COEFFICIENTS FOR
THE FARRELL-JONES CONJECTURE

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ABSTRACT. We introduce the Farrell-Jones Conjecture with coefficients in an additive category with G-action. This is a variant of the Farrell-Jones Conjecture about the algebraic K- or L-Theory of a group ring RG. It allows to treat twisted group rings and crossed product rings. The conjecture with coefficients is stronger than the original conjecture but it has better inheritance properties. Since known proofs using controlled algebra carry over to the set-up with coefficients we obtain new results about the original Farrell-Jones Conjecture. The conjecture with coefficients implies the fibered version of the Farrell-Jones Conjecture.

1. INTRODUCTION

The Farrell-Jones Conjecture predicts that the algebraic K- or L-theory of a group ring RG can be described in terms of the K- respectively L-theory of group rings RH, where H ranges over the family of virtually cyclic subgroups of G, compare [FJ93]. More formally the conjecture says that the assembly map

$$H_*^G(E_{\mathcal{F}}G; \mathbf{K}_R) \to K_*(RG),$$

which assembles $K_*(RG)$ from the pieces $K_*(RH)$, $H \in \mathcal{F}$, is an isomorphism if $\mathcal{F}$ is the family of virtually cyclic subgroups of G. Here $H_*^G(-; \mathbf{K}_R)$ is a G-equivariant homology theory and $E_{\mathcal{F}}G$ denotes the classifying space for the family of subgroups $\mathcal{F}$. A group is called virtually cyclic if it contains a cyclic subgroup of finite index. For more explanations see [LR05]. There is a similar formulation for L-theory.

The goal of this paper is to define, and prove in many cases, a "Farrell-Jones Conjecture with coefficients in an additive category with G-action". A precise formulation is given in Section 3 and makes essential use of Definition 2.1. The conjecture with coefficients is a generalization of the original conjecture. The special case where the additive category with G-action is the category of finitely generated free R-modules equipped with the trivial G-action corresponds to the usual Farrell-Jones Conjecture.

The reason for considering this sort of generalization of the Farrell-Jones Conjecture is twofold. First, the conjecture with coefficients has better inheritance properties.

Theorem 1.1. *The Farrell-Jones Conjecture with coefficients in an additive category with G-action passes to arbitrary subgroups and more generally it "pulls back" under arbitrary group homomorphisms, where of course the family needs to be pulled back as well. But even more is true: the injectivity and surjectivity part of the conjecture have these inheritance properties independently.*

For a precise statement see Conjecture 3.2, Corollary 4.3 and Theorem 4.5.

Second, we obtain as a special case of the conjecture with coefficients the correct conjecture for the K-theory of twisted group rings and more generally crossed product rings. For example let G operate through ring homomorphisms on the ring

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R, i.e. we have a group homomorphism $\alpha\colon G \to \mathrm{Aut}(R)$, and let $R_\alpha G$ denote the twisted group ring. Then of course one expects that $K_*(R_\alpha G)$ can be assembled from the $K_*(R_\alpha H)$, where H ranges over the virtually cyclic subgroups of G. Our conjecture makes this precise, compare Conjecture 6.18.

Recall that crossed product rings play an important role in Moody's Induction Theorem, see [Moo87] and Chapter 8 in [Pas89].

Of course all this is only useful if there are techniques which prove the more general conjecture. Many results on the Farrell-Jones Conjecture (without coefficients) use the concept of controlled algebra and the description of the assembly map as a "forget-control map". In Section 7 we extend these concepts to the case with coefficients and formulate a "forget-control version" of the Farrell-Jones Conjecture with Coefficients. By simply inspecting existing proofs in the literature, see Section 8, one obtains results about the Farrell-Jones Conjecture with Coefficients. Combined with the inheritance properties we obtain the following new results about the original Farrell-Jones Conjecture.

Corollary 1.2. *Let G be the fundamental group of a closed Riemannian manifold of strictly negative sectional curvature and let Γ be a subgroup of G. Then for every ring R the assembly map*

$$H_*^\Gamma(E_{\mathcal{VC}yc}\Gamma; \mathbf{K}_R) \to K_*(R\Gamma)$$

is an isomorphism.

This is an extension of the main result from [BR05]. It follows from Theorem 4.5, Remark 3.3 and Theorem 8.1.

Corollary 1.3. *Let Γ be a subgroup of a hyperbolic group in the sense of Gromov and let R be a ring. Then the assembly map*

$$H_*^\Gamma(E_{\mathcal{F}in}\Gamma; \mathbf{K}_R) \to K_*(R\Gamma)$$

is split injective.

This generalizes [Ros04] and uses the fact proven in [RS04] that hyperbolic groups in the sense of Gromov satisfy the assumptions of Theorem 8.2.

Corollary 1.4. *Let G be group of finite asymptotic dimension that admits a finite model for the classifying space BG. Let Γ be a subgroup of G. Then for every ring R the assembly map*

$$H_*^\Gamma(E\Gamma; \mathbf{K}_R) \to K_*(R\Gamma)$$

is split injective.

This is a generalization of [Bar03] and follows because of the inheritance properties immediately from Theorem 8.3.

In [FJ93] Farrell and Jones develop the Fibred Isomorphism Conjecture, a different generalization of the Farrell-Jones Conjecture, which also has better inheritance properties. The fibred version is however not so well adapted to proofs which use controlled algebra as opposed to controlled topology. The precise relationship between the Fibered Farrell-Jones Conjecture and the Farrell-Jones Conjecture with Coefficients is discussed in Remark 4.4.

In the context of topological K-theory of C^*-algebras there is an analog to the Farrell-Jones Conjecture with Coefficients 3.2, the Baum-Connes Conjecture with Coefficients. Of course the development of a Farrell-Jones Conjecture with Coefficients was motivated by this analogy. The reader should be warned that the Baum-Connes Conjecture with Coefficients is know to be wrong [HLS02]. At present it is not clear whether the Farrell-Jones Conjecture with Coefficients 3.2 fails for the groups considered in [HLS02].

We would like to thank David Rosenthal for helpful discussions on his paper [Ros04].

The paper is organized as follows:

2. The category $\mathcal{A} *_G T$

In the following we will consider additive categories $\mathcal{A}$ with a right G-action, i.e. to every group element g we assign an additive covariant functor $g^* \colon \mathcal{A} \to \mathcal{A}$, such that $1^* = \mathrm{id}$ and composition of functors (denoted $\circ$) relates to multiplication in the group via $g^* \circ h^* = (hg)^*$.

Definition 2.1. Let $\mathcal{A}$ be an additive category with a right G-action and let T be a left G-set. We define a new additive category denoted $\mathcal{A} *_G T$ as follows. An object A in $\mathcal{A} *_G T$ is a family

$$A = (A_t)_{t \in T}$$

of objects in $\mathcal{A}$ where we require that $\{t \in T \mid A_t \neq 0\}$ is a finite set. A morphism $\phi \colon A \to B$ is a collection of morphisms

$$\phi = (\phi_{g,t})_{(g,t) \in G \times T},$$

where

$$\phi_{g,t} \colon A_t \to g^*(B_{gt})$$

is a morphism in $\mathcal{A}$. We require that the set of pairs $(g,t) \in G \times T$ with $\phi_{g,t} \neq 0$ is finite. Addition of morphisms is defined componentwise. Composition of morphisms is defined as follows. Let $\phi = (\phi_{g,t}) \colon A \to B$ and $\phi' = (\phi'_{g,t}) \colon B \to C$ be given then the composition $\psi = \phi' \circ \phi \colon A \to C$ has components

$$(2.2) \qquad \psi_{g,t} = \sum_{\substack{h,k \in G \\ g = kh}} h^*(\phi'_{k,ht}) \circ \phi_{h,t}.$$

The reader could now pass immediately to Section 3 in order to see how the Farrell-Jones Conjecture with Coefficients is formulated.

Remark 2.3 (Naturality of $\mathcal{A} *_G T$)**.** The construction $\mathcal{A} *_G T$ is natural in $\mathcal{A}$, i.e. if $F \colon \mathcal{A} \to \mathcal{A}'$ is an additive functor which is equivariant with respect to the G-action then

$$(F *_G T(A))_t = F(A_t) \quad \text{and} \quad (F *_G T(\phi))_{g,t} = F(\phi_{g,t})$$

defines a functor $F *_G T \colon \mathcal{A} *_G T \to \mathcal{A}' *_G T$. If the functor F is an equivalence of categories, then $F *_G T$ is an equivalence of categories.

If $f \colon T \to T'$ is a G-equivariant map then

$$(\mathcal{A} *_G f(A))_{t'} = \oplus_{t \in f^{-1}(t')} A_t \quad \text{and} \quad (\mathcal{A} *_G f(\phi))_{g,t'} = \oplus_{t \in f^{-1}(t')} \phi_{g,t}$$

defines (almost) a functor $\mathcal{A} *_G f \colon \mathcal{A} *_G T \to \mathcal{A} *_G T'$. The minor problem, that this definition involves the choice of a direct sum, can be resolved by redefining an object in $\mathcal{A} *_G T$ to be an object A as before together with a choice of direct sum $\oplus_{s \in S} A_s$ for every subset S of T. We prefer to ignore this problem.

Example 2.4 (Trivial action). A ring R can be considered as a category with one object. Let $R_\oplus$ denote the additive category obtained from R by formally adding sums, compare [Mac71, Exercise 5 on p.194]. This is a small model for the category of finitely generated free R-modules. If we equip $R_\oplus$ with the trivial right G-action then there is a canonical identification

$$R_\oplus *_G T = R\mathcal{G}^G(T)_\oplus,$$

where the right hand side is constructed in [DL98, Section 2] and plays an important role in the construction of the assembly map by Davis and Lück. This identification is natural with respect to maps of G-sets $T \to T'$. Let us specialize to the case where $T = G/H$. Then the inclusion of the full subcategory consisting of objects $A = (A_{gH})$, with $A_{gH} = 0$ for $gH \neq eH$, induces an equivalence

$$(2.5) \qquad\qquad \mathcal{F}^f(RH) \simeq R_\oplus *_G G/H.$$

Here $\mathcal{F}^f(RH)$ denotes the category of finitely generated free RH-modules.

Example 2.6 (Twisted group rings). Suppose the group G acts via ring homomorphisms on R, i.e. we are given a group homomorphism $\alpha\colon G \to \mathrm{Aut}(R)$. Then the twisted group ring $R_\alpha G$ is defined to be RG as an abelian group with the twisted multiplication determined by $gr = \alpha(r)g$ for $r \in R$ and $g \in G$. There is a right G-operation defined on the category of R-modules, where $g^*M = \mathrm{res}_{\alpha(g)} M$, i.e. g^*M has the same underlying abelian group but the R-module structure is twisted by α, i.e. $r \cdot_{\mathrm{res}_{\alpha(g)} M} m = \alpha(g)(r)m$ for $r \in R$ and $m \in M$. Let $\mathcal{F}^f(R)$ denote a small model for the category of finitely generated free right R-modules. One can arrange that $\mathcal{F}^f(R)$ is closed under the G-operation. We show in Section 6 that there is an equivalence of categories

$$\mathcal{F}^f(R) *_G \mathrm{pt} \simeq \mathcal{F}^f(R_\alpha G)$$

and that more generally

$$\mathcal{F}^f(R) *_G G/H \simeq \mathcal{F}^f(R_{\alpha|_H} H).$$

Example 2.7 (Group extensions). Suppose K is a normal subgroup of Γ and let $p\colon \Gamma \to \Gamma/K = G$ denote the quotient homomorphism. If the group extension $1 \to K \to \Gamma \to G \to 1$ splits we can choose a group homomorphism $s\colon G \to \Gamma$ such that $p \circ s = \mathrm{id}$. If we define $\alpha(\gamma)\colon RK \to RK$ as conjugation with $s(\gamma)$ we see that the group ring $R\Gamma$ can be written as a twisted group ring $RK_\alpha G$, compare Example 2.6. If however the extension is non-split and s is only a set-theoretical section (with $s(1) = 1$) then the group ring $R\Gamma$ is a crossed product ring

$$R\Gamma = RK_{\alpha,\tau}G,$$

and no longer a twisted group ring, compare [Pas89] and Section 6 below. In particular $\gamma \mapsto \alpha(\gamma)$ no longer defines an action of $G = \Gamma/K$ on RK and correction terms, expressible in terms of the cocycle $\tau(\gamma, \gamma') = s(\gamma)s(\gamma')s(\gamma\gamma')^{-1}$ will appear.

The language developed above absorbs these extra difficulties. We will see that we can work with an honest action if we are working in the context of actions on additive categories. By (2.5) we have

$$\mathcal{F}^f(RK) \xrightarrow{\simeq} R_\oplus *_\Gamma \Gamma/K.$$

The category on the right should be thought of as a "fat" version of the category $\mathcal{F}^f(RK)$ of finitely generated free RK-modules, which has the advantage that it carries an honest naturally defined right Γ/K-action. Now applying $- *_{\Gamma/K} \mathrm{pt}$

should be thought of as forming a "fattened" twisted group ring, compare Example 2.6. In fact we have an additive equivalence

$$(R_\oplus *_\Gamma \Gamma/K) *_{\Gamma/K} \mathrm{pt} \xrightarrow{\cong} R_\oplus *_{\Gamma \times \Gamma/K} (\Gamma/K \times \mathrm{pt})$$
$$\xrightarrow{\cong} R_\oplus *_\Gamma \mathrm{pt} = R\Gamma_\oplus$$

by an application of Proposition 2.8(i) respectively (ii) below.

Let K and G be groups. If $\mathcal{A}$ is an additive category with right K-action and S is a K-G biset. Then $\mathcal{A} *_K S$ can be equipped with a right G-action as follows. If $A = (A_s)_{s \in S}$ is an object and $\phi = (\phi_{k,s})_{(k,s) \in K \times S}$ is a morphism in $\mathcal{A} *_K S$ then for $g \in G$ we define g^*A and $g^*\phi$ by

$$(g^*A)_s = A_{sg^{-1}} \quad \text{and} \quad (g^*\phi)_{k,s} = \phi_{k,sg^{-1}}.$$

Proposition 2.8. (i) *Let K and G be groups. Suppose $\mathcal{A}$ is an additive category with right K-action, let S be a K-G biset and let T be a left G-set. Then there is an additive isomorphism of additive categories*

$$(\mathcal{A} *_K S) *_G T \xrightarrow{\cong} \mathcal{A} *_{K \times G} (S \times T).$$

Here in order to form the category on the right hand side we let $K \times G$ act from the right on $\mathcal{A}$ via the projection to K, and from the left on the set $S \times T$ by $(k,g)(s,t) = (ksg^{-1}, gt)$ for $(k,g) \in K \times G$ and $(s,t) \in S \times T$.

(ii) *Let N be a normal subgroup of G. Let $\mathcal{A}$ be an additive category with a right G-action such that N acts trivially. Let T be a left G-set such that N operates freely. Then there is an additive functor which is an equivalence of categories*

$$\mathcal{A} *_G T \xrightarrow{\cong} \mathcal{A} *_{G/N} (N \backslash T).$$

(iii) *Let H be a subgroup of G and $\mathcal{A}$ be an additive category with right G-action. We denote by $\mathrm{res}_H \mathcal{A}$ the additive category $\mathcal{A}$ considered with the H action obtained by restriction. Then for an H-set T the map $T \to G \times_H T$ defined by $t \mapsto (1_G, t)$ induces an equivalence of additive categories*

$$(\mathrm{res}_H \mathcal{A}) *_H T \to \mathcal{A} *_G (G \times_H T).$$

Proof. (i) The functor $F \colon (\mathcal{A} *_K S) *_G T \to \mathcal{A} *_{K \times G} (S \times T)$ is given by

$$(F(A))_{(s,t)} = (A_t)_s \quad \text{and} \quad (F(\phi))_{(k,g)(s,t)} = (\phi_{g,t})_{k,s}.$$

Note that if $\phi \colon A \to B$ is a morphism in $(\mathcal{A} *_K S) *_G T$ then $\phi_{g,t} \colon A_t \to g^*(B_{gt})$ is a morphism in $\mathcal{A} *_K S$ and $(\phi_{g,t})_{k,s} \colon (A_t)_s \to k^*((g^*B_{gt})_{ks})$ is a morphism in $\mathcal{A}$. The target of this last morphism is

$$k^*((g^*B_{gt})_{ks}) = k^*((B_{gt})_{ksg^{-1}}) = k^*(F(B)_{(ksg^{-1},gt)}) = (k,g)^*(F(B)_{(k,g)(s,t)})$$

by definition of the G-action on $\mathcal{A} *_K S$ and the $K \times G$-action on $\mathcal{A}$ and $S \times T$. In particular this is a correct target for $F(\phi)_{(k,g)(s,t)}$. Clearly, F is an isomorphism of categories. To verify that F is indeed an additive functor is lengthy but straightforward.

(ii) Let $p \colon T \to N \backslash T$ denote the projection and consider $\mathcal{A} *_G p \colon \mathcal{A} *_G T \to \mathcal{A} *_G (N \backslash T)$, see Remark 2.3. Next we define an additive functor $F \colon \mathcal{A} *_G (N \backslash T) \to \mathcal{A} *_{G/N} (N \backslash T)$. For both these categories objects are given by sequences $(A_t)_{t \in T}$ of objects in $\mathcal{A}$ indexed by T and we define F as the identity on objects. Let $\phi \colon A \to B$ be a morphism in $\mathcal{A} *_G (N \backslash T)$. For $g \in G$, $n \in N$ and $t \in T$ $\phi_{gn,Nt}$ is a morphism

$$A_{Nt} \to (gn)^* B_{gnNt} = g^* B_{gNt}$$

and we define F on morphisms by

$$(F(\phi))_{gN,Nt} = \sum_{n \in N} \phi_{gn,Nt}.$$

Then the composition $F \circ (\mathcal{A} *_G p)$ is full and faithful and hence an equivalence of additive categories.

(iii) It is straight forward to check that this functor is full and faithful. $\qquad\square$

Let $\Phi \colon K \to G$ be a group homomorphism. For a given additive category $\mathcal{A}$ with right K-action we define $\mathrm{ind}_\Phi \mathcal{A}$, a category with right G-action, as

$$\mathrm{ind}_\Phi \mathcal{A} = \mathcal{A} *_K \mathrm{res}_\Phi G.$$

This is a special case of the construction discussed before Proposition 2.8. Here $\mathrm{res}_\Phi G$ denotes G considered as K-G biset via Φ. Our main motivation for Proposition 2.8 was the following corollary which will play a key role when we will study inheritance properties for Isomorphism Conjectures.

Corollary 2.9. *Let $\mathcal{A}$ be an additive category with a right K-action. For a group homomorphism $\Phi \colon K \to G$ and a G-set T there is an additive functor which is an equivalence of categories*

$$(\mathrm{ind}_\Phi \mathcal{A}) *_G T \xrightarrow{\cong} \mathcal{A} *_K (\mathrm{res}_\Phi T).$$

Proof. According to Proposition 2.8(i) and (ii) we have an isomorphism respectively an equivalence

$$(\mathcal{A} *_K \mathrm{res}_\Phi G) *_G T \xrightarrow{\cong} \mathcal{A} *_{K \times G} (G \times T) \xrightarrow{\cong} \mathcal{A} *_K G \times_G T = \mathcal{A} *_K \mathrm{res}_\Phi T.$$

$\qquad\square$

3. The Farrell-Jones Conjecture with coefficients

Let $\mathbb{K}^{-\infty} \colon \mathsf{Add\ Cat} \to \mathsf{Sp}$ be the functor that associates the non-connective K-theory spectrum to an additive category (using the split exact structure). This functor is constructed in [PW85]. See [BFJR04, Section 2.1 and 2.5] for a brief review of this functor and its properties. Let G be a group and $\mathrm{Or}G$ be the orbit category of G, whose objects are transitive G-sets of the form G/H and whose morphisms are G-equivariant maps. For any $\mathrm{Or}G$-spectrum $\mathbf{E}$, i.e. for any functor $\mathbf{E} \colon \mathrm{Or}G \to \mathsf{Sp}$, Davis and Lück construct a G-equivariant homology theory for G-CW-complexes by

$$H_*^G(X; \mathbf{E}) = \pi_*(X_+ \underset{\mathrm{Or}G}{\wedge} \mathbf{E}),$$

where $X_+ \wedge_{\mathrm{Or}G} \mathbf{E}$ denotes the balanced smash product of $X_+ = \mathrm{map}_G(?, X_+)$ considered as a contravariant $\mathrm{Or}G$-space and the covariant $\mathrm{Or}G$-spectrum $\mathbf{E}$. For more details see [DL98, Section 4]. For a group G and a family $\mathcal{F}$ of subgroups, i.e. a collection of subgroups that is closed under subconjugation, there is a G-CW-complex $E_\mathcal{F}G$ with the property that for a subgroup H of G the set of fixed points $E_\mathcal{F}G^H$ is empty if $H \notin \mathcal{F}$ and contractible if $H \in \mathcal{F}$, see for example [Lüc04]. The triple $(\mathbf{E}, \mathcal{F}, G)$ is said to satisfy the Isomorphism Conjecture if the so called assembly map

$$H_*^G(E_\mathcal{F}G; \mathbf{E}) \to H_*^G(\mathrm{pt}; \mathbf{E}) = \pi_*(\mathbf{E}(G/G))$$

induced by the projection $E_\mathcal{F}G \to \mathrm{pt}$ is an isomorphism, see [DL98, Definition 5.1]. In this paper we will use the following $\mathrm{Or}G$-spectra.

Definition 3.1. Let $\mathcal{A}$ be an additive category with right G-action. The $\mathrm{Or}G$-spectrum $\mathbf{K}_\mathcal{A}$ is defined by

$$\mathbf{K}_\mathcal{A}(T) = \mathbb{K}^{-\infty}(\mathcal{A} *_G T).$$

Conjecture 3.2 (Algebraic K-Theory Farrell-Jones-Conjecture with Coefficients)**.**
*Let G be a group and let $\mathcal{VC}yc$ be the family of virtually cyclic subgroups of G. Let
$\mathcal{A}$ be an additive category with a right G-action. Then the assembly map*

$$H_*^G(E_{\mathcal{VC}yc}G; \mathbf{K}_\mathcal{A}) \to H_*^G(pt; \mathbf{K}_\mathcal{A})$$

is an isomorphism.

Remark 3.3. If $\mathcal{A} = R_\oplus$ then by Example 2.4 $\mathbf{K}_\mathcal{A}$ can be identified with the func-
tor introduced in [DL98, Section 2]. In particular Conjecture 3.2 implies the original
conjecture by Farrell and Jones in [FJ93]. (The formulation of Davis and Lück has
been identified with the original formulation of Farrell and Jones in [HP04].)

4. Inheritance properties

By definition a family of subgroups of a group G is a collection of subgroups
closed under taking subgroups and conjugation. If $\Phi\colon K \to G$ is group homomor-
phism and $\mathcal{F}$ is a family of subgroups of G then we define a family of subgroups of
K by setting

$$\Phi^*\mathcal{F} = \{H \subset G \mid H \text{ is a subgroup of } K \text{ and } \Phi(H) \in \mathcal{F}\}.$$

Remark 4.1. The K-CW-complex $\operatorname{res}_\Phi E_\mathcal{F}G$ is a model for the classifying space
$E_{\Phi^*\mathcal{F}}K$, because it satisfies the characterizing property concerning the fixed point
sets.

Proposition 4.2. *Let $\Phi\colon K \to G$ be a group homomorphism. Let $\mathcal{A}$ be an additive
category with right K-action and let $\mathcal{F}$ be a family of subgoups of G. Then the
assembly map*

$$H_*^G(E_\mathcal{F}G; \mathbf{K}_{\operatorname{ind}_\Phi \mathcal{A}}) \to H_*^G(pt; \mathbf{K}_{\operatorname{ind}_\Phi \mathcal{A}})$$

is equivalent to the assembly map

$$H_*^K(E_{\Phi^*\mathcal{F}}K; \mathbf{K}_\mathcal{A}) \to H_*^K(pt; \mathbf{K}_\mathcal{A}).$$

Proof. Because of Corollary 2.9 and since $\mathbb{K}^{-\infty}$ preserves equivalences we have
equivalences of $\mathrm{Or}G$-spectra $\mathbf{K}_{\operatorname{ind}_\Phi \mathcal{A}} \xrightarrow{\cong} \mathbf{K}_\mathcal{A} \circ \operatorname{res}_\Phi$ and therefore for every G-space
X a natural isomorphism

$$H_*^G(X; \mathbf{K}_{\operatorname{ind}_\Phi \mathcal{A}}) \cong H_*^G(X; \mathbf{K}_\mathcal{A} \circ \operatorname{res}_\phi)$$

For a K-space Y define $\operatorname{ind}_\Phi Y$ to be the quotient of $G \times Y$ by the right K action
given by $(g,y)k = (g\Phi(k), k^{-1}y)$. For every G-space X there is an isomorphism

$$
\begin{aligned}
\operatorname{res}_\Phi X_+^? \;&=\; \operatorname{map}_K(?, \operatorname{res}_\Phi X)_+ \\
&\cong\; \operatorname{map}_G(\operatorname{ind}_\Phi(?), X)_+ \\
&\cong\; \operatorname{map}_G(??, X)_+ \underset{\mathrm{Or}G}{\wedge} \operatorname{map}_G(\operatorname{ind}_\Phi(?), ??)_+ \\
&=\; X_+^{??} \underset{\mathrm{Or}G}{\wedge} \operatorname{map}_G(\operatorname{ind}_\Phi(?), ??)_+
\end{aligned}
$$

of contravariant pointed $\mathrm{Or}K$-spaces. Here and in the next display ? denotes func-
toriality in $\mathrm{Or}K$ and ?? denotes functoriality in $\mathrm{Or}G$. For every covariant functor
$\mathbf{F}$ from K-sets to spectra there is an isomorphism of covariant $\mathrm{Or}G$-spectra

$$
\begin{aligned}
\operatorname{map}_G(\operatorname{ind}_\Phi(?), ??)_+ \underset{\mathrm{Or}K}{\wedge} \mathbf{F}(?) \;&\cong\; \operatorname{map}_K(?, \operatorname{res}_\Phi(??))_+ \underset{\mathrm{Or}K}{\wedge} \mathbf{F}(?) \\
&\cong\; \mathbf{F} \circ \operatorname{res}_\Phi(??).
\end{aligned}
$$

Combining these isomorphisms with associativity of balanced smash products we
obtain an isomorphism of spectra

$$\operatorname{res}_\Phi X_+ \underset{\mathrm{Or}K}{\wedge} \mathbf{F} \cong X_+ \underset{\mathrm{Or}G}{\wedge} \mathbf{F} \circ \operatorname{res}_\Phi$$

and in particular a natural isomorphism

$$H^K(\mathrm{res}_\phi X; \mathbf{K}_\mathcal{A}) \cong H^G(X; \mathbf{K}_\mathcal{A} \circ \mathrm{res}_\Phi).$$

It remains to observe that $\mathrm{res}_\Phi \, \mathrm{pt} = \mathrm{pt}$ and that $\mathrm{res}_\Phi E_\mathcal{F} G$ is a model for $E_{\Phi^* \mathcal{F}} K$.
$\square$

Proposition 4.2 has the following immediate consequence.

Corollary 4.3. *Let $\Phi \colon K \to G$ be a group homomorphism. Let $\mathcal{F}$ be a family of subgroups of G. Suppose that for every additive category $\mathcal{A}$ with G-action the assembly map*

$$H_*^G(E_\mathcal{F} G; \mathbf{K}_\mathcal{A}) \to H_*^G(pt; \mathbf{K}_\mathcal{A})$$

is injective. Then for every additive category $\mathcal{B}$ with K-action the assembly map

$$H_*^K(E_{\Phi^* \mathcal{F}} K; \mathbf{K}_\mathcal{B}) \to H_*^K(pt; \mathbf{K}_\mathcal{B})$$

is injective. The same statement holds with injectivity replaced by surjectivity in assumption and conclusion.

Remark 4.4 (With coefficients is stronger than fibered)**.** The fibered version of the Farrell-Jones Conjecture in algebraic K-theory for a group G (and a ring R), [FJ93, Section 1.7] can be formulated as follows: for every group homeomorphism $\Phi \colon K \to G$ the assembly map $H_*^K(E_{\Phi^* \mathcal{VC}_{yc}} K; \mathbf{K}_R) \to H_*^K(pt; \mathbf{K}_R)$ is an isomorphism, see Section 6 and in particular Remark 6.6 in [BL04]. Therefore by Corollary 4.3 the Farrell-Jones Conjecture with Coefficients 3.2 implies the Fibered Farrell-Jones Conjecture.

Corollary 4.3 implies in particular the following theorem about the Farrell-Jones Conjecture with Coefficients 3.2.

Theorem 4.5. *Let H be a subgroup of G. Suppose that for every additive category $\mathcal{A}$ with G-action the assembly map*

$$H_*^G(E_{\mathcal{VC}yc} G; \mathbf{K}_\mathcal{A}) \to H_*^G(pt; \mathbf{K}_\mathcal{A})$$

is injective or surjective respectively. Then for every additive category $\mathcal{B}$ with right H-action the assembly map

$$H_*^H(E_{\mathcal{VC}yc} H; \mathbf{K}_\mathcal{B}) \to H_*^H(pt; \mathbf{K}_\mathcal{B})$$

is injective or surjective respectively. In particular, if the Farrell-Jones Conjecture with Coefficients 3.2 holds for a group G, then it holds for every subgroup of G.

Similar as for rings there exists a suspension category and hence results that hold without a condition on the coefficient category can always be shifted down. More precisely the following holds.

Proposition 4.6. *For every additive category $\mathcal{A}$ with G-action there is an additive category $\Sigma\mathcal{A}$ with G-action such that for every family of subgroups $\mathcal{F}$ and every $n \in \mathbb{Z}$ the assembly map*

$$H_n^G(E_\mathcal{F} G; \mathbf{K}_\mathcal{A}) \to H_n^G(pt; \mathbf{K}_\mathcal{A})$$

is isomorphic to the assembly map

$$H_{n-1}^G(E_\mathcal{F} G; \mathbf{K}_{\Sigma\mathcal{A}}) \to H_{n-1}^G(pt; \mathbf{K}_{\Sigma\mathcal{A}}).$$

Proof. We use a construction of $\Sigma\mathcal{A}$ that is similar to the construction from [PW85]. For a given $\mathcal{A}$ there is a natural construction of a Karoubi filtration of additive categories $\mathcal{A}' \subset \Lambda\mathcal{A}$ whose quotient we denote by $\Sigma\mathcal{A}$. Here $\mathcal{A}'$ is naturally equivalent to $\mathcal{A}$ and there is an Eilenberg swindle on $\Lambda\mathcal{A}$, see Example 7.2. Therefore $\mathcal{A} \to \Lambda\mathcal{A} \to \Sigma\mathcal{A}$ induces a fibration sequence in (non-connective) K-theory by [CP95, Theorem 1.28] and $K_*\Lambda\mathcal{A} = 0$. Because the construction is natural, there

are G-actions on $\Lambda\mathcal{A}$ and $\Sigma\mathcal{A}$. Both, the Karoubi filtration and the Eilenberg swindle are preserved by the passage from $\mathcal{A}$ to $\mathcal{A} *_G T$. Therefore we have a fibration sequence of $\mathrm{Or}G$-spectra,

$$\mathbf{K}_\mathcal{A} \to \mathbf{K}_{\Lambda\mathcal{A}} \to \mathbf{K}_{\Sigma\mathcal{A}}$$

that gives long exact sequences of the associated homology groups for every G-space X. By the Eilenberg swindle on $\mathcal{A} *_G T$ the groups $H_*^G(X; \mathbf{K}_{\Lambda\mathcal{A}})$ vanish and the boundary map in the long exact sequence yields the desired identification of assembly maps. $\square$

From Proposition 4.6 we obtain the following analog of [BFJR04, Corollary 7.3].

Corollary 4.7. *Let $\mathcal{F}$ be a family of subgroups of the group G. If for every additive category $\mathcal{A}$ with right G-action the assembly map*

$$H_*^G(E_\mathcal{F}G; \mathbf{K}_\mathcal{A}) \to H_*^G(pt; \mathbf{K}_\mathcal{A})$$

is injective or surjective respectively in a fixed degree $* = n$, then this assembly map is injective or surjective respectively in all degrees $* = j$ with $j \le n$.

5. L-THEORY

Everything we did for algebraic K-theory has an analog in L-theory and we will state the corresponding conjecture and inheritance result here quickly. An additive category with involution is an additive category $\mathcal{A}$ together with an additive contravariant functor $\# = (-)^\# \colon \mathcal{A} \to \mathcal{A}$ such that $\# \circ \# = \mathrm{id}$. We consider now additive categories with involution and right G-action, where we require in addition that for every $g \in G$ the covariant functor g^* is compatible with the involution $\#$, i.e. $\# \circ g^* = g^* \circ \#$. If T is a G-set then

$$(A^\#)_t = (A_t)^\# \quad \text{and} \quad (\phi^\#)_{g,t} = g^*((\phi_{g^{-1},gt})^\#)$$

defines an involution on $\mathcal{A} *_G T$. There is a functor $\mathbb{L}^{-\infty} \colon \mathsf{Add\ Cat\ Inv} \to \mathsf{Sp}$ that associates the L-theory spectrum to an additive category with involution constructed by Ranicki [Ran92]. We consider the $\mathrm{Or}G$-spectrum $\mathbf{L}_\mathcal{A}$ defined by

$$\mathbf{L}_\mathcal{A}(T) = \mathbb{L}^{-\infty}(\mathcal{A} *_G T).$$

Conjecture 5.1 (L-Theory Farrell-Jones-Conjecture with Coefficients)**.** *Let G be a group and let $\mathcal{VC}yc$ be the family of virtually cyclic subgroups of G. Let $\mathcal{A}$ be an additive category with involution with a right G-action. Then the assembly map*

$$H^G(E_{\mathcal{VC}yc}G); \mathbf{L}_\mathcal{A}) \to H^G(pt; \mathbf{L}_\mathcal{A})$$

is an isomorphism.

The only property of the functor $\mathbb{K}^{-\infty}$ that was used in the proof of Proposition 4.2 is that it sends equivalences of categories to equivalences of spectra. Because this property holds also for the functor $\mathbb{L}^{-\infty}$ there is also the L-theory version of Proposition 4.2. Therefore there are also L-theory versions of Corollary 4.3, Remark 4.4 and Theorem 4.5. We spell out only the analog of Theorem 4.5.

Theorem 5.2. *Let H be a subgroup of G. Suppose that for every additive category $\mathcal{A}$ with involution with G-action the assembly map*

$$H_*^G(E_{\mathcal{VC}yc}G; \mathbf{L}_\mathcal{A}) \to H_*^G(pt; \mathbf{L}_\mathcal{A})$$

is injective or surjective respectively. Then for every additive category $\mathcal{B}$ with involution with right H-action the assembly map

$$H_*^H(E_{\mathcal{VC}yc}H; \mathbf{L}_\mathcal{B}) \to H_*^H(pt; \mathbf{L}_\mathcal{B})$$

is injective or surjective respectively. In particular, if the L-Theory Farrell-Jones Conjecture with Coefficients 5.1 holds for a group G, then it also holds for every subgroup of G.

6. CROSSED PRODUCTS

In this section we show that the Farrell-Jones Conjecture with Coefficients 3.2 covers crossed product rings. We first recall the notion of a crossed product ring, compare [Pas89].

Let R be a ring, G be a group and $\alpha\colon G \to \mathrm{Aut}(R), g \mapsto \alpha_g$ and $\tau\colon G{\times}G \to R^\times, (g,h) \mapsto \tau_{g,h}$ be maps. Here $R^\times$ are the units of R and $\mathrm{Aut}(R)$ denotes the group of ring-automorphisms of R. We require that

$$(6.1) \qquad \tau_{g,h}\tau_{gh,k} \;=\; \alpha_g(\tau_{h,k})\tau_{g,hk}$$

$$(6.2) \qquad \tau_{g,h}\alpha_{gh}(r) \;=\; (\alpha_g \circ \alpha_h)(r)\tau_{g,h}$$

for $g,h,k \in G$, $r \in R$. We will also assume that $\alpha_e = \mathrm{id}_R$, where e denotes the unit element in G.

The crossed product ring $R_{\alpha,\tau}G$ is as an additive group RG, but is equipped with a twisted multiplication $\cdot_{\alpha,\tau}$ where

$$(6.3) \qquad (rg) \cdot_{\alpha,\tau} (sh) = r\alpha_g(s)\tau_{g,h}gh$$

for $r,s \in R$ and $g,h \in G$. The element $1_R e$ is the unit of $R_{\alpha,\tau}G$.

Setting $g = e$ or $h = e$ in (6.2) we conclude that for $g \in G$,

$$(6.4) \qquad \tau_{e,g} \text{ and } \tau_{g,e} \text{ lie in the center of } R.$$

Example 6.5. The notion of a crossed product ring naturally appears in the following situation. Consider an extension of groups

$$1 \to K \to \Gamma \xrightarrow{p} G \to 1.$$

Choose a set-theoretical section s of p such that $s(1) = 1$. Let S be a ring, let $R = SK$ and set $\alpha_g(r) = s(g)rs(g)^{-1}$ and $\tau_{g,h} = s(g)s(h)s(gh)^{-1}$. Then

$$R_{\alpha,\tau}G \cong S\Gamma.$$

Our aim is now to define an additive category $\mathcal{A}_{\alpha,\tau}$ with a right G-action such that the category $\mathcal{A} *_G \mathrm{pt}$ is equivalent to the category of finitely generated free $R_{\alpha,\tau}G$-modules.

We start with the category $\mathcal{F}^f(R)$ of finitely generated free R-modules. If φ is an automorphism of R, then we define a functor $M \mapsto \mathrm{res}_\varphi M$ where the latter is the R-module obtained by twisting the R-module structure by φ, i.e. $r\cdot_{\mathrm{res}_\varphi M}v = \varphi(r)\cdot_M v$. This defines a right action of $\mathrm{Aut}(R)$ on $\mathcal{F}^f(R)$. (If we want a small category we can restrict attention to modules of the form $\mathrm{res}_\varphi R^n$.)

We digress for a moment and discuss the special case where $\tau \equiv 1_R$. Then α is a group homomorphism and we obtain an action of G on $\mathcal{F}^f(R)$. This is the desired category with G-action in this special case. The equivalence to the category of finitely generated free $R_\alpha G$-modules sends the morphism $\phi\colon M \to N$ in $\mathcal{F}^f(R) *_G \mathrm{pt}$ with components

$$\phi_g \colon M \to \mathrm{res}_{\alpha_g} N$$

to the $R_\alpha G$-linear map

$$R_\alpha G \otimes_R M \to R_\alpha G \otimes_R N, \quad x \otimes v \mapsto xg^{-1} \otimes \phi_g(v).$$

We continue with the explanation of the general case. In general, $L_{\tau_{g,h}}$ defines a natural transformation from $\mathrm{res}_{\alpha_{gh}}$ to $\mathrm{res}_{\alpha_h} \circ \mathrm{res}_{\alpha_g}$. (Here we denoted the map $v \mapsto rv$ for $r \in R$ by L_r. The expression rv is formed with respect to the the original module multiplication on M.)

The category $\mathcal{A}_{\alpha,\tau}$ is now obtained by rigidifying this operation as follows. Objects of $\mathcal{A}_{\alpha,\tau}$ are pairs (M,g) where M is a finitely generated free R-module and $g \in G$. Morphisms from (M,g) to (N,h) are R-linear maps $\varphi\colon \mathrm{res}_{\alpha_g} M \to \mathrm{res}_{\alpha_h} N$ and composition is composition of linear maps. The right action of $\gamma \in G$ is defined by $(M,g) \mapsto (M,g\gamma)$ on objects and by

$$(6.6) \qquad \varphi \mapsto \gamma^*\varphi = L_{\tau_{h,\gamma}}^{-1} \circ \varphi \circ L_{\tau_{g,\gamma}}$$

for a morphism $\varphi\colon (M,g) \to (N,h)$. The only thing one has to check is that $\delta^*(\gamma^*\varphi) = (\gamma\delta)^*\varphi$ for a morphism $\varphi\colon (M,g) \to (N,h)$. Recall that such a morphism is given by an additive map $\varphi\colon M \to N$ for which $\varphi \circ L_{\alpha_g(r)} = L_{\alpha_h(r)} \circ \varphi$ for $v \in M$, $r \in R$. Using this and (6.1) we can compute

$$
\begin{aligned}
\delta^*(\gamma^*\varphi) &= L_{\tau_{h\gamma,\delta}}^{-1} \circ L_{\tau_{h,\gamma}}^{-1} \circ \varphi \circ L_{\tau_{g,\gamma}} \circ L_{\tau_{g\gamma,\delta}} \\
&= L_{\tau_{h,\gamma\delta}}^{-1} \circ L_{\alpha_h(\tau_{\gamma,\delta})}^{-1} \circ \varphi \circ L_{\alpha_g(\tau_{\gamma,\delta})} \circ L_{\tau_{g,\gamma\delta}} \\
&= L_{\tau_{h,\gamma\delta}}^{-1} \circ L_{\alpha_h(\tau_{\gamma,\delta})}^{-1} \circ L_{\alpha_h(\tau_{\gamma,\delta})} \circ \varphi \circ L_{\tau_{g,\gamma\delta}} \\
&= L_{\tau_{h,\gamma\delta}}^{-1} \circ \varphi \circ L_{\tau_{g,\gamma\delta}} \\
&= (\gamma\delta)^*\varphi.
\end{aligned}
$$

Proposition 6.7. *The categories $\mathcal{A}_{\alpha,\tau} *_G \mathrm{pt}$ and the category $\mathcal{F}^f(R_{\alpha,\tau}G)$ of finitely generated free $R_{\alpha,\tau}G$ modules are equivalent as additive categories.*

Proof. We start by listing a number of useful consequences of (6.1), (6.2) and $\alpha_e = \mathrm{id}_R$,

$$(6.8) \qquad \alpha_a(\tau_{b,c}) = \tau_{a,b}\tau_{ab,c}\tau_{a,bc}^{-1}$$

$$(6.9) \qquad \alpha_a^{-1}(r) = \tau_{a^{-1},a}^{-1}\alpha_{a^{-1}}(r)\tau_{a^{-1},a}$$

$$(6.10) \qquad (\alpha_a \circ \alpha_b)(r) = \tau_{a,b}\alpha_{ab}(r)\tau_{a,b}^{-1}$$

for a, b, $c \in G$ and $r \in R$. From the definition of the product $\cdot = \cdot_{\alpha,\tau}$ in (6.3) we recall

$$(6.11) \qquad a \cdot r = \alpha_a(r) \cdot a$$

$$(6.12) \qquad r \cdot a = a \cdot \alpha_a^{-1}(r)$$

$$(6.13) \qquad a \cdot b = \tau_{a,b} \cdot ab$$

for $a, b \in G$ and $r \in R$.

Denote by $(\mathcal{A}_{\alpha,\tau} *_G \mathrm{pt})_0$ the full subcategory of $\mathcal{A}_{\alpha,\tau} *_G \mathrm{pt}$ whose objects are of the form (M,e). It is easy to check that the inclusion $(\mathcal{A}_{\alpha,\tau} *_G \mathrm{pt})_0 \to \mathcal{A}_{\alpha,\tau} *_G \mathrm{pt}$ is an equivalence of additive categories. We define a functor $F\colon (\mathcal{A}_{\alpha,\tau} *_G \mathrm{pt})_0 \to \mathcal{F}^f(R_{\alpha,\tau}G)$ as follows. For an object (M,e) in $(\mathcal{A}_{\alpha,\tau} *_G \mathrm{pt})_0$ let $F(M,e) = R_{\alpha,\tau}G \otimes_R M$. A morphism $\phi\colon (M,e) \to (N,e)$ in $(\mathcal{A}_{\alpha,\tau} *_G \mathrm{pt})$ is by definition a sequence $(\phi_\gamma)_{\gamma \in G}$ where $\phi_\gamma\colon M \to \mathrm{res}_{\alpha_\gamma} N$ is an R-linear map. Because we can add morphisms in additive categories it will suffice to discuss morphisms for which $\phi_\gamma = 0$ for all but one $\gamma \in G$; we write (φ,g) for the morphism given by $\phi_\gamma = \varphi$ if $\gamma = g$ and $\phi_\gamma = 0$ otherwise, in particular φ is an additive map $M \to N$ for which

$$(6.14) \qquad \varphi(rv) = \alpha_g(r)\varphi(v) \quad \text{for all} \quad r \in R, v \in M.$$

We define $F(\varphi,g)\colon F(M,e) \to F(N,e)$ as the linear map

$$(6.15) \qquad x \otimes v \mapsto x \cdot g^{-1} \cdot \tau_{g,g^{-1}}^{-1} \otimes \varphi(v).$$

Note that

$$
\begin{aligned}
xr \cdot g^{-1} \cdot \tau_{g,g^{-1}}^{-1} &= x \cdot g^{-1} \cdot \alpha_{g^{-1}}^{-1}(r)\tau_{g,g^{-1}}^{-1} && \text{by (6.12)} \\
&= x \cdot g^{-1} \cdot \tau_{g,g^{-1}}^{-1}\alpha_g(r) && \text{by (6.9)}
\end{aligned}
$$

Because of (6.14) this means that $F(\varphi, g)$ defines indeed a well defined map on the tensor product. (This explains the appearance of $\tau_{g,g^{-1}}^{-1}$ in (6.15); without this term the map is ill defined.)

Next we check that F is compatible with composition, a somewhat tedious calculation. Let $(\psi, h) \colon (N, e) \to (L, e)$ be a second morphism in $(\mathcal{A}_{\alpha,\tau} *_G \mathrm{pt})_0$, in particular ψ is an additive map $N \to L$ for which

$$(6.16) \qquad \psi(rv) = \alpha_h(r)\psi(v) \quad \text{for all} \quad r \in R, v \in N.$$

Then $F(\psi, h) \circ F(\varphi, g)$ maps $x \otimes v$ to

$$
\begin{aligned}
& x \cdot g^{-1} \cdot \tau_{g,g^{-1}}^{-1} \cdot h^{-1} \cdot \tau_{h,h^{-1}}^{-1} \otimes \psi(\varphi(v)) && \\
&= x \cdot g^{-1} \cdot h^{-1} \cdot \alpha_{h^{-1}}^{-1}(\tau_{g,g^{-1}}^{-1})\tau_{h,h^{-1}}^{-1} \otimes \psi(\varphi(v)) && \text{by (6.12)} \\
&= x \cdot \tau_{g^{-1},h^{-1}} \cdot (hg)^{-1} \cdot \alpha_{h^{-1}}^{-1}(\tau_{g,g^{-1}}^{-1})\tau_{h,h^{-1}}^{-1} \otimes \psi(\varphi(v)) && \text{by (6.13)} \\
&= x \cdot (hg)^{-1} \cdot \alpha_{(hg)^{-1}}^{-1}(\tau_{g^{-1},h^{-1}})\alpha_{h^{-1}}^{-1}(\tau_{g,g^{-1}}^{-1})\tau_{h,h^{-1}}^{-1} \otimes \psi(\varphi(v)) && \text{by (6.12)} \\
&= x \cdot (hg)^{-1} \cdot A \otimes \psi(\varphi(v))
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \alpha_{(hg)^{-1}}^{-1}(\tau_{g^{-1},h^{-1}})\alpha_{h^{-1}}^{-1}(\tau_{g,g^{-1}}^{-1})\tau_{h,h^{-1}}^{-1} \\
&= \left(\tau_{hg,(hg)^{-1}}^{-1}\alpha_{hg}(\tau_{g^{-1},h^{-1}})\tau_{hg,(hg)^{-1}} \right) \\
&\quad \left(\tau_{h,h^{-1}}^{-1}\alpha_h(\tau_{g,g^{-1}}^{-1})\tau_{h,h^{-1}} \right)\tau_{h,h^{-1}}^{-1} && \text{by (6.9)} \\
&= \tau_{hg,(hg)^{-1}}^{-1}\left(\tau_{h,g}^{-1}\alpha_h(\alpha_g(\tau_{g^{-1},h^{-1}}))\tau_{h,g} \right) \\
&\quad \tau_{hg,(hg)^{-1}}\tau_{h,h^{-1}}^{-1}\alpha_h(\tau_{g,g^{-1}}^{-1}) && \text{by (6.10)} \\
&= \tau_{hg,(hg)^{-1}}^{-1}\tau_{h,g}^{-1}\left(\alpha_h(\tau_{g,g^{-1}}\tau_{e,h^{-1}}\tau_{g,(hg)^{-1}}^{-1}) \right) \\
&\quad \tau_{h,g}\tau_{hg,(hg)^{-1}}\tau_{h,h^{-1}}^{-1}\alpha_h(\tau_{g,g^{-1}}^{-1}) && \text{by (6.8)} \\
&= \tau_{hg,(hg)^{-1}}^{-1}\tau_{h,g}^{-1}\left(\alpha_h(\tau_{g,g^{-1}})\alpha_h(\tau_{e,h^{-1}})\alpha_h(\tau_{g,(hg)^{-1}})^{-1} \right) \\
&\quad \tau_{h,g}\tau_{hg,(hg)^{-1}}\tau_{h,h^{-1}}^{-1}\alpha_h(\tau_{g,g^{-1}})^{-1} \\
&= \tau_{hg,(hg)^{-1}}^{-1}\tau_{h,g}^{-1}\left(\tau_{h,g}\tau_{hg,g^{-1}}\tau_{h,e}^{-1} \right)\left(\tau_{h,e}\tau_{h,h^{-1}}\tau_{h,h^{-1}}^{-1} \right) \\
&\quad \left(\tau_{h,g}\tau_{hg,(hg)^{-1}}\tau_{h,h^{-1}}^{-1} \right)^{-1}\tau_{h,g}\tau_{hg,(hg)^{-1}}\tau_{h,h^{-1}}^{-1}\left(\tau_{h,g}\tau_{hg,g^{-1}}\tau_{h,e}^{-1} \right)^{-1} && \text{by (6.8)} \\
&= \tau_{hg,(hg)^{-1}}^{-1}\tau_{hg,g^{-1}}\tau_{h,e}\tau_{hg,g^{-1}}^{-1}\tau_{h,g}^{-1} \\
&= \tau_{hg,(hg)^{-1}}^{-1}\tau_{h,g}^{-1}\tau_{h,e} && \text{by (6.4)}.
\end{aligned}
$$

Compute the composition in $(\mathcal{A}_{\alpha,\tau} *_G \mathrm{pt})_0$ as follows

$$
\begin{aligned}
(\psi, h) \circ (\varphi, g) &= (g^*\psi \circ \varphi, hg) && \text{by (2.2)} \\
&= (L_{\tau_{h,g}}^{-1} \circ \psi \circ L_{\tau_{e,g}} \circ \varphi, hg) && \text{by (6.6)}.
\end{aligned}
$$

Therefore $F((\psi, h) \circ (\varphi, g))$ maps $x \otimes v$ to

$$
\begin{aligned}
& (x \cdot (hg)^{-1} \cdot \tau_{hg,(hg)^{-1}}^{-1} \otimes \tau_{h,g}^{-1}\psi(\tau_{e,g}\varphi(v)) \\
&= x \cdot (hg)^{-1} \cdot \tau_{hg,(hg)^{-1}}^{-1} \otimes \tau_{h,g}^{-1}\alpha_h(\tau_{e,g})\psi(\varphi(v)) && \text{by (6.16)} \\
&= x \cdot (hg)^{-1} \otimes B\psi(\varphi(v))
\end{aligned}
$$

where

$$B = \tau^{-1}_{hg,(hg)^{-1}} \tau^{-1}_{h,g} \alpha_h(\tau_{e,g})$$
$$= \tau^{-1}_{hg,(hg)^{-1}} \tau^{-1}_{h,g} \tau_{h,e} \tau_{h,g} \tau^{-1}_{h,g} \qquad \text{by (6.8)}$$
$$= \tau^{-1}_{hg,(hg)^{-1}} \tau^{-1}_{h,g} \tau_{h,e}.$$

Thus $A = B$ and this shows $F(\psi, h) \circ F(\varphi, g) = F((\psi, h) \circ (\varphi, g))$. Thus F is indeed a functor. It is straight forward to check that F is full and faithful, i.e. an equivalence of categories. $\qquad \square$

The following sharpening of Proposition 6.7 is obtained by formal arguments.

Corollary 6.17. *Suppose we are given a crossed product situation*

$$R, \quad \alpha \colon G \to \operatorname{Aut}(R), \quad \tau \colon G \times G \to R^{\times}.$$

Then there exists an additive category $\mathcal{A}_{\alpha,\tau}$ with a right G-action, such that for every orbit G/H the category

$$\mathcal{A}_{\alpha,\tau} *_G G/H \quad \text{and the category} \quad \mathcal{F}^f(R_{\alpha|,\tau|}H)$$

of finitely generated $R_{\alpha|,\tau|}H$-modules are equivalent. Here $\alpha|$ and $\tau|$ denote the restriction of α and τ to H respectively $H \times H$. In particular there is for every G/H and every $n \in \mathbb{Z}$ an isomorphism

$$K_n(\mathcal{A}_{\alpha,\tau} *_G G/H) \cong K_n(R_{\alpha|,\tau|}H).$$

Proof. We have a chain of equivalences

$$\mathcal{A}_{\alpha,\tau} *_G G/H \quad \simeq \quad (\operatorname{res}_H \mathcal{A}_{\alpha,\tau}) *_H \operatorname{pt}$$
$$\simeq \quad \mathcal{A}_{\alpha|,\tau|} *_H \operatorname{pt}$$
$$\simeq \quad R_{\alpha|,\tau|}H_{\oplus}.$$

Here the first equivalence is a special case of Proposition 2.8 (iii) and the last follows immediately from the previous Proposition 6.7. The second equivalence is induced from the H-equivariant inclusion $\mathcal{A}_{\alpha|,\tau|} \to \operatorname{res}_H \mathcal{A}_{\alpha,\tau}$ which sends (M, h) to the same element considered as an object of $\mathcal{A}_{\alpha,\tau}$. This inclusion is clearly full and faithful and every object (M, g) in the target is isomorphic to $(\operatorname{res}_{\alpha_g} M, e)$. One then checks that in general an H-equivariant equivalence $\mathcal{A} \to \mathcal{B}$ induces an equivalence $\mathcal{A} *_H T \to \mathcal{B}_H * T$ for every G-set T. $\qquad \square$

Observe that in particular the G-equivariant homology theory $H^G_*(-; \mathbf{K}_{\mathcal{A}_{\alpha,\tau}})$ evaluated on an orbit G/H is isomorphic to $K_*(R_{\alpha|,\tau|}H)$. The following special case of Conjecture 3.2 hence makes precise the idea that $K_*(R_{\alpha,\tau}G)$ should be assembled from the pieces $K_*(R_{\alpha|,\tau|}H)$, where H ranges over the virtually cyclic subgroups of G.

Conjecture 6.18. *Suppose $R_{\alpha,\tau}G$ is a crossed product ring, then the assembly map*

$$H^G_*(E_{\mathcal{V}\mathcal{C}yc}G; \mathbf{K}_{\mathcal{A}_{\alpha,\tau}}) \to H^G_*(pt; \mathbf{K}_{\mathcal{A}_{\alpha,\tau}}) \cong K_*(R_{\alpha,\tau}G)$$

induced from $E_{\mathcal{V}\mathcal{C}yc}G \to pt$ is an isomorphism.

7. Controlled algebra

Many results on the Farrell-Jones conjecture (without coefficients) use the concept of controlled algebra. In this section we briefly indicate how the fundamental concepts of controlled algebra extend from rings to additive categories with group actions.

The following generalizes the definitions in [BFJR04, Section 2].

Definition 7.1. Let $\mathcal{A}$ be an additive category with a right G-action and let X be a free G-space. Define the additive category with right G-action

$$\mathcal{C}(X; \mathcal{A})$$

as follows. Objects are families $A = (A_x)_{x \in X}$ of objects in $\mathcal{A}$ such that $\operatorname{supp} A = \{x \in X \mid A_x \neq 0\}$ is locally finite. A morphism $\phi \colon A \to B$ is a family $(\phi_{y,x})_{(y,x) \in X \times X}$, where $\phi_{y,x} \colon A_x \to B_y$ is a morphism in $\mathcal{A}$ and for fixed x the set of y with $\phi_{y,x} \neq 0$ is finite and for fixed y the set of x with $\phi_{y,x} \neq 0$ is finite. The composition $\psi = \phi' \circ \phi$ is defined to be

$$\psi_{z,x} = \sum_{y \in X} \phi'_{z,y} \circ \phi_{y,x}.$$

The element $g \in G$ acts via the covariant additive functor g^* which is given by

$$(g^*A)_x = g^*(A_{gx}) \quad \text{and} \quad (g^*\phi)_{y,x} = g^*(\phi_{gy,gx}).$$

It now makes sense to consider the fixed category $\mathcal{C}(X; \mathcal{A})^G$. An object A and a morphism ϕ in the fixed category satisfy

$$A_x = g^*(A_{gx}) \quad \text{and} \quad \phi_{y,x} = g^*(\phi_{gy,gx}).$$

Observe that in the case where the category $\mathcal{A}$ is $R_\oplus$ for some ring R equipped with the trivial G-action we obtain the category which was denoted $\mathcal{C}^G(X; R)$ in [BFJR04, Section 2], compare also Example 2.4.

7.1. **Support conditions.** As usual we define the support of an object A, respectively a morphism ϕ as

$$\operatorname{supp} A = \{x \in X \mid A_x \neq 0\} \quad \text{and} \quad \operatorname{supp} \phi = \{(x,y) \in X \times X \mid \phi_{y,x} \neq 0\}.$$

If a set $\mathcal{E}$ of subsets of $X \times X$ and a set $\mathcal{F}$ of subsets of X satisfies the conditions (i)-(iv) listed in [BFJR04, Subsection 2.3] then we speak of $\mathcal{E}$ respectively $\mathcal{F}$ as morphism and object support conditions and define

$$\mathcal{C}(X; \mathcal{E}, \mathcal{F}; \mathcal{A})$$

to be the subcategory of $\mathcal{C}(X; \mathcal{A})$ consisting of objects A for which there exists an $F \in \mathcal{F}$ such that $\operatorname{supp} A \subset F$ and morphisms ϕ for which there exists an $E \in \mathcal{E}$ such that $\operatorname{supp} \phi \subset E$. Observe that for $g \in G$ we have

$$\operatorname{supp} g^*A = g^{-1}(\operatorname{supp} A) \quad \text{and} \quad \operatorname{supp} g^*\phi = g^{-1}(\operatorname{supp} \phi).$$

We say that $\mathcal{E}$ is G-invariant if for every $g \in G$, $E \in \mathcal{E}$ we have $g(E) \in \mathcal{E}$, where G acts diagonally on $X \times X$. We say that $\mathcal{F}$ is G-invariant if for every $g \in G$, $F \in \mathcal{F}$ we have $g(F) \in \mathcal{F}$. For G-invariant object and morphism support conditions $\mathcal{E}$ and $\mathcal{F}$ there is a G-action on $\mathcal{C}(X; \mathcal{E}, \mathcal{F}; \mathcal{A})$ and we can consider the corresponding fixed category, which we denote

$$\mathcal{C}^G(X; \mathcal{E}, \mathcal{F}; \mathcal{A}).$$

The following example was used in Proposition 4.6.

Example 7.2. Let $X = [0, \infty)$. Let $\mathcal{E} = \{E_\alpha \mid \alpha > 0\}$, where $E_\alpha = \{(x,y) \in [0,\infty)^{\times 2} \mid |x - y| < \alpha\}$ and $\mathcal{F} = \{[0, r] \mid r \in \mathbb{R}\}$. Then there is an Eilenberg swindle on $\Lambda \mathcal{A} = \mathcal{C}([0, \infty); \mathcal{E}; \mathcal{A})$ induced by the map $t \mapsto t + 1$ on $[0, \infty)$ and $\mathcal{A}' = \mathcal{C}([0, \infty); \mathcal{E}, \mathcal{F}; \mathcal{A}) \subset \Lambda \mathcal{A} = \mathcal{C}([0, \infty); \mathcal{E}; \mathcal{A})$ is a Karoubi filtration, see [CP95, Definition 1.27].

7.2. **Assembly as Forget-Control.** From this point on it is clear that every construction and every proof in [BFJR04] which treats the category of finitely generated R-modules in a formal way does have an analog in our context. In particular there is a category

$$\mathcal{D}^G(X; \mathcal{A})$$

defined analogously to the category $\mathcal{D}^G(X)$ from Subsection 3.2 in [BFJR04] and this construction is functorial in the G-space X. The functor

$$X \mapsto \mathbb{K}^{-\infty}\mathcal{D}^G(X).$$

is a G-equivariant homology theory on the category of G-CW complexes, compare [BFJR04, Section 4].

We will now identify this controlled version of a G-equivariant homology theory with the G-equivariant homology theory that we defined in Section 3 via the OrG-spectrum $\mathbf{K}_\mathcal{A}$ from Definition 3.1.

Theorem 7.3. *There is an isomorphism between the functors $X \mapsto H_*^G(X; \mathbf{K}_\mathcal{A})$ and $X \mapsto \pi_{*+1}(\mathbb{K}^{-\infty}\mathcal{D}(X; \mathcal{A})^G)$ from G-CW-complexes to graded abelian groups. In particular, the map*

$$K_{*+1}(\mathcal{D}(E_\mathcal{F}G, \mathcal{A})^G) \to K_{*+1}(\mathcal{D}(pt, \mathcal{A})^G)$$

is a model for the assembly map

$$H_*^G(E_\mathcal{F}G, \mathbf{K}_\mathcal{A}) \to H_*^G(pt, \mathbf{K}_\mathcal{A}).$$

Proof. Without twisted coefficients this was done in [BFJR04, Section 6]. The proof in the case with twisted coefficients is essentially the same. The only step in the proof where the argument needs to be rethought is Step (ii) in the proof of [BFJR04, Proposition 6.2]. This step is redone in lemma 7.4 below. $\square$

Lemma 7.4. *Let T be a G-set. Let $\mathcal{F}_{Gc}$ the object support condition on $T \times G$ that contains exactly the G-compact subsets. Let $\mathcal{E}_\Delta$ be the morphism control condition on T that contains only the diagonal of T. Let $p \colon T \times G \to T$ denote the projection. There is an additive functor*

$$F \colon \mathcal{A} *_G T \to \mathcal{C}(T \times G, p^{-1}\mathcal{E}_\Delta, \mathcal{F}_{Gc}; \mathcal{A})^G$$

which yields an equivalence of categories. This functor is natural in T.

Proof. It is straightforward to check that

$$F(A)_{(t,g^{-1})} = g^*(A_{gt}), \quad F(\phi)_{(t',k^{-1}),(t,g^{-1})} = \begin{cases} g^*(\phi_{kg^{-1},gt}) & \text{if } t' = t \\ 0 & \text{if } t' \neq t \end{cases}$$

defines an additive functor. Here, in order to check that $F(A)$ satisfies the object support condition observe that $G(t,g) \mapsto g^{-1}t$ is a bijection between the orbits of the left G-set $T \times G$ and the set T. Because of the object support condition an object in the target category can be written as a direct sum of objects supported on a single orbit of $T \times G$. Because of the G-invariance an object $C = (C_{(t,g)})$ supported on a single orbit $G(t,g)$ is determined by its value at one point of the orbit together with the G-action on the category $\mathcal{A}$. Now the object $A \in \mathcal{A} *_G T$ supported on the single point $\{t\}$ which is given by $A_t = C_{(t,e)}$ maps to C under the functor F. Since the functor is additive we conclude that every object in the target category is isomorphic to an object in the image of the functor F. The functor is easily seen to be faithful. It remains to prove that it is full, i.e. surjective on morphism sets. If $f = (f_{(t',k^{-1}),(t,g^{-1})})$ is a morphism in the target category then because of the $p^{-1}\mathcal{E}_\Delta$-condition $f_{(t',k^{-1}),(t,g^{-1})}$ is non-trivial only if $t' = t$. If one defines a morphism ϕ in $\mathcal{A} *_G T$ by $\phi_{k,t} = f_{(t,k^{-1})(t,e)}$ then one can use the G-invariance of f in order to check that $F(\phi) = f$. $\square$

8. APPLICATIONS

As already mentioned many arguments in controlled algebra treat the category $\mathcal{A}$ as a formal variable. Consequently existing proofs for results about the Farrell-Jones Conjecture without coefficients can often be carried over to the context with coefficients. We will state three results obtained in this way.

The following is a generalization of the main Theorem in [BR05].

Theorem 8.1. *Let G be the fundamental group of a closed Riemannian manifold of strictly negative sectional curvature. Then the algebraic K-theory Farrell-Jones Conjecture with Coefficients 3.2 holds for G.*

Proof. Even though the original proof (injectivity in [BFJR04] and surjectivity in [BR05]) is quite lengthy, one can quickly check that the only places in the proof where the arguments need to be rethought are the following.

(i) The functor ind appearing in Proposition 8.3 in [BFJR04] needs to be promoted to a functor $\mathrm{ind}\colon \mathcal{D}^H(X; \mathrm{res}_H \mathcal{A}) \to \mathcal{D}^G(G \times_H X; \mathcal{A})$, where X is an H-space, H is a subgroup of G and $\mathcal{A}$ is an additive category with G-action. The new formulas for the functor ind are

$$(\mathrm{ind}\, M)_{[g,x]} = (g^{-1})^* M_x \quad \text{and} \quad (\mathrm{ind}\, \phi)_{[g',x],[g,x]} = (g^{-1})^*(\phi_{g^{-1}g'x',x})$$

if $g^{-1}g' \in H$ and 0 otherwise.

(ii) The proof of injectivity uses the injectivity result for the assembly with respect to the trivial family from [CP95], compare (iii) in Subsection 10.3 in [BFJR04]. We hence need the version with coefficients of that result. It is a special case of Theorem 8.2 below.

(iii) In order to define the "Nil"-spectra denoted $\mathbb{N}_i$ in Subsection 10.2 in [BFJR04] one needs that the assembly map for the infinite cyclic group with respect to the trivial family is split injective. This is the special case of Theorem 8.2 below, where G is the infinite cyclic group.

(iv) The construction of the transfer functor and the proof of its properties in Section 5 of [BR05] need to be adapted to the set-up with coefficients. This will occupy the rest of this proof.

It will be convenient to restrict this discussion to connective K-theory because we use Waldhausen categories. This suffices by Corollary 4.7. (On the other hand, this discussion can be extended to non-connective K-theory, by giving an adhoc definition of non-connective K-theory for the Waldhausen categories we encounter in the following, compare [BR05, Remark 5.3].)

First we need a replacement for the category of homotopy finite chain complexes, defined in Subsection 5.2 and 8.1 in [BR05]. Given a category $\mathcal{A}$ with a right G-action and an infinite cardinal number κ (chosen large enough) we construct below in Lemma 9.2 a category with right G-action $\mathcal{A}^\kappa$. Analogously to $\overline{\mathcal{C}}^G(X; \mathcal{E})$ from [BR05] we define $\overline{\mathcal{C}}^G(X; \mathcal{E}; \mathcal{A}^\kappa)$ by allowing objects $M = (M_x)_{x \in X}$, where M_x is an object in $\mathcal{A}^\kappa$ and the support of M is an arbitrary subset of X. The category $\overline{\mathcal{C}}^G(X; \mathcal{E}; \mathcal{A}^\kappa)$ plays the role of the category that is (unfortunately) called $\overline{\mathcal{A}}$ in Subsection 8.1 in [BR05]. Hence the category $\mathrm{ch}_{hf}\, \mathcal{C}^G(X; \mathcal{E}; \mathcal{A})$ is defined to be the "homotopy closure" of the category $\mathrm{ch}_f\, \mathcal{C}(X; \mathcal{E}; \mathcal{A})$ inside $\mathrm{ch}\, \overline{\mathcal{C}}^G(X; \mathcal{E}; \mathcal{A}^\kappa)$. The fibre complex F and its variants from Subsection 5.3 in [BR05] can be considered as objects in $\mathrm{ch}\, \overline{\mathcal{C}}^G(\tilde{E} \times \mathbb{T}; \mathcal{E}; \mathcal{F}^\kappa(\mathbb{Z}))$, which is defined analogous to $\mathrm{ch}\, \overline{\mathcal{C}}^G(X; \mathcal{E}; \mathcal{A}^\kappa)$. Here $\mathcal{F}^\kappa(\mathbb{Z})$ denotes a small model for the category of those free $\mathbb{Z}$-modules which admit a basis of cardinality less than or equal to κ. The category $\mathcal{F}^\kappa(\mathbb{Z})$ carries the trivial G-action. The "tensor product" $- \otimes -\colon \mathcal{A}^\kappa \times \mathcal{F}^\kappa(\mathbb{Z}) \to \mathcal{A}^\kappa$ from Lemma 9.2 (iii) now allows to construct the the transfer functor $M \mapsto M \otimes F$,

$\phi \mapsto \phi \otimes \nabla$ as before. The proof carries over without change until the end of Subsection 5.4 in [BR05]. In Proposition 5.9 the action of the Swan group $\mathrm{Sw}(G; \mathbb{Z})$ on $K_n(RG)$ needs to be replaced with the action of the Swan group on $K_n(\mathcal{A} *_G \mathrm{pt})$ that we describe below in Section 9. The proof of Proposition 5.9 in [BR05] remains unchanged until one reaches diagram (5.12). That diagram now gets replaced with the following diagram

$$
\begin{array}{ccc}
\mathcal{C}^G(Gb_0; \mathcal{A}) & \xrightarrow{-\otimes \nabla} & \mathrm{ch}_{hf}\, \mathcal{C}^G(Gb_0; \mathcal{A}) \\
{\scriptstyle F} \big\uparrow & & \big\uparrow {\scriptstyle \mathrm{ch}_{hf}\, F} \\
\mathcal{A} *_G \mathrm{pt} & \xrightarrow{-\otimes F_0} & \mathrm{ch}_{hf}(\mathcal{A} *_G \mathrm{pt}).
\end{array}
$$

Here $\mathrm{ch}_{hf}(\mathcal{A} *_G \mathrm{pt})$ is the homotopy closure of $\mathrm{ch}_f(\mathcal{A} *_G \mathrm{pt})$ in $\mathrm{ch}(\mathcal{A}^\kappa *_G \mathrm{pt})$. The functor F is a special case of the equivalence from Lemma 7.4 and is given by $F(A)_{g^{-1}b_0} = g^*(A)$ and $F(\phi)_{k^{-1}b_0, g^{-1}b_0} = g^*(\phi_{kg^{-1}})$. The functor $- \otimes \nabla$ is given by $(M_{gb_0}) \otimes \nabla = (M_{gb_0} \otimes F_0)$ and $(\phi_{gb_0, hb_0}) \otimes \nabla = (\phi_{gb_0, hb_0} \otimes \nabla_{gb_0, hb_0})$. We can equip F_0 with a G-action in such way that $\nabla_{gb_0, hb_0} \colon F_0 \to F_0$ corresponds to $l_{gh^{-1}}$, i.e. to left multiplication with gh^{-1}. With this notation the functor $- \otimes F_0$ is defined as follows. The object A maps to $A \otimes F_0$ the morphism $\phi = (\phi_g)$ maps to $(\phi_g \otimes l_g)$. As opposed to the original diagram in [BR05] the diagram now commutes. Let $\mathrm{inc} \colon \mathcal{A} *_G \mathrm{pt} \to \mathrm{ch}_{hf}(\mathcal{A} *_G \mathrm{pt})$ denote the inclusion. It follows from the discussion of the Swan group action in Section 9 below that on the level of K-theory $\mathrm{inc}^{-1} \circ (-\otimes F_0)$ corresponds to multiplication with $[F_0] = \sum_i (-1)^i [H_i(F_0)] \in \mathrm{Sw}^{\mathrm{ch}}(G; \mathbb{Z}) \cong \mathrm{Sw}(G; \mathbb{Z})$. $\qquad\square$

The following is a generalization of a result of Rosenthal [Ros04].

Theorem 8.2. *Let G be a group. Suppose that there is a model for $E_{\mathcal{F}in}G$ that is a finite G-CW-complex and admits a compactification X such that*

- *(i) the G-action extends to X;*
- *(ii) X is metrizable;*
- *(iii) X^F is contractible for every $F \in \mathcal{F}in$;*
- *(iv) $(E_{\mathcal{F}in}G)^F$ is dense in X for every $F \in \mathcal{F}in$;*
- *(v) compact subsets of $E_{\mathcal{F}in}G$ become small near $Y = X - E_{\mathcal{F}in}G$. That is, for every compact subset of $E_{\mathcal{F}in}G$ and for every neighborhood $U \subset X$ of $y \in Y$, there exists a neighborhood $V \subset X$ of y such that $g \in G$ and $gK \cap V \neq \emptyset$ implies $gK \subset U$.*

Let $\mathcal{A}$ be an additive category with right G-action. Then the assembly map

$$
H_*^G(E_{\mathcal{F}in}G; \mathbf{K}_{\mathcal{A}}) \to H_*^G(pt; \mathbf{K}_{\mathcal{A}})
$$

is split injective.

The following is a generalization of the main result from [Bar03].

Theorem 8.3. *Let G be group of finite asymptotic dimension that admits a finite model for the classifying space BG. Let $\mathcal{A}$ be an additive category with right G-action. Then the assembly map*

$$
H_*^G(EG; \mathbf{K}_{\mathcal{A}}) \to H_*^G(pt; \mathbf{K}_{\mathcal{A}})
$$

is split injective.

Both, the proof of Theorem 8.2 and of Theorem 8.3 are trivial modifications of the original proofs in [Ros04] respectively [Bar03]. Everywhere in these proofs the category of R-modules is treated as a formal variable and can simply be replaced by the additive category with G-action $\mathcal{A}$.

9. The Swan group action

In the proof of Theorem 8.1 we used some facts about Swan group actions which we are going to prove now. The Swan group $\mathrm{Sw}(G; \mathbb{Z})$ is the K_0-group of the category of $\mathbb{Z}G$-modules that are finitely generated as $\mathbb{Z}$-modules. Recall from Subsection 8.2 of [BR05] that there are version $Sw^{\mathrm{fr}}(G; \mathbb{Z})$ and $\mathrm{Sw}^{\mathrm{ch}}(G; \mathbb{Z})$ defined using $\mathbb{Z}G$-modules that are finitely generated free as $\mathbb{Z}$-modules, respectively bounded below chain complexes of $\mathbb{Z}G$-modules that are free as $\mathbb{Z}$-modules and whose homology is finitely generated as a $\mathbb{Z}G$-module. We have shown in Proposition 8.3 in [BR05] that the natural maps $j \colon \mathrm{Sw}^{\mathrm{fr}}(G; \mathbb{Z}) \to \mathrm{Sw}^{\mathrm{ch}}(G; \mathbb{Z})$ and $i \colon \mathrm{Sw}^{\mathrm{fr}}(G; \mathbb{Z}) \to \mathrm{Sw}(G; \mathbb{Z})$ are isomorphisms.

Lemma 9.1. *Let $\mathcal{A}$ be an additive category with right G-action. There exists for $n \geq 1$ a commutative diagram*

$$
\begin{array}{ccc}
K_n(\mathcal{A} *_G pt) \otimes_{\mathbb{Z}} \mathrm{Sw}^{fr}(G; \mathbb{Z}) & \longrightarrow & K_n(\mathcal{A} *_G pt) \\
\Big\downarrow{\mathrm{id}\, \otimes j} & & \Big\downarrow{\mathrm{inc}} \\
K_n(\mathcal{A} *_G pt) \otimes_{\mathbb{Z}} \mathrm{Sw}^{\mathrm{ch}}(G; \mathbb{Z}) & \longrightarrow & K_n(\mathrm{ch}_{hf}(\mathcal{A} *_G pt)),
\end{array}
$$

where both vertical arrows are isomorphisms. In this way $K_n(\mathcal{A} *_G pt)$ becomes a module over the Swan ring $\mathrm{Sw}^{fr}(G; \mathbb{Z}) \cong \mathrm{Sw}^{\mathrm{ch}}(G; \mathbb{Z}) \cong \mathrm{Sw}(G; \mathbb{Z})$.

Using the suspension category $\Sigma\mathcal{A}$ from Proposition 4.6 it is possible to formulate a version of the above Lemma that applies to all n. However, for our purposes the above formulation suffices.

Proof. Choose a small model $\mathcal{F}^f(\mathbb{Z})$ for the category of finitely generated free $\mathbb{Z}$-modules such that the underlying $\mathbb{Z}$-module of every $\mathbb{Z}G$-module that is used in the construction of $\mathrm{Sw}^{\mathrm{fr}}(G; \mathbb{Z})$ is contained in $\mathcal{F}^f(\mathbb{Z})$.

We replace $\mathcal{A}$ by the equivalent category $\mathcal{A}^f$ from Lemma 9.2 but refer to it as $\mathcal{A}$ in the following. This is justified because $- *_G pt$ respects equivalences, compare Remark 2.3. We can hence assume that there exists a tensor product $- \otimes - \colon \mathcal{A} \times \mathcal{F}^f(\mathbb{Z}) \to \mathcal{A}$ with the expected properties. Then define for a Swan module M the additive functor $- \otimes M \colon \mathcal{A} *_G pt \to \mathcal{A} *_G pt$ by

$$
A \otimes M = A \otimes UM, \quad \text{and} \quad (\phi \otimes M)_g = \phi_g \otimes l_g.
$$

Here UM denotes the underlying $\mathbb{Z}$-module of M and l_g denotes left multiplication by g. Note that the $\mathbb{Z}G$-module structure of M enters only in the morphisms. A morphism $f \colon M \to N$ of Swan modules induces a natural transformation $\tau(f, A) \colon A \otimes M \to A \otimes N$ that is given by $\tau(f, A)_g = 0$ if $g \neq e$ and $\tau(f, A)_e = \mathrm{id}_A \otimes f$. A short exact sequence $L \to M \to N$ of Swan-modules leads to a short exact sequence of functors, because the underlying sequence $UL \to UM \to UN$ always splits and being a short exact sequence of functors is checked objectwise. One uses [Wal85, 1.3.2 (4)] in order to check that one obtains the $\mathrm{Sw}^{\mathrm{fr}}(G; \mathbb{Z})$-action.

For the $\mathrm{Sw}^{\mathrm{ch}}(G; \mathbb{Z})$-action arrange that the underlying $\mathbb{Z}$-modules of all Swan modules that appear in the chain complexes lie in a small model $\mathcal{F}^\kappa(\mathbb{Z})$ of the category of finitely generated free modules that admit a basis B of cardinality $\mathrm{card}(B) \leq \kappa$. (Strictly speaking we have to have a cardinality assumption when we define the category of chain complexes that leads to $\mathrm{Sw}^{\mathrm{ch}}(G; \mathbb{Z})$.) By Lemma 9.2 there exists an inclusion $\mathcal{A} \to \mathcal{A}^\kappa$ and the "tensor product" we used so far extends to a tensor product $- \otimes - \colon \mathcal{A}^\kappa \otimes \mathcal{F}^\kappa(\mathbb{Z}) \to \mathcal{A}^\kappa$.

Define $\mathrm{ch}_{\mathrm{hf}}(\mathcal{A} *_G pt)$ to be the category of chain complexes in $\mathrm{ch}(\mathcal{A}^\kappa *_G pt)$ that are chain homotopy equivalent to a bounded below and above chain complex in $\mathrm{ch}(\mathcal{A} *_G pt)$. Similarly let $\mathrm{ch}_{\mathrm{hf}} \mathcal{F}^f(\mathbb{Z})$ denote the category of chain complexes

in $\mathrm{ch}\,\mathcal{F}^{\kappa}(\mathbb{Z})$ which are chain homotopy equivalent to a finite complex in $\mathrm{ch}\,\mathcal{F}^{f}(\mathbb{Z})$, compare [BR05, Section 8.1]. The right hand vertical arrow induced by the inclusion $\mathcal{A} *_{G} \mathrm{pt} \to \mathrm{ch}_{\mathrm{hf}}(\mathcal{A} *_{G} \mathrm{pt})$ is an equivalence by [BR05, Lemma 8.1]. Now for a chain complex $C_{\bullet}$ which represents an element in $\mathrm{Sw}^{\mathrm{ch}}(G; \mathbb{Z})$ one defines a functor

$$- \otimes C_{\bullet} \colon \mathcal{A} *_{G} \mathrm{pt} \to \mathrm{ch}_{\mathrm{hf}}(\mathcal{A} *_{G} \mathrm{pt})$$

analogously to $- \otimes M$ above by $A \otimes C_{\bullet} = A \otimes UC_{\bullet}$. This is well defined because $UC_{\bullet}$ is by definition of $\mathrm{Sw}^{\mathrm{ch}}(G; \mathbb{Z})$ a bounded below complex of finitely generated free $\mathbb{Z}$-modules whose homology is concentrated in finitely many degrees and each of its homology groups is a finitely generated $\mathbb{Z}$-module. Such a complex is homotopy equivalent to a complex $C'_{\bullet}$ of finitely generated free $\mathbb{Z}$-modules which is concentrated in finitely many degrees. (In order to prove this assume that $C_{\bullet}$ is concentrated in non-negative degrees and use induction over the largest number m such that $H_{m}(C_{\bullet}) \neq 0$. In the case $m = 0$ the complex $UC_{\bullet}$ is a resolution of the finitely generated $\mathbb{Z}$-module $H_{0}(C_{\bullet})$ and is hence homotopy equivalent to a finite resolution. For $m \geq 1$ choose a finite resolution $D_{\bullet}$ of $H_{m}(C_{\bullet})$ and construct a H_{m}-isomorphism $f \colon D_{\bullet} \to UC_{\bullet}$. Factorize f over its mapping cylinder $\mathrm{cyl}(f) \simeq UC_{\bullet}$ and study the sequence $D_{\bullet} \to \mathrm{cyl}(f) \to \mathrm{cone}(f)$.) Consequently $A \otimes UC_{\bullet}$ is homotopy equivalent to $A \otimes C'_{\bullet}$ and hence lies in $\mathrm{ch}_{\mathrm{hf}}\,\mathcal{A} *_{G} \mathrm{pt}$. Again a short exact sequence of chain complexes leads to a short exact sequence of functors, because the objects depend only on the underlying $\mathbb{Z}$-chain complexes. A homology equivalence $C_{\bullet} \to D_{\bullet}$ can be considered as a homotopy equivalence $UC_{\bullet} \to UD_{\bullet}$ and hence induces a homotopy equivalence $A \otimes UC_{\bullet} \to A \otimes UD_{\bullet}$. $\qquad\square$

In the proof above and in the proof of Theorem 8.1 we used the following lemma.

Lemma 9.2. *Let $\mathcal{A}$ be a small additive category with a G-action by additive functors. Let κ be a fixed infinite cardinal. Denote by $\mathcal{F}^{\kappa}(\mathbb{Z})$ some small model for the category of all free $\mathbb{Z}$-modules which admit a basis B with $\mathrm{card}(B) \leq \kappa$. Equip $\mathcal{F}^{\kappa}(\mathbb{Z})$ with a tensorproduct functor $- \otimes_{\mathbb{Z}} -$. (This of course involves choices.) Let $\mathcal{F}^{f}(\mathbb{Z})$ be the full subcategory of $\mathcal{F}^{\kappa}(\mathbb{Z})$ that consists of finitely generated free $\mathbb{Z}$-modules.*

There exist additive categories $\mathcal{A}^{f}$ and $\mathcal{A}^{\kappa}$ with G-action and G-equivariant additive inclusion functors

$$\mathcal{A} \to \mathcal{A}^{f} \to \mathcal{A}^{\kappa}$$

such that the following conditions hold.

(i) *The inclusion $\mathcal{A} \to \mathcal{A}^{f}$ is an equivalence of categories.*

(ii) *In $\mathcal{A}^{\kappa}$ there exist categorical sums over indexing sets J with $\mathrm{card}(J) \leq \kappa$. Hence if A_{j}, $j \in J$ is a family of objects in $\mathcal{A}^{\kappa}$ then $\bigoplus_{j \in J} A_{j}$ exists. Moreover one can make a choice for these sum objects such that for every $g \in G$ we have an equality*

$$g^{*}\left(\bigoplus_{j \in J} A_{j}\right) = \bigoplus_{j \in J} g^{*}(A_{j}).$$

(iii) *There exists a bilinear bifunctor*

$$- \otimes - \colon \mathcal{A}^{\kappa} \times \mathcal{F}^{\kappa}(\mathbb{Z}) \to \mathcal{A}^{\kappa},$$

which restricts to

$$- \otimes - \colon \mathcal{A}^{f} \times \mathcal{F}^{f}(\mathbb{Z}) \to \mathcal{A}^{f}.$$

The functor is compatible with direct sums in the sense that for a family of objects A_{j}, $j \in J$ in $\mathcal{A}^{\kappa}$ with $\mathrm{card}(J) \leq \kappa$ and a $\mathbb{Z}$-module $F \in \mathcal{F}^{\kappa}(\mathbb{Z})$

there exists a natural isomorphism

$$\left(\bigoplus_{j \in J} A_j\right) \otimes F \cong \bigoplus_{j \in J} (A_j \otimes F)$$

(The analogous statement for finite direct sums holds in both variables because of the bilinearity.) The bifunctor is also compatible with the G-action in the sense that for every $g \in G$ and all morphisms $\phi \colon A \to B$ in $\mathcal{A}^\kappa$ and $f \colon F \to F'$ in $\mathcal{F}^\kappa(\mathbb{Z})$ we have equalities

$$g^*(A \otimes F) = (g^* A) \otimes F \quad and \quad g^*(f \otimes \phi) = g^*(f) \otimes \phi.$$

(iv) For objects A in $\mathcal{A}^\kappa$ and F, F' in $\mathcal{F}^\kappa(\mathbb{Z})$ we have a natural isomorphism

$$(A \otimes F) \otimes F' \cong A \otimes (F \otimes F').$$

Proof. The construction of the categories $\mathcal{A}^f$ and $\mathcal{A}^\kappa$ makes use of the following two elementary constructions. First construction: let $\mathcal{B}$ and $\mathcal{C}$ be two Ab-categories, i.e. categories enriched over abelian groups, compare [Mac71, I.8]. Then we define the Ab-category $\mathcal{B} \otimes \mathcal{C}$ as follows. Objects are pairs of objects which we denote $B \otimes C$, where B is an object in $\mathcal{B}$ and C an object in $\mathcal{C}$. We set

$$\mathrm{mor}_{\mathcal{B} \otimes \mathcal{C}}(B \otimes C, B' \otimes C') = \mathrm{mor}_{\mathcal{B}}(B, B') \otimes_{\mathbb{Z}} \mathrm{mor}_{\mathcal{C}}(C, C').$$

Composition and identities are defined in the obvious way. The construction is functorial with respect to additive functors in $\mathcal{B}$ and $\mathcal{C}$ and preserves additive equivalences.

Second construction: given an Ab-category $\mathcal{D}$ and a set I we define the category $\mathcal{D}(I)$ to be the category whose objects are families $D = (D(i))_{i \in I}$ of objects in $\mathcal{D}$ and where a morphism $f \colon D \to D'$ is a family of morphisms $f(j, i) \colon D(i) \to D(j)$, $i, j \in I$ subject to the condition that for a fixed $i \in I$ there are only finitely many $j \in I$ such that $f(j, i) \neq 0$. Composition is the usual matrix multiplication, where the components of $f' \circ f$ are given by

$$(f' \circ f)(k, i) = \sum_j f'(k, j) \circ f(j, i).$$

Now set $\mathcal{A}^f = \mathcal{A} \otimes \mathcal{F}^f(\mathbb{Z})$. The inclusion $\mathcal{A} \to \mathcal{A}^f$ is given by $A \mapsto A \otimes \mathbb{Z}^1$, where $\mathbb{Z}^1$ is some 1-dimensional free $\mathbb{Z}$-module in $\mathcal{F}^f(\mathbb{Z})$. It is not difficult to check that this is an equivalence of categories. We define a "tensor product" $- \otimes - \colon \mathcal{A}^f \times \mathcal{F}^f(\mathbb{Z}) \to \mathcal{A}^f$ by $(A \otimes F) \otimes F' = A \otimes (F \otimes F')$. The G-action on $\mathcal{A}^f$ is defined by $g^*(A \otimes F) = (g^* A) \otimes F$.

Next we choose a set I of cardinality κ and set

$$\mathcal{A}^\kappa = \left(\mathcal{A}^f \otimes \mathcal{F}^\kappa(\mathbb{Z})\right)(I).$$

The inclusion functor $\mathcal{A}^f \to \mathcal{A}^\kappa$ sends $A \otimes F$ to the object which at some fixed index i_0 is given by $A \otimes F$ and is zero everywhere else. The G-action extends via $g^*((A(i) \otimes F(i))_{i \in I}) = ((g^* A(i)) \otimes F(i))_{i \in I}$. The "tensor product" extends by $((A(i) \otimes F(i))_{i \in I}) \otimes F = (A(i) \otimes (F(i) \otimes F))_{i \in I}$, where of course the $A(i)$ are objects in $\mathcal{A}$ and the $F(i)$ and F are objects in $\mathcal{F}^\kappa(\mathbb{Z})$. The existence of the required direct sums is a consequence of the fact that $\mathrm{card}(I \times I) = \mathrm{card}(I)$ for an infinite set I, see for example [Lan02, Appendix 2, § 3, Theorem 3.6]. $\square$

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