h_t and h'_t are homotopies from $f \circ g$ to id_M respectively from $g \circ f$ to id_N then the tracks are defined to be the paths in M that are given by $t \mapsto h_t(x)$ for $x \in M$ and $t \mapsto f \circ h'_t(y)$ for $y \in N$. In both cases, for pseudoisotopies and for homotopy equivalences, the tracks can be used to define ε -control.

2.8.8 Assembly as Forget Control

If instead of a single problem over M one defines a family of problems over $M \times [1, \infty)$ and requires the control to tend to zero for $t \rightarrow \infty$ in a suitable sense, then one obtains something which is a homology theory in M . Relaxing the control to just bounded families yields the classical assembly map. This idea appears in [252] in the context of pseudoisotopies and in a more categorical fashion suitable for higher algebraic K -theory in [55] and [241]. We spell out some details in the case of algebraic K -theory, i.e. for geometric modules.

Let M be a Riemannian manifold with fundamental group G and let $\mathcal{S}(1/t)$ be the space of all functions $[1, \infty) \rightarrow [0, \infty)$, $t \mapsto \delta_t$ such that $t \mapsto t \cdot \delta_t$ is bounded. Similarly let $\mathcal{S}(1)$ be the space of all functions $t \mapsto \delta_t$ which are bounded. Note that $\mathcal{S}(1/t) \subset \mathcal{S}(1)$. A G -invariant morphism ϕ over $\tilde{M} \times [1, \infty)$ is \mathcal{S} -controlled for $\mathcal{S} = \mathcal{S}(1)$ or $\mathcal{S}(1/t)$ if there exists an $\alpha > 0$ and a $\delta_t \in \mathcal{S}$ (both depending on the morphism) such that $\phi_{(x,t),(x',t')} \neq 0$ implies that $|t-t'| \leq \alpha$ and $d_{\tilde{M}}(x, x') \leq \delta_{\min\{t,t'\}}$. We denote by $\mathcal{C}^G(\tilde{M} \times [1, \infty), \mathcal{S}; R)$ the category of all \mathcal{S} -controlled morphisms. Furthermore $\mathcal{C}^G(\tilde{M} \times [1, \infty), \mathcal{S}; R)^\infty$ denotes the quotient category which has the same objects, but where two morphisms are identified, if their difference factorizes over an object which lives over $\tilde{M} \times [1, N]$ for some large but finite number N . This passage to the quotient category is called “taking germs at infinity”. It is a special case of a Karoubi quotient, compare [51].

Theorem 167: (*Classical Assembly as Forget Control.*) Suppose M is aspherical, i.e. M is a model for BG , then for all $n \in \mathbb{Z}$ the map

$$\pi_n(\mathbf{KC}^G(\tilde{M} \times [1, \infty), \mathcal{A}(1/t); R)^\infty) \rightarrow \pi_n(\mathbf{KC}^G(\tilde{M} \times [1, \infty), \mathcal{A}(1); R)^\infty)$$

can be identified up to an index shift with the classical assembly map that appears in Conjecture 11, i.e. with

$$H_{n-1}(BG; \mathbf{K}(R)) \rightarrow K_{n-1}(RG).$$

Note that the only difference between the left and the right hand side is that on the left morphism are required to become smaller in a $1/t$ -fashion, whereas on the right hand side they are only required to stay bounded in the $[1, \infty)$ -direction.

Using so called equivariant continuous control (see [7] and [19, sect. 2] for the equivariant version) one can define an equivariant homology theory which applies to arbitrary G -CW-complexes. This leads to a “forget-control description” for the generalized assembly maps that appear in the Farrell–Jones Conjecture 58. Alternatively one can use stratified spaces and stratified Riemannian manifolds in order to describe generalized assembly maps in terms of metric control. Compare [111, 3.6 on p. 270] and [252, appendix].

Methods to Improve Control

From the above description of assembly maps we learn that the problem of proving surjectivity results translates into the problem of improving control. A combination of many different techniques is used in order to achieve such control-improvements. We discuss some prototypical arguments which go back to [98] and [102] and again restrict attention to K -theory. Of course this can only give a very incomplete impression of the whole program which is mainly due to Farrell–Hsiang and Farrell–Jones. The reader should consult [120] and [162] for a more detailed survey.

We restrict to the basic case, where M is a compact Riemannian manifold with negative sectional curvature. In order to explain a contracting property of the geodesic flow $\Phi: \mathbb{R} \times \tilde{S}M \rightarrow \tilde{S}M$ on the unit sphere bundle $\tilde{S}M$, we introduce the notion of foliated control. We think of $\tilde{S}M$ as a manifold equipped with the one-dimensional foliation by the flow lines of Φ and equip it with its natural Riemannian metric. Two vectors v and w in $\tilde{S}M$ are called foliated (α, δ) -controlled if there exists a path of length α inside one flow line such that v lies within distance $\delta/2$ of the starting point of that path and w lies within distance $\delta/2$ of its endpoint.

Two vectors v and $w \in \tilde{S}M$ are called asymptotic if the distance between their associated geodesic rays is bounded. These will then determine the same point on the sphere at infinity which can be introduced to compactify \tilde{M} to a disk. Recall that the universal covering of a negatively curved manifold is diffeomorphic to \mathbb{R}^n .

Suppose v and w are α -controlled asymptotic vectors, i.e. their distance is smaller than $\alpha > 0$. As a consequence of negative sectional curvature the vectors $\Phi_t(v)$ and $\Phi_t(w)$ are foliated $(C\alpha, \delta_t)$ -controlled, where $C > 1$ is a constant and $\delta_t > 0$ tends to zero when t tends to ∞ . So roughly speaking the flow contracts the directions transverse to the flow lines and leaves the flow direction as it is, at least if we only apply it to asymptotic vectors.

This property can be used in order to find foliated (α, δ) -controlled representatives of K -theory classes with arbitrary small δ if one is able to define a suitable transfer from M to \widetilde{SM} , which yields representatives whose support is in an asymptotic starting position for the flow. Here one needs to take care of the additional problem that in general such a transfer may not induce an isomorphism in K -theory.

Finally one is left with the problem of improving foliated control to ordinary control. Corresponding statements are called “Foliated Control Theorems”. Compare [18, 101, 103, 104] and [108].

If such an improvement were possible without further hypothesis, we could prove that the classical assembly map, i.e. the assembly map with respect to the trivial family is surjective. We know however that this is not true in general. It fails for example in the case of topological pseudoisotopies or for algebraic K -theory with arbitrary coefficients. In fact the geometric arguments that are involved in a “Foliated Control Theorem” need to exclude flow lines in \widetilde{SM} which correspond to “short” closed geodesic loops in SM . But the techniques mentioned above can be used in order to achieve ε -control for arbitrary small $\varepsilon > 0$ outside of a suitably chosen neighbourhood of “short” closed geodesics. This is the right kind of control for the source of the assembly map which involves the family of cyclic subgroups. (Note that a closed loop in M determines the conjugacy class of a maximal infinite cyclic subgroup inside $G = \pi_1(M)$.) We see that even in the torsionfree case the class of cyclic subgroups of G naturally appears during the proof of a surjectivity result.

Another source for processes which improve control are expanding self-maps. Think for example of an n -torus $\mathbb{R}^n/\mathbb{Z}^n$ and the self-map f_s which is induced by $m_s: \mathbb{R}^n \rightarrow \mathbb{R}^n, x \rightarrow sx$ for a large positive integer s . If one pulls an automorphism back along such a map one can improve control, but unfortunately the new automorphism describes a different K -theory class. Additional algebraic arguments nevertheless make this technique very successful. Compare for example [98]. Sometimes a clever mixture between flows and expanding self-maps is needed in order to achieve the goal, compare [105]. Recent work of Farrell–Jones (see [114–116] and [161]) makes use of a variant of the Cheeger–Fukaya–Gromov collapsing theory.

Remark 168: (*Algebraicizing the Farrell–Jones Approach.*) In this Subsection we sketched some of the geometric ideas which are used in order to obtain control over an h -cobordism, a pseudoisotopy or an automorphism of a geometric module representing a single class in K_1 . In Sect. 2.8.8 we used families over the cone

$M \times [1, \infty)$ in order to describe the whole algebraic K -theory assembly map at once in categorical terms without ever referring to a single K -theory element. The recent work [21] shows that the geometric ideas can be adapted to this more categorical set-up, at least in the case where the group is the fundamental group of a Riemannian manifold with strictly negative curvature. However serious difficulties had to be overcome in order to achieve this. One needs to formulate and prove a Foliated Control Theorem in this context and also construct a transfer map to the sphere bundle for higher K -theory which is in a suitable sense compatible with the control structures.

The Descent Principle

2.8.10

In Theorem 167 we described the classical assembly map as a forget control map using G -invariant geometric modules over $\tilde{M} \times [1, \infty)$. If in that context one does not require the modules and morphisms to be invariant under the G -action one nevertheless obtains a forget control functor between additive categories for which we introduce the notation

$$\mathcal{D}(1/t) = \mathcal{C}(\tilde{M} \times [1, \infty), \mathcal{S}(1/t); R)^\infty \rightarrow \mathcal{D}(1) = \mathcal{C}(\tilde{M} \times [1, \infty), \mathcal{S}(1); R)^\infty .$$

Applying K -theory yields a version of a “coarse” assembly map which is the algebraic K -theory analogue of the map described in Sect. 2.5.1. A crucial feature of such a construction is that the left hand side can be interpreted as a locally finite homology theory evaluated on \tilde{M} . It is hence an invariant of the proper homotopy type of M . Compare [7] and [326]. It is usually a lot easier to prove that this coarse assembly map is an equivalence. Suppose for example that M has non-positive curvature, choose a point $x_0 \in M$ (this will destroy the G -invariance) and with increasing $t \in [1, \infty)$ move the modules along geodesics towards x_0 . In this way one can show that the coarse assembly map is an isomorphism. Such coarse assembly maps exist also in the context of algebraic L -theory and topological K -theory, compare [151, 263].

Results about these maps (compare e.g. [17, 55, 336, 338]) lead to injectivity results for the classical assembly map by the “descent principle” (compare [52, 55, 263]) which we will now briefly describe in the context of algebraic K -theory. (We already stated an analytic version in Sect. 2.5.1.) For a spectrum E with G -action we denote by E^{hG} the homotopy fixed points. Since there is a natural map from fixed points to homotopy fixed points we obtain a commutative diagram

$$\begin{array}{ccc} \mathbf{K}(\mathcal{D}(1/t))^G & \longrightarrow & \mathbf{K}(\mathcal{D}(1))^G \\ \downarrow & & \downarrow \\ \mathbf{K}(\mathcal{D}(1/t))^{hG} & \longrightarrow & \mathbf{K}(\mathcal{D}(1))^{hG} . \end{array}$$

If one uses a suitable model K -theory commutes with taking fixed points and hence the upper horizontal map can be identified with the classical assembly map by Theorem 167. Using that K -theory commutes with infinite products [54], one can show by an induction over equivariant cells, that the vertical map on the left is an equivalence. Since we assume that the map $\mathbf{K}(\mathcal{D}(1/t)) \rightarrow \mathbf{K}(\mathcal{D}(1))$ is an equivalence, a standard property of the homotopy fixed point construction implies that the lower horizontal map is an equivalence. It follows that the upper horizontal map and hence the classical assembly map is split injective. A version of this argument which involves the assembly map for the family of finite subgroups can be found in [271].

2.8.11 Comparing to Other Theories

Every natural transformation of G -homology theories leads to a comparison between the associated assembly maps. For example one can compare topological K -theory to periodic cyclic homology [72], i.e. for every Banach algebra completion $\mathcal{A}(G)$ of $\mathbb{C}G$ inside $C_r^*(G)$ there exists a commutative diagram

$$\begin{CD} K_*(BG) @>>> K_*(\mathcal{A}(G)) \\ @VVV @VVV \\ H_*(BG; HP_*(\mathbb{C})) @>>> HP_*(\mathcal{A}(G)) . \end{CD}$$

This is used in [72] to prove injectivity results for word hyperbolic groups. Similar diagrams exist for other cyclic theories (compare for example [246]).

A suitable model for the cyclotomic trace $trc: K_n(RG) \rightarrow TC_n(RG)$ from (connective) algebraic K -theory to topological cyclic homology [38] leads for every family \mathcal{F} to a commutative diagram

$$\begin{CD} H_n(E_{\mathcal{F}}(G); \mathbf{K}_{\mathbb{Z}}^{con}) @>>> K_n^{con}(\mathbb{Z}G) \\ @VVV @VVV \\ H_n(E_{\mathcal{F}}(G); \mathbf{TC}_{\mathbb{Z}}) @>>> TC_n(\mathbb{Z}G) . \end{CD}$$

Injectivity results about the left hand and the lower horizontal map lead to injectivity results about the upper horizontal map. This is the principle behind Theorem 142 and 143.

2.9 Computations

Our ultimate goal is to compute K - and L -groups such as $K_n(RG)$, $L_n^{(-\infty)}(RG)$ and $K_n(C_r^*(G))$. Assuming that the Baum–Connes Conjecture 59 or the Farrell–Jones Conjecture 58 is true, this reduces to the computation of the left hand side of the

corresponding assembly map, i.e. to $H_n^G(E_{\mathcal{F}, \mathcal{L}, \mathcal{N}}(G); \mathbf{K}^{\text{top}})$, $H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G); \mathbf{K}_R)$ and $H_n^G(E_{\mathcal{V}\mathcal{C}\mathcal{Y}}(G); \mathbf{L}_R^{(-\infty)})$. This is much easier since here we can use standard methods from algebraic topology such as spectral sequences, Mayer–Vietoris sequences and Chern characters. Nevertheless such computations can be pretty hard. Roughly speaking, one can obtain a general reasonable answer after rationalization, but integral computations have only been done case by case and no general pattern is known.

K - and L -Groups for Finite Groups

In all these computations the answer is given in terms of the values of $K_n(RG)$, $L_n^{(-\infty)}(RG)$ and $K_n(C_r^*(G))$ for finite groups G . Therefore we briefly recall some of the results known for finite groups focusing on the case $R = \mathbb{Z}$

Topological K -Theory for Finite Groups

Let G be a finite group. By $r_F(G)$, we denote the number of isomorphism classes of irreducible representations of G over the field F . By $r_{\mathbb{R}}(G; \mathbb{R})$, $r_{\mathbb{R}}(G; \mathbb{C})$, respectively $r_{\mathbb{R}}(G; \mathbb{H})$ we denote the number of isomorphism classes of irreducible real G -representations V , which are of real, complex respectively of quaternionic type, i.e. $\text{aut}_{\mathbb{R}G}(V)$ is isomorphic to the field of real numbers \mathbb{R} , complex numbers \mathbb{C} or quaternions \mathbb{H} . Let $RO(G)$ respectively $R(G)$ be the *real* respectively the *complex representation ring*.

Notice that $\mathbb{C}G = l^1(G) = C_r^*(G) = C_{\max}^*(G)$ holds for a finite group, and analogous for the real versions.

Proposition 169 Let G be a finite group.

(i) We have

$$K_n(C_r^*(G)) \cong \begin{cases} R(G) \cong \mathbb{Z}^{r_{\mathbb{C}}(G)} & \text{for } n \text{ even;} \\ 0 & \text{for } n \text{ odd.} \end{cases}$$

(ii) There is an isomorphism of topological K -groups

$$K_n(C_r^*(G; \mathbb{R})) \cong K_n(\mathbb{R})^{r_{\mathbb{R}}(G; \mathbb{R})} \times K_n(\mathbb{C})^{r_{\mathbb{R}}(G; \mathbb{C})} \times K_n(\mathbb{H})^{r_{\mathbb{R}}(G; \mathbb{H})}.$$

Moreover $K_n(\mathbb{C})$ is 2-periodic with values $\mathbb{Z}, 0$ for $n = 0, 1$, $K_n(\mathbb{R})$ is 8-periodic with values $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}, 0, 0, 0$ for $n = 0, 1, \dots, 7$ and $K_n(\mathbb{H}) = K_{n+4}(\mathbb{R})$ for $n \in \mathbb{Z}$.

Proof One gets isomorphisms of C^* -algebras

$$C_r^*(G) \cong \prod_{j=1}^{r_{\mathbb{C}}(G)} M_{n_j}(\mathbb{C})$$

and

$$C_r^*(G; \mathbb{R}) \cong \prod_{i=1}^{r_{\mathbb{R}}(G; \mathbb{R})} M_{m_i}(\mathbb{R}) \times \prod_{i=1}^{r_{\mathbb{R}}(G; \mathbb{C})} M_{n_i}(\mathbb{C}) \times \prod_{i=1}^{r_{\mathbb{R}}(G; \mathbb{H})} M_{p_i}(\mathbb{H})$$

from [283, theorem 7 on page 19, corollary 2 on page 96, page 102, page 106]. Now the claim follows from Morita invariance and the well-known values for $K_n(\mathbb{R})$, $K_n(\mathbb{C})$ and $K_n(\mathbb{H})$ (see for instance [301, page 216]).

To summarize, the values of $K_n(C_r^*(G))$ and $K_n(C_r^*(G; \mathbb{R}))$ are explicitly known for finite groups G and are in the complex case in contrast to the real case always torsion free.

Algebraic K -Theory for Finite Groups

Here are some facts about the algebraic K -theory of integral group rings of finite groups.

170 Proposition 170 Let G be a finite group.

- (i) $K_n(\mathbb{Z}G) = 0$ for $n \leq -2$.
- (ii) We have

$$K_{-1}(\mathbb{Z}G) \cong \mathbb{Z}^r \oplus (\mathbb{Z}/2)^s,$$

where

$$r = 1 - r_{\mathbb{Q}}(G) + \sum_{p \mid |G|} r_{\mathbb{Q}_p}(G) - r_{\mathbb{F}_p}(G)$$

and the sum runs over all primes dividing the order of G . (Recall that $r_F(G)$ denotes the number of isomorphism classes of irreducible representations of G over the field F .) There is an explicit description of the integer s in terms of global and local Schur indices [58]. If G contains a normal abelian subgroup of odd index, then $s = 0$.

- (iii) The group $\tilde{K}_0(\mathbb{Z}G)$ is finite.
- (iv) The group $\text{Wh}(G)$ is a finitely generated abelian group and its rank is $r_{\mathbb{R}}(G) - r_{\mathbb{Q}}(G)$.
- (v) The groups $K_n(\mathbb{Z}G)$ are finitely generated for all $n \in \mathbb{Z}$.
- (vi) We have $K_{-1}(\mathbb{Z}G) = 0$, $\tilde{K}_0(\mathbb{Z}G) = 0$ and $\text{Wh}(G) = 0$ for the following finite groups $G = \{1\}, \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/2 \times \mathbb{Z}/2, D_6, D_8$, where D_m is the dihedral group of order m .

If p is a prime, then $K_{-1}(\mathbb{Z}[\mathbb{Z}/p]) = K_{-1}(\mathbb{Z}[\mathbb{Z}/p \times \mathbb{Z}/p]) = 0$.

We have

$$K_{-1}(\mathbb{Z}[\mathbb{Z}/6]) \cong \mathbb{Z}, \quad \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/6]) = 0, \quad \text{Wh}(\mathbb{Z}/6) = 0$$

$$K_{-1}(\mathbb{Z}[D_{12}]) \cong \mathbb{Z}, \quad \tilde{K}_0(\mathbb{Z}[D_{12}]) = 0, \quad \text{Wh}(D_{12}) = 0.$$

(vii) Let $\text{Wh}_2(G)$ denote the cokernel of the assembly map

$$H_2(BG; \mathbf{K}(\mathbb{Z})) \rightarrow K_2(\mathbb{Z}G) .$$

We have $\text{Wh}_2(G) = 0$ for $G = \{1\}, \mathbb{Z}/2, \mathbb{Z}/3$ and $\mathbb{Z}/4$. Moreover $|\text{Wh}_2(\mathbb{Z}/6)| \leq 2$, $|\text{Wh}_2(\mathbb{Z}/2 \times \mathbb{Z}/2)| \geq 2$ and $\text{Wh}_2(D_6) = \mathbb{Z}/2$.

Proof (i) and (ii) are proved in [58].

(iii) is proved in [298, proposition 9.1 on page 573].

(iv) This is proved for instance in [232].

(v) See [181, 248].

(vi) and (vii) The computation $K_{-1}(\mathbb{Z}G) = 0$ for $G = \mathbb{Z}/p$ or $\mathbb{Z}/p \times \mathbb{Z}/p$ can be found in [22, theorem 10.6, p. 695] and is a special case of [58].

The vanishing of $\tilde{K}_0(\mathbb{Z}G)$ is proven for $G = D_6$ in [262, theorem 8.2] and for $G = D_8$ in [262, theorem 6.4]. The cases $G = \mathbb{Z}/2, \mathbb{Z}/3, \mathbb{Z}/4, \mathbb{Z}/6$, and $(\mathbb{Z}/2)^2$ are treated in [79, corollary 5.17]. Finally, $\tilde{K}_0(\mathbb{Z}D_{12}) = 0$ follows from [79, theorem 50.29 on page 266] and the fact that $\mathbb{Q}D_{12}$ as a \mathbb{Q} -algebra splits into copies of \mathbb{Q} and matrix algebras over \mathbb{Q} , so its maximal order has vanishing class group by Morita equivalence.

The claims about $\text{Wh}_2(\mathbb{Z}/n)$ for $n = 2, 3, 4, 6$ and for $\text{Wh}_2((\mathbb{Z}/2)^2)$ are taken from [85, proposition 5], [89, p. 482] and [296, pp. 218 and 221].

We get $K_2(\mathbb{Z}D_6) \cong (\mathbb{Z}/2)^3$ from [296, theorem 3.1]. The assembly map $H_2(B\mathbb{Z}/2; \mathbf{K}(\mathbb{Z})) \rightarrow K_2(\mathbb{Z}[\mathbb{Z}/2])$ is an isomorphism by [89, theorem on p. 482]. Now construct a commutative diagram

$$\begin{array}{ccc} H_2(B\mathbb{Z}/2; \mathbf{K}(\mathbb{Z})) & \xrightarrow{\cong} & H_2(BD_6; \mathbf{K}(\mathbb{Z})) \\ \cong \downarrow & & \downarrow \\ K_2(\mathbb{Z}[\mathbb{Z}/2]) & \longrightarrow & K_2(\mathbb{Z}D_6) \end{array}$$

whose lower horizontal arrow is split injective and whose upper horizontal arrow is an isomorphism by the Atiyah–Hirzebruch spectral sequence. Hence the right vertical arrow is split injective and $\text{Wh}_2(D_6) = \mathbb{Z}/2$.

Let us summarize. We already mentioned that a complete computation of $K_n(\mathbb{Z})$ is not known. Also a complete computation of $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$ for arbitrary primes p is out of reach (see [221, pp. 29,30]). There is a complete formula for $K_{-1}(\mathbb{Z}G)$ and $K_n(\mathbb{Z}G) = 0$ for $n \leq -2$ and one has a good understanding of $\text{Wh}(G)$ (see [232]). We have already mentioned Borel’s formula for $K_n(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ for all $n \in \mathbb{Z}$ (see Remark 14). For more rational information see also 177.

Algebraic L -Theory for Finite Groups

Here are some facts about L -groups of finite groups.

171 **Proposition 171** Let G be a finite group. Then

- (i) For each $j \leq 1$ the groups $L_n^{(j)}(\mathbb{Z}G)$ are finitely generated as abelian groups and contain no p -torsion for odd primes p . Moreover, they are finite for odd n .
- (ii) For every decoration (j) we have

$$L_n^{(j)}(\mathbb{Z}G)[1/2] \cong L_n^{(j)}(\mathbb{R}G)[1/2] \cong \begin{cases} \mathbb{Z}[1/2]^{r_{\mathbb{R}}(G)} & n \equiv 0 \pmod{4}; \\ \mathbb{Z}[1/2]^{r_{\mathbb{C}}(G)} & n \equiv 2 \pmod{4}; \\ 0 & n \equiv 1, 3 \pmod{4}. \end{cases}$$

- (iii) If G has odd order and n is odd, then $L_n^\varepsilon(\mathbb{Z}G) = 0$ for $\varepsilon = p, h, s$.

Proof (i) See for instance [143].

(ii) See [258, proposition 22.34 on page 253].

(iii) See [13] or [143].

The L -groups of $\mathbb{Z}G$ are pretty well understood for finite groups G . More information about them can be found in [143].

2.9.2 Rational Computations for Infinite Groups

Next we state what is known rationally about the K - and L -groups of an infinite (discrete) group, provided the Baum–Connes Conjecture 59 or the relevant version of the Farrell–Jones Conjecture 58 is known.

In the sequel let $(\mathcal{F} \mathcal{C} \mathcal{Y})$ be the set of conjugacy classes (C) for finite cyclic subgroups $C \subseteq G$. For $H \subseteq G$ let $N_G H = \{g \in G \mid gHg^{-1} = H\}$ be its *normalizer*, let $Z_G H = \{g \in G \mid ghg^{-1} = h \text{ for } h \in H\}$ be its *centralizer*, and put

$$W_G H := N_G H / (H \cdot Z_G H),$$

where $H \cdot Z_G H$ is the normal subgroup of $N_G H$ consisting of elements of the form hu for $h \in H$ and $u \in Z_G H$. Notice that $W_G H$ is finite if H is finite.

Recall that the *Burnside ring* $A(G)$ of a finite group is the Grothendieck group associated to the abelian monoid of isomorphism classes of finite G -sets with respect to the disjoint union. The ring multiplication comes from the cartesian product. The zero element is represented by the empty set, the unit is represented by $G/G = \text{pt}$. For a finite group G the abelian groups $K_q(C_r^*(G))$, $K_q(RG)$ and $L^{(-\infty)}(RG)$ become modules over $A(G)$ because these functors come with a Mackey structure and $[G/H]$ acts by $\text{ind}_H^G \circ \text{res}_G^H$.

We obtain a ring homomorphism

$$\chi^G: A(G) \rightarrow \prod_{(H) \in \mathcal{FAN}} \mathbb{Z}, \quad [S] \mapsto (|S^H|)_{(H) \in \mathcal{FAN}},$$

which sends the class of a finite G -set S to the element given by the cardinalities of the H -fixed point sets. This is an injection with finite cokernel. This leads to an isomorphism of \mathbb{Q} -algebras

$$\chi_{\mathbb{Q}}^G := \chi^G \otimes_{\mathbb{Z}} \text{id}_{\mathbb{Q}}: A(G) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{(H) \in (\mathcal{FAN})} \mathbb{Q}. \quad (2.47)$$

For a finite cyclic group C let

$$\theta_C \in A(C) \otimes_{\mathbb{Z}} \mathbb{Z}[1/|C|] \quad (2.48)$$

be the element which is sent under the isomorphism $\chi_{\mathbb{Q}}^C: A(C) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \prod_{(H) \in \mathcal{FAN}} \mathbb{Q}$ of (2.47) to the element, whose entry is one if $(H) = (C)$ and is zero if $(H) \neq (C)$. Notice that θ_C is an idempotent. In particular we get a direct summand $\theta_C \cdot K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \mathbb{Q}$ in $K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \mathbb{Q}$ and analogously for $K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L^{(-\infty)}(RG) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Rationalized Topological K -Theory for Infinite Groups

The next result is taken from [203, theorem 0.4 and page 127]. Recall that Λ^G is the ring $\mathbb{Z} \subseteq \Lambda^G \subseteq \mathbb{Q}$ which is obtained from \mathbb{Z} by inverting the orders of the finite subgroups of G .

Theorem 172: (*Rational Computation of Topological K -theory for Infinite Groups.*) Suppose that the group G satisfies the Baum–Connes Conjecture 59. Then there is an isomorphism

$$\bigoplus_{p+q=n} \bigoplus_{(C) \in (\mathcal{FCY})} K_p(BZ_G C) \otimes_{\mathbb{Z}[W_G C]} \theta_C \cdot K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\cong} K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \Lambda^G.$$

If we tensor with \mathbb{Q} , we get an isomorphism

$$\bigoplus_{p+q=n} \bigoplus_{(C) \in (\mathcal{FCY})} H_p(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C \cdot K_q(C_r^*(C)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Rationalized Algebraic K-Theory for Infinite Groups

Recall that for algebraic K-theory of the integral group ring we know because of Proposition 73 that in the Farrell–Jones Conjecture we can reduce to the family of finite subgroups. A reduction to the family of finite subgroups also works if the coefficient ring is a regular \mathbb{Q} -algebra, compare 70. The next result is a variation of [201, theorem 0.4].

173 **Theorem 173:** (*Rational Computation of Algebraic K-theory.*) Suppose that the group G satisfies the Farrell–Jones Conjecture 58 for the algebraic K-theory of RG , where either $R = \mathbb{Z}$ or R is a regular ring with $\mathbb{Q} \subset R$. Then we get an isomorphism

$$\bigoplus_{p+q=n} \bigoplus_{(C) \in (\mathcal{F} \mathcal{C} \mathcal{Y})} H_p(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C \cdot K_q(RC) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} K_n(RG) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

174 **Remark 174** If in Theorem 173 we assume the Farrell–Jones Conjecture for the algebraic K-theory of RG but make no assumption on the coefficient ring R , then we still obtain that the map appearing there is split injective.

Example 175. (*The Comparison Map from Algebraic to Topological K-theory.*) If we consider $R = \mathbb{C}$ as coefficient ring and apply $- \otimes_{\mathbb{Z}} \mathbb{C}$ instead of $- \otimes_{\mathbb{Z}} \mathbb{Q}$, the formulas simplify. Suppose that G satisfies the Baum–Connes Conjecture 59 and the Farrell–Jones Conjecture 58 for algebraic K-theory with \mathbb{C} as coefficient ring. Recall that $\text{con}(G)_f$ is the set of conjugacy classes (g) of elements $g \in G$ of finite order. We denote for $g \in G$ by $\langle g \rangle$ the cyclic subgroup generated by g .

Then we get the following commutative square, whose horizontal maps are isomorphisms and whose vertical maps are induced by the obvious change of theory homomorphisms (see [201, theorem 0.5])

$$\begin{array}{ccc} \bigoplus_{p+q=n} \bigoplus_{(g) \in \text{con}(G)_f} H_p(Z_G \langle g \rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \xrightarrow{\cong} & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\ \downarrow & & \downarrow \\ \bigoplus_{p+q=n} \bigoplus_{(g) \in \text{con}(G)_f} H_p(Z_G \langle g \rangle; \mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\text{top}}(\mathbb{C}) & \xrightarrow{\cong} & K_n(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

The Chern character appearing in the lower row of the commutative square above has already been constructed by different methods in [26]. The construction in [201] works also for \mathbb{Q} (and even smaller rings) and other theories like algebraic K- and L-theory. This is important for the proof of Theorem 98 and to get the commutative square above.

Example 176. (*A Formula for $K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$.*) Suppose that the Farrell–Jones Conjecture is true rationally for $K_0(\mathbb{Z}G)$, i.e. the assembly map

$$A_{\mathcal{V}\mathcal{C}\mathcal{Y}}: H_0^G(\text{Evcy}(G); \mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism. Then we obtain

$$K_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q} \cong$$

$$K_0(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \oplus \bigoplus_{(C) \in (\mathcal{F}\mathcal{C}\mathcal{Y})} H_1(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C \cdot K_{-1}(RC) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Notice that $\tilde{K}_0(\mathbb{Z}G) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains only contributions from $K_{-1}(\mathbb{Z}C) \otimes_{\mathbb{Z}} \mathbb{Q}$ for finite cyclic subgroups $C \subseteq G$.

Remark 177 Note that these statements are interesting already for finite groups. For instance Theorem 172 yields for a finite group G and $R = \mathbb{C}$ an isomorphism

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$$\bigoplus_{(C) \in (\mathcal{F}\mathcal{C}\mathcal{Y})} \Lambda_G \otimes_{\Lambda_G[W_G C]} \theta_C \cdot R(C) \otimes_{\mathbb{Z}} \Lambda_G \cong R(G) \otimes_{\mathbb{Z}} \Lambda_G.$$

which in turn implies Artin’s Theorem discussed in Remark 82.

Rationalized Algebraic L -Theory for Infinite Groups

Here is the L -theory analogue of the results above. Compare [201, theorem 0.4].

Theorem 178: (*Rational Computation of Algebraic L -theory for Infinite Groups.*) Suppose that the group G satisfies the Farrell–Jones Conjecture 58 for L -theory. Then we get for all $j \in \mathbb{Z}, j \leq 1$ an isomorphism

178

$$\bigoplus_{p+q=n} \bigoplus_{(C) \in (\mathcal{F}\mathcal{C}\mathcal{Y})} H_p(BZ_G C; \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C \cdot L_q^{(j)}(RC) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} L_n^{(j)}(RG) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Remark 179: (*Separation of Variables.*) Notice that in Theorem 172, 173 and 178 we see again the principle we called *separation of variables* in Remark 13. There is a group homology part which is independent of the coefficient ring R and the K - or L -theory under consideration and a part depending only on the values of the theory under consideration on RC or $C_r^*(C)$ for all finite cyclic subgroups $C \subseteq G$.

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Integral Computations for Infinite Groups

2.9.3

As mentioned above, no general pattern for integral calculations is known or expected. We mention at least one situation where a certain class of groups can

be treated simultaneously. Let \mathcal{MFL} be the subset of \mathcal{FLN} consisting of elements in \mathcal{FLN} which are maximal in \mathcal{FLN} . Consider the following assertions on the group G .

- (M) $M_1, M_2 \in \mathcal{MFL}, M_1 \cap M_2 \neq 1 \Rightarrow M_1 = M_2$;
- (NM) $M \in \mathcal{MFL} \Rightarrow N_G M = M$;
- (VCL_I) If V is an infinite virtually cyclic subgroup of G , then V is of type I (see Lemma 71);
- (FJK_N) The Isomorphism Conjecture of Farrell–Jones for algebraic K -theory 58 is true for $\mathbb{Z}G$ in the range $n \leq N$ for a fixed element $N \in \mathbb{Z} \amalg \{\infty\}$, i.e. the assembly map $A: \mathcal{H}_n^G(E_{\text{Vcyc}}(G); \mathbf{K}_R) \xrightarrow{\cong} K_n(RG)$ is bijective for $n \in \mathbb{Z}$ with $n \leq N$.

Let $\tilde{K}_n(C_r^*(H))$ be the cokernel of the map $K_n(C_r^*(\{1\})) \rightarrow K_n(C_r^*(H))$ and $\tilde{L}_n^{(j)}(RG)$ be the cokernel of the map $L_n^{(j)}(R) \rightarrow L_n^{(j)}(RG)$. This coincides with $\tilde{L}_n^{(j)}(R)$, which is the cokernel of the map $L_n^{(j)}(\mathbb{Z}) \rightarrow L_n^{(j)}(R)$ if $R = \mathbb{Z}$ but not in general. Denote by $\text{Wh}_n^R(G)$ the n -th Whitehead group of RG which is the $(n - 1)$ -th homotopy group of the homotopy fiber of the assembly map $BG_+ \wedge \mathbf{K}(R) \rightarrow \mathbf{K}(RG)$. It agrees with the previous defined notions if $R = \mathbb{Z}$. The next result is taken from [83, theorem 4.1].

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Theorem 180 Let $\mathbb{Z} \subseteq \Lambda \subseteq \mathbb{Q}$ be a ring such that the order of any finite subgroup of G is invertible in Λ . Let (\mathcal{MFL}) be the set of conjugacy classes (H) of subgroups of G such that H belongs to \mathcal{MFL} . Then:

- (i) If G satisfies (M), (NM) and the Baum–Connes Conjecture 59, then for $n \in \mathbb{Z}$ there is an exact sequence of topological K -groups

$$0 \rightarrow \bigoplus_{(H) \in (\mathcal{MFL})} \tilde{K}_n(C_r^*(H)) \rightarrow K_n(C_r^*(G)) \rightarrow K_n(G \setminus E_{\mathcal{FLN}}(G)) \rightarrow 0,$$

which splits after applying $- \otimes_{\mathbb{Z}} \Lambda$.

- (ii) If G satisfies (M), (NM), (VCL_I) and the L -theory part of the Farrell–Jones Conjecture 58, then for all $n \in \mathbb{Z}$ there is an exact sequence

$$\begin{aligned} \dots \rightarrow H_{n+1}(G \setminus E_{\mathcal{FLN}}(G); \mathbf{L}^{(-\infty)}(R)) &\rightarrow \bigoplus_{(H) \in (\mathcal{MFL})} \tilde{L}_n^{(-\infty)}(RH) \\ &\rightarrow L_n^{(-\infty)}(RG) \rightarrow H_n(G \setminus E_{\mathcal{FLN}}(G); \mathbf{L}^{(-\infty)}(R)) \rightarrow \dots \end{aligned}$$

It splits after applying $- \otimes_{\mathbb{Z}} \Lambda$, more precisely

$$L_n^{(-\infty)}(RG) \otimes_{\mathbb{Z}} \Lambda \rightarrow H_n(G \setminus E_{\mathcal{FLN}}(G); \mathbf{L}^{(-\infty)}(R)) \otimes_{\mathbb{Z}} \Lambda$$

is a split-surjective map of Λ -modules.

- (iii) If G satisfies the assertions (M), (NM) and the Farrell–Jones Conjecture 58 for $L_n(RG)[1/2]$, then the conclusion of assertion (ii) still holds if we invert

2 everywhere. Moreover, in the case $R = \mathbb{Z}$ the sequence reduces to a short exact sequence

$$0 \rightarrow \bigoplus_{(H) \in (\mathcal{MFJ})} \tilde{L}_n^{(j)}(\mathbb{Z}H) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow L_n^{(j)}(\mathbb{Z}G) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow H_n(G \backslash E_{\mathcal{FJN}}(G); \mathbb{L}(\mathbb{Z})) \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow 0,$$

which splits after applying $-\otimes_{\mathbb{Z}[\frac{1}{2}]} \Lambda[\frac{1}{2}]$.

(iv) If G satisfies (M), (NM), and (FJ) K_N , then there is for $n \in \mathbb{Z}, n \leq N$ an isomorphism

$$H_n(E_{\text{vcy}}(G), E_{\mathcal{FJN}}(G); \mathbf{K}_R) \oplus \bigoplus_{(H) \in (\mathcal{MFJ})} \text{Wh}_n^R(H) \xrightarrow{\cong} \text{Wh}_n^R(G),$$

where $\text{Wh}_n^R(H) \rightarrow \text{Wh}_n^R(G)$ is induced by the inclusion $H \rightarrow G$.

Remark 181: (Role of $G \backslash E_{\mathcal{FJN}}(G)$.) Theorem 180 illustrates that for such computations a good understanding of the geometry of the orbit space $G \backslash E_{\mathcal{FJN}}(G)$ is necessary.

181

Remark 182 In [83] it is explained that the following classes of groups do satisfy the assumption appearing in Theorem 180 and what the conclusions are in the case $R = \mathbb{Z}$. Some of these cases have been treated earlier in [34, 212].

182

- Extensions $1 \rightarrow \mathbb{Z}^n \rightarrow G \rightarrow F \rightarrow 1$ for finite F such that the conjugation action of F on \mathbb{Z}^n is free outside $0 \in \mathbb{Z}^n$;
- Fuchsian groups F ;
- One-relator groups G .

Theorem 180 is generalized in [204] in order to treat for instance the semi-direct product of the discrete three-dimensional Heisenberg group by $\mathbb{Z}/4$. For this group $G \backslash E_{\mathcal{FJN}}(G)$ is S^3 .

A calculation for 2-dimensional crystallographic groups and more general cocompact NEC-groups is presented in [212] (see also [236]). For these groups the orbit spaces $G \backslash E_{\mathcal{FJN}}(G)$ are compact surfaces possibly with boundary.

Example 183. Let F be a cocompact Fuchsian group with presentation

$$F = \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_t \mid c_1^{\gamma_1} = \dots = c_t^{\gamma_t} = c_1^{-1} \cdots c_t^{-1} [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

for integers $g, t \geq 0$ and $\gamma_i > 1$. Then $G \backslash E_{\mathcal{FJN}}(G)$ is a closed orientable surface of genus g . The following is a consequence of Theorem 180 (see [212] for more details).

- There are isomorphisms

$$K_n(C_r^*(F)) \cong \begin{cases} (2 + \sum_{i=1}^t (\gamma_i - 1)) \cdot \mathbb{Z} & n = 0 ; \\ (2g) \cdot \mathbb{Z} & n = 1 . \end{cases}$$

- The inclusions of the maximal subgroups $\mathbb{Z}/\gamma_i = \langle c_i \rangle$ induce an isomorphism

$$\bigoplus_{i=1}^t \text{Wh}_n(\mathbb{Z}/\gamma_i) \xrightarrow{\cong} \text{Wh}_n(F)$$

for $n \leq 1$.

- There are isomorphisms

$$L_n(\mathbb{Z}F)[1/2] \cong \begin{cases} (1 + \sum_{i=1}^t [\frac{\gamma_i}{2}]) \cdot \mathbb{Z}[1/2] & n \equiv 0 \ (4) ; \\ (2g) \cdot \mathbb{Z}[1/2] & n \equiv 1 \ (4) ; \\ (1 + \sum_{i=1}^t [\frac{\gamma_i-1}{2}]) \cdot \mathbb{Z}[1/2] & n \equiv 2 \ (4) ; \\ 0 & n \equiv 3 \ (4) , \end{cases}$$

where $[r]$ for $r \in \mathbb{R}$ denotes the largest integer less than or equal to r .

From now on suppose that each γ_i is odd. Then the number m above is odd and we get for $\varepsilon = p$ and s

$$L_n^\varepsilon(\mathbb{Z}F) \cong \begin{cases} \mathbb{Z}/2 \oplus (1 + \sum_{i=1}^t \frac{\gamma_i-1}{2}) \cdot \mathbb{Z} & n \equiv 0 \ (4) ; \\ (2g) \cdot \mathbb{Z} & n \equiv 1 \ (4) ; \\ \mathbb{Z}/2 \oplus (1 + \sum_{i=1}^t \frac{\gamma_i-1}{2}) \cdot \mathbb{Z} & q \equiv 2 \ (4) ; \\ (2g) \cdot \mathbb{Z}/2 & n \equiv 3 \ (4) . \end{cases}$$

For $\varepsilon = h$ we do not know an explicit formula. The problem is that no general formula is known for the 2-torsion contained in $\tilde{L}_{2q}^h(\mathbb{Z}[\mathbb{Z}/m])$, for m odd, since it is given by the term $\widehat{H}^2(\mathbb{Z}/2; \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/m]))$, see [14, theorem 2].

Information about the left hand side of the Farrell–Jones assembly map for algebraic K -theory in the case where G is $SL_3(\mathbb{Z})$ can be found in [306].

2.9.4 Techniques for Computations

We briefly outline some methods that are fundamental for computations and for the proofs of some of the theorems above.

Equivariant Atiyah–Hirzebruch Spectral Sequence

Let \mathcal{H}_*^G be a G -homology theory with values in Λ -modules. Then there are two spectral sequences which can be used to compute it. The first one is the rather

obvious equivariant version of the Atiyah–Hirzebruch spectral sequence. It converges to $\mathcal{H}_n^G(X)$ and its E^2 -term is given in terms of Bredon homology

$$E_{p,q}^2 = H_p^G(X; \mathcal{H}_q^G(G/H))$$

of X with respect to the coefficient system, which is given by the covariant functor $\text{Or}(G) \rightarrow \Lambda\text{-MODULES}$, $G/H \mapsto \mathcal{H}_q^G(G/H)$. More details can be found for instance in [82, theorem 4.7].

p -Chain Spectral Sequence

There is another spectral sequence, the p -chain spectral sequence [83]. Consider a covariant functor $\mathbf{E}: \text{Or}(G) \rightarrow \text{SPECTRA}$. It defines a G -homology theory $\mathcal{H}_*^G(-; \mathbf{E})$ (see Proposition 156). The p -chain spectral sequence converges to $\mathcal{H}_n^G(X)$ but has a different setup and in particular a different E^2 -term than the equivariant Atiyah–Hirzebruch spectral sequence. We describe the E^1 -term for simplicity only for a proper G -CW-complex.

A p -chain is a sequence of conjugacy classes of finite subgroups

$$(H_0) < \dots < (H_p)$$

where $(H_{i-1}) < (H_i)$ means that H_{i-1} is subconjugate, but not conjugate to (H_i) . Notice for the sequel that the group of automorphism of G/H in $\text{Or}(G)$ is isomorphic to NH/H . To such a p -chain there is associated the NH_p/H_p - NH_0/H_0 -set

$$\begin{aligned} S((H_0) < \dots < (H_p)) \\ = \text{map}(G/H_{p-1}, G/H_p)^G \times_{NH_{p-1}/H_{p-1}} \dots \times_{NH_1/H_1} \text{map}(G/H_0, G/H_1)^G. \end{aligned}$$

The E^1 -term $E_{p,q}^1$ of the p -chain spectral sequence is

$$\bigoplus_{(H_0) < \dots < (H_p)} \pi_q \left((X^{H_p} \times_{NH_p/H_p} S((H_0) < \dots < (H_p)))_+ \wedge_{NH_0/H_0} \mathbf{E}(G/H_0) \right),$$

where Y_+ means the pointed space obtained from Y by adjoining an extra base point. There are many situations where the p -chain spectral sequence is much more useful than the equivariant Atiyah–Hirzebruch spectral sequence. Sometimes a combination of both is necessary to carry through the desired calculation.

Equivariant Chern Characters

Equivariant Chern characters have been studied in [201] and [203] and allow to compute equivariant homology theories for proper G -CW-complexes. The existence of the equivariant Chern character says that under certain conditions the Atiyah–Hirzebruch spectral sequence collapses and, indeed, the source of the equivariant Chern character is canonically isomorphic to $\bigoplus_{p+q} E_{p,q}^2$, where $E_{p,q}^2$ is the E^2 -term of the equivariant Atiyah–Hirzebruch spectral sequence.

The results of Sect. 2.9.2 are essentially proved by applying the equivariant Chern character to the source of the assembly map for the family of finite subgroups.

References

1. H. Abels. Parallelizability of proper actions, global K -slices and maximal compact subgroups. *Math. Ann.*, 212:1–19, 1974/75.
2. H. Abels. A universal proper G -space. *Math. Z.*, 159(2):143–158, 1978.
3. U. Abresch. Über das Glätten Riemannscher Metriken. Habilitationsschrift, Bonn, 1988.
4. J. F. Adams. *Stable homotopy and generalised homology*. University of Chicago Press, Chicago, Ill., 1974. Chicago Lectures in Mathematics.
5. C. Anantharaman and J. Renault. Amenable groupoids. In *Groupoids in analysis, geometry, and physics (Boulder, CO, 1999)*, volume 282 of *Contemp. Math.*, pages 35–46. Amer. Math. Soc., Providence, RI, 2001.
6. C. Anantharaman-Delaroche and J. Renault. *Amenable groupoids*, volume 36 of *Monographies de L'Enseignement Mathématique*. Geneva, 2000. With a foreword by Georges Skandalis and Appendix B by E. Germain.
7. D. R. Anderson, F. X. Connolly, S. C. Ferry, and E. K. Pedersen. Algebraic K -theory with continuous control at infinity. *J. Pure Appl. Algebra*, 94(1):25–47, 1994.
8. D. R. Anderson and W. C. Hsiang. The functors K_{-i} and pseudo-isotopies of polyhedra. *Ann. of Math. (2)*, 105(2):201–223, 1977.
9. C. S. Aravinda, F. T. Farrell, and S. K. Roushon. Surgery groups of knot and link complements. *Bull. London Math. Soc.*, 29(4):400–406, 1997.
10. C. S. Aravinda, F. T. Farrell, and S. K. Roushon. Algebraic K -theory of pure braid groups. *Asian J. Math.*, 4(2):337–343, 2000.
11. M. F. Atiyah. Global theory of elliptic operators. In *Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)*, pages 21–30. Univ. of Tokyo Press, Tokyo, 1970.
12. M. F. Atiyah. Elliptic operators, discrete groups and von Neumann algebras. *Astérisque*, 32-33:43–72, 1976.
13. A. Bak. Odd dimension surgery groups of odd torsion groups vanish. *Topology*, 14(4):367–374, 1975.
14. A. Bak. The computation of surgery groups of finite groups with abelian 2-hyerelementary subgroups. In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pages 384–409. Lecture Notes in Math., Vol. 551. Springer-Verlag, Berlin, 1976.
15. M. Banagl and A. Ranicki. Generalized Arf invariants in algebraic L -theory. arXiv: math.AT/0304362, 2003.
16. A. Bartels. On the domain of the assembly map in algebraic K -theory. *Algebr. Geom. Topol. (electronic)*, 3:1037–1050, 2003.
17. A. Bartels. Squeezing and higher algebraic K -theory. *K-Theory*, 28(1):19–37, 2003.

18. A. Bartels, F. T. Farrell, L. E. Jones, and H. Reich. A foliated squeezing theorem for geometric modules. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*. 2003. Proceedings of the summer school “High dimensional manifold theory” in Trieste May/June 2001.
19. A. Bartels, F. T. Farrell, L. E. Jones, and H. Reich. On the isomorphism conjecture in algebraic K -theory. *Topology*, 43(1):157–231, 2004.
20. A. Bartels and W. Lück. Isomorphism conjecture for homotopy K -theory and groups acting on trees. Preprintreihe SFB 478 – Geometrische Strukturen in der Mathematik, Heft 342, Münster, arXiv:math.KT/0407489, 2004.
21. A. Bartels and H. Reich. On the Farrell-Jones conjecture for higher algebraic K -theory. Preprintreihe SFB 478 – Geometrische Strukturen in der Mathematik, Heft 282, Münster, to appear in JAMS, 2003.
22. H. Bass. *Algebraic K-theory*. W. A. Benjamin, Inc., New York-Amsterdam, 1968.
23. H. Bass. Some problems in “classical” algebraic K -theory. In *Algebraic K-theory, II: “Classical” algebraic K-theory and connections with arithmetic (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 3–73. Lecture Notes in Math., Vol. 342. Springer-Verlag, Berlin, 1973.
24. H. Bass. Euler characteristics and characters of discrete groups. *Invent. Math.*, 35:155–196, 1976.
25. H. Bass, A. Heller, and R. G. Swan. The Whitehead group of a polynomial extension. *Inst. Hautes Études Sci. Publ. Math.*, (22):61–79, 1964.
26. P. Baum and A. Connes. Chern character for discrete groups. In *A fête of topology*, pages 163–232. Academic Press, Boston, MA, 1988.
27. P. Baum and A. Connes. Geometric K -theory for Lie groups and foliations. *Enseign. Math. (2)*, 46(1-2):3–42, 2000.
28. P. Baum, A. Connes, and N. Higson. Classifying space for proper actions and K -theory of group C^* -algebras. In *C^* -algebras: 1943–1993 (San Antonio, TX, 1993)*, pages 240–291. Amer. Math. Soc., Providence, RI, 1994.
29. P. Baum and R. G. Douglas. K -homology and index theory. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, pages 117–173. Amer. Math. Soc., Providence, R.I., 1982.
30. P. Baum, N. Higson, and R. Plymen. Representation theory of p -adic groups: a view from operator algebras. In *The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998)*, pages 111–149. Amer. Math. Soc., Providence, RI, 2000.
31. P. Baum and M. Karoubi. On the Baum–Connes conjecture in the real case. *Q. J. Math.*, 55(3):231–235, 2004.
32. P. Baum, S. Millington, and R. Plymen. Local-global principle for the Baum–Connes conjecture with coefficients. *K-Theory*, 28(1):1–18, 2003.
33. E. Berkove, F. T. Farrell, D. Juan-Pineda, and K. Pearson. The Farrell-Jones isomorphism conjecture for finite covolume hyperbolic actions and the algebraic K -theory of Bianchi groups. *Trans. Amer. Math. Soc.*, 352(12):5689–5702, 2000.

34. E. Berkove, D. Juan-Pineda, and K. Pearson. The lower algebraic K -theory of Fuchsian groups. *Comment. Math. Helv.*, 76(2):339–352, 2001.
35. A. J. Berrick. *An approach to algebraic K-theory*. Pitman (Advanced Publishing Program), Boston, Mass., 1982.
36. A. J. Berrick, I. Chatterji, and G. Mislin. From acyclic groups to the Bass conjecture for amenable groups. Preprint, to appear in *Math. Annalen*, 2001.
37. B. Blackadar. *K-theory for operator algebras*. Springer-Verlag, New York, 1986.
38. M. Bökstedt, W. C. Hsiang, and I. Madsen. The cyclotomic trace and algebraic K -theory of spaces. *Invent. Math.*, 111(3):465–539, 1993.
39. A. Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.
40. A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Springer-Verlag, Berlin, 1972. Lecture Notes in Mathematics, Vol. 304.
41. W. Browder. *Surgery on simply-connected manifolds*. Springer-Verlag, New York, 1972. *Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 65*.
42. M. Burger and A. Valette. Idempotents in complex group rings: theorems of Zalesskii and Bass revisited. *J. Lie Theory*, 8(2):219–228, 1998.
43. D. Burghelea and R. Lashof. Stability of concordances and the suspension homomorphism. *Ann. of Math. (2)*, 105(3):449–472, 1977.
44. S. Cappell, A. Ranicki, and J. Rosenberg, editors. *Surveys on surgery theory. Vol. 1*. Princeton University Press, Princeton, NJ, 2000. Papers dedicated to C. T. C. Wall.
45. S. Cappell, A. Ranicki, and J. Rosenberg, editors. *Surveys on surgery theory. Vol. 2*. Princeton University Press, Princeton, NJ, 2001. Papers dedicated to C. T. C. Wall.
46. S. E. Cappell. Mayer-Vietoris sequences in hermitian K -theory. In *Algebraic K-theory, III: Hermitian K-theory and geometric applications (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 478–512. Lecture Notes in Math., Vol. 343. Springer-Verlag, Berlin, 1973.
47. S. E. Cappell. Splitting obstructions for Hermitian forms and manifolds with $Z_2 \subset \pi_1$. *Bull. Amer. Math. Soc.*, 79:909–913, 1973.
48. S. E. Cappell. Manifolds with fundamental group a generalized free product. I. *Bull. Amer. Math. Soc.*, 80:1193–1198, 1974.
49. S. E. Cappell. Unitary nilpotent groups and Hermitian K -theory. I. *Bull. Amer. Math. Soc.*, 80:1117–1122, 1974.
50. S. E. Cappell and J. L. Shaneson. On 4-dimensional s -cobordisms. *J. Differential Geom.*, 22(1):97–115, 1985.
51. M. Cárdenas and E. K. Pedersen. On the Karoubi filtration of a category. *K-Theory*, 12(2):165–191, 1997.
52. G. Carlsson. Deloopings in algebraic K -theory. In this Handbook.
53. G. Carlsson. A survey of equivariant stable homotopy theory. *Topology*, 31(1):1–27, 1992.

54. G. Carlsson. On the algebraic K -theory of infinite product categories. *K-theory*, 9:305–322, 1995.
55. G. Carlsson and E. K. Pedersen. Controlled algebra and the Novikov conjectures for K - and L -theory. *Topology*, 34(3):731–758, 1995.
56. G. Carlsson, E. K. Pedersen, and W. Vogell. Continuously controlled algebraic K -theory of spaces and the Novikov conjecture. *Math. Ann.*, 310(1):169–182, 1998.
57. D. W. Carter. Localization in lower algebraic K -theory. *Comm. Algebra*, 8(7):603–622, 1980.
58. D. W. Carter. Lower K -theory of finite groups. *Comm. Algebra*, 8(20):1927–1937, 1980.
59. J. Chabert and S. Echterhoff. Permanence properties of the Baum-Connes conjecture. *Doc. Math.*, 6:127–183 (electronic), 2001.
60. J. Chabert, S. Echterhoff, and R. Nest. The Connes-Kasparov conjecture for almost connected groups and for linear p -adic groups. Preprintreihe SFB 478 – Geometrische Strukturen in der Mathematik, Heft 251, 2003.
61. S. Chang. On conjectures of Mathai and Borel. *Geom. Dedicata*, 106:161–167, 2004.
62. S. Chang and S. Weinberger. On invariants of Hirzebruch and Cheeger-Gromov. *Geom. Topol.*, 7:311–319 (electronic), 2003.
63. I. Chatterji and K. Ruane. Some geometric groups with rapid decay. preprint, arXiv:math.GT/0310356, 2004.
64. J. Cheeger and M. Gromov. Bounds on the von Neumann dimension of L^2 -cohomology and the Gauss-Bonnet theorem for open manifolds. *J. Differential Geom.*, 21(1):1–34, 1985.
65. J. Cheeger and M. Gromov. On the characteristic numbers of complete manifolds of bounded curvature and finite volume. In *Differential geometry and complex analysis*, pages 115–154. Springer-Verlag, Berlin, 1985.
66. P.-A. Cherix, M. Cowling, P. Jolissaint, P. Julg, and A. Valette. *Groups with the Haagerup property*, volume 197 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2001.
67. G. Cliff and A. Weiss. Moody’s induction theorem. *Illinois J. Math.*, 32(3):489–500, 1988.
68. T. D. Cochran. Noncommutative knot theory. *Algebr. Geom. Topol.*, 4:347–398, 2004.
69. T. D. Cochran, K. E. Orr, and P. Teichner. Knot concordance, Whitney towers and L^2 -signatures. *Ann. of Math. (2)*, 157(2):433–519, 2003.
70. M. M. Cohen. *A course in simple-homotopy theory*. Springer-Verlag, New York, 1973. Graduate Texts in Mathematics, Vol. 10.
71. A. Connes. *Noncommutative geometry*. Academic Press Inc., San Diego, CA, 1994.
72. A. Connes and H. Moscovici. Cyclic cohomology, the Novikov conjecture and hyperbolic groups. *Topology*, 29(3):345–388, 1990.
73. F. X. Connolly and M. O. M. da Silva. The groups $N^r K_0(\mathbb{Z}\pi)$ are finitely generated $\mathbb{Z}[N^r]$ -modules if π is a finite group. *K-Theory*, 9(1):1–11, 1995.

74. F. X. Connolly and T. Koźniewski. *Nil groups in K -theory and surgery theory*. *Forum Math.*, 7(1):45–76, 1995.
75. F. X. Connolly and S. Prassidis. On the exponent of the cokernel of the forget-control map on K_0 -groups. *Fund. Math.*, 172(3):201–216, 2002.
76. F. X. Connolly and S. Prassidis. On the exponent of the NK_0 -groups of virtually infinite cyclic groups. *Canad. Math. Bull.*, 45(2):180–195, 2002.
77. F. X. Connolly and A. Ranicki. On the calculation of $UNIL_*$. arXiv:math.AT/0304016v1, 2003.
78. J. Cuntz. K -theoretic amenability for discrete groups. *J. Reine Angew. Math.*, 344:180–195, 1983.
79. C. W. Curtis and I. Reiner. *Methods of representation theory. Vol. II*. John Wiley & Sons Inc., New York, 1987. With applications to finite groups and orders, A Wiley-Interscience Publication.
80. K. R. Davidson. *C^* -algebras by example*, volume 6 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 1996.
81. J. F. Davis. Manifold aspects of the Novikov conjecture. In *Surveys on surgery theory, Vol. 1*, pages 195–224. Princeton Univ. Press, Princeton, NJ, 2000.
82. J. F. Davis and W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in K - and L -theory. *K-Theory*, 15(3):201–252, 1998.
83. J. F. Davis and W. Lück. The p -chain spectral sequence. Preprintreihe SFB 478 – Geometrische Strukturen in der Mathematik, Heft 257, Münster. To appear in a special issue of *K-Theory* dedicated to H. Bass, 2002.
84. P. de la Harpe. Groupes hyperboliques, algèbres d’opérateurs et un théorème de Jolissaint. *C. R. Acad. Sci. Paris Sér. I Math.*, 307(14):771–774, 1988.
85. R. K. Dennis, M. E. Keating, and M. R. Stein. Lower bounds for the order of $K_2(FG)$ and $Wh_2(G)$. *Math. Ann.*, 223(2):97–103, 1976.
86. S. K. Donaldson. Irrationality and the h -cobordism conjecture. *J. Differential Geom.*, 26(1):141–168, 1987.
87. A. N. Dranishnikov, G. Gong, V. Lafforgue, and G. Yu. Uniform embeddings into Hilbert space and a question of Gromov. *Canad. Math. Bull.*, 45(1):60–70, 2002.
88. E. Dror Farjoun. Homotopy and homology of diagrams of spaces. In *Algebraic topology (Seattle, Wash., 1985)*, pages 93–134. Springer-Verlag, Berlin, 1987.
89. M. J. Dunwoody. $K_2(\mathbb{Z}\pi)$ for π a group of order two or three. *J. London Math. Soc. (2)*, 11(4):481–490, 1975.
90. W. Dwyer, T. Schick, and S. Stolz. Remarks on a conjecture of Gromov and Lawson. In *High-dimensional manifold topology*, pages 159–176. World Sci. Publishing, River Edge, NJ, 2003.
91. W. Dwyer, M. Weiss, and B. Williams. A parametrized index theorem for the algebraic K -theory Euler class. *Acta Math.*, 190(1):1–104, 2003.
92. B. Eckmann. Cyclic homology of groups and the Bass conjecture. *Comment. Math. Helv.*, 61(2):193–202, 1986.
93. B. Eckmann. Projective and Hilbert modules over group algebras, and finitely dominated spaces. *Comment. Math. Helv.*, 71(3):453–462, 1996.

94. D. S. Farley. Proper isometric actions of Thompson's groups on Hilbert space. *Int. Math. Res. Not.*, (45):2409–2414, 2003.
95. F. T. Farrell. The nonfiniteness of Nil. *Proc. Amer. Math. Soc.*, 65(2):215–216, 1977.
96. F. T. Farrell. The exponent of $UNil$. *Topology*, 18(4):305–312, 1979.
97. F. T. Farrell and W. C. Hsiang. On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, Proc. Sympos. Pure Math., XXXII, pages 325–337. Amer. Math. Soc., Providence, R.I., 1978.
98. F. T. Farrell and W. C. Hsiang. The topological-Euclidean space form problem. *Invent. Math.*, 45(2):181–192, 1978.
99. F. T. Farrell and W. C. Hsiang. On Novikov's conjecture for nonpositively curved manifolds. I. *Ann. of Math. (2)*, 113(1):199–209, 1981.
100. F. T. Farrell and W. C. Hsiang. The Whitehead group of poly-(finite or cyclic) groups. *J. London Math. Soc. (2)*, 24(2):308–324, 1981.
101. F. T. Farrell and L. E. Jones. h -cobordisms with foliated control. *Bull. Amer. Math. Soc. (N.S.)*, 15(1):69–72, 1986.
102. F. T. Farrell and L. E. Jones. K -theory and dynamics. I. *Ann. of Math. (2)*, 124(3):531–569, 1986.
103. F. T. Farrell and L. E. Jones. Foliated control with hyperbolic leaves. *K-Theory*, 1(4):337–359, 1987.
104. F. T. Farrell and L. E. Jones. Foliated control theory. I, II. *K-Theory*, 2(3):357–430, 1988.
105. F. T. Farrell and L. E. Jones. The surgery L -groups of poly-(finite or cyclic) groups. *Invent. Math.*, 91(3):559–586, 1988.
106. F. T. Farrell and L. E. Jones. Negatively curved manifolds with exotic smooth structures. *J. Amer. Math. Soc.*, 2(4):899–908, 1989.
107. F. T. Farrell and L. E. Jones. A topological analogue of Mostow's rigidity theorem. *J. Amer. Math. Soc.*, 2(2):257–370, 1989.
108. F. T. Farrell and L. E. Jones. Foliated control without radius of injectivity restrictions. *Topology*, 30(2):117–142, 1991.
109. F. T. Farrell and L. E. Jones. Rigidity in geometry and topology. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 653–663, Tokyo, 1991. Math. Soc. Japan.
110. F. T. Farrell and L. E. Jones. Stable pseudoisotopy spaces of compact non-positively curved manifolds. *J. Differential Geom.*, 34(3):769–834, 1991.
111. F. T. Farrell and L. E. Jones. Isomorphism conjectures in algebraic K -theory. *J. Amer. Math. Soc.*, 6(2):249–297, 1993.
112. F. T. Farrell and L. E. Jones. Topological rigidity for compact non-positively curved manifolds. In *Differential geometry: Riemannian geometry (Los Angeles, CA, 1990)*, pages 229–274. Amer. Math. Soc., Providence, RI, 1993.
113. F. T. Farrell and L. E. Jones. The lower algebraic K -theory of virtually infinite cyclic groups. *K-Theory*, 9(1):13–30, 1995.

114. F. T. Farrell and L. E. Jones. Collapsing foliated Riemannian manifolds. *Asian J. Math.*, 2(3):443–494, 1998.
115. F. T. Farrell and L. E. Jones. Rigidity for aspherical manifolds with $\pi_1 \subset GL_m(\mathbb{R})$. *Asian J. Math.*, 2(2):215–262, 1998.
116. F. T. Farrell and L. E. Jones. Local collapsing theory. *Pacific J. Math.*, 210(1):1–100, 2003.
117. F. T. Farrell and P. A. Linnell. K -theory of solvable groups. *Proc. London Math. Soc.* (3), 87(2):309–336, 2003.
118. F. T. Farrell and P. A. Linnell. Whitehead groups and the Bass conjecture. *Math. Ann.*, 326(4):723–757, 2003.
119. F. T. Farrell and S. K. Roushon. The Whitehead groups of braid groups vanish. *Internat. Math. Res. Notices*, (10):515–526, 2000.
120. T. Farrell. The Borel conjecture. In T. Farrell, L. Götsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 225–298. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school “High dimensional manifold theory” in Trieste May/June 2001, Number 1. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.
121. T. Farrell, L. Götsche, and W. Lück, editors. *High dimensional manifold theory*. Number 9 in ICTP Lecture Notes. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school “High dimensional manifold theory” in Trieste May/June 2001, Number 1. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.
122. T. Farrell, L. Götsche, and W. Lück, editors. *High dimensional manifold theory*. Number 9 in ICTP Lecture Notes. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school “High dimensional manifold theory” in Trieste May/June 2001, Number 2. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.
123. T. Farrell, L. Jones, and W. Lück. A caveat on the isomorphism conjecture in L -theory. *Forum Math.*, 14(3):413–418, 2002.
124. S. Ferry. The homeomorphism group of a compact Hilbert cube manifold is an *anr*. *Ann. Math.* (2), 106(1):101–119, 1977.
125. S. Ferry. A simple-homotopy approach to the finiteness obstruction. In *Shape theory and geometric topology (Dubrovnik, 1981)*, pages 73–81. Springer-Verlag, Berlin, 1981.
126. S. Ferry and A. Ranicki. A survey of Wall’s finiteness obstruction. In *Surveys on surgery theory, Vol. 2*, volume 149 of *Ann. of Math. Stud.*, pages 63–79. Princeton Univ. Press, Princeton, NJ, 2001.
127. S. C. Ferry, A. A. Ranicki, and J. Rosenberg. A history and survey of the Novikov conjecture. In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, pages 7–66. Cambridge Univ. Press, Cambridge, 1995.

128. S. C. Ferry, A. A. Ranicki, and J. Rosenberg, editors. *Novikov conjectures, index theorems and rigidity. Vol. 1*. Cambridge University Press, Cambridge, 1995. Including papers from the conference held at the Mathematisches Forschungsinstitut Oberwolfach, Oberwolfach, September 6–10, 1993.
129. S. C. Ferry and S. Weinberger. Curvature, tangentiality, and controlled topology. *Invent. Math.*, 105(2):401–414, 1991.
130. Z. Fiedorowicz. The Quillen–Grothendieck construction and extension of pairings. In *Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977)*, I, pages 163–169. Springer-Verlag, Berlin, 1978.
131. M. H. Freedman. The topology of four-dimensional manifolds. *J. Differential Geom.*, 17(3):357–453, 1982.
132. M. H. Freedman. The disk theorem for four-dimensional manifolds. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983)*, pages 647–663, Warsaw, 1984. PWN.
133. S. M. Gersten. On the spectrum of algebraic K -theory. *Bull. Amer. Math. Soc.*, 78:216–219, 1972.
134. É. Ghys. Groupes aléatoires (d’après Misha Gromov,...). *Astérisque*, (294):viii, 173–204, 2004.
135. D. Grayson. Higher algebraic K -theory. II (after Daniel Quillen). In *Algebraic K-theory (Proc. Conf., Northwestern Univ., Evanston, Ill., 1976)*, pages 217–240. Lecture Notes in Math., Vol. 551. Springer-Verlag, Berlin, 1976.
136. M. Gromov. Large Riemannian manifolds. In *Curvature and topology of Riemannian manifolds (Katata, 1985)*, pages 108–121. Springer-Verlag, Berlin, 1986.
137. M. Gromov. Asymptotic invariants of infinite groups. In *Geometric group theory, Vol. 2 (Sussex, 1991)*, pages 1–295. Cambridge Univ. Press, Cambridge, 1993.
138. M. Gromov. Spaces and questions. *Geom. Funct. Anal.*, (Special Volume, Part I):118–161, 2000. GAFA 2000 (Tel Aviv, 1999).
139. M. Gromov. Random walk in random groups. *Geom. Funct. Anal.*, 13(1):73–146, 2003.
140. E. Guentner, N. Higson, and S. Weinberger. The Novikov conjecture for linear groups. preprint, 2003.
141. I. Hambleton. Algebraic K - and L -theory and applications to the topology of manifolds. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 299–369. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school “High dimensional manifold theory” in Trieste May/June 2001, Number 1. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.
142. I. Hambleton and E. K. Pedersen. Identifying assembly maps in K - and L -theory. *Math. Annalen*, 328(1):27–58, 2004.
143. I. Hambleton and L. R. Taylor. A guide to the calculation of the surgery obstruction groups for finite groups. In *Surveys on surgery theory, Vol. 1*, pages 225–274. Princeton Univ. Press, Princeton, NJ, 2000.

144. A. E. Hatcher. Concordance spaces, higher simple-homotopy theory, and applications. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, pages 3–21. Amer. Math. Soc., Providence, R.I., 1978.
145. J.-C. Hausmann. On the homotopy of nonnilpotent spaces. *Math. Z.*, 178(1):115–123, 1981.
146. L. Hesselholt and I. Madsen. On the K -theory of nilpotent endomorphisms. In *Homotopy methods in algebraic topology (Boulder, CO, 1999)*, volume 271 of *Contemp. Math.*, pages 127–140. Amer. Math. Soc., Providence, RI, 2001.
147. N. Higson. The Baum-Connes conjecture. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, pages 637–646 (electronic), 1998.
148. N. Higson. Bivariant K -theory and the Novikov conjecture. *Geom. Funct. Anal.*, 10(3):563–581, 2000.
149. N. Higson and G. Kasparov. E -theory and KK -theory for groups which act properly and isometrically on Hilbert space. *Invent. Math.*, 144(1):23–74, 2001.
150. N. Higson, V. Lafforgue, and G. Skandalis. Counterexamples to the Baum-Connes conjecture. *Geom. Funct. Anal.*, 12(2):330–354, 2002.
151. N. Higson, E. K. Pedersen, and J. Roe. C^* -algebras and controlled topology. *K-Theory*, 11(3):209–239, 1997.
152. N. Higson and J. Roe. On the coarse Baum-Connes conjecture. In *Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993)*, pages 227–254. Cambridge Univ. Press, Cambridge, 1995.
153. N. Higson and J. Roe. Amenable group actions and the Novikov conjecture. *J. Reine Angew. Math.*, 519:143–153, 2000.
154. N. Higson and J. Roe. *Analytic K -homology*. Oxford University Press, Oxford, 2000. Oxford Science Publications.
155. B. Hu. Retractions of closed manifolds with nonpositive curvature. In *Geometric group theory (Columbus, OH, 1992)*, volume 3 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 135–147. de Gruyter, Berlin, 1995.
156. B. Z. Hu. Whitehead groups of finite polyhedra with nonpositive curvature. *J. Differential Geom.*, 38(3):501–517, 1993.
157. K. Igusa. The stability theorem for smooth pseudoisotopies. *K-Theory*, 2(1-2):vi+355, 1988.
158. H. Inassaridze. *Algebraic K -theory*, volume 311 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1995.
159. M. Joachim. K -homology of C^* -categories and symmetric spectra representing K -homology. *Math. Annalen*, 328:641–670, 2003.
160. E. Jones. A paper for F.T. Farrell on his 60-th birthday. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, pages 200–260. 2003. Proceedings of the summer school “High dimensional manifold theory” in Trieste May/June 2001.

161. L. Jones. A paper for F.T. Farrell on his 60-th birthday. preprint, to appear in the Proceedings of the school/conference on “High-dimensional Manifold Topology” in Trieste, May/June 2001.
162. L. Jones. Foliated control theory and its applications. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 405– 460. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school “High dimensional manifold theory” in Trieste May/June 2001, Number 2. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.
163. P. Julg. Remarks on the Baum–Connes conjecture and Kazhdan’s property T . In *Operator algebras and their applications (Waterloo, ON, 1994/1995)*, volume 13 of *Fields Inst. Commun.*, pages 145–153. Amer. Math. Soc., Providence, RI, 1997.
164. P. Julg. Travaux de N. Higson et G. Kasparov sur la conjecture de Baum–Connes. *Astérisque*, (252):Exp. No. 841, 4, 151–183, 1998. Séminaire Bourbaki. Vol. 1997/98.
165. P. Julg. La conjecture de Baum–Connes à coefficients pour le groupe $Sp(n, 1)$. *C. R. Math. Acad. Sci. Paris*, 334(7):533–538, 2002.
166. P. Julg and G. Kasparov. Operator K -theory for the group $SU(n, 1)$. *J. Reine Angew. Math.*, 463:99–152, 1995.
167. M. Karoubi. Bott periodicity in topological, algebraic and hermitian K -theory. In this Handbook.
168. G. Kasparov. Novikov’s conjecture on higher signatures: the operator K -theory approach. In *Representation theory of groups and algebras*, pages 79–99. Amer. Math. Soc., Providence, RI, 1993.
169. G. Kasparov and G. Skandalis. Groupes “boliques” et conjecture de Novikov. *C. R. Acad. Sci. Paris Sér. I Math.*, 319(8):815–820, 1994.
170. G. G. Kasparov. Equivariant KK -theory and the Novikov conjecture. *Invent. Math.*, 91(1):147–201, 1988.
171. G. G. Kasparov. K -theory, group C^* -algebras, and higher signatures (conspetus). In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, pages 101–146. Cambridge Univ. Press, Cambridge, 1995.
172. G. G. Kasparov and G. Skandalis. Groups acting on buildings, operator K -theory, and Novikov’s conjecture. *K-Theory*, 4(4):303–337, 1991.
173. M. A. Kervaire. Le théorème de Barden–Mazur–Stallings. *Comment. Math. Helv.*, 40:31–42, 1965.
174. N. Keswani. Homotopy invariance of relative eta-invariants and C^* -algebra K -theory. *Electron. Res. Announc. Amer. Math. Soc.*, 4:18–26 (electronic), 1998.
175. N. Keswani. Von Neumann eta-invariants and C^* -algebra K -theory. *J. London Math. Soc. (2)*, 62(3):771–783, 2000.
176. R. C. Kirby and L. C. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations*. Princeton University Press, Princeton, N.J., 1977. With notes by J. Milnor and M. F. Atiyah, *Annals of Mathematics Studies*, No. 88.

177. M. Kolster. K -theory and arithmetics. Lectures - ICTP Trieste 2002.
178. M. Kreck. Surgery and duality. *Ann. of Math. (2)*, 149(3):707–754, 1999.
179. M. Kreck and W. Lück. *The Novikov Conjecture: Geometry and Algebra*, volume 33 of *Oberwolfach Seminars*. Birkhäuser, 2004.
180. P. H. Kropholler and G. Mislin. Groups acting on finite-dimensional spaces with finite stabilizers. *Comment. Math. Helv.*, 73(1):122–136, 1998.
181. A. O. Kuku. K_n, SK_n of integral group-rings and orders. In *Applications of algebraic K -theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983)*, pages 333–338. Amer. Math. Soc., Providence, RI, 1986.
182. A. O. Kuku and G. Tang. Higher K -theory of group-rings of virtually infinite cyclic groups. *Math. Ann.*, 325(4):711–726, 2003.
183. V. Lafforgue. Une démonstration de la conjecture de Baum-Connes pour les groupes réductifs sur un corps p -adique et pour certains groupes discrets possédant la propriété (T). *C. R. Acad. Sci. Paris Sér. I Math.*, 327(5):439–444, 1998.
184. V. Lafforgue. A proof of property (RD) for cocompact lattices of $SL(3, \mathbf{R})$ and $SL(3, \mathbf{C})$. *J. Lie Theory*, 10(2):255–267, 2000.
185. V. Lafforgue. Banach KK -theory and the Baum-Connes conjecture. In *European Congress of Mathematics, Vol. II (Barcelona, 2000)*, volume 202 of *Progr. Math.*, pages 31–46. Birkhäuser, Basel, 2001.
186. V. Lafforgue. K -théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes. *Invent. Math.*, 149(1):1–95, 2002.
187. T. Y. Lam. *A first course in noncommutative rings*. Springer-Verlag, New York, 1991.
188. E. C. Lance. *Hilbert C^* -modules*. Cambridge University Press, Cambridge, 1995. A toolkit for operator algebraists.
189. R. Lee and R. H. Szczarba. The group $K_3(\mathbf{Z})$ is cyclic of order forty-eight. *Ann. of Math. (2)*, 104(1):31–60, 1976.
190. L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.
191. P. A. Linnell. Decomposition of augmentation ideals and relation modules. *Proc. London Math. Soc. (3)*, 47(1):83–127, 1983.
192. P. A. Linnell. Division rings and group von Neumann algebras. *Forum Math.*, 5(6):561–576, 1993.
193. E. Lluís-Puebla, J.-L. Loday, H. Gillet, C. Soulé, and V. Snaith. *Higher algebraic K -theory: an overview*, volume 1491 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1992.
194. J.-L. Loday. K -théorie algébrique et représentations de groupes. *Ann. Sci. École Norm. Sup. (4)*, 9(3):309–377, 1976.
195. J. Lott. The zero-in-the-spectrum question. *Enseign. Math. (2)*, 42(3-4):341–376, 1996.
196. W. Lück. The geometric finiteness obstruction. *Proc. London Math. Soc. (3)*, 54(2):367–384, 1987.

197. W. Lück. *Transformation groups and algebraic K-theory*. Springer-Verlag, Berlin, 1989. *Mathematica Gottingensis*.
198. W. Lück. Dimension theory of arbitrary modules over finite von Neumann algebras and L^2 -Betti numbers. II. Applications to Grothendieck groups, L^2 -Euler characteristics and Burnside groups. *J. Reine Angew. Math.*, 496:213–236, 1998.
199. W. Lück. The type of the classifying space for a family of subgroups. *J. Pure Appl. Algebra*, 149(2):177–203, 2000.
200. W. Lück. A basic introduction to surgery theory. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 1–224. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school “High dimensional manifold theory” in Trieste May/June 2001, Number 1. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.
201. W. Lück. Chern characters for proper equivariant homology theories and applications to K - and L -theory. *J. Reine Angew. Math.*, 543:193–234, 2002.
202. W. Lück. *L^2 -invariants: theory and applications to geometry and K-theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.
203. W. Lück. The relation between the Baum-Connes conjecture and the trace conjecture. *Invent. Math.*, 149(1):123–152, 2002.
204. W. Lück. K - and L -theory of the semi-direct product discrete three-dimensional Heisenberg group by $\mathbb{Z}/4$. in preparation, 2003.
205. W. Lück. L^2 -invariants from the algebraic point of view. Preprintreihe SFB 478 – Geometrische Strukturen in der Mathematik, Heft 295, Münster, [arXiv:math.GT/0310489](https://arxiv.org/abs/math.GT/0310489) v1, 2003.
206. W. Lück. Survey on classifying spaces for families of subgroups. Preprintreihe SFB 478 – Geometrische Strukturen in der Mathematik, Heft 308, Münster, [arXiv:math.GT/0312378](https://arxiv.org/abs/math.GT/0312378) v1, 2004.
207. W. Lück and D. Meinrup. On the universal space for group actions with compact isotropy. In *Geometry and topology: Aarhus (1998)*, pages 293–305. Amer. Math. Soc., Providence, RI, 2000.
208. W. Lück, H. Reich, J. Rognes, and M. Varisco. *In preparation*, 2003.
209. W. Lück and M. Rørdam. Algebraic K -theory of von Neumann algebras. *K-Theory*, 7(6):517–536, 1993.
210. W. Lück and T. Schick. Approximating L^2 -signatures by their finite-dimensional analogues. Preprintreihe SFB 478 – Geometrische Strukturen in der Mathematik, Heft 190, Münster, to appear in *Forum*, 2001.
211. W. Lück and T. Schick. Various L^2 -signatures and a topological L^2 -signature theorem. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, pages 200–260. 2003. Proceedings of the summer school “High dimensional manifold theory” in Trieste May/June 2001.

212. W. Lück and R. Stamm. Computations of K - and L -theory of cocompact planar groups. *K-Theory*, 21(3):249–292, 2000.
213. I. Madsen and M. Rothenberg. Equivariant pseudo-isotopies and K_{-1} . In *Transformation groups (Osaka, 1987)*, volume 1375 of *Lecture Notes in Math.*, pages 216–230. Springer-Verlag, Berlin, 1989.
214. V. Mathai. Spectral flow, eta invariants, and von Neumann algebras. *J. Funct. Anal.*, 109(2):442–456, 1992.
215. M. Matthey and G. Mislin. Equivariant K -homology and restriction to finite cyclic subgroups. *K-Theory*, 32(2):167–179, 2004.
216. D. Meintrup. *On the Type of the Universal Space for a Family of Subgroups*. PhD thesis, Westfälische Wilhelms-Universität Münster, 2000.
217. D. Meintrup and T. Schick. A model for the universal space for proper actions of a hyperbolic group. *New York J. Math.*, 8:1–7 (electronic), 2002.
218. J. Milnor. On manifolds homeomorphic to the 7-sphere. *Ann. of Math. (2)*, 64:399–405, 1956.
219. J. Milnor. *Lectures on the h -cobordism theorem*. Princeton University Press, Princeton, N.J., 1965.
220. J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.*, 72:358–426, 1966.
221. J. Milnor. *Introduction to algebraic K -theory*. Princeton University Press, Princeton, N.J., 1971. Annals of Mathematics Studies, No. 72.
222. I. Mineyev and G. Yu. The Baum-Connes conjecture for hyperbolic groups. *Invent. Math.*, 149(1):97–122, 2002.
223. A. S. Miščenko and A. T. Fomenko. The index of elliptic operators over C^* -algebras. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(4):831–859, 967, 1979. English translation in *Math. USSR-Izv.* 15 (1980), no. 1, 87–112.
224. G. Mislin. Wall’s finiteness obstruction. In *Handbook of algebraic topology*, pages 1259–1291. North-Holland, Amsterdam, 1995.
225. G. Mislin and A. Valette. *Proper group actions and the Baum-Connes Conjecture*. Advanced Courses in Mathematics CRM Barcelona. Birkhäuser, 2003.
226. S. A. Mitchell. On the Lichtenbaum-Quillen conjectures from a stable homotopy-theoretic viewpoint. In *Algebraic topology and its applications*, volume 27 of *Math. Sci. Res. Inst. Publ.*, pages 163–240. Springer-Verlag, New York, 1994.
227. J. A. Moody. Brauer induction for G_0 of certain infinite groups. *J. Algebra*, 122(1):1–14, 1989.
228. G. J. Murphy. *C^* -algebras and operator theory*. Academic Press Inc., Boston, MA, 1990.
229. J. Neukirch, A. Schmidt, and K. Wingberg. *Cohomology of number fields*. Springer-Verlag, Berlin, 2000.
230. A. J. Nicas. On the higher Whitehead groups of a Bieberbach group. *Trans. Amer. Math. Soc.*, 287(2):853–859, 1985.
231. R. Oliver. Projective class groups of integral group rings: a survey. In *Orders and their applications (Oberwolfach, 1984)*, volume 1142 of *Lecture Notes in Math.*, pages 211–232. Springer-Verlag, Berlin, 1985.

232. R. Oliver. *Whitehead groups of finite groups*. Cambridge University Press, Cambridge, 1988.
233. H. Oyono-Oyono. Baum-Connes Conjecture and extensions. *J. Reine Angew. Math.*, 532:133–149, 2001.
234. H. Oyono-Oyono. Baum-Connes conjecture and group actions on trees. *K-Theory*, 24(2):115–134, 2001.
235. D. S. Passman. *Infinite crossed products*. Academic Press Inc., Boston, MA, 1989.
236. K. Pearson. Algebraic K -theory of two-dimensional crystallographic groups. *K-Theory*, 14(3):265–280, 1998.
237. E. Pedersen and C. Weibel. A non-connective delooping of algebraic K -theory. In *Algebraic and Geometric Topology; proc. conf. Rutgers Uni., New Brunswick 1983*, volume 1126 of *Lecture notes in mathematics*, pages 166–181. Springer-Verlag, 1985.
238. E. K. Pedersen. On the K_{-i} -functors. *J. Algebra*, 90(2):461–475, 1984.
239. E. K. Pedersen. On the bounded and thin h -cobordism theorem parameterized by \mathbf{R}^k . In *Transformation groups, Poznań 1985*, volume 1217 of *Lecture Notes in Math.*, pages 306–320. Springer-Verlag, Berlin, 1986.
240. E. K. Pedersen. Controlled methods in equivariant topology, a survey. In *Current trends in transformation groups*, volume 7 of *K-Monogr. Math.*, pages 231–245. Kluwer Acad. Publ., Dordrecht, 2002.
241. E. K. Pedersen and C. A. Weibel. K -theory homology of spaces. In *Algebraic topology (Arcata, CA, 1986)*, pages 346–361. Springer-Verlag, Berlin, 1989.
242. G. K. Pedersen. *C^* -algebras and their automorphism groups*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, 1979.
243. M. Pimsner and D. Voiculescu. K -groups of reduced crossed products by free groups. *J. Operator Theory*, 8(1):131–156, 1982.
244. M. V. Pimsner. KK -groups of crossed products by groups acting on trees. *Invent. Math.*, 86(3):603–634, 1986.
245. R. T. Powers. Simplicity of the C^* -algebra associated with the free group on two generators. *Duke Math. J.*, 42:151–156, 1975.
246. M. Puschnigg. The Kadison-Kaplansky conjecture for word-hyperbolic groups. *Invent. Math.*, 149(1):153–194, 2002.
247. D. Quillen. On the cohomology and K -theory of the general linear groups over a finite field. *Ann. of Math. (2)*, 96:552–586, 1972.
248. D. Quillen. Finite generation of the groups K_i of rings of algebraic integers. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 179–198. Lecture Notes in Math., Vol. 341. Springer-Verlag, Berlin, 1973.
249. D. Quillen. Higher algebraic K -theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer-Verlag, Berlin, 1973.
250. F. Quinn. A geometric formulation of surgery. In *Topology of Manifolds (Proc. Inst., Univ. of Georgia, Athens, Ga., 1969)*, pages 500–511. Markham, Chicago, Ill., 1970.

251. F. Quinn. Ends of maps. I. *Ann. of Math.* (2), 110(2):275–331, 1979.
252. F. Quinn. Ends of maps. II. *Invent. Math.*, 68(3):353–424, 1982.
253. A. Ranicki. Foundations of algebraic surgery. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 491–514. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school “High dimensional manifold theory” in Trieste May/June 2001, Number 2. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.
254. A. A. Ranicki. Algebraic L -theory. II. Laurent extensions. *Proc. London Math. Soc.* (3), 27:126–158, 1973.
255. A. A. Ranicki. Algebraic L -theory. III. Twisted Laurent extensions. In *Algebraic K-theory, III: Hermitian K-theory and geometric application (Proc. Conf. Seattle Res. Center, Battelle Memorial Inst., 1972)*, pages 412–463. Lecture Notes in Mathematics, Vol. 343. Springer-Verlag, Berlin, 1973.
256. A. A. Ranicki. *Exact sequences in the algebraic theory of surgery*. Princeton University Press, Princeton, N.J., 1981.
257. A. A. Ranicki. The algebraic theory of finiteness obstruction. *Math. Scand.*, 57(1):105–126, 1985.
258. A. A. Ranicki. *Algebraic L-theory and topological manifolds*. Cambridge University Press, Cambridge, 1992.
259. A. A. Ranicki. *Lower K- and L-theory*. Cambridge University Press, Cambridge, 1992.
260. A. A. Ranicki. On the Novikov conjecture. In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, pages 272–337. Cambridge Univ. Press, Cambridge, 1995.
261. H. Reich. L^2 -Betti numbers, isomorphism conjectures and noncommutative localization. Preprintreihe SFB 478 – Geometrische Strukturen in der Mathematik, Heft 250, Münster, to appear in the Proceedings of the Workshop on Noncommutative Localization - ICMS Edinburgh, April 2002, 2003.
262. I. Reiner. Class groups and Picard groups of group rings and orders. pages iv+44, Providence, R. I., 1976. American Mathematical Society. Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics, No. 26.
263. J. Roe. *Index theory, coarse geometry, and topology of manifolds*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996.
264. J. Rognes. $K_4(\mathbb{Z})$ is the trivial group. *Topology*, 39(2):267–281, 2000.
265. J. Rognes and C. Weibel. Two-primary algebraic K -theory of rings of integers in number fields. *J. Amer. Math. Soc.*, 13(1):1–54, 2000. Appendix A by Manfred Kolster.
266. J. Rosenberg. K -theory and geometric topology. In this Handbook.
267. J. Rosenberg. C^* -algebras, positive scalar curvature and the Novikov conjecture. III. *Topology*, 25:319–336, 1986.
268. J. Rosenberg. *Algebraic K-theory and its applications*. Springer-Verlag, New York, 1994.

269. J. Rosenberg. Analytic Novikov for topologists. In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, pages 338–372. Cambridge Univ. Press, Cambridge, 1995.
270. J. Rosenberg and S. Stolz. A “stable” version of the Gromov-Lawson conjecture. In *The Čech centennial (Boston, MA, 1993)*, pages 405–418. Amer. Math. Soc., Providence, RI, 1995.
271. D. Rosenthal. Splitting with continuous control in algebraic K -theory. *K-Theory*, 32(2):139–166, 2004.
272. C. P. Rourke and B. J. Sanderson. *Introduction to piecewise-linear topology*. Springer-Verlag, Berlin, 1982. Reprint.
273. L. H. Rowen. *Ring theory. Vol. II*. Academic Press Inc., Boston, MA, 1988.
274. R. Roy. The trace conjecture—a counterexample. *K-Theory*, 17(3):209–213, 1999.
275. R. Roy. A counterexample to questions on the integrality property of virtual signature. *Topology Appl.*, 100(2-3):177–185, 2000.
276. J. Sauer. K -theory for proper smooth actions of totally disconnected groups. Ph.D. thesis, 2002.
277. T. Schick. Finite group extensions and the Baum-Connes conjecture. in preparation.
278. T. Schick. A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture. *Topology*, 37(6):1165–1168, 1998.
279. T. Schick. Operator algebras and topology. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 571–660. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school “High dimensional manifold theory” in Trieste May/June 2001, Number 2. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.
280. P. Schneider. Über gewisse Galoiscohomologiegruppen. *Math. Z.*, 168(2):181–205, 1979.
281. H. Schröder. *K -theory for real C^* -algebras and applications*. Longman Scientific & Technical, Harlow, 1993.
282. J.-P. Serre. Cohomologie des groupes discrets. In *Prospects in mathematics (Proc. Sympos., Princeton Univ., Princeton, N.J., 1970)*, pages 77–169. Ann. of Math. Studies, No. 70. Princeton Univ. Press, Princeton, N.J., 1971.
283. J.-P. Serre. *Linear representations of finite groups*. Springer-Verlag, New York, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.
284. J. L. Shaneson. Wall’s surgery obstruction groups for $G \times Z$. *Ann. of Math.* (2), 90:296–334, 1969.
285. W.-X. Shi. Deforming the metric on complete Riemannian manifolds. *J. Differential Geom.*, 30(1):223–301, 1989.
286. J. R. Silvester. *Introduction to algebraic K -theory*. Chapman & Hall, London, 1981. Chapman and Hall Mathematics Series.
287. G. Skandalis. Une notion de nucléarité en K -théorie (d’après J. Cuntz). *K-Theory*, 1(6):549–573, 1988.

288. G. Skandalis. Progrès récents sur la conjecture de Baum-Connes. Contribution de Vincent Lafforgue. *Astérisque*, (276):105–135, 2002. Séminaire Bourbaki, Vol. 1999/2000.
289. G. Skandalis, J. L. Tu, and G. Yu. The coarse Baum-Connes conjecture and groupoids. *Topology*, 41(4):807–834, 2002.
290. C. Soulé. On higher p -adic regulators. In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 372–401. Springer-Verlag, Berlin, 1981.
291. C. Soulé. Éléments cyclotomiques en K -théorie. *Astérisque*, (147-148):225–257, 344, 1987. Journées arithmétiques de Besançon (Besançon, 1985).
292. V. Srinivas. *Algebraic K-theory*. Birkhäuser Boston Inc., Boston, MA, 1991.
293. C. Stark. Topological rigidity theorems. In R. Daverman and R. Sher, editors, *Handbook of Geometric Topology*, chapter 20. Elsevier, 2002.
294. C. W. Stark. Surgery theory and infinite fundamental groups. In *Surveys on surgery theory, Vol. 1*, volume 145 of *Ann. of Math. Stud.*, pages 275–305. Princeton Univ. Press, Princeton, NJ, 2000.
295. N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967.
296. M. R. Stein. Excision and K_2 of group rings. *J. Pure Appl. Algebra*, 18(2):213–224, 1980.
297. S. Stolz. Manifolds of positive scalar curvature. In T. Farrell, L. Göttsche, and W. Lück, editors, *High dimensional manifold theory*, number 9 in ICTP Lecture Notes, pages 661–708. Abdus Salam International Centre for Theoretical Physics, Trieste, 2002. Proceedings of the summer school “High dimensional manifold theory” in Trieste May/June 2001, Number 2. http://www.ictp.trieste.it/~pub_off/lectures/vol9.html.
298. R. G. Swan. Induced representations and projective modules. *Ann. of Math.* (2), 71:552–578, 1960.
299. R. G. Swan. *K-theory of finite groups and orders*. Springer-Verlag, Berlin, 1970. Lecture Notes in Mathematics, Vol. 149.
300. R. G. Swan. Higher algebraic K -theory. In *K-theory and algebraic geometry: connections with quadratic forms and division algebras (Santa Barbara, CA, 1992)*, volume 58 of *Proc. Sympos. Pure Math.*, pages 247–293. Amer. Math. Soc., Providence, RI, 1995.
301. R. M. Switzer. *Algebraic topology—homotopy and homology*. Springer-Verlag, New York, 1975. Die Grundlehren der mathematischen Wissenschaften, Band 212.
302. R. W. Thomason. Beware the phony multiplication on Quillen’s $\mathcal{A}^{-1}\mathcal{A}$. *Proc. Amer. Math. Soc.*, 80(4):569–573, 1980.
303. T. tom Dieck. Orbittypen und äquivariante Homologie. I. *Arch. Math. (Basel)*, 23:307–317, 1972.
304. T. tom Dieck. *Transformation groups*. Walter de Gruyter & Co., Berlin, 1987.
305. J.-L. Tu. The Baum-Connes conjecture and discrete group actions on trees. *K-Theory*, 17(4):303–318, 1999.

306. S. Upadhyay. Controlled algebraic K -theory of integral group ring of $SL(3, \mathbb{Z})$. *K-Theory*, 10(4):413–418, 1996.
307. A. Valette. *Introduction to the Baum–Connes conjecture*. Birkhäuser Verlag, Basel, 2002. From notes taken by Indira Chatterji, With an appendix by Guido Mislin.
308. K. Varadarajan. *The finiteness obstruction of C. T. C. Wall*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1989. A Wiley-Interscience Publication.
309. W. Vogell. Algebraic K -theory of spaces, with bounded control. *Acta Math.*, 165(3-4):161–187, 1990.
310. W. Vogell. Boundedly controlled algebraic K -theory of spaces and its linear counterparts. *J. Pure Appl. Algebra*, 76(2):193–224, 1991.
311. W. Vogell. Continuously controlled A -theory. *K-Theory*, 9(6):567–576, 1995.
312. J. B. Wagoner. Delooping classifying spaces in algebraic K -theory. *Topology*, 11:349–370, 1972.
313. F. Waldhausen. Algebraic K -theory of generalized free products. I, II. *Ann. of Math. (2)*, 108(1):135–204, 1978.
314. F. Waldhausen. Algebraic K -theory of topological spaces. I. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, Proc. Sympos. Pure Math., XXXII, pages 35–60. Amer. Math. Soc., Providence, R.I., 1978.
315. F. Waldhausen. Algebraic K -theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, pages 318–419. Springer-Verlag, Berlin, 1985.
316. F. Waldhausen. Algebraic K -theory of spaces, concordance, and stable homotopy theory. In *Algebraic topology and algebraic K -theory (Princeton, N.J., 1983)*, pages 392–417. Princeton Univ. Press, Princeton, NJ, 1987.
317. C. T. C. Wall. Finiteness conditions for CW -complexes. *Ann. of Math. (2)*, 81:56–69, 1965.
318. C. T. C. Wall. Finiteness conditions for CW complexes. II. *Proc. Roy. Soc. Ser. A*, 295:129–139, 1966.
319. C. T. C. Wall. Poincaré complexes. I. *Ann. of Math. (2)*, 86:213–245, 1967.
320. C. T. C. Wall. *Surgery on compact manifolds*. American Mathematical Society, Providence, RI, second edition, 1999. Edited and with a foreword by A. A. Ranicki.
321. N. E. Wegge-Olsen. *K -theory and C^* -algebras*. The Clarendon Press Oxford University Press, New York, 1993. A friendly approach.
322. C. Weibel. Algebraic K -theory of rings of integers in local and global fields. In this Handbook.
323. C. Weibel. An introduction to algebraic K -theory. In preparation, see <http://math.rutgers.edu/~weibel/Kbook.html>.
324. C. A. Weibel. K -theory and analytic isomorphisms. *Invent. Math.*, 61(2):177–197, 1980.

325. C. A. Weibel. Mayer-Vietoris sequences and module structures on NK_* . In *Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980)*, volume 854 of *Lecture Notes in Math.*, pages 466–493. Springer-Verlag, Berlin, 1981.
326. M. Weiss. Excision and restriction in controlled K -theory. *Forum Math.*, 14(1):85–119, 2002.
327. M. Weiss and B. Williams. Automorphisms of manifolds and algebraic K -theory. I. *K-Theory*, 1(6):575–626, 1988.
328. M. Weiss and B. Williams. Assembly. In *Novikov conjectures, index theorems and rigidity, Vol. 2 (Oberwolfach, 1993)*, pages 332–352. Cambridge Univ. Press, Cambridge, 1995.
329. M. Weiss and B. Williams. Automorphisms of manifolds. In *Surveys on surgery theory, Vol. 2*, volume 149 of *Ann. of Math. Stud.*, pages 165–220. Princeton Univ. Press, Princeton, NJ, 2001.
330. G. W. Whitehead. *Elements of homotopy theory*. Springer-Verlag, New York, 1978.
331. B. Wilking. Rigidity of group actions on solvable Lie groups. *Math. Ann.*, 317(2):195–237, 2000.
332. B. Williams. K -theory and surgery. In this Handbook.
333. M. Yamasaki. L -groups of crystallographic groups. *Invent. Math.*, 88(3):571–602, 1987.
334. G. Yu. On the coarse baum-connes conjecture. *K-theory*, 9:199–221, 1995.
335. G. Yu. Localization algebras and the coarse Baum-Connes conjecture. *K-Theory*, 11(4):307–318, 1997.
336. G. Yu. Asymptotic Fredholm modules, vector bundles with small variation and a nonvanishing theorem for K -theoretic indices. *K-Theory*, 13(4):331–346, 1998.
337. G. Yu. The Novikov conjecture for groups with finite asymptotic dimension. *Ann. of Math. (2)*, 147(2):325–355, 1998.
338. G. Yu. The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. *Invent. Math.*, 139(1):201–240, 2000.
339. G. L. Yu. Baum-Connes conjecture and coarse geometry. *K-Theory*, 9(3):223–231, 1995.
340. G. L. Yu. Coarse Baum-Connes conjecture. *K-Theory*, 9(3):199–221, 1995.