ON THE K- AND L-THEORY OF THE ALGEBRA OF OPERATORS AFFILIATED TO A FINITE VON NEUMANN ALGEBRA

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Abstract. We construct a real valued dimension for arbitrary modules over the algebra of operators affiliated to a finite von Neumann algebra. Moreover we determine the algebraic K_0- and K_1-group and the L-groups of such an algebra.

1. Introduction

The aim of this paper is twofold. First we would like to promote the algebra of operators affiliated to a finite von Neumann algebra as a convenient tool in understanding L^2-invariants. In particular we extend the notion of a real valued dimension from finitely generated projective modules over such an algebra to arbitrary modules (Theorem 3.11). This is motivated by the analogous result for von Neumann algebras in [14]. Second we would like to collect and complete the known results about K- and L-theory of finite von Neumann algebras and their algebras of affiliated operators (Theorem 6.1, Theorem 6.3 and Theorem 7.1).
Let us define our objects of study. A von Neumann algebra $\mathcal{A}$ is a weakly closed $*$-invariant subalgebra of the algebra $\mathcal{B}(H)$ of bounded linear operators on some Hilbert space H. It is called finite if it admits a normal faithful trace $\mathrm{tr} : \mathcal{A} \to \mathbb{C}$. Our favourite example is the group von Neumann algebra of a discrete group $\mathcal{N}\Gamma$ obtained by completing the left regular representation of the complex group ring $\mathbb{C}\Gamma$ on $l^2\Gamma$. A not necessarily bounded operator a is affiliated to $\mathcal{A}$ if $ba \subset ab$ for all operators $b \in \mathcal{A}'$. Here $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$ in $\mathcal{B}(H)$, and $ba \subset ab$ means that restricted to the possibly smaller domain of ba the two operators coincide.

Notation 1.1. Given a finite von Neumann algebra $\mathcal{A}$ let $\mathcal{U}$ denote the set of all closed densely defined operators affiliated to $\mathcal{A}$. If $\mathcal{A}$ happens to be a group von Neumann algebra $\mathcal{N}\Gamma$ we write $\mathcal{U}\Gamma$.

It was shown by Murray and von Neumann [19] that these operators indeed form an algebra, when addition and multiplication is defined as the closure of the naive addition and composition. Note that there is no reasonable topology on the algebra of affiliated operators. It turns out that $\mathcal{U}$ is a localization of $\mathcal{A}$ (in the sense of ring theory). The simple-minded analogy with the passage from abelian groups to $\mathbb{Q}$-vectorspaces is often very helpful. We now describe the main results of this paper.

Date: August 26, 2003.
2000 *Mathematics Subject Classification.* 18f25, 46L10.
Key words and phrases. L-Theory,von Neumann algebras.

Theorem 1.2. *There is a well behaved notion of dimension for arbitrary $\mathcal{U}$-modules.*

For a more precise statement see Theorem 3.11.
In the context of L^2-invariants the notion of dimension for $\mathcal{N}\Gamma$-modules is used to define the so called L^2-Betti numbers. In Section 5 below we explain how $\mathcal{U}\Gamma$ can be used to define L^2-Betti numbers. Computing L^2-Betti numbers via $\mathcal{U}\Gamma$-modules is analogous to computing ordinary Betti numbers via homology with rational coefficients instead of working with integral homology. The passage from $\mathcal{A}$-modules to $\mathcal{U}$-modules is the algebraic analogue of the passage from unreduced to reduced L^2-homology in the Hilbert space set-up. In Section 4 we investigate in more detail the effect of the passage from $\mathcal{A}$ to $\mathcal{U}$ for the corresponding module categories and compare two competing defintions of torsion modules.
Building on the results about $K_1(\mathcal{A})$ from [17] and the comparison of the L-theory of $\mathcal{A}$ with topological K-theory in [27] we completely determine the algebraic K_0- and K_1-groups and the L-theory groups of $\mathcal{A}$ and $\mathcal{U}$. Among other results we prove in Sections 6 and 7:

Theorem 1.3. *Let K_0 and K_1 denote the algebraic K-theory groups. Let L_p and L_h denote the L-groups based on projective respectively free modules.*

(i) *The passage from $\mathcal{A}$ to $\mathcal{U}$ does not affect K_0, L_h or L_p. In particular the corresponding relative L-groups vanish.*
(ii) *The odd L_p-groups of $\mathcal{A}$ and $\mathcal{U}$ vanish.*

Moreover if Γ is a finitely generated group which is not virtually abelian, then we have:

(iii) *$K_1(\mathcal{U}\Gamma) = 0$ and all odd L-groups of $\mathcal{N}\Gamma$ and $\mathcal{U}\Gamma$ vanish.*
(iv) *The relative K-group $K_0(\mathcal{N}\Gamma \to \mathcal{U}\Gamma)$ which can be identified with K_0 of the category of finitely presented torsion $\mathcal{A}$-modules vanishes.*

Note that any Γ-CW-complex with compact quotient space, whose isotropy groups are all finite, gives rise to a finite dimensional complex of finitely generated projective $\mathcal{N}\Gamma$-modules. The space is L^2-acyclic, i.e. all its L^2-Betti numbers vanish, if and only if the corresponding $\mathcal{U}\Gamma$-complex is acyclic. Therefore an L^2-acyclic space naturally gives rise to an element in the relative K-group $K_0(\mathcal{N}\Gamma \to \mathcal{U}\Gamma)$. Similarly an L^2-acyclic manifold naturally defines an element in a relative L-group. Unfortunately the results above show that these groups vanish if Γ is a finitely generated group which is not virtually abelian.
The algebra $\mathcal{U}\Gamma$ of operators affiliated to a group von Neumann algebra plays a central role in Linnell's work [12] on the Atiyah Conjecture about rationality of L^2-Betti numbers and the related Zero-Divisor Conjecture. Recently Cochran, Orr and Teichner used the algebra $\mathcal{U}\Gamma$ to construct via L^2-signatures new invariants detecting knots that are non-slice [4].
In both cases it is important that the algebra $\mathcal{U}\Gamma$ is large enough to contain a semisimple ring or even a skew field $\mathcal{D}\Gamma$ which contains the integral group ring $\mathbb{Z}\Gamma$, but $\mathcal{D}\Gamma$ does not fit into the group von Neumann algebra $\mathcal{N}\Gamma$, i.e.

we have inclusions

$$\begin{array}{ccc} \mathbb{Z}\Gamma & \hookrightarrow & \mathcal{N}\Gamma \\ \downarrow & & \downarrow \\ \mathcal{D}\Gamma & \hookrightarrow & \mathcal{U}\Gamma. \end{array}$$

The general philosophy is that invariants are defined over $\mathcal{D}\Gamma$ but they become accessible only after passage to $\mathcal{U}\Gamma$. In the simple case where $\Gamma = \mathbb{Z}$ is the infinite cyclic group the above diagram can be identified with

$$\begin{array}{ccc} \mathbb{Z}\left[x^{\pm 1}\right] & \hookrightarrow & \mathrm{L}^{\infty}(\mathrm{S}^1) \\ \downarrow & & \downarrow \\ \mathbb{Q}\left(x\right) & \hookrightarrow & \mathrm{L}(\mathrm{S}^1), \end{array}$$

where the rings are the Laurent-polynomials, the field of rational functions in one variable, the essentially bounded functions on S^1 and the measurable functions on S^1. Note that Laurent polynomials with zeros on S^1 are not invertible as essentially bounded functions but become invertible as measurable functions.

Most of the results below are contained in the author's thesis [26]. We would like to thank Wolfgang Lück for discussions and encouragement. The K- and L-theory computations will also be included in his book [15].

2. Review of some fundamental properties of $\mathcal{U}$

In this section we briefly review a few known properties of the algebra of operators affiliated to a finite von Neumann algebra. As above let $\mathcal{A}$ be a finite von Neumann algebra and denote by $\mathcal{U}$ the closed densely defined operators affiliated to $\mathcal{A}$. It was observed in [13] that $\mathcal{A}$ has astonishingly good ring-theoretical properties. Namely it is a semihereditary ring, i.e. every finitely generated submodule of a projective $\mathcal{A}$-module is projective. As a consequence the category of finitely presented $\mathcal{A}$-modules is an abelian category. The ring $\mathcal{U}$ has even better properties:

Proposition 2.1. *We have the following facts about $\mathcal{U}$.*

(i) *The algebra $\mathcal{U}$ is a von Neumann regular ring, i.e. for every $\mathcal{U}$-module the functor $-\otimes_{\mathcal{U}} M$ is exact.*
(ii) *Every finitely presented $\mathcal{U}$-module is finitely generated projective and the category of finitely generated projective $\mathcal{U}$-modules is abelian.*
(iii) *Every operator $b \in \mathcal{U}$ can be written in the form $b = at^{-1}$ with a, $t \in \mathcal{A}$ and t invertible in $\mathcal{U}$. Every non-zero-divisor in $\mathcal{A}$ becomes invertible in $\mathcal{U}$. The ring $\mathcal{A}$ fulfills both Ore-conditions with respect to the set of all non-zerodivisors and $\mathcal{U}$ is isomorphic to the Ore localization of $\mathcal{A}$.*
(iv) *The functor $-\otimes_{\mathcal{A}} \mathcal{U}$ is exact.*
(v) *The ring $\mathcal{U}$ is unit-regular, i.e. for every element $a \in \mathcal{U}$ there exists an invertible element $b \in \mathcal{U}^{\times}$ with $aba = a$.*
(vi) *The ring $\mathcal{U}$ is $*$-regular, i.e. it is a von Neumann regular ring in which $a^*a = 0$ implies $a = 0$.*

Proof. (i) The original definition of von Neumann regularity was in terms of elements: for every element x of the ring there exists an element y such that $xyx = x$. That this definition is equivalent to the definition given above and that (ii) holds for von Neumann regular rings can be found for example in [28, Lemma 4.15, Theorem 4.16, Theorem 9.15] and [32, Theorem 4.2.9]. We therefore show (v). Let $a = us$ be the polar decomposition with partial isometry u. Then $p = uu^*$ and $q = u^*u$ are Murray von Neumann equivalent projections. Since $\mathcal{A}$ is a finite von Neumann algebra there exists a partial isometry v with $1 - p = vv^*$ and $1 - q = v^*v$. Now $u^* + v^*$ is an isometry and in particular invertible. Moreover we have $(u^* + v^*)u = u^*u$ and with $b = t(u^* + v^*)$ we get

$$aba = ust(u^* + v^*)us = usts.$$

It remains to be found an invertible $t \in \mathcal{U}$ with $sts = s$. This can easily be achieved using the functional calculus. (iii) Let $b = us$ the polar decomposition. Replace s by the invertible selfadjoint operator $p + s$ with $p = 1 - u^*u$ and use the functional calculus to write $p + s$ as a fraction. The right Ore condition follows and its left handed version follows by using the antiautomorphism $* : \mathcal{U} \to \mathcal{U}$. (vi) Compare [21, Theorem 5.1.9]. □

Lemma 2.2. *Let $a \in \mathcal{U}$ be an operator. The following statements are equivalent.*

(i) *a is invertible in $\mathcal{U}$.*
(ii) *a is injective as an operator, i.e. $\ker(a : \mathrm{dom}(a) \to H) = 0$.*
(iii) *a has dense image, i.e. $\overline{\mathrm{im}(a)} = \overline{a(\mathrm{dom}(a))} = H$.*
(iv) *$l_a : \mathcal{U} \to \mathcal{U}$ given by $b \mapsto ab$ is an isomorphism of right $\mathcal{U}$-modules.*

If moreover $a \in \mathcal{A} \subset \mathcal{U}$ the above statements are also equivalent to:

(v) *a is a non-zerodivisor in $\mathcal{A}$.*
(vi) *$l_a : \mathcal{A} \to \mathcal{A}$, $b \mapsto ab$ is injective.*

Proof. (i) ⇔ (ii) ⇔ (iii): We only show (ii) ⇒ (i) and (iii) ⇒ (i). Let $a = us$ be the polar decomposition of a. Here u is a partial isometry and hence $p = u^*u$ and $q = uu^*$ are projections. We have

$$\overline{\mathrm{im}(a)} = \mathrm{im}(u) = \mathrm{im}(uu^*) = \mathrm{im}(q), \qquad \text{and}$$
$$\ker(a) = \ker(u) = \ker(u^*u) = \ker(p).$$

Now if $\overline{\mathrm{im}(a)} = H$, then $q = \mathrm{id}$ and $0 = \mathrm{tr}_{\mathcal{A}}(\mathrm{id}) - \mathrm{tr}_{\mathcal{A}}(q) = \mathrm{tr}_{\mathcal{A}}(\mathrm{id}) - \mathrm{tr}_{\mathcal{A}}(p) = \mathrm{tr}_{\mathcal{A}}(\mathrm{id} - p)$. Since the trace is faithful this implies $p = \mathrm{id}$ and therefore u is a unitary operator and in particular invertible. In the case $\ker(a) = 0$ we argue similarly. Now since $\ker(s) = \ker(u) = 0$ we can use the functional calculus to define an inverse $f(s)u^*$ with $f(\lambda) = \frac{1}{\lambda}$ for $\lambda \neq 0$. (i) ⇔ (iv) is clear. (i) ⇒ (v) and (v) ⇒ (vi) are easy. It remains to show (vi) ⇒ (ii). Suppose the bounded operator $a \in \mathcal{A}$ has a nontrivial kernel. If $p_{\ker(a)}$ denotes the projection onto the kernel we have $ap_{\ker(a)} = 0$ and we see that l_a is not injective. □

Note that similar statements hold for matrices since $M_n(\mathcal{A})$ is again a finite von Neumann algebra and $M_n(\mathcal{U})$ is isomorphic to its algebra of affiliated operators.

3. Dimensions

The main aim of this section is to prove that there is a well-behaved notion of dimension for arbitrary $\mathcal{U}$-modules (Theorem 3.11), and that $-\otimes_{\mathcal{A}}\mathcal{U}$ induces an isomorphism in K_0 (Theorem 3.7). Essentially certain properties of the lattice of projections of a finite von Neumann algebra are responsible for these facts. Recall that a lattice is a partially ordered set $Latt$, where for any two elements $x, y \in Latt$ the greatest lower bound $\inf(\{x, y\})$ and the least upper bound $\sup(\{x, y\})$ exist. If the corresponding property holds also for arbitrary subsets instead of two-element subsets the lattice is said to be complete. For more on lattices we refer to [30, Chapter III].
We will consider the lattice $Latt_{Proj}(\mathcal{A})$ of projections in $\mathcal{A}$ with the order given by $p \leq q$ iff $qp = p$. This is a complete lattice ([31, Chapter V, Proposition 1.1]). Given a right R-module M_R one can consider the set $Latt_{ds}(M_R)$ of those submodules of M which are direct summands. Note that for R von Neumann regular and M_R finitely generated projective a submodule is a direct summand if and only if it is finitely generated.

Proposition 3.1. *Given a finite von Neumann algebra $\mathcal{A}$ and its algebra of affiliated operators $\mathcal{U}$, all partially ordered sets in the following commutative square are complete lattices, and all maps are order isomorphisms and therefore lattice isomorphisms.*

$$\begin{array}{ccc} Latt_{Proj}(\mathcal{A}) & \longrightarrow & Latt_{ds}(\mathcal{A}_{\mathcal{A}}) \\ \downarrow & & \downarrow \\ Latt_{Proj}(\mathcal{U}) & \longrightarrow & Latt_{ds}(\mathcal{U}_{\mathcal{U}}) \end{array}$$

The maps are given as follows:

$$\begin{array}{ll} Latt_{Proj}(\mathcal{A}) \to Latt_{Proj}(\mathcal{U}) & p \mapsto p \\ Latt_{Proj}(\mathcal{A}) \to Latt_{ds}(\mathcal{A}_{\mathcal{A}}) & p \mapsto p\mathcal{A} \\ Latt_{Proj}(\mathcal{A}) \to Latt_{ds}(\mathcal{U}_{\mathcal{U}}) & p \mapsto p\mathcal{U} \\ Latt_{ds}(\mathcal{A}_{\mathcal{A}}) \to Latt_{ds}(\mathcal{U}_{\mathcal{U}}) & I \mapsto I\mathcal{U}. \end{array}$$

Here $I\mathcal{U}$ is the right $\mathcal{U}$-module generated by I in $\mathcal{U}$.

Proof. Commutativity of the diagram is obvious. Once we have proven that all lattices are isomorphic completeness follows from the completeness of $Latt_{Proj}(\mathcal{A})$. In order to prove that a map is a lattice isomorphism it is sufficient to show that it is an order isomorphism. By commutativity of the square it is sufficient to deal only with three maps. Since projections in $\mathcal{U}$ are bounded operators they already lie in $\mathcal{A}$, so the lattices $Latt_{Proj}(\mathcal{A})$ and $Latt_{Proj}(\mathcal{U})$ coincide. Given a finitely generated right ideal $I_{\mathcal{U}}$ in the $*$-regular ring $\mathcal{U}$ there is a unique projection $p \in \mathcal{U}$ such that $p\mathcal{U} = I_{\mathcal{U}}$, compare [20, Part II, Chapter IV, Theorem 4.5]. This leads to the bijection $Latt_{Proj}(\mathcal{U}) \to Latt_{ds}(\mathcal{U}_{\mathcal{U}})$. If $p \leq q$, then multiplying $p\mathcal{U} \subset \mathcal{U}$ from the left by q leads to $p\mathcal{U} \subset q\mathcal{U}$. Note that in general an order preserving bijection need not be an order isomorphism. But of course if $p\mathcal{U} \subset q\mathcal{U}$, then multiplying from the left by $1 - q$ yields $p \leq q$. It remains to prove that $Latt_{Proj}(\mathcal{A}) \to Latt_{ds}(\mathcal{A}_{\mathcal{A}})$ is surjective; then injectivity follows from the commutativity of the diagram. So given a right ideal I in $\mathcal{A}$ which is a direct summand, there is an idempotent e such that $e\mathcal{A} = I$. We have to replace the idempotent by a projection. The following lemma finishes the proof. That the lemma applies can be found in [9, Theorem 2.7.8, page 158]. □

The following lemma is [3, Prop. 4.6.2].

Lemma 3.2. *In a $*$-ring R where every element of the form $1 + a^*a$ is invertible the following holds: given an idempotent e there always exists a projection p, such that $pR = eR$.*

Proof. Set $z = 1 - (e^* - e)^2 = 1 + (e^* - e)^*(e^* - e)$. Then $z = z^*$ and $ze = ez = ee^*e$ and also $z^{-1}e = ez^{-1}$. If we now set $p = ee^*z^{-1}$ then p is a projection and $pe = e$ and $ep = p$. This leads to $pR = eR$. □

We are not primarily interested in these lattices of submodules, but rather in the set of isomorphism classes of such modules. This is the first step in passing from embedded submodules to abstract finitely generated projective modules and then later to arbitrary modules. The point is that over a unit-regular ring isomorphism of submodules can be expressed in lattice theoretic terms.

Lemma 3.3. *Let R be a unit-regular ring and let R_R be the ring considered as a right R-module. Two finitely generated submodules L and M are isomorphic if and only if they have a common complement in R_R, i.e. a submodule N exists with $R_R = M \oplus N$ and $R_R = L \oplus N$.*

Proof. Compare Corollary 4.4 and Theorem 4.5 in [5]. □

We obtain the following refined information on the diagram in Proposition 3.1.

Proposition 3.4. *The lattice isomorphisms in Proposition 3.1 induce bijections of isomorphism classes, where isomorphism of projections $p \cong q$ means there exist elements x and y in $\mathcal{A}$ respectively $\mathcal{U}$ such that $p = xy$ and $q = yx$.*

Proof. Again we only have to deal with three of the four maps. The statement for the map $Latt_{ds}(\mathcal{A}_{\mathcal{A}}) \to Latt_{ds}(\mathcal{U}_{\mathcal{U}})$ will follow from the commutativity of the square. We begin with the maps $Latt_{Proj}(\mathcal{A}) \to Latt_{ds}(\mathcal{A}_{\mathcal{A}})$ and $Latt_{Proj}(\mathcal{U}) \to Latt_{ds}(\mathcal{U}_{\mathcal{U}})$. Let R be an arbitrary ring. If $p = xy$ and $q = yx$, then left multiplication by x respectively y yield mutually inverse homomorphisms between pR and qR. On the other hand given such mutually inverse homomorphisms the image of p respectively q under these homomorphisms are possible choices for x and y. Next we handle the map $Latt_{Proj}(\mathcal{A}) \to Latt_{Proj}(\mathcal{U})$. The only difficulty is to show that if p and q are isomorphic (alias algebraically equivalent) inside $\mathcal{U}$, then they are already isomorphic in $\mathcal{A}$. The converse is obviously true. From Lemma 3.3 above we know that isomorphic finitely generated ideals in $\mathcal{U}$ have a common complement. We have thus expressed isomorphism in lattice theoretic terms. Since we already know from Proposition 3.1 that the map is a lattice isomorphism, p and q have a common complement in $Latt_{Proj}(\mathcal{A})$. The following lemma, which is due to Kaplansky, finishes the proof for the map

$Latt_{Proj}(\mathcal{A}) \to Latt_{Proj}(\mathcal{U})$ since partial isometries are bounded and therefore in $\mathcal{A}$. The lemma also tells us that projections p and q in $\mathcal{A}$ are isomorphic or algebraically equivalent if and only if they are Murray von Neumann equivalent. □

Lemma 3.5. *If two projections p and q in a von Neumann algebra $\mathcal{A}$ have a common complement in the lattice of projections $Latt_{Proj}(\mathcal{A})$, then they are already Murray von Neumann equivalent, i.e. there is a partial isometry $u \in \mathcal{A}$ such that $p = u^*u$ and $q = uu^*$.*

Proof. See [11, Theorem 6.6(b)]. There it is proven more generally for AW^*-algebras. □

Since $M_n(\mathcal{A})$ is again a finite von Neumann algebra and its algebra of affiliated operators is isomorphic to $M_n(\mathcal{U})$ one can apply the above results to matrix algebras.

Corollary 3.6. *There is a commutative diagram of complete lattices and lattice isomorphisms, where all the maps are compatible with the different notions of isomorphism for the elements of the lattices.*

$$\begin{array}{ccccc}
Latt_{Proj}(M_n(\mathcal{A})) & \longrightarrow & Latt_{ds}(M_n(\mathcal{A})_{M_n(\mathcal{A})}) & \longrightarrow & Latt_{ds}(\mathcal{A}^n_{\mathcal{A}}) \\
\downarrow & & \downarrow & & \downarrow \\
Latt_{Proj}(M_n(\mathcal{U})) & \longrightarrow & Latt_{ds}(M_n(\mathcal{U})_{M_n(\mathcal{U})}) & \longrightarrow & Latt_{ds}(\mathcal{U}^n_{\mathcal{U}})
\end{array}$$

There are stabilization maps

$$\begin{array}{rcl}
Latt_{Proj}(M_n(\mathcal{A})) & \to & Latt_{Proj}(M_{n+1}(\mathcal{A})) \\
 & \cdots & \\
Latt_{ds}(\mathcal{U}^n_{\mathcal{U}}) & \to & Latt_{ds}(\mathcal{U}^{n+1}_{\mathcal{U}})
\end{array}$$

and these maps are compatible with the above lattice isomorphisms and the different notions of isomorphism for the elements of the lattices.

Proof. The map $Latt_{ds}(M_n(\mathcal{A})_{M_n(\mathcal{A})}) \to Latt_{ds}(\mathcal{A}^n_{\mathcal{A}})$ is given by $- \otimes_{M_n(\mathcal{A})} \mathcal{A}^n_{\mathcal{A}}$ followed by the map induced from the natural isomorphism of right $\mathcal{A}$-modules $M_n(\mathcal{A}) \otimes_{M_n(\mathcal{A})} \mathcal{A}^n_{\mathcal{A}} \cong \mathcal{A}^n_{\mathcal{A}}$. Morita equivalence tells us that $- \otimes_{\mathcal{A}} \mathcal{A}^n_{M_n(\mathcal{A})}$ followed by a natural isomorphism is an inverse of this map. The same argument applies to $\mathcal{U}$. The vertical map on the right is given by mapping a submodule $M \subset \mathcal{A}^n$ to the $\mathcal{U}$-module it generates inside $\mathcal{U}^n$. Since the diagram commutes, this map is also a lattice isomorphism. □

An immediate consequence of the above is the following.

Theorem 3.7. *The functor $- \otimes_{\mathcal{A}} \mathcal{U}$ induces an isomorphism of the monoids of isomorphism classes of finitely generated projective modules. In particular the natural map*

$$K_0(\mathcal{A}) \to K_0(\mathcal{U})$$

is an isomorphism.

Proof. The maps $Latt_{ds}(\mathcal{A}^n_{\mathcal{A}}) \to Latt_{ds}(\mathcal{U}^n_{\mathcal{U}})$ are compatible with isomorphism, stabilization and direct sums. □

We now apply the standard procedure to obtain a dimension function. Fix a faithful normal trace $\mathrm{tr}_{\mathcal{A}} : \mathcal{A} \to \mathbb{C}$. Extending the trace to matrices by summing over the diagonal yields maps $\dim : Latt_{Proj}(\mathrm{M}_n(\mathcal{A})) \to \mathbb{R} \quad p \mapsto \mathrm{tr}_{\mathcal{A}}(p)$ which are compatible with the stabilization maps. Of course we normalize the traces such that $\mathrm{tr}_{\mathrm{M}_n(\mathcal{A})}(1_n) = n$. Because of the trace property dim is well-defined on isomorphism classes of projections. Given a finitely generated projective module over $\mathcal{A}$ or $\mathcal{U}$ there is always an isomorphic module M which is a direct summand in $\mathcal{A}^n$ respectively $\mathcal{U}^n$ for some $n \in \mathbb{N}$. Sending these modules through the diagram of 3.6 to their corresponding projections in $\mathrm{M}_n(\mathcal{A})$ and taking the trace gives a real number: the dimension $\dim_{\mathcal{U}}(M)$ of M. For reference purposes we summarize its properties in the following proposition.

Proposition 3.8. *Let M be a finitely generated projective $\mathcal{U}$-module.*

(i) $\dim_{\mathcal{U}}(M)$ *depends only on the isomorphism class of M.*
(ii) $\dim_{\mathcal{U}}(M \oplus N) = \dim_{\mathcal{U}}(M) + \dim_{\mathcal{U}}(N)$.
(iii) $\dim_{\mathcal{U}}(M \otimes_{\mathcal{A}} \mathcal{U}) = \dim_{\mathcal{A}}(M)$ *if M is a finitely generated projective $\mathcal{A}$-module.*
(iv) $M = 0$ *if and only if* $\dim_{\mathcal{U}}(M) = 0$.

Proof. (i) and (iii) follow immediately from 3.6. Up to isomorphism and stabilization a direct sum of modules corresponds to the block diagonal sum of projections, this yields (ii). Faithfulness of the trace implies (iv). □

So far we have not used the fact that the lattices are complete. We will see that this will enable us to extend the notion of dimension to arbitrary $\mathcal{U}$-modules. The following definition is completely analoguous to the definition of the dimension for $\mathcal{A}$-modules given in [14].

Definition 3.9. Let M be an arbitrary $\mathcal{U}$-module. Define $\dim'_{\mathcal{U}}(M) \in [0, \infty]$ as

$$\dim'_{\mathcal{U}}(M) = \sup\{\dim_{\mathcal{U}}(P) \,|\, P \subset M, P \text{ fin gen. projective submodule}\}.$$

The next lemma is the main technical point in proving that this dimension is well-behaved. It uses the completeness of the lattices. If K is a submodule of the finitely generated projective module M we define

$$\overline{K} = \bigcap_{K \subset Q \subset M} Q \quad \subset \quad M,$$

where the intersection is over all finitely generated submodules Q of M, which contain K.

Lemma 3.10. *Let K be a submodule of $\mathcal{U}^n$. Since the lattice $Latt_{ds}(\mathcal{U}^n_{\mathcal{U}})$ is complete the supremum of the set $\{P \,|\, P \subset K,\, P$ finitely generated$\}$ exists.*

(i) *We have* $\overline{K} = \sup\{P \,|\, P \subset K,\, P \text{ finitely generated}\}$ *and this module is finitely generated and therefore projective.*
(ii) *We have* $\dim'_{\mathcal{U}}(K) = \dim_{\mathcal{U}}(\overline{K})$.

Proof. (i) Let $\{P_i \,|\, i \in I\}$ be the system of finitely generated submodules of K and $\{Q_j \,|\, j \in J\}$ be the system of finitely generated modules containing K. Since every element of K generates a finitely generated submodule of K we know that $K \subset \sup\{P_i \,|\, i \in I\}$. Since the lattice is complete $\sup\{P_i \,|\, i \in$

$I\}$ is one of the finitely generated modules containing K in the definition of $\overline{K}$. We get $\overline{K} \subset \sup\{P_i \,|\, i \in I\}$. Since $P_i \subset Q_j$ for i, j arbitrary it follows that $\sup\{P_i \,|\, i \in I\} \subset Q_j$ for all $j \in J$ and therefore $\sup\{P_i \,|\, i \in I\} \subset \overline{K}$.
(ii) From (i) we know that $\overline{K}$ is finitely generated projective. Let p be the projection corresponding to $\overline{K}$ and p_i be those corresponding to the P_i, then p is the limit of the increasing net p_i and normality of the trace implies the result. □

Let $\dim_{\mathcal{A}}$ be the dimension for $\mathcal{A}$-modules considered in [14].

Theorem 3.11. *The dimension* $\dim'_{\mathcal{U}}$ *defined above has the following properties.*

(i) *Invariance under isomorphisms:* $\dim'_{\mathcal{U}}(M)$ *depends only on the isomorphism class of* M*.*
(ii) *Extension: If* Q *is a finitely generated projective module, then*
$$\dim'_{\mathcal{U}}(Q) = \dim_{\mathcal{U}}(Q).$$
(iii) *Additivity: Given an exact sequence of modules*
$$0 \to M_0 \to M_1 \to M_2 \to 0$$
we have
$$\dim'_{\mathcal{U}}(M_1) = \dim'_{\mathcal{U}}(M_0) + \dim'_{\mathcal{U}}(M_2).$$
(iv) *Cofinality: Let* $M = \bigcup_{i \in I} M_i$ *be a directed union of submodules (i.e. given* $i, j \in I$ *there always exists a* $k \in I$*, such that* $M_i, M_j \subset M_k$*) then*
$$\dim'_{\mathcal{U}}(M) = \sup\{\dim'_{\mathcal{U}}(M_i) \,|\, i \in I\}.$$

These four properties determine $\dim'_{\mathcal{U}}$ *uniquely. Moreover the following holds.*

(v) *If* M *is an* $\mathcal{A}$*-module, then* $\dim'_{\mathcal{U}}(M \otimes_{\mathcal{A}} \mathcal{U}) = \dim_{\mathcal{A}}(M)$*.*
(vi) *If* M *is finitely generated projective, then* $\dim'_{\mathcal{U}}(M) = 0$ *if and only if* $M = 0$*.*
(vii) *Monotony:* $M \subset N$ *implies* $\dim'_{\mathcal{U}}(M) \leq \dim'_{\mathcal{U}}(N)$*.*

Notation 3.12. After having established the proof we will write $\dim_{\mathcal{U}}$ instead of $\dim'_{\mathcal{U}}$. This is justified by (ii).

Proof. (i) Invariance under isomorphisms: This follows from the definition and the corresponding property 3.8 (i) for finitely generated projective modules. (ii) Extension property: Let P be a finitely generated projective submodule of the finitely generated projective module Q, then since $\mathcal{U}$ is von Neumann regular P is a direct summand of Q. The additivity for finitely generated projective modules 3.8 (ii) implies that $\dim_{\mathcal{U}}(P) \leq \dim_{\mathcal{U}}(Q)$. The claim follows. (iv) Cofinality: Let $M = \bigcup_{i \in I} M_i$ be a directed union. It is obvious from the definition that $\dim'_{\mathcal{U}}(M_i) \leq \dim'_{\mathcal{U}}(M)$ and therefore $\sup\{\dim'_{\mathcal{U}}(M_i) \,|\, i \in I\} \leq \dim'_{\mathcal{U}}(M)$. Let now $P \subset M$ be finitely generated projective. Since the system is directed there is an $i \in I$ such that M_i contains all generators of P and therefore P itself. It follows that $\dim_{\mathcal{U}}(P) \leq \dim'_{\mathcal{U}}(M_i) \leq \sup\{\dim'_{\mathcal{U}}(M_i) \,|\, i \in I\}$ and finally
$$\sup\{\dim_{\mathcal{U}}(P) \,|\, P \subset M, P \text{ fin. gen. projective}\} \leq \sup\{\dim'_{\mathcal{U}}(M_i) \,|\, i \in I\}.$$

(vii) Monotony: $M \subset N$ implies $\dim'_{\mathcal{U}}(M) \leq \dim'_{\mathcal{U}}(N)$, since on the left one has to take the supremum over a smaller set of numbers. (iii) Additivity: The proof is completely analoguous to the proof of additivity for the dimension in [14]. (vi) Was already proven in 3.8.
Let us now prove uniqueness. This is done in several steps. Step1: The extension property determines $\dim'_{\mathcal{U}}$ uniquely on finitely generated projective modules. Step2: Let now $K \subset Q$ be a submodule of a finitely generated projective module. The module K is the directed union of its finitely generated submodules $K = \bigcup_{i \in I} K_i$. Since $\mathcal{U}$ is von Neumann regular the K_i are projective (semihereditary would be sufficient here). Now $\dim'_{\mathcal{U}}(K)$ is uniquely determined by cofinality and Step1. Step3: If M is finitely generated there is an exact sequence $0 \to K \to \mathcal{U}^n \to M \to 0$. Additivity together with Step2 implies the result for finitely generated modules. Step4: An arbitrary module is the directed union of its finitely generated submodules and again one applies cofinality. (v) The proof follows the same pattern as the proof of uniqueness. Note that it is shown in [14] that the dimension $\dim_{\mathcal{A}}$ for $\mathcal{A}$-modules also has the properties (i) to (iv). Step1: For finitely generated projective $\mathcal{A}$-modules this is the content of 3.8(iii). Step2: A submodule K of a finitely generated projective $\mathcal{A}$-module is the directed union of its finitely generated submodules K_i which are projective since $\mathcal{A}$ is semihereditary. Since $- \otimes_{\mathcal{A}} \mathcal{U}$ is exact and commutes with colimits $K \otimes_{\mathcal{A}} \mathcal{U}$ is the directed union of the $K_i \otimes_{\mathcal{A}} \mathcal{U}$. Now apply cofinality of $\dim_{\mathcal{A}}$ and $\dim'_{\mathcal{U}}$ and use Step1. Step3: For a finitely generated module M applying $- \otimes_{\mathcal{A}} \mathcal{U}$ to the exact sequence $0 \to K \to \mathcal{A}^n \to M \to 0$ yields an exact sequence. Now use Step2 and the additivity of $\dim_{\mathcal{A}}$ respectively $\dim'_{\mathcal{U}}$. Step4: An arbitrary module is the directed union of its finitely generated submodules. Proceed as in Step2 and use Step3. □

4. Torsion modules

Thinking of $\mathcal{U}$ as a localization of $\mathcal{A}$ it is no surprise that on the one hand we lose information by passing to $\mathcal{U}$-modules, but on the other hand $\mathcal{U}$-modules have better properties. For example, every finitely presented $\mathcal{U}$-module is finitely generated projective and the category of finitely generated projective $\mathcal{U}$-modules is abelian. We will now investigate this passage systematically.

Definition 4.1. For an $\mathcal{A}$-module M we define its torsion submodule $\mathbf{t}M$ as

$$\mathbf{t}M = \ker(M \to M \otimes_{\mathcal{A}} \mathcal{U}).$$

A module M is called a **torsion module** if $M \otimes_{\mathcal{A}} \mathcal{U} = 0$ or equivalently $\mathbf{t}M = M$. A module is called **torsionfree** if $\mathbf{t}M = 0$.

This is consistent with the terminology for example in [30, Chapter II, page 57] because $\mathcal{U}$ is isomorphic to the classical ring of fractions of $\mathcal{A}$. An element $m \in M$ lies in $\mathbf{t}M$ if and only if it is a torsion element in the following sense: there exists a non-zerodivisor $s \in \mathcal{A}$, such that $ms = 0$. Compare [30, Chapter II, Corollary 3.3]. The module $M/\mathbf{t}M$ is torsionfree since $\overline{m}s = \overline{ms} = 0 \in M/\mathbf{t}M$ implies the existence of s' with $mss' = 0$ and therefore $m \in \mathbf{t}M$.
On the other hand, following [14, page 146] we make the following definition.

Definition 4.2. Let M be an $\mathcal{A}$-module, then

$$\mathbf{T}M = \bigcup N,$$

where the union is over all $N \subset M$ with $\dim_{\mathcal{A}}(N) = \dim_{\mathcal{U}}(N \otimes_{\mathcal{A}} \mathcal{U}) = 0$. We denote by $\mathbf{P}M$ the cokernel of the inclusion $\mathbf{T}M \subset M$.

This is indeed a submodule because for two submodules N, $N' \subset M$ with $\dim_{\mathcal{A}}(N) = \dim_{\mathcal{A}}(N') = 0$ the additivity of the dimension together with

$$N + N'/N \cong N'/N' \cap N$$

implies $\dim_{\mathcal{A}}(N + N') = 0$. Note that $\dim_{\mathcal{A}}(\mathbf{T}M) = 0$ by cofinality and $\mathbf{T}M$ is the largest submodule with vanishing dimension.

Finitely generated projective modules. We have seen in 3.7 that isomorphism classes of finitely generated projective $\mathcal{A}$- and $\mathcal{U}$-modules are in bijective correspondence via $- \otimes_{\mathcal{A}} \mathcal{U}$, and by adding a suitable complement one verifies that the natural map $M \to M \otimes_{\mathcal{A}} \mathcal{U}$ is injective. A finitely generated projective module over $\mathcal{A}$ or $\mathcal{U}$ is trivial if and only if its dimension vanishes by 3.8. Therefore $\mathbf{t}M = \mathbf{T}M = 0$ for finitely generated projective $\mathcal{A}$-modules.

Finitely presented modules. The category of finitely presented $\mathcal{A}$-modules was investigated in [13]. Since the ring $\mathcal{A}$ is semihereditary it is an abelian category.

Proposition 4.3. *Let M be a finitely presented $\mathcal{A}$-module. Then*

(i) *$M \otimes_{\mathcal{A}} \mathcal{U}$ is finitely generated projective.*
(ii) *$\mathbf{T}M = \mathbf{t}M$.*
(iii) *M is a torsion module if and only if $\dim_{\mathcal{A}}(M) = 0$.*
(iv) *$\mathbf{P}M = M/\mathbf{T}M$ is projective and $M \cong \mathbf{P}M \oplus \mathbf{T}M$.*
(v) *Under the isomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{U})$ the class $[\mathbf{P}M]$ corresponds to $[M \otimes_{\mathcal{A}} \mathcal{U}]$.*

Proof. (i) By right exactness of the tensor product $M \otimes_{\mathcal{A}} \mathcal{U}$ is finitely presented. Over the von Neumann regular ring $\mathcal{U}$ this implies being finitely generated projective. (iv) is proven in [13]. Now $M \otimes_{\mathcal{A}} \mathcal{U} \cong \mathbf{P}M \otimes_{\mathcal{A}} \mathcal{U} \oplus \mathbf{T}M \otimes_{\mathcal{A}} \mathcal{U} \cong \mathbf{P}M$ because $\mathbf{T}M \cong \operatorname{coker}(f)$ for some weak isomorphism $f : \mathcal{A}^n \to \mathcal{A}^n$ and by 2.2 and right exactness of the tensor product we get $\mathbf{T}M \otimes_{\mathcal{A}} \mathcal{U} \cong \operatorname{coker}(f) \otimes_{\mathcal{A}} \mathcal{U} \cong \operatorname{coker}(f \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{U}}) \cong 0$. The rest follows. □

Arbitrary modules. In general $\mathbf{t}M$ and $\mathbf{T}M$ differ. Counterexamples can already be realized by finitely generated modules. More precisely the following holds.

Proposition 4.4. *Let M be an $\mathcal{A}$-module.*

(i) *$\mathbf{t}M \subset \mathbf{T}M$ and torsion modules have vanishing dimensions.*
(ii) *There exists a finitely generated $\mathcal{A}$-module with $\mathbf{t}M = 0$ and $\mathbf{T}M = M$. For this module $M \otimes_{\mathcal{A}} \mathcal{U} \neq 0$ but $\dim_{\mathcal{A}}(M) = \dim_{\mathcal{U}}(M \otimes_{\mathcal{A}} \mathcal{U}) = 0$.*
(iii) *M is a torsion module if and only if it is the directed union of quotients of finitely presented torsion modules.*
(iv) *If M is finitely generated then $\mathbf{P}M = M/\mathbf{T}M$ is projective and therefore $M \cong \mathbf{P}M \oplus \mathbf{T}M$.*

Proof. (i) It suffices to show that $\dim_{\mathcal{A}}(\mathbf{t}M) = \dim_{\mathcal{U}}(\mathbf{t}M \otimes_{\mathcal{A}} \mathcal{U}) = 0$. Now $\mathbf{t}M$ consists of torsion elements and therefore $\mathbf{t}M \otimes_{\mathcal{A}} \mathcal{U} = 0$. (ii) We will give an example of such a module below. (iv) was proven in [14]. Let us prove (iii): A quotient of a torsion module is a torsion module, and a directed union of torsion modules is again a torsion module. On the other hand suppose M is a torsion module. $M = \bigcup_{i \in I} M_i$ is the directed union of its finitely generated submodules. Since any submodule of a torsion module is a torsion module it remains to be shown that a finitely generated torsion module N is always a quotient of a finitely presented torsion module. Choose a surjection $p : \mathcal{A}^n \to N$ and let $K = \ker(p)$ be the kernel. Since $N \cong \mathcal{A}^n/K$ is a torsion module, for every $a \in \mathcal{A}^n$ there exists a non-zerodivisor $s \in \mathcal{A}$ such that $as \in K$. Let $e_i = (0, \ldots, 1, \ldots, 0)^{tr}$ be the standard basis for $\mathcal{A}^n$ and choose s_i with $e_i s_i \in K$. Note that $\operatorname{diag}(0, \ldots, s_i, \ldots, 0)e_i = e_i s_i$, where $\operatorname{diag}(b_1, \ldots, b_n)$ denotes the diagonal matrix with the corresponding entries. Since K is a right $\mathcal{A}$-module we have for an arbitrary vector $(a_1, \ldots, a_n)^{tr} \in \mathcal{A}^n$ that

$$\operatorname{diag}(s_1, \ldots, s_n)(a_1, \ldots, a_n)^{tr} = \sum e_i s_i a_i \in K.$$

Let S denote the right linear map $\mathcal{A}^n \to \mathcal{A}^n$ corresponding to the diagonal matrix $\operatorname{diag}(s_1, \ldots, s_n)$, then $\operatorname{im}(S) \subset K$ and $N \cong \mathcal{A}^n/K$ is a quotient of the finitely presented module $\mathcal{A}^n/\operatorname{im}(S)$. Moreover $\mathcal{A}^n/\operatorname{im}(S) \cong \bigoplus \mathcal{A}/s_i\mathcal{A}$ is a torsion module by Proposition 4.3(iii) and 2.2. □

Note 4.5. In [16, Definition 2.1] we defined a module to be cofinal-measurable if all its finitely generated submodules are quotients of finitely presented zero-dimensional modules. By Proposition 4.3(iii) and 4.4(iii) we see that a module is cofinal-measurable if and only if it is a torsion module.

Example 4.6. We will now give a counterexample to finish the proof of the above proposition. Let I_λ, $\lambda \in \Lambda$ be a directed family of right ideals in $\mathcal{A}$ such that each I_λ is a direct summand, $\dim_{\mathcal{A}}(I_\lambda) < 1$ and $\sup_{\lambda \in \Lambda}(\dim_{\mathcal{A}}(I_\lambda)) = 1$. Note that $I = \bigcup_{\lambda \in \Lambda} I_\lambda \neq \mathcal{A}$, because $1 \in I_\lambda$ for some λ would contradict $\dim_{\mathcal{A}}(I_\lambda) < 1$. Since I_λ is a direct summand $\mathcal{A}/I_\lambda \to (\mathcal{A}/I_\lambda) \otimes_{\mathcal{A}} \mathcal{U}$ is injective. Using that $- \otimes_{\mathcal{A}} \mathcal{U}$ is exact and commutes with colimits one verifies that

$$\mathcal{A}/I \to (\mathcal{A}/I) \otimes_{\mathcal{A}} \mathcal{U} \cong \mathcal{A} \otimes_{\mathcal{A}} \mathcal{U}/I \otimes_{\mathcal{A}} \mathcal{U} \cong \mathcal{A} \otimes_{\mathcal{A}} \mathcal{U}/\cup_\lambda (I_\lambda \otimes_{\mathcal{A}} \mathcal{U})$$

is injective as well. Therefore $\mathbf{t}(\mathcal{A}/I) = 0$. On the other hand additivity and cofinality of the dimension imply $\dim_{\mathcal{A}}(\mathcal{A}/I) = \dim_{\mathcal{U}}((\mathcal{A}/I) \otimes_{\mathcal{A}} \mathcal{U}) = 0$. Here is a concrete example where such a situation arises: Take $\mathcal{A} = \mathrm{L}^\infty(\mathrm{S}^1, \mu)$, the essentially bounded functions on the unit circle with respect to the normalized Haar measure μ on S^1. Let X_i, $i \in \mathbb{N}$ be an increasing sequence of measurable subsets of S^1 such that $\mu(X_i) < 1$ and $\sup_{i \in \mathbb{N}}(\mu(X_i)) = 1$. The corresponding characteristic functions χ_{X_i} generate ideals in $\mathcal{A}$ with the desired properties, since $\dim_{\mathcal{A}}(\chi_{X_i}\mathcal{A}) = \mu(X_i)$.

5. L^2-Invariants

We will now briefly summarize how the notions developed above apply in order to provide alternative descriptions of L^2-invariants. Recall the following definitions from [14, Section 4]. Let Γ be a group. Given an arbitrary

Γ-space X one defines the singular homology of X with twisted coefficients in the $\mathbb{Z}\Gamma$-module $\mathcal{N}\Gamma$ as

$$H_p^\Gamma(X;\mathcal{N}\Gamma) = H_p(C_*^{sing}(X) \otimes_{\mathbb{Z}\Gamma} \mathcal{N}\Gamma),$$

where $C_*^{sing}(X)$ is the singular chain complex of X considered as a complex of right $\mathbb{Z}\Gamma$-modules. Here $H_p^\Gamma(X;\mathcal{N}\Gamma)$ is still a right $\mathcal{N}\Gamma$-module and therefore

$$b_p^{(2)}(X) = \dim_{\mathcal{A}}(H_p^\Gamma(X;\mathcal{N}\Gamma))$$

makes sense. This definition extends the definition of L^2-Betti numbers via Hilbert $\mathcal{N}\Gamma$-modules for regular coverings of CW-complexes of finite type to arbitrary Γ-spaces. Completely analogous we define

$$H_p^\Gamma(X;\mathcal{U}\Gamma) = H_p(C_*^{sing}(X) \otimes_{\mathbb{Z}\Gamma} \mathcal{U}\Gamma).$$

Proposition 5.1. *Let X be an arbitrary Γ-space.*

(i) $H_p^\Gamma(X;\mathcal{N}\Gamma) \otimes_{\mathcal{N}\Gamma} \mathcal{U}\Gamma \cong H_p^\Gamma(X;\mathcal{U}\Gamma)$ *as $\mathcal{U}\Gamma$-modules.*

(ii) $b_p^{(2)}(X) = \dim_{\mathcal{U}\Gamma}(H_p^\Gamma(X;\mathcal{U}\Gamma))$.

(iii) $\mathbf{t}H_p^\Gamma(X;\mathcal{N}\Gamma) = \ker(H_p^\Gamma(X;\mathcal{N}\Gamma) \to H_p^\Gamma(X;\mathcal{U}\Gamma))$.

If X is a regular covering of a CW-complex of finite type, then

(iv) *The module $H_p^\Gamma(X;\mathcal{N}\Gamma)$ is finitely presented, the module $H_p^\Gamma(X;\mathcal{U}\Gamma)$ is finitely generated projective and under the isomorphism $K_0(\mathcal{N}\Gamma) \to K_0(\mathcal{U}\Gamma)$ the class $\left[\mathbf{P}H_p^\Gamma(X;\mathcal{N}\Gamma)\right]$ corresponds to $\left[H_p^\Gamma(X;\mathcal{U}\Gamma)\right]$.*

(v) $\mathbf{t}H_p^\Gamma(X;\mathcal{N}\Gamma) = \mathbf{T}H_p^\Gamma(X;\mathcal{N}\Gamma) = \ker(H_p^\Gamma(X;\mathcal{N}\Gamma) \to H_p^\Gamma(X;\mathcal{U}\Gamma))$.

Proof. Since $-\otimes_{\mathcal{N}\Gamma}\mathcal{U}\Gamma$ is exact it commutes with homology. If X is a regular covering of finite type the singular chain complex is quasi-isomorphic to the cellular chain complex which consists of finitely generated free $\mathbb{Z}\Gamma$-modules. Since finitely presented $\mathcal{N}\Gamma$-modules form an abelian category the homology modules $H_p^\Gamma(X;\mathcal{N}\Gamma)$ are again finitely presented. The rest follows from the preceding results about $\mathbf{t}$, $\mathbf{T}$ and the dimension. □

As far as dimension is concerned one can therefore work with $\mathcal{U}$-modules. Tensoring with $\mathcal{U}$ is the algebraic analogue of the passage from unreduced to reduced L^2-homology in the Hilbert space set-up. If one is interested in finer invariants like the Novikov-Shubin invariants the passage to $\mathcal{U}$-modules is too harsh. The torsion submodule $\mathbf{t}H_p^\Gamma(X;\mathcal{N}\Gamma)$ carries the information of the Novikov-Shubin invariants in case X is a regular covering of a CW-complex of finite type and seems to be the right candidate to carry similar information in general. Compare [16] and Note 4.5.

6. K-Theory

In this section we will determine the algebraic K_0 and K_1 groups of finite von Neumann algebras and their algebras of affiliated operators. Every von Neumann algebra can be decomposed into a direct sum of algebras of type I_{fin}, I_∞, II_1, II_∞ and III, compare [31, Chapter V, Theorem 1.19]. Since we are considering finite von Neumann algebras only the types I_{fin} and II_1 can occur. A von Neumann algebra of type I_n is isomorphic to $\mathrm{M}_n(\mathrm{L}^\infty(X;\mu))$, where $\mathrm{L}^\infty(X;\mu)$ is the algebra of essentially bounded complex valued functions on some compact measure space (X,μ) with finite measure. The algebra of affiliated operators in this case is $\mathrm{M}_n(\mathrm{L}(X;\mu))$. Here $\mathrm{L}(X;\mu)$ denotes the algebra of all measurable functions on X. A general algebra of type I_{fin} is of the form

$$\tilde{\prod}_{n=1}^{\infty}\mathrm{M}_n(\mathrm{L}^\infty(X_n;\mu_n)).$$

Here the tilde denotes the restricted product consisting of all sequences of operators which have a uniform bound on their norm. Using the characterization as an Ore localization one shows that the corresponding algebra of affiliated operators is given by the (unrestricted) product algebra

$$\prod_{n=1}^{\infty}\mathrm{M}_n(\mathrm{L}(X_n;\mu_n)).$$

The group von Neumann algebra of a finitely generated group is of type I_{fin} if the group is virtually abelian and of type II_1 otherwise. If the group is not finitely generated mixed types can occur. These facts are discussed in [8].

Let us start with K_0. As usual we fix a normalized trace $\mathrm{tr}_{\mathcal{A}}$ on $\mathcal{A}$. We denote the center of $\mathcal{A}$ by $\mathrm{Z}(\mathcal{A})$. The center valued trace for $\mathcal{A}$ is a linear map

$$\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}:\mathcal{A}\to\mathrm{Z}(\mathcal{A}),$$

which is uniquely determined by $\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(ab)=\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(ba)$ for all $a,b\in\mathcal{A}$ and $\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(c)=c$ for all $c\in\mathrm{Z}(\mathcal{A})$, see [10, Proposition 8.2.8, p.517]. Our fixed trace $\mathrm{tr}_{\mathcal{A}}:\mathcal{A}\to\mathbb{R}$ factorizes over the center valued trace, i.e. we have $\mathrm{tr}_{\mathcal{A}}(a)=\mathrm{tr}_{\mathcal{A}}(\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(a))$ for all $a\in\mathcal{A}$, compare [10, Proposition 8.3.10, p.525].

Proposition 6.1. *Let $\mathcal{A}$ be a finite von Neumann algebra with center $\mathrm{Z}(\mathcal{A})$ and let $\mathcal{U}$ be the associated algebra of affiliated operators. Let $\mathrm{Z}(\mathcal{A})_{sa}$ denote the vector space of selfadjoint elements in $\mathrm{Z}(\mathcal{A})$ and let $\mathrm{Z}(\mathcal{A})_{pos}$ denote the cone of nonnegative elements. Let $\mathrm{Proj}(\mathcal{A})$ denote the monoid of isomorphism classes of finitely generated projective $\mathcal{A}$-modules.*

(i) *The natural map $\mathcal{A}\to\mathcal{U}$ induces an isomorphism $K_0(\mathcal{A})\cong K_0(\mathcal{U})$.*
(ii) *There is a commutative diagram*

$$\begin{array}{ccccc} \mathrm{Proj}(\mathcal{A}) & \longrightarrow & K_0(\mathcal{A}) & & \\ \downarrow & & \downarrow{\scriptstyle \dim_{\mathrm{Z}(\mathcal{A})}} & \searrow^{\dim_{\mathcal{A}}} & \\ \mathrm{Z}(\mathcal{A})_{pos} & \longrightarrow & \mathrm{Z}(\mathcal{A})_{sa} & \longrightarrow & \mathbb{R}. \end{array}$$

The map $\dim_{\mathrm{Z}(\mathcal{A})}$ is induced by the center valued trace. All maps in the square are injective.
(iii) *If $\mathcal{A}$ is of type II_1, then $\dim_{\mathrm{Z}(\mathcal{A})}$ is an isomorphism.*
(iv) *If $\mathcal{A}=\tilde{\prod}_{n=1}^{\infty}\mathrm{M}_n(\mathrm{L}^\infty(X_n;\mu_n))$ is of type I_{fin}, then the image of $\dim_{\mathrm{Z}(\mathcal{A})}$ is given by*

$$\tilde{\prod}_{n=1}^{\infty}\mathrm{L}^\infty(X_n;\frac{1}{n}\mathbb{Z};\mu_n),$$

where $\mathrm{L}^\infty(X_n; \frac{1}{n}\mathbb{Z}; \mu_n)$ *denotes the essentially bounded functions with values in* $\frac{1}{n}\mathbb{Z}$.

Proof. (i) This has already been proven in Theorem 3.7. (ii) In Section 3 we normalized the traces $\mathrm{tr}_{\mathrm{M}_n(\mathcal{A})}$ such that $\mathrm{tr}_{\mathrm{M}_n(\mathcal{A})}(1_n) = n$. We extend the center valued trace to matrices by summing over the diagonal. This gives maps $\mathrm{tr}_n : \mathrm{M}_n(\mathcal{A}) \to \mathrm{Z}(\mathcal{A})$ which are compatible with the stabilization maps $\mathrm{M}_n(\mathcal{A}) \to \mathrm{M}_{n+1}(\mathcal{A})$. Since $\frac{1}{n}\mathrm{tr}_n$ has the characteristic properties of a center valued trace if we identify $\mathrm{Z}(\mathcal{A})$ with $\mathrm{Z}(\mathrm{M}_n(\mathcal{A}))$ we have $\mathrm{tr}_n = n \cdot \mathrm{tr}_{\mathrm{Z}(\mathrm{M}_n(\mathcal{A}))}$. Two projections p and q are Murray von Neumann equivalent and therefore isomorphic (compare Proposition 3.4) if and only if $\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(p) = \mathrm{tr}_{\mathrm{Z}(\mathcal{A})}(q)$. The center valued trace of a projection is nonnegative and selfadjoint, see [10, Theorem 8.4.3, p.532]. Therefore we get injective maps

$$Latt_{Proj}(\mathrm{M}_n(\mathcal{A}))/_{\cong} \to \mathrm{Z}(\mathcal{A})_{pos}$$

which are compatible with the stabilization maps. This gives an injective map $\mathrm{Proj}(\mathcal{A}) \to \mathrm{Z}(\mathcal{A})_{pos}$. Since the block sum of projections corresponds to an ordinary sum of traces we see that it is a map of monoids and that $\mathrm{Proj}(\mathcal{A})$ satisfies cancellation. Therefore the map $\mathrm{Proj}(\mathcal{A}) \to K_0(\mathcal{A})$ is injective. This implies that also the induced map $\dim_{\mathrm{Z}(\mathcal{A})} : K_0(\mathcal{A}) \to \mathrm{Z}(\mathcal{A})_{sa}$ is injective. (iii) By [10, Theorem 8.4.4 (i), p.533] we know that the image of the center valued trace in the case of a type II_1 algebra is the set

$$\{a \in \mathrm{Z}(\mathcal{A}) \,|\, a \text{ selfadjoint, nonnegative, } |a| \leq 1\}.$$

Because of the factor n in tr_n we get the result. (iv) Note that the product of the individual center valued traces restricted to the restricted product satisfies the characteristic properties of a center valued trace. The result follows from [10, Theorem 8.4.4 (ii), p.533]. □

In [17] the algebraic K_1-groups of von Neumann algebras are determined. Of course they depend on the type of the von Neumann algebra. Moreover, the authors compute a modified K_1-group $K_1^w(\mathcal{A})$ which is defined in terms of injective endomorphisms between finitely generated free modules ([17, Def. 1.1]). Here w stands for weak isomorphism, compare Lemma 2.2. It turns out that the group $K_1^w(\mathcal{A})$ admits a very natural interpretation.

Proposition 6.2. *There is a natural isomorphism* $K_1^w(\mathcal{A}) \cong K_1(\mathcal{U})$.

Proof. If we apply Lemma 2.2 to matrix algebras we see that an endomorphism f between finitely generated free $\mathcal{A}$-modules is injective if and only if $f \otimes \mathrm{id}_{\mathcal{U}}$ is an isomorphism. Therefore $f \mapsto [f \otimes \mathrm{id}_{\mathcal{U}}]$ gives a well-defined map since the relations analogous to [17, Def. 1.1 (i)–(iii)] hold in $K_1(\mathcal{U})$. To define an inverse we change the point of view and consider $K_1(\mathcal{U})$ as $\mathrm{GL}(\mathcal{U})_{ab}$. Let C be an invertible matrix over $\mathcal{U}$. Let $T \subset \mathcal{A}$ be the set of non-zerodivisors in $\mathcal{A}$. From Proposition 2.1 we know that $\mathcal{U} = \mathcal{A}T^{-1}$. It follows that $M_n(\mathcal{U}) = M_n(\mathcal{A})(T \cdot 1_n)^{-1}$ is also an Ore localization. Therefore we can find a matrix A over $\mathcal{A}$ and $s \in \mathcal{A}$ such that $C = As^{-1}1_n$. Note that A and $s1_n$ have to be injective endomorphisms because they become invertible over $\mathcal{U}$. One uses the Ore condition and the relation [17, Def. 1.1 (ii)] to show that $C \mapsto [A] - [s1_n]$ is well-defined. Since $K_1^w(\mathcal{A})$ is abelian the map factorizes over $K_1(\mathcal{U})$. That the maps are mutually inverse follows from the relations (i)–(iii) in [17, Def 1.1]. □

An invertible operator $a \in \mathrm{GL}_n(\mathcal{A})$ has a gap in the spectrum near zero, therefore we can define the Fuglede-Kadison determinant via the functional calculus as

$$\det_{FK}(a) = \exp\left(\frac{1}{2}\mathrm{tr}_{\mathrm{Z}(\mathcal{A})}\left(\log(a^*a)\right)\right).$$

Let $\mathcal{A}$ be of type I_n. Define a bijection $\eta_n : \mathrm{Z}(\mathcal{A}) \to \mathrm{Z}(\mathcal{A})$ by $a = us \mapsto us^{\frac{1}{n}}$, where $a = us$ is the polar decomposition with u the partial isometry. Define the normalized determinant

$$\det_{norm} : \mathrm{M}_k(\mathrm{M}_n(\mathrm{Z}(\mathcal{A}))) = \mathrm{M}_{kn}(\mathrm{Z}(\mathcal{A})) \to \mathrm{Z}(\mathcal{A})$$

as the composition $\eta_n \circ \det$, where $\det$ is the ordinary determinant. For a general type I_{fin} algebra let $\det_{norm}$ be the product of the normalized determinants. The normalization is necessary to assure that the determinant respects the restricted product.
The results from [17] can now be rephrased as follows.

Proposition 6.3. *Let $\mathcal{A}$ be a finite von Neumann algebra and let $\mathcal{U}$ be the associated algebra of affiliated operators.*

(i) *If $\mathcal{A}$ is of type II_1, then $K_1(\mathcal{U}) = 0$ and the Fuglede Kadison determinant gives an isomorphism*

$$\det_{FK} : K_1(\mathcal{A}) \xrightarrow{\cong} \mathrm{Z}(\mathcal{A})^{\times}_{pos},$$

where $\mathrm{Z}(\mathcal{A})^{\times}_{pos}$ denotes the group of positive invertible elements in the center of $\mathcal{A}$.

(ii) *If $\mathcal{A}$ is of type I_{fin}, then we have the following commutative diagram*

$$\begin{array}{ccc} K_1(\mathcal{A}) & \longrightarrow & K_1(\mathcal{U}) \\ \downarrow{\scriptstyle \det_{norm}} & & \downarrow{\scriptstyle \det_{norm}} \\ \mathrm{Z}(\mathcal{A})^{\times} & \longrightarrow & \mathrm{Z}(\mathcal{U})^{\times}, \end{array}$$

where the vertical maps are isomorphisms.

Note that

$$\mathrm{Z}(\tilde{\prod}_{n=1}^{\infty}\mathrm{M}_n(\mathrm{L}^{\infty}(X_n;\mu_n))) = \tilde{\prod}_{n=1}^{\infty}\mathrm{L}^{\infty}(X_n;\mu_n)$$

and

$$\mathrm{Z}(\prod_{n=1}^{\infty}\mathrm{M}_n(\mathrm{L}(X_n;\mu_n))) = \prod_{n=1}^{\infty}\mathrm{L}(X_n;\mu_n).$$

Proof. Almost everything follows from [17]. It remains only to remark that the normalized determinant extends to the algebra of affiliated operators and that K_1 as well as the normalized determinant is compatible with products. □

We should also mention that for $R = \mathcal{A}$ or $\mathcal{U}$ the natural map $(R^\times)_{ab} \to K_1(R)$ induced by the map which sends an invertible element to the corresponding 1×1-matrix is an isomorphism. For $\mathcal{A}$ this is a special case of [7, Theorem 7], since a finite von Neumann algebra is a finite AW^*-algebra. Since $\mathcal{U}$ is a unit-regular ring the statement for $\mathcal{U}$ follows from [18] and [6]. Localizations yield exact sequences in K-theory. See for example [1], [2] or [29]. We will use such a sequence to compute K_0 of the category of finitely presented torsion $\mathcal{A}$-modules.

Definition 6.4. Let $\mathcal{T}$ denote the category of finitely presented torsion $\mathcal{A}$-modules equipped with the standard exact structure where exact sequences are not required to be split.

Note that since $\mathcal{A}$ is semihereditary such modules admit a one-dimensional resolution by finitely generated projective modules. There is an exact sequence of algebraic K-groups

$$K_1(\mathcal{A}) \to K_1(\mathcal{U}) \to K_0(\mathcal{T}) \to K_0(\mathcal{A}) \to K_0(\mathcal{U}).$$

If we apply this to a finite von Neumann algebra $\mathcal{A}$ we can compute K_0 of the category of finitely presented $\mathcal{A}$-torsion modules.

Proposition 6.5. *Let $\mathcal{A}$ be a finite von Neumann algebra and let $\mathcal{U}$ be the associated algebra of affiliated operators.*

(i) *If $\mathcal{A}$ is of type II_1, then $K_0(\mathcal{T}) = 0$.*
(ii) *If $\mathcal{A} = \tilde{\prod} \mathrm{M}_n(\mathrm{L}^\infty(X_n;\mu_n))$ is of type I_{fin}, then*

$$K_0(\mathcal{T}) = \mathrm{Z}(\mathcal{U})^\times / \mathrm{Z}(\mathcal{A})^\times = \prod \mathrm{L}(X_n;\mu_n)^\times / (\tilde{\prod} \mathrm{L}^\infty(X_n;\mu_n))^\times.$$

Proof. Since $K_0(\mathcal{A}) \to K_0(\mathcal{U})$ is an isomorphism by 3.7 this follows immediately from the above results about K_1 in 6.3. □

There is an alternative description of the group $K_0(\mathcal{T})$ where elements are represented by finite complexes of finitely generated projective $\mathcal{A}$-modules that become acyclic after localization (compare [33]). Note that any Γ-CW-complex X with compact quotient space, whose isotropy groups are all finite, gives rise to a finite dimensional complex of finitely generated projective $\mathcal{N}\Gamma$-modules. If all its L^2-Betti numbers vanish it gives rise to an element in the corresponding $K_0(\mathcal{T})$. Short: every L^2-acyclic space naturally defines an element in $K_0(\mathcal{T})$.

7. L-Theory

We now determine the L-theory of a finite von Neumann algebra $\mathcal{A}$ and its algebra of affiliated operators $\mathcal{U}$. Recall from [24] that an element in the n-th symmetric L-theory group is represented by a symmetric algebraic Poincaré complex (SAPC). Two such are identified if they are homotopic or their difference is homotopic to the boundary of an $n+1$ dimensional symmetric algebraic complex (which need not be a Poincaré complex). Similar for quadratic L-theory. Since in our case $\frac{1}{2}$ is contained in the ring an n-dimensional SAPC is a complex C_* together with a self-dual chain map $\phi_0 : C^{n-*} \to C_*$. Working with complexes of finitely generated projective modules leads to projective L-theory $L_p^*(R)$. Working with free modules

we get $L^*_h(R)$. If additionally the modules are based and we require ϕ_0 to have trivial torsion in $\tilde{K}_1(R) = K_1(R)/\langle \pm 1 \rangle$ we denote the corresponding L-group by $L^*_s(R)$.
Let $\phi : P \to P^*$ be a symmetric form representing an element in $L^0_p(\mathcal{A})$. Choose a projection $\overline{p} \in \mathrm{M}_n(\mathcal{A})$ with $P = \mathrm{im}(\overline{p} : \mathcal{A}^n \to \mathcal{A}^n)$ let $p : \mathcal{A}^n \to P$ be the corresponding projection and define $\Phi = ip^*\phi p$, where $i : (\mathcal{A}^n)^* \to \mathcal{A}^n$ is the standard isomorphism. Sending ϕ to $\left[\chi_{(-\infty,0)}(\Phi)\right] - \left[\chi_{(0,\infty)}(\Phi)\right]$ using the functional calculus gives a map sign : $L^0_p(\mathcal{A}) \to K_0(\mathcal{A})$. It is proven in [27] that this map is an isomorphism. This is the starting point for the following results. To state the results it is convenient to decompose a type I_{fin} algebra further into $\mathcal{A} = \mathcal{A}_{even} \times \mathcal{A}_{odd}$, where $\mathcal{A}_{even}$ is the restricted product of all the I_n constituents with n even. Note that in general L_h and L_s are not compatible with products, whereas L_p is compatible with products.

Theorem 7.1. *Let $\mathcal{A}$ be a finite von Neumann algebra and $\mathcal{U}$ its algebra of affiliated operators.*
General results.

(i) *For $\mathcal{A}$ and $\mathcal{U}$ the symmetrization map from quadratic to symmetric L-theory is an isomorphism.*
(ii) *For both algebras L-theory is 2-periodic.*
(iii) *For decorations $\epsilon = p$ or h the natural map $L^*_\epsilon(\mathcal{A}) \to L^*_\epsilon(\mathcal{U})$ is an isomorphism. In particular all relative L-groups vanish.*

Results about L_p.

(iv) *All maps in the following commutative square are isomorphisms.*

$$\begin{array}{ccc} L^0_p(\mathcal{A}) & \xrightarrow{\text{sign}} & K_0(\mathcal{A}) \\ \downarrow & & \downarrow \\ L^0_p(\mathcal{U}) & \xrightarrow{\text{sign}} & K_0(\mathcal{U}) \end{array}$$

(v) *We have $L^1_p(\mathcal{A}) = L^1_p(\mathcal{U}) = 0$.*

Results about L_h.

(vi) *If $\mathcal{A}$ is of type II_1 then $L^1_h(\mathcal{A}) = L^1_h(\mathcal{U}) = 0$ and there is a short exact sequence*
$$0 \to \mathbb{Z}/2 \to L^0_h(\mathcal{A}) \to L^0_p(\mathcal{A}) \to 0$$
and an analogous sequence for $\mathcal{U}$.
(vii) *If $\mathcal{A}$ is of type I_{fin} then $L^1_h(\mathcal{A}) = L^1_h(\mathcal{U}) = 0$. If the $\mathcal{A}_{odd}$ part of $\mathcal{A}$ is nontrivial there is an exact sequence*
$$0 \longrightarrow L^0_h(\mathcal{A}) \longrightarrow L^0_p(\mathcal{A}) \xrightarrow{\overline{\text{sign}}} K_0(\mathcal{A})/\langle [\mathcal{A}], 2K_0(\mathcal{A})\rangle \longrightarrow 0.$$
otherwise there is an exact sequence
$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow L^0_h(\mathcal{A}) \longrightarrow L^0_p(\mathcal{A}) \xrightarrow{\overline{\text{sign}}} K_0(\mathcal{A})/2K_0(\mathcal{A}) \longrightarrow 0.$$
There are analogous sequences for $\mathcal{U}$. Remember that $K_0(\mathcal{A}) = K_0(\mathcal{U})$.

Results about L_s.

(ix) *If $\mathcal{A}$ is of type II_1 then $L^1_s(\mathcal{A}) = L^1_s(\mathcal{U}) = 0$ and all the maps in the commutative square*

$$\begin{array}{ccc} L^0_s(\mathcal{A}) & \longrightarrow & L^0_h(\mathcal{A}) \\ \downarrow & & \downarrow \\ L^0_s(\mathcal{U}) & \longrightarrow & L^0_h(\mathcal{U}) \end{array}$$

are isomorphisms.

(x) *If $\mathcal{A}$ is of type I_{even} then $L^1_s(\mathcal{A}) = L^1_s(\mathcal{U}) = 0$ and there is an exact sequence*

$$0 \longrightarrow L^0_s(\mathcal{A}) \longrightarrow L^0_h(\mathcal{A}) \xrightarrow{\overline{\det_{norm} \circ \tau}} \mathrm{Z}(\mathcal{A})^\times_{sa}/\mathrm{Z}(\mathcal{A})^\times_{pos} \longrightarrow 0.$$

There is an analogous sequence for $\mathcal{U}$. Moreover

$$\mathrm{Z}(\mathcal{A})^\times_{sa}/\mathrm{Z}(\mathcal{A})^\times_{pos} = \mathrm{Z}(\mathcal{U})^\times_{sa}/\mathrm{Z}(\mathcal{U})^\times_{pos} = \{f \in \mathrm{Z}(\mathcal{A}) | f^2 = 1\}$$

(xi) *If $\mathcal{A}$ is of type I_{fin} and the $\mathcal{A}_{odd}$ part of $\mathcal{A}$ is nontrivial, then $L^1_s(\mathcal{A}) = L^1_s(\mathcal{U}) = \mathbb{Z}/2$ and there is an exact sequence*

$$0 \longrightarrow L^0_s(\mathcal{A}) \longrightarrow L^0_h(\mathcal{A}) \xrightarrow{\overline{\det_{norm} \circ \tau}}$$
$$\mathrm{Z}(\mathcal{A})^\times_{sa}/\left\langle \mathrm{Z}(\mathcal{A})^\times_{pos}, \det_{norm}(-1)\right\rangle \longrightarrow 0.$$

There is an analogous sequence for $\mathcal{U}$.

Proof. General results: Since both algebras contain $\frac{1}{2}$ and $\sqrt{-1}$ (i) and (ii) follow from [23, Prop.3.3] resp. [23, 4.4]. (iii) will follow from the computations below.
About L_p: Since $\mathcal{A}$ is a C^*-algebra it is proven in [27, Theorem 1.6] that sign is an isomorphism. The map sending $[P]-[Q]$ to the form represented by $\mathrm{id}_P \oplus -\mathrm{id}_Q$ is an explicit inverse. Since there is a functional calculus for selfadjoint unbounded operators the signature map makes sense for $\mathcal{U}$ and the same argument can be applied. From the K-theory isomorphism 6.1(i) we get the square of isomorphisms.
The comparison result [27, Theorem 1.8] asserts that $L^p_1(\mathcal{A}) = K^{top}_1(\mathcal{A})$ because $\mathcal{A}$ is a C^*-algebra. It is well known that the topological K_1-group of a von Neumann algebra vanishes, see e.g. [3, Example 8.1.2]. Odd quadratic L-groups of semisimple rings vanish as is shown in [22]. Investigating the proof one realizes that it is actually sufficient that a finitely generated submodule of a projective module always splits as a direct summand. This is one possible characterization of a von Neumann regular ring and therefore $L^{odd}_p(\mathcal{U}) = 0$. Alternatively observe that over a von Neumann regular ring a chain complex is homotopic to its homology and an odd dimensional SAPC all whose differentials are trivial is the boundary of the SAC which is given by the lower half of the complex. Alltogether this implies the comparison result (iii) for $\epsilon = p$.

About L_h: For any ring R with involution there is a Rothenberg sequence

$$\cdots \longrightarrow L^1_h(R) \longrightarrow L^1_p(R) \longrightarrow \hat{H}^1(\tilde{K}_0(R))$$

$$\longrightarrow L^0_h(R) \longrightarrow L^0_p(R) \xrightarrow{l_0} \hat{H}^0(\tilde{K}_0(R)) \longrightarrow \cdots.$$

Compare Example 3.11 in [25]. Here $\hat{H}^*(M)$ is the $\mathbb{Z}/2$-Tate cohomology of the $\mathbb{Z}[\mathbb{Z}/2]$-module M. The sequence is natural in R. Therefore the comparison result for K_0 and L_p together with the five-lemma imply (iii) in the $\epsilon = h$ case.
(vi) The results about K_0 in 6.1 imply that the involution on $K_0(\mathcal{A})$ is trivial and $\frac{1}{2} \in K_0(\mathcal{A})$. Therefore $\hat{H}^*(K_0(\mathcal{A})) = 0$. The long exact sequence in Tate cohomology associated to $0 \to K_0(\mathbb{Z}) \to K_0(\mathcal{A}) \to \tilde{K}_0(\mathcal{A}) \to 0$ computes $\hat{H}^0(\tilde{K}_0(\mathcal{A})) = 0$ and $\hat{H}^1(\tilde{K}_0(\mathcal{A})) = \mathbb{Z}/2$ and the Rothenberg sequence above implies the result.
(vii) By 6.1 the involution on $K_0(\mathcal{A})$ is trivial and multiplication by 2 injective. Therefore the long exact sequence in Tate cohomology gives us

$$0 \longrightarrow \hat{H}^1(\tilde{K}_0(\mathcal{A})) \longrightarrow \mathbb{Z}/2 \xrightarrow{\eta}$$

$$K_0(\mathcal{A})/2K_0(\mathcal{A}) \xrightarrow{q} \hat{H}^0(\tilde{K}_0(\mathcal{A})) \longrightarrow 0.$$

Here η maps the generator to $[\mathcal{A}]$ and is injective if and only if $\mathcal{A}$ contains a nontrivial $\mathcal{A}_{odd}$ part.
The map l_0 in the Rothenberg sequence sends an SAPC (C, ϕ) to $[C]$ considered as an element in $\hat{H}^0(\tilde{K}_0(\mathcal{A}))$. Since $[P]-[Q] = [P]+[Q]$ in $K_0(\mathcal{A})/2K_0(\mathcal{A})$ we have a commutative square

$$\begin{array}{ccccc} \cdots \longrightarrow & L^0_p(\mathcal{A}) & \xrightarrow{l_0} & \hat{H}^0(\tilde{K}_0(\mathcal{A})) & \longrightarrow \cdots \\ & \cong \big\downarrow \text{sign} & & \big\uparrow q & \\ & K_0(\mathcal{A}) & \xrightarrow{p} & \hat{H}^0(K_0(\mathcal{A})) = K_0(\mathcal{A})/2K_0(\mathcal{A}). & \end{array}$$

We see that l_0 is surjective, hence $L^1_h(\mathcal{A}) = 0$, and $\eta = 0$ or $\eta \neq 0$ decides what type of exact sequence we get.
About L_s: There is a similar Rothenberg sequence relating L_h and L_s.

$$\cdots \longrightarrow L^1_s(R) \longrightarrow L^1_h(R) \longrightarrow \hat{H}^1(\tilde{K}_1(R))$$

$$\longrightarrow L^0_s(R) \longrightarrow L^0_h(R) \xrightarrow{l_0} \hat{H}^0(\tilde{K}_1(R)) \longrightarrow \cdots.$$

Here $\tilde{K}_1(R)$ is $K_1(R)$ modulo the natural image of $K_1(\mathbb{Z}) = \langle \pm 1 \rangle$.
(ix) From 6.3 we know that $K_1(\mathcal{U}) = 0$. The formula for the Fuglede-Kadison determinant tells us that the involution on $K_1(\mathcal{A}) = \mathrm{Z}(\mathcal{A})^{\times}_{pos}$ is trivial and $K_1(\mathcal{A}) = \tilde{K}_1(\mathcal{A})$. Since elements in $\mathrm{Z}(\mathcal{A})^{\times}_{pos}$ admit a unique square root we see that $\hat{H}^*(\tilde{K}_1(\mathcal{A})) = \hat{H}^*(\tilde{K}_1(\mathcal{U})) = 0$. The result follows from the Rothenberg sequence.
(x) We know that $K_1(\mathcal{A}) = \mathrm{Z}(\mathcal{A})^{\times}$ via the normalized determinant. The involution is the standard involution and therefore $\hat{H}^1(K_1(\mathcal{A})) = 0$ and

$\hat{H}^0(K_1(\mathcal{A})) = \mathrm{Z}(\mathcal{A})^{\times}_{sa}/\mathrm{Z}(\mathcal{A})^{\times}_{pos}$. If $\mathcal{A}$ is of type I_{even} then $\det_{norm}(-1) = 1$ and therefore $K_1(\mathcal{A}) = \tilde{K}_1(\mathcal{A})$. The map l_0 is surjective because every element in $\mathrm{Z}(\mathcal{A})^{\times}_{sa}$ admits a preimage in $\mathrm{GL}_1(\mathcal{A})^{\mathbb{Z}/2}$ under the normalized determinant. Such an element can be interpreted as an element in $L^0_h(\mathcal{A})$. The result follows from the Rothenberg sequence.
(xi) If the $\mathcal{A}_{odd}$ part is nontrivial then the long exact sequence in Tate cohomology tells us $\hat{H}^1(\tilde{K}_1(\mathcal{A})) = 0$ and gives us an exact sequence

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow \hat{H}^0(K_1(\mathcal{A})) \xrightarrow{r} \hat{H}^0(\tilde{K}_1(\mathcal{A})) \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

It follows from Example 3.11 in [25] that the map l_0 sends an SAPC (C, ϕ) to the torsion $-\tau(\phi_0)$ considered as an element in $\hat{H}^0(\tilde{K}_1(\mathcal{A}))$. Since we can assume that ϕ_0 is symmetric it follows that $\mathrm{im}(l_0) \subset \mathrm{im}(r)$. Since every selfadjoint element in $\mathrm{Z}(\mathcal{A})^{\times}$ admits a symmetric preimage in $\mathrm{GL}_1(\mathcal{A})$ under the normalized determinant we see that $\mathrm{im}(r) = \mathrm{im}(l_0)$ is surjective. The result now follows from the Rothenberg sequence. The proof for $\mathcal{U}$ is completely analogous. □

Remark 7.2. For integral group rings one usually defines $L^s_*(\mathbb{Z}\Gamma)$ requiring the torsion to vanish in the Whiteheadgroup $Wh(\Gamma) = K_1(\mathbb{Z}\Gamma)/\langle \pm\gamma \rangle$. If therefore one defines $L^S_n(\mathcal{N}\Gamma)$ by requiring the torsion to lie in the subgroup of $K_1(\mathcal{N}\Gamma)$ generated by the image of -1 and the group elements we get a map $L^s_n(\mathbb{Z}\Gamma) \to L^S_n(\mathcal{N}\Gamma)$. In the II_1-case, i.e. if Γ is finitely generated and not virtually abelian there is no difference between $L^S_n(\mathcal{N}\Gamma)$ and $L^s_n(\mathcal{N}\Gamma)$ because the Fuglede-Kadison determinant of a group-element vanishes, compare 6.3(i).

Version: March 13, 2001

References

[1] **Bass, Hyman**: "*Algebraic K-theory*", W. A. Benjamin, Inc., New York-Amsterdam (1968)

[2] **Berrick, A. J. and Keating, M. E.**: "*The localization sequence in K-theory*", *K*-Theory 9, 577–589 (1995)

[3] **Blackadar, Bruce**: "*K-theory for operator algebras*", Cambridge University Press, Cambridge, 2nd edn. (1998)

[4] **Cochran, T.D.and Orr, K.E. and Teichner, P.**: "*Knot Concordance, Whitney Towers and L^2-signatures*", http://www.lanl.gov/abs/math/9908117

[5] **Goodearl, K. R.**: "*von Neumann regular rings*", Pitman (Advanced Publishing Program), Boston, Mass. (1979)

[6] **Handelman, David**: "*K_1 of noncommutative von Neumann regular rings*", J. Pure Appl. Algebra 8, 105–118 (1976)

[7] **Handelman, David**: "*Stable range in AW* algebras*", Proc. Amer. Math. Soc. 76, 241–249 (1979)

[8] **de la Harpe, Pierre**: "*Operator algebras, free groups and other groups*", Astérisque 121–153, Recent advances in operator algebras (Orléans, 1992) (1995)

[9] **Kadison, Richard V. and Ringrose, John R.**: "*Fundamentals of the theory of operator algebras. Vol. I*", Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, Elementary theory (1983)

[10] **Kadison, Richard V. and Ringrose, John R.**: "*Fundamentals of the theory of operator algebras. Vol. II*", Academic Press Inc., Orlando, FL, Advanced theory (1986)

[11] **Kaplansky, Irving**: "*Projections in Banach algebras*", Ann. of Math. (2) 53, 235–249 (1951)

[12] **Linnell, Peter A.**: "*Division rings and group von Neumann algebras*", Forum Math. 5, 561–576 (1993)

[13] **Lück, Wolfgang**: "*Hilbert modules and modules over finite von Neumann algebras and applications to L^2-invariants*", Math. Ann. 309, 247–285 (1997)

[14] **Lück, Wolfgang**: "*Dimension theory of arbitrary modules over finite von Neumann algebras and L^2-Betti numbers. I. Foundations*", J. Reine Angew. Math. 495, 135–162 (1998)

[15] **Lück, Wolfgang**: "*L^2-invariants and K-theory*", to appear, Springer (2002)

[16] **Lück, Wolfgang, Reich, Holger, and Schick, Thomas**: "*Novikov-Shubin invariants for arbitrary group actions and their positivity*", in: *Tel Aviv Topology Conference: Rothenberg Festschrift (1998)*, 159–176, Amer. Math. Soc., Providence, RI (1999)

[17] **Lück, Wolfgang and Rørdam, Mikael**: "*Algebraic K-theory of von Neumann algebras*", K-Theory 7, 517–536 (1993)

[18] **Menal, Pere and Moncasi, Jaume**: "*K_1 of von Neumann regular rings*", J. Pure Appl. Algebra 33, 295–312 (1984)

[19] **Murray, F.J. and von Neumann, J.**: "*On Rings of Operators*", Ann. of Math. (2) 37, 116–229 (1936)

[20] **von Neumann, John**: "*Continuous geometry*", Princeton University Press, Princeton, N.J. (1960)

[21] **Pedersen, Gert K.**: "*Analysis now*", Springer-Verlag, New York (1989)

[22] **Ranicki, Andrew**: "*On the algebraic L-theory of semisimple rings*", J. Algebra 50, 242–243 (1978)

[23] **Ranicki, Andrew**: "*The algebraic theory of surgery. I. Foundations*", Proc. London Math. Soc. (3) 40, 87–192 (1980)

[24] **Ranicki, Andrew**: "*Exact sequences in the algebraic theory of surgery*", Princeton University Press, Princeton, N.J. (1981)

[25] **Ranicki, Andrew**: "*Algebraic L-theory and topological manifolds*", Cambridge University Press, Cambridge (1992)

[26] **Reich, Holger**: "*Group von Neumann algebras and related algebras*", Ph.D. thesis, University of Göttingen (1999)

[27] **Rosenberg, Jonathan**: "*Analytic Novikov for topologists*", in: *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, 338–372, Cambridge Univ. Press, Cambridge (1995)

[28] **Rotman, Joseph J.**: "*An introduction to homological algebra*", Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York (1979)

[29] **Schofield, A. H.**: "*Representation of rings over skew fields*", LMS Lecture Note Series Vol.92, Cambridge University Press, Cambridge (1985)

[30] **Stenström, Bo**: "*Rings of quotients*", Springer-Verlag, New York, Die Grundlehren der Mathematischen Wissenschaften, Band 217, An introduction to methods of ring theory (1975)

[31] **Takesaki, Masamichi**: "*Theory of operator algebras. I*", Springer-Verlag, New York (1979)

[32] **Weibel, Charles**: "*An introduction to homological algebra*", Cambridge University Press, Cambridge (1994)

[33] **Weibel, Charles and Yao, Dongyuan**: "*Localization for the K-theory of noncommutative rings*", in: *Algebraic K-theory, commutative algebra, and algebraic geometry (Santa Margherita Ligure, 1989)*, 219–230, Amer. Math. Soc., Providence, RI (1992)

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