BERLIN SUMMER SCHOOL TALKS
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MICHAEL WEISS

Talk notes and exercises. Much of the research is joint work with Rui Reis. The exercises are probably too hard and too numerous to be a good basis for the tutorial sessions, but perhaps they can provide some inspiration.

I. First talk: Rational Pontryagin classes and smoothing theory

It is known (Thom, Novikov, Kirby-Siebenmann and others) that

$$B\mathrm{O} \simeq_{\mathbb{Q}} B\mathrm{TOP}$$

where $B\mathrm{TOP} = \bigcup_n B\mathrm{TOP}(n)$ and $\mathrm{TOP}(n)$ is the (topological or simplicial) group of homeomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^n$. It is not true that $B\mathrm{O}(n) \simeq_{\mathbb{Q}} B\mathrm{TOP}(n)$ for all or most finite n. The cohomology $H^*(B\mathrm{TOP}(n); \mathbb{Q})$ is well understood only in the range $* < 4n/3$ approximately (Farrell-Hsiang). It follows from the Farrell-Hsiang computations that $B\mathrm{TOP}(n)$ is not rationally equivalent to $B\mathrm{O}(n)$ for sufficiently large odd n.

Hypothesis A: *For even n, we have $e^2 = p_{n/2}$ in $H^{2n}(BS\mathrm{TOP}(n); \mathbb{Q})$.*

Here $e \in H^n(BS\mathrm{TOP}(n); \mathbb{Z})$ is the Euler class (basic obstruction for splitting off trivial line bundles) and $p_{n/2}$ is the Pontryagin class, i.e., the unique class in $H^{2n}(B\mathrm{TOP}; \mathbb{Q})$ which maps to the Pontryagin class in $H^{2n}(B\mathrm{O}; \mathbb{Q})$. The S in $S\mathrm{TOP}(n)$ is for orientation-preserving homeomorphisms. Note that hypo A implies that $p_{n/2} = 0 \in H^{2n}(BS\mathrm{TOP}(m); \mathbb{Q})$ for $m < n$, and also $p_{n/2} = 0 \in H^{2n}(B\mathrm{TOP}(m); \mathbb{Q})$ for $m < n$.

Smoothing theory in the tradition of C Morlet (see Kirby-Siebenmann for an exposition) provides a homotopy pullback square

$$\begin{tikzcd}
\mathrm{Diff}_\partial(D^n) \arrow[r, "\nabla"] \arrow[d] & \Omega^n(\mathrm{O}(n)) \arrow[d] \\
\mathrm{Homeo}_\partial(D^n) \arrow[r, "\nabla"] & \Omega^n(\mathrm{TOP}(n))
\end{tikzcd}$$

The vertical arrows are obvious inclusions and the horizontal maps are obtained by taking derivatives of diffeomorphisms (and homeomorphisms ...). By Ω^n of a based space X we mean *space of maps* $(D^n, S^{n-1}) \to (X, \star)$.
Since $\mathrm{Homeo}_\partial(D^n)$ is contractible (Alexander trick), it follows that

$$\mathrm{Diff}_\partial(D^n) \simeq \Omega^{n+1}(\mathrm{TOP}(n)/\mathrm{O}(n)).$$

This explains why $B\mathrm{TOP}(n)$ is important in *differential topology* and suggests that we can learn something about $B\mathrm{TOP}(n)$ by exploring $\mathrm{Diff}_\partial(D^n)$. (In this connection, see also recent work of Watanabe.)

Date: May 31.

It is a tradition from concordance theory to explore the spaces $\mathrm{Diff}_\partial(D^n)$ and therefore the spaces $B\mathrm{TOP}(n)$ by means of homotopy fiber sequences

$$\Omega\mathrm{Diff}_\partial(D^n) \longrightarrow \mathrm{Diff}_\partial(D^n \times D^1) \longrightarrow \mathcal{R}(n,1)$$

where $\mathcal{R}(n,1)$ is the space of *regular* (=nonsingular) smooth maps from $D^n \times D^1$ to $\mathbb{R}$ which on the boundary of $D^n \times D^1$ agree with the second projection $D^n \times D^1 \to D^1 \subset \mathbb{R}$. This led to the Farrell-Hsiang computations mentioned above. In connection with hypothesis A, it is more appropriate to set up homotopy fiber sequences

$$\Omega^2\mathrm{Diff}_\partial(D^n) \longrightarrow \mathrm{Diff}_\partial(D^n \times D^2) \longrightarrow \mathcal{R}(n,2)$$

where $\mathcal{R}(n,2)$ is the space of regular smooth maps from $D^n \times D^2$ to $\mathbb{R}^2$ which on the boundary of $D^n \times D^2$ agree with the second projection $D^n \times D^2 \to D^2 \subset \mathbb{R}^2$. There is an action of $S^1 = S\mathrm{O}(2)$ on $\mathcal{R}(n,2)$ by conjugation. There is a "derivative" map ∇ from $\mathcal{R}(n,2)$ to the base point component of Ω^{n+2} of the based space of surjective linear maps $\mathbb{R}^n \times \mathbb{R}^2 \to \mathbb{R}^2$. We write informally

$$\nabla\colon \mathcal{R} \longrightarrow \Omega_0^{n+2}(\mathrm{O}(n+2)/\mathrm{O}(n)).$$

It is an S^1-map for a suitable action of S^1 on the target. For even $n \geq 4$, the target is rationally an Eilenberg-MacLane space $K(\mathbb{Z}, n-3)$.

Hypothesis B. *For even $n \geq 4$, the cohomology class in Borel cohomology $H^{n-3}_{S^1}(\mathcal{R};\mathbb{Q})$ defined by the above ∇ is zero.*

By smoothing theory, hypothesis B has the following translation. We have

$$\mathcal{R} \simeq \Omega^{n+2}\mathrm{hofiber}[\mathrm{O}(n+2)/\mathrm{O}(n) \to \mathrm{TOP}(n+2)/\mathrm{TOP}(n)]$$

and the map ∇ out of $\mathcal{R}$ turns into the obvious forgetful map

$$\Omega^{n+2}\mathrm{hofiber}[\mathrm{O}(n+2)/\mathrm{O}(n) \to \mathrm{TOP}(n+2)/\mathrm{TOP}(n)] \longrightarrow \Omega^{n+2}(\mathrm{O}(n+2)/\mathrm{O}(n))\ .$$

Therefore hypothesis B is related to the problem of finding a (rational) homotopy left inverse for the inclusion-induced map $\mathrm{O}(n+2)/\mathrm{O}(n) \to \mathrm{TOP}(n+2)/\mathrm{TOP}(n)$. The expressions $\mathrm{O}(n+2)/\mathrm{O}(n)$ and $\mathrm{TOP}(n+2)/\mathrm{TOP}(n)$ are reminiscent of second rates of change for "functors" $n \mapsto B\mathrm{O}(n)$ and $n \mapsto B\mathrm{TOP}(n)$. This will be taken up in talk 2, but then with functors $V \mapsto B\mathrm{O}(V)$ and $V \mapsto B\mathrm{TOP}(V)$ where V is a finite dimensional real vector space with inner product.

The implication $A \Rightarrow B$ is easy. Some indications will be given in exercises.

II. **Second talk: Characteristic classes for homotopy theorists**

The following is due to Ioan James. Let $v\colon \mathbb{R}P^{n-1} \longrightarrow \mathrm{O}(n)$ be the inclusion of the subspace of reflections. Then v has a stable splitting; in other words there is a map

$$\omega\colon \mathrm{O}(n) \longrightarrow \Omega^\infty\Sigma^\infty(\mathbb{R}P^{n-1}_+)$$

such that ωv is the inclusion $\mathbb{R}P^{n-1} \to \Omega^\infty\Sigma^\infty(\mathbb{R}P^{n-1}_+)$. This splitting and variations were very important in Adams' work on the vector-fields-on-spheres problem.

The splitting map ω can be composed with linearization

$$\lambda\colon \Omega^\infty\Sigma^\infty(\mathbb{R}P^{n-1}_+) \longrightarrow \Omega^\infty(\mathbf{H}\mathbb{Z}/2 \wedge \mathbb{R}P^{n-1}_+) \ \simeq\ \prod_{j=0}^{n-1} K(\mathbb{Z}/2, j).$$

The composition $\lambda\omega$ is therefore a collection of cohomology classes on $\mathrm{O}(n)$. These are the looped Stiefel-Whitney classes $\Omega w_1, \dots, \Omega w_n$.

Later Kahn and Priddy found that ω extends to $\mathrm{G}(n)$, the space of homotopy automorphisms of S^{n-1}, and there is compatibility for the different n, so that there is a map

$$\omega\colon \mathrm{G} = \bigcup_n \mathrm{G}(n) \longrightarrow \Omega^\infty\Sigma^\infty(\mathbb{R}P^\infty_+)\ .$$

Again $\lambda\omega$ is the collection of the looped Stiefel-Whitney classes. But it is more interesting to note, as Kahn-Priddy did, that the composition of the new ω with the transfer

$$\Omega^\infty\Sigma^\infty(\mathbb{R}P^\infty_+) \longrightarrow \Omega^\infty\Sigma^\infty S^0$$

(a map of infinite loop spaces) is the inclusion of G in $\Omega^\infty\Sigma^\infty S^0$ up to a translation. So it is a homotopy equivalence after discarding some components of the target. Therefore the transfer map is split onto on homotopy groups in positive dimensions.

Soon after, M Crabb (LMS book) found that the maps ω of James and Kahn-Priddy could be delooped. The delooping is not exactly a map with source $B\mathrm{O}$ or $B\mathrm{G}$, but a section of a fibration on $B\mathrm{O}$ or $B\mathrm{G}$ where each fiber has an infinite loop space structure (which includes a preferred "zero" element) and is equivalent as such to $\Omega^{\infty-1}(\mathbb{R}P^\infty_+)$. After composition with linearization λ as above, it turns into an honest map, the total Stiefel-Whitney class. Crabb's construction is therefore a "nonlinear" improvement on the traditional total Stiefel-Whitney class. Crabb also constructed a similar nonlinear improvement on the total Chern class of a complex vector bundle.

Much later Weiss and Williams (1988) formalized the James and Priddy constructions (but not Crabb's delooping) in the following manner. Let $\mathcal{J}$ be the category of finite dimensional real vector spaces with inner product, where morphisms are isometric linear injections. Let F be a covariant continuous functor from $\mathcal{J}$ to topological groups (or grouplike associative topological monoids). Let $F(\mathbb{R}^\infty)$ be the homotopy colimit (telescope) of the groups $F(\mathbb{R}^n)$, for $n \to \infty$. Then there is a map

$$F(\mathbb{R}^\infty) \longrightarrow \Omega^\infty(\Theta F \wedge_{\mathbb{Z}/2} E\mathbb{Z}/2_+)$$

where ΘF is a certain spectrum constructed from F. It is made up of the spaces

$$F(\mathbb{R}^{n+1})/F(\mathbb{R}^n) := \text{hofiber}[\ BF(\mathbb{R}^n) \to BF(\mathbb{R}^{n+1})\ ].$$

For example, if $F(V) = \mathrm{TOP}(V)$, then ΘF is made up of the spaces $\mathrm{TOP}(n+1)/\mathrm{TOP}(n)$ which, as we have seen, are closely related to diffeomorphisms of disks and more specifically to the spaces $\mathcal{R}(n,1)$ from concordance theory. The spectrum ΘF therefore turns out to be a form of algebraic K-theory, better known as Waldhausen's $\mathbf{A}(\star)$.

This led to the orthogonal calculus (Weiss 1995). Let F be any covariant continuous functor from $\mathcal{J}$ to spaces (based spaces for simplicity). There is a way to extract from such a functor a sequence of spectra $\Theta^{(i)}F$ with action of $\mathrm{O}(i)$, where $i = 1, 2, 3, 4, \dots$. They serve as the coefficient spectra for something like Stiefel-Whitney classes (case $i = 1$) and Pontryagin classes (case $i = 2$) etc. on $F(\mathbb{R}^\infty)$. More precisely, there is a characteristic class, analogous to Crabb's nonlinear total Stiefel-Whitney class, on $F(\mathbb{R}^\infty)$. It is represented by a section σ_1 of a fibration on $F(\mathbb{R}^\infty)$ whose fibers are infinite loop spaces (each with a base point, alias zero element). These infinite loop spaces are equivalent as such to

$$\Omega^{\infty-1}(\Theta^{(1)}F \wedge_{\mathrm{O}(1)} E\mathrm{O}(1)_+)$$

On the *vanishing locus* of this section (space of vertical paths from σ_1 to zero section), a second characteristic class is defined, analogous to the total Pontraygin class of real vector bundles. It is represented by a section σ_2 of a fibration whose fibers are infinite loop spaces equivalent to

$$\Omega^{\infty-1}(\Theta^{(2)}F \wedge_{\mathrm{O}(2)} E\mathrm{O}(2)_+).$$

And so on.
There is not enough time to give many details here, but it is important to say what this has to do with functor calculus *and* what it has to do with hypotheses A and B. Many types of functor calculus are about covariant functors from a fixed category $\mathcal{C}$ to spaces. They start by making sense of the concept *polynomial functor of degree* $\le k$ from $\mathcal{C}$ to spaces. Given *any* functor F from $\mathcal{C}$ to spaces, we try to find a natural transformation $\eta_k : F \to T_kF$ where T_kF is polynomial of degree $\le k$ and η_k has a universal (initial) property. If that can be done, then the universal properties of the η_k for all k lead to a tower diagram (Taylor tower) of natural transformations

$$\begin{tikzcd}
 & {} \arrow[d, dotted] \\
 & T_3F \arrow[d] \\
 & T_2F \arrow[d] \\
 & T_1F \arrow[d] \\
F \arrow[r, "\eta_0"'] \arrow[ur, "\eta_1"'] \arrow[uur, "\eta_2"'] \arrow[uuur, "\eta_3"'] & T_0F
\end{tikzcd}$$

If it makes sense to speak of the homotopy fiber(s) of $T_kF \to T_{k-1}F$, then these should be *homogeneous functors of degree* k and one hopes to have a classification theorem for them.

Definition. A continuous functor E from $\mathcal{J}$ to spaces is polynomial of degree $\le k$ if for every V in $\mathcal{J}$ the canonical map

$$E(V) \longrightarrow \underset{0 \ne U \subset \mathbb{R}^{k+1}}{\operatorname{holim}} F(V \oplus U)$$

is a homotopy equivalence.

The homotopy (inverse) limit is taken over the poset of all nonzero linear subspaces of $\mathbb{R}^{k+1}$. It is a topological poset. The topology is built into the definition of the homotopy limit.

Example. If E is polynomial of degree 0, then $E(V) \to E(V \oplus \mathbb{R})$ induced by the inclusion $V \to V \oplus \mathbb{R}$ is always a homotopy equivalence; it follows that $E(0) \to E(V)$ induced by the inclusion $0 \to V$ is always a homotopy equivalence, and so E is essentially constant.
Example. A functor E which is polynomial of degree ≤ 5, say, has strong extrapolation properties: if we know how it behaves on the full subcategory of $\mathcal{J}$ consisting of all vector spaces of dimensions 10,11,12,13,14,15, then we know how it behaves on all vector spaces of degree 9, too ... and by iteration, how it behaves on all vector spaces of dimension ≤ 15.

For a continuous F from $\mathcal{J}$ to spaces, best polynomial approximations $\eta_k : F \to T_kF$ exist. If F has values in based spaces, we can speak of the homotopy fiber of $T_kF \to T_{k-1}F$, which is a *homogeneous* functor of degree k with values in based spaces. The classification theorem for these is as follows.

Theorem. *For any homogeneous functor E of degree $k > 0$ from $\mathcal{J}$ to based spaces, there exists a spectrum Θ with action of $\mathrm{O}(k)$ such that E is equivalent (...) to the functor*

$$V \mapsto \Omega^\infty(((V \otimes \mathbb{R}^k)^c \wedge \Theta)_{h\mathrm{O}(k)})$$

where $(\ldots)^c$ *is a one-point compactification and* $(\ldots)_{h\mathrm{O}(k)}$ *means a homotopy orbit* (*Borel*) *construction.*

So a continuous F from $\mathcal{J}$ to based spaces determines a Taylor tower with homogeneous layers of degree $k = 1, 2, 3, \ldots$, and each of these homogeneous layers determines a spectrum $\Theta = \Theta^{(k)}F$ with action of $\mathrm{O}(k)$. We think of them as the derivatives of F at *infinity.* Their homotopy type only depends on the behavior of F on vector spaces of high dimension: indeed, $\Theta^{(k)}F$ in degree kn is the based space

$$\mathrm{hofiber}\Big[\ E(\mathbb{R}^n) \longrightarrow \operatorname*{holim}_{0\neq U\subset\mathbb{R}^k}\ F(\mathbb{R}^n \oplus U)\ \Big].$$

Therefore it is most interesting to evaluate the Taylor tower of F at an object maximally far away from infinity. The object 0 recommends itself! The Taylor tower is

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & T_3F(0) \arrow[d] \\
 & T_2F(0) \arrow[d] \\
 & T_1F(0) \arrow[d] \\
F(0) \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & T_0F(0)
\end{tikzcd}$$

Here $T_0F(0) = F(\mathbb{R}^\infty) = \mathrm{hocolim}_n F(\mathbb{R}^n)$. The map $T_1F(0) \to T_0F(0)$, converted to a fibration, is a *principal twisted H-fibration* where H is the infinite loop space $\Omega^{\infty-1}(\Theta^{(1)}F_{h\mathrm{O}(1)})$. As such it is classified by a generalized cohomology class $[\sigma_1]$ of degree 1 on $T_0F(0)$ with twisted coefficients in the spectrum

$$(\Theta^{(1)}F)_{h\mathrm{O}(1)}\ .$$

Similarly the map $T_2F(0) \to T_1F(0)$ is classified by a generalized cohomology class $[\sigma_2]$ of degree 1 on $T_1F(0)$ with twisted coefficients in the spectrum $(\Theta^{(2)}F)_{h\mathrm{O}(2)}$. And so on. In the cases $F(V) = B\mathrm{O}(V)$ and $F(V) = B\mathrm{G}(V)$, the class $[\sigma_1]$ is the nonlinear total Stiefel-Whitney class of Crabb, described above. For general F, we can still think of it as a generalized total Stiefel-Whitney class. Likewise, think of $[\sigma_2]$ as a generalized total Pontryagin class.

Now we discuss the implication $B \Rightarrow A$. We compare the functors $V \mapsto E(V) = B\mathrm{O}(V)$ and $V \mapsto F(V) = B\mathrm{TOP}(V)$ on $\mathcal{J}$. Assuming hypothesis B, we try to make a (rational) $F \to E$ which is homotopy left inverse to the obvious $E \to F$. This implies hypothesis A because it is easy to check that such a hypothetical left inverse $F \to E$ must take the Pontryagin classes for E to the Pontryagin classes for F, and the Euler classes for E to the Euler classes for F. The functor E is rationally polynomial of degree 2. Furthermore the inclusion $T_0E(0) \to T_0F(0)$ is a rational equivalence, since it amounts to $B\mathrm{O} \simeq_{\mathbb{Q}} B\mathrm{TOP}$. In that way ... it turns out to be enough to show that the maps of spectra

$$\Theta^{(k)}E \to \Theta^{(k)}F$$

induced by $E \to F$ admit rational homotopy left inverses with enough $\mathrm{O}(k)$ invariance for $k = 1, 2$. For $k = 1$ this is well known: the inclusion $\mathbf{S}^0 \to \mathbf{A}(\star)$ has such a left inverse, and not just rationally. In the case $k = 2$ we want to deduce it from hypothesis B. For that we use smoothing theory to identify the source and target of the map ∇ in hypo B as unstable forms (or sufficiently good approximations) of the forgetful map

$$\text{hofiber}[\Theta^{(2)}E \to \Theta^{(2)}F] \longrightarrow \Theta^{(2)}E\ .$$

III. Third talk: Singularity theory, multisingularity theory, h-principles and generalizations

In this talk the order of topics and their relative weight is not what the title suggests.

The best known h-principle is the main theorem of immersion theory. Let M and N be smooth manifolds, without boundary for simplicity, $\dim(N) > \dim(M)$. Let $\mathbf{imm}(M,N)$ be the space of smooth immersions and let $\mathbf{fimm}(M,N)$ be the space of formal immersions: an element is a pair (f,λ) where $f\colon M \to N$ is a continuous map and λ is a vector bundle monomorphism $TM \to f^*TN$, over M.

Theorem. *The embedding* $\mathbf{imm}(M,N) \to \mathbf{fimm}(M,N)$ *taking* f *to* (f,df) *is a homotopy equivalence.* (Here $\mathbf{imm}(M,N)$ is the question while $\mathbf{fimm}(M,N)$ is the answer. See exercises.)

Experts in h-principles will tell you that it has something to do with sheaves: $\mathbf{imm}(-,N)$ and $\mathbf{fimm}(-,N)$ are sheaves and the two almost obviously *agree on stalks*, up to homotopy equivalence (see exercises). Therefore you should always have expected this theorem, although it took some great minds by surprise long ago (1959).

For another formulation, we need some abstract homotopy nonsense. Let $\mathcal{C}$ be a small category and let X, Y be contravariant functors from $\mathcal{C}$ to spaces (cautious interpretation of the word *space*). There is a space $\mathrm{nat}(X,Y) = \mathrm{nat}_{\mathcal{C}}(X,Y)$ of natural transformations. It is known to have bad homotopy properties. If $f\colon X \to X'$ induces homotopy equivalences $X(c) \to X'(c)$ for all c or $g\colon Y \to Y'$ induces homotopy equivalences $Y(c) \to Y'(c)$ for all c, it does not follow that $f^*\colon \mathrm{nat}(X',Y) \to \mathrm{nat}(X,Y)$ is a homotopy equivalence, nor does it follow that $g_*\colon \mathrm{nat}(X,Y) \to \mathrm{nat}(X,Y')$ is a homotopy equivalence. There is a substitute $\mathrm{honat}(X,Y) = \mathrm{honat}_{\mathcal{C}}(X,Y)$ which does not have these failings, with a natural comparison map $\mathrm{nat}(X,Y) \to \mathrm{honat}(X,Y)$. As a functor of the variable X alone, it is sufficiently characterized by these properties and the following additional ones: the map $\mathrm{nat}(X,Y) \to \mathrm{honat}(X,Y)$ is a homotopy equivalence for representable X and $\mathrm{honat}(-,Y)$ takes homotopy pushout squares to homotopy pullback squares (oops, and an axiom for disjoint unions, also infinite ones).

Let $\mathcal{M}an$ be the category of smooth d-dimensional manifolds (fixed d) where the morphisms are smooth embeddings. We think of this as enriched, that is, each morphism set is a space. Let $\mathcal{D}_1$ be the full subcategory spanned by the objects $\mathbb{R}^d$ and (optionally) $\emptyset$. For M in $\mathcal{M}an$ and fixed smooth N of dimension $> d$, every immersion $f\colon M \to N$ determines a natural transformation of functors on $\mathcal{D}_1$,

$$\mathbf{emb}(-,M) \longrightarrow \mathbf{imm}(-,N)$$

by composition with f. It is easy to show that the resulting map

$$\mathbf{imm}(M,N) \to \mathrm{nat}_{\mathcal{D}_1}(\mathbf{emb}(-,N), \mathbf{imm}(-,N))$$

is a bijection and with appropriate definitions it is a homeomorphism, due to the sheaf property of $\mathbf{imm}(-,N)$. The following is much less obvious; it is another formulation of the immersion theorem.

Theorem bis. *The composite map*

$$\mathbf{imm}(M,N) \to \cdots \to \mathrm{honat}_{\mathcal{D}_1}(\mathbf{emb}(-,M), \mathbf{imm}(-,N))$$

is a homotopy equivalence for every M in $\mathcal{M}an$.

If you accept that the analogous statement holds for $\mathbf{fimm}(M,N)$, which is easier, then you can easily prove the standard formulation of the immersion theorem, as above, assuming the new-fangled formulation. Assuming the new-fangled means: it is enough to check that the comparison map $\mathbf{imm}(M,N) \to \mathbf{fimm}(M,N)$ is a homotopy equivalence for M in $\mathcal{D}_1$. See exercises.

In the "bis" formulation the theorem has some very interesting variants. For example let $\mathbf{emb}(M,N)$ be the space of smooth embeddings from M to N, for M in $\mathcal{M}an$. For $k \geq 0$ let $\mathcal{D}_k$ be the full subcategory of $\mathcal{M}an$ with objects $\mathbb{R}^d \times \{1,2,\dots,\ell\}$ where $0 \leq \ell \leq k$. (Therefore $\mathcal{D}_k$ has exactly $k+1$ objects.)

Theorem. *The composite map*

$$\mathbf{emb}(M,N) \to \cdots \to \mathrm{honat}_{\mathcal{D}_k}(\mathbf{emb}(-,M), \mathbf{emb}(-,N))$$

is $((n-d-2)k - const)$-connected.

This is embedding theory in the spirit of immersion theory. The formal deduction of this from *multiple disjunction theorems for embeddings* can be found in Weiss and Goodwillie-Weiss (around 1999), in a slightly different language. Status of the multiple disjunction theorems: work of Goodwillie and Klein, most of it available on the web, but not as published as you might think. Monumental at any rate.
It is interesting that we seem to have given up on the idea that h-principles must be closely related to sheaf theory. For this theorem about embeddings resembles an h-principle but $\mathbf{emb}(-,N)$ is not a sheaf, or what? See exercises.

Before we look at other examples let me introduce some notation. Given a continuous contravariant functor F from $\mathcal{M}an$ to spaces, M in $\mathcal{M}an$ and $k \geq 0$, let

$$T_kF(M) := \mathrm{honat}_{\mathcal{D}_k}(\mathbf{emb}(-,M), F(-)) \ .$$

We have as before $F(M) \to T_kF(M)$. This is very reminiscent of notation in the previous talk and indeed we get a tower of contravariant functors on $\mathcal{M}an$,

$$\begin{tikzcd}
 & \vdots \arrow[d, dotted] \\
 & T_3F \arrow[d] \\
 & T_2F \arrow[d] \\
 & T_1F \arrow[d] \\
F \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & T_0F \ .
\end{tikzcd}$$

This is the *Taylor tower in manifold calculus* (context free set-up). But I do not want to emphasize the functor calculus point of view (again) in this talk.

Example. For $d \geq 1$ and M in $\mathcal{M}an$ let $F(M)$ be the space of smooth maps $M \to \mathbb{R}$ which have no singularities other than Morse and birth-death singularities. (A birth-death singularity is, roughly, a pair of Morse singularities in collision. If you have read Milnor's book on the h-cobordism theorem, you have seen them.) Or for $d \geq 2$ and M in $\mathcal{M}an$ let $F(M)$ be the space of smooth maps $M \to \mathbb{R}^2$ which have no singularities other than folds, cusps, lips, beak-to-beak and swallowtail. (Here folds and cusps are among the oldest and best known singularities of maps to the plane, studied already by Whitney; the other three types are three ways in which two cusps can collide and cancel out.) As part of a more general theorama, V Vassiliev showed the following (for both choices of F).

Theorem. *The canonical map $F(M) \to T_1F(M)$ induces an isomorphism in homology.*

In the case where M is open (no compact components), that map is a homotopy equivalence and the statement is part of an older theorama due to Gromov, generalizing the immersion theorem. But Vassiliev emphasizes the case where M is closed, and it does not help in understanding his proof to look for a close analogy with immersion theory. Instead, let's look for an analogy with embedding theory. The point is this. Let λ (*linearization*) be a functor such as Thom-Dold's infinite symmetric product, taking a space X to a based space λX with abelian monoid structure such that the homotopy groups of λX are the homology groups of X. We are trying to show that the canonical map $F \to T_1F$ induces an equivalence $\lambda F \to \lambda T_1F$. Now it is remarkable that λF does not have the slightest chance of being in any way identifiable with $T_1(\lambda F)$, unlike F which is supposed to resemble T_1F. For example F will take disjoint unions $M_1 \amalg M_2$ to products: $F(M_1 \amalg M_2) = F(M_1) \times F(M_2)$, and T_1F has the same property by inspection, and $T_1(\lambda F)$ has the same property almost (exercise), but $\lambda F(M_1 \amalg M_2)$ fails spectacularly to be similar to $\lambda F(M_1) \times \lambda F(M_2)$. Why? Because $\lambda F(M_1 \amalg M_2) \cong \lambda(F(M_1) \times F(M_2))$ is more like the *tensor product* of $\lambda F(M_1)$ and $\lambda F(M_2)$, stupid. Therefore it is a good idea to try $T_k\lambda F$ and $T_k\lambda T_1F$, and this is what Vassiliev does. He has some very beautiful arguments to show that the vertical arrows in

$$\begin{array}{ccc} \lambda F & \longrightarrow & \lambda(T_1F) \\ \downarrow & & \downarrow \\ T_k(\lambda F) & \longrightarrow & T_k(\lambda(T_1F)) \end{array}$$

are highly connected (connectivity $\to \infty$ as $k \to \infty$). It happens to be obvious that the lower horizontal arrow is an equivalence. (End of proof.) This interpretation/appreciation of Vassiliev's proof comes from Reis-Weiss III ... combined with Boavida-Weiss.

We have seen that the sheaf property in a narrow sense is not something we need to insist on in discussing h-principles and generalizations. The F in Vassiliev's theorem *is* a sheaf. Can we generalize Vassiliev's theorem by allowing some F which are not sheaves? Here is an example. Define another F in such a way that $F(M)$ for M in $\mathcal{M}an$ is the space of smooth functions $M \to \mathbb{R}$ which have only Morse and birth-death singularities, but never more than one birth-death singularity *for each critical value*. This F is not a sheaf. We do not want to compare with T_1F because for example T_1F shows simple behavior on disjoint unions while F does not. But we still have a theorem about this F. Indeed, much of Vassiliev's proof survives and so it is still true that the comparison maps

$$\begin{array}{c} \lambda F \\ \downarrow \\ T_k(\lambda F) \end{array}$$

are highly connected (connectivity $\to \infty$ as $k \to \infty$). This gives valuable insight into the homology of $F(M)$ for all or some M.

IV. **Fourth talk: Spaces of smooth maps to the plane and applications to Pontryagin classes**

Our strategy for the proof of hypothesis B can be motivated by a summary of Cerf's approach to $\mathcal{R}(n,1)$. Let $\mathcal{W}(n,1)$ be the space of generalized Morse functions (only Morse and birth-death singularities allowed) from $D^n \times D^1$ to $\mathbb{R}$ with usual boundary conditions (must agree with the standard projection on or near boundary). Cerf notes that $\mathcal{W}(n,1)$ is connected. Then he works hard to show that $\pi_1(\mathcal{W}(n,1),\mathcal{R}(n,1))$ is trivial. Hence $\pi_0\mathcal{R}(n,1)$ is trivial (an important special case of the statement he is after). Generalizations by Hatcher-Wagoner and correction by Igusa and big generalizations by Waldhausen/Igusa.

We define $\mathcal{W} = \mathcal{W}(n,2)$ as follows. It is a subspace of the space of all smooth maps from $D^n \times D^2$ to $\mathbb{R}^2$. The remaining conditions on $f \in \mathcal{W}$ are as follows.

- Must agree with the standard projection $D^n \times D^2 \to D^2 \subset \mathbb{R}^2$ on the boundary.
- The only singularity types allowed are fold, cusp, lips, beak-to-beak and swallowtail. (For the singularities in this list the differential has rank 1, so each determines a line in $\mathbb{R}^2$, the *tangent in the target*.)
- For each critical value (in the target) there are only finitely many critical points (in the source).
- Where a critical value y has associated critical points $x_1,\dots x_r$ and corresponding tangents-in-target $\ell_1,\dots\ell_r$, at most one of the x_i is not a fold. If one of the x_i is not a fold, then all ℓ_i are distinct. If all the x_i are folds, then at most two of the ℓ_i can be the same, in which case the corresponding fold curves in the target make a single tangency at y.

Yes, pictures would be good. Yes, we deviate from Cerf by imposing conditions on singularities *in the target*, not just *in the source*.

And recall that $\mathcal{R} = \mathcal{R}(n,2)$ is the subspace of $\mathcal{W}$ consisting of the $f \in \mathcal{W}$ which are everywhere regular. Our strategy is then:

- Show that $H^{n-3}_{S^1}(\mathcal{W},\star;\mathbb{Q}) \longrightarrow H^{n-3}_{S^1}(\mathcal{R},\star;\mathbb{Q})$ induced by inclusion $\mathcal{R} \to \mathcal{W}$ is zero.
- Show by geometric means that the cocycle ∇ extends from $\mathcal{R}_{hS^1}$ to $\mathcal{W}_{hS^1}$.

First bullet: the main tool is manifold calculus analysis of $\mathcal{W}$ as in talk 3. (No details.) Second bullet: read on. We use an approximation

$$f\colon (M,L) \longrightarrow (\mathcal{W},\mathcal{R})$$

where M is smooth without boundary, with free S^1 action, f is an S^1-map, and L is a closed S^1-invariant subset of M and smooth codim 0 submanifold with boundary. M can be noncompact. We can assume that f is highly connected. In that case, of course, the orbit manifold M_{S^1} is a good model for $\mathcal{R}_{hS^1}$. Let

$$f^{\mathrm{ad}}\colon M \times_{S^1} (D^n \times D^2) \longrightarrow M \times_{S^1} D^2$$

be the adjoint, a map over M_{S^1}. (Strictly speaking the target should be $M \times_{S^1} \mathbb{R}^2$ but it can be arranged as written.) We can assume that f^{ad} is smooth and generic. (No details.) We can assume that there are no critical values in the boundary of $M \times_{S^1} D^2$. The target $M \times_{S^1} D^2$ is then stratified by "number and type of critical points in preimage". For example, there is one stratum containing the *regular* values, another for those having *one fold* as the only critical point in their preimage, another for *five folds and one cusp, all making different directions in target*, another for *three folds and one beak-to-beak, all making different directions in target*, another for *two kissing folds and three other folds making three other directions in target*, and so on. All strata are manifolds without boundary except for the stratum of regular values U which has $\partial U = M \times_{S^1} \partial D^2$. We have two theorems about this stratification, specifically about the stratum U.

1) *It is possible to attach an ideal boundary $\partial_{\mathrm{in}} U^\omega$ to U, in addition to the existing ∂U, to obtain a manifold U^ω with boundary $\partial_{\mathrm{in}} U^\omega \amalg \partial U$ and a* proper *map $U^\omega \to M \times_{S^1} D^2$ which extends the inclusion of U. Note that this extension must map $\partial_{\mathrm{in}} U^\omega$ onto the critical value set of f^{ad} .*

2) *The* S^1*-bundle on* $\partial_{\mathrm{in}}U^\omega$ *obtained by pulling back the* S^1*-bundle* $M \times D_2 \to M \times_{S^1} D^2$ *along* $\partial_{\mathrm{in}}U^\omega \to M \times_{S^1} D^2$ *is not far from being trivial. Its squared Euler class is zero* (*away from a few small primes*).

The dénouement based on these two theorems is as follows. We need to show that the class $[\nabla]$ extends from $H^{n-3}_{S^1}(\mathcal{R})$ to $H^{n-3}_{S^1}(\mathcal{W})$. For $H^{n-3}_{S^1}(\mathcal{W})$ we substitute

$$H^{n-3}(M_{S^1}) \;\cong\; H^{n-1}(M \times_{S^1} D^2, M \times_{S^1} \partial D^2)$$

and then, wanting to do even better, we go up to

$$H^{n-1}(U^\omega, \partial U^\omega)\ .$$

For $H^{n-3}_{S^1}(\mathcal{R})$ we substitute

$$H^{n-1}(L \times_{S^1} D^2, L \times_{S^1} \partial D^2)\ .$$

So the situation is that an $(n-1)$-cocycle is already prescribed in/on a region of U^ω far away from $\partial_{\mathrm{in}}U^\omega$: on $L \times_{S^1} D^2 \,\cup\, M \times_{S^1} \partial D^2$, and we look for an extension to all of U^ω, rel $\partial_{\mathrm{in}}U^\omega$. The cocycle, where already prescribed, is based on the relation $e^2 = p_{n/2}$ and independent vanishing arguments for e and $p_{n/2}$ applied to the fiberwise tangent bundle of

$$f^{\mathrm{ad}} \colon M \times_{S^1} (D^n \times D^2) \longrightarrow M \times_{S^1} D^2\ .$$

This description makes an extension of the cocycle to all of U^ω straightforward. But we need to know that the resulting cohomology class of dim $n-1$ on

$$\partial_{\mathrm{in}}U^\omega \cap (\text{union of connected components of } U^\omega \text{ making contact with } \partial U)$$

is zero. That cohomology class is pulled back from a class $[\gamma]$ on $\mathcal{K}_{hS^1}$ where $\mathcal{K}$ is the space of smooth homotopy n-disks in $D^n \times D^2$, with standard boundary $\partial D^n \times 0$. So a statement like $[\gamma] = 0$ would allow us to complete the argument. Because of the symmetry-breaking virtues of $\partial_{\mathrm{in}}U^\omega$, theorem 2) just above, and because we land in the base point component of $\mathcal{K}_{hS^1}$, it turns out that we can get away with a much weaker statement. Etc. Further details in preparation.

(*Much of the above is joint work with Rui Reis*).

V. Exercises 1

1a. Explain the equation $e^2 = p_{n/2}$ for oriented real n-dimensional vector bundles, in ordinary integer cohomology.

1b. Explain the equation $e^2 = p_{n/2}$ at the cocycle level in Chern-Weil theory. This is supposed to take place in the deRham complex of a smooth manifold M equipped with an orientable n-dimensional Riemannian vector bundle V.

1c. Recall the Alexander trick, saying that $\mathrm{Homeo}_\partial(D^n)$ is contractible.

1d. Give reasons why there is a homotopy fiber sequence

$$\Omega\mathrm{Diff}_\partial(D^n) \longrightarrow \mathrm{Diff}_\partial(D^n \times D^1) \longrightarrow \mathcal{R}(n,1)$$

and show that $\mathcal{R}(n,1)$ is homotopy equivalent to the space of *concordances of* D^n, that is, the space of diffeomorphisms from $D^n \times D^1$ to $D^n \times D^1$ which extend the identity on $D^n \times \{-1\}$ union $\partial D^n \times D^1$. (If you have questions on how to topologize spaces of smooth maps: don't look for a topology, make them into simplicial sets.)

1e. The homotopy equivalence of $\mathcal{R}(n,1)$ with the space of concordances of D^n in the previous problem ... what would you say is the analogue for $\mathcal{R}(n,2)$?

1f. The map of hypothesis B has an analogue

$$\nabla : \mathcal{R}(n,1) \longrightarrow \Omega^{n+1}(\mathrm{O}(n+1)/\mathrm{O}(n))$$

which is an $\mathrm{O}(1)$-map. (The description of the target is a bit informal; say what the correct definition should be.) Show that this map is nullhomotopic with *derived* $\mathrm{O}(1)$-invariance. That means: it is nullhomotopic with $\mathrm{O}(1)$-invariance if you enlarge the source to $\mathcal{R}(n,1) \times E\mathrm{O}(1)$, with the diagonal $\mathrm{O}(1)$-action.

1g. Explain why the base point component of $\Omega^{n+2}(\mathrm{O}(n+2)/\mathrm{O}(n))$ is rationally an Eilenberg-MacLane space $K(\mathbb{Z}, n-3)$ for even $n \geq 4$. What can you say about $\mathrm{O}(n+2)/\mathrm{O}(n)$ for odd n and what would you conclude for the base point component of $\Omega^{n+2}(\mathrm{O}(n+2)/\mathrm{O}(n))$ in that case?

1h. The space $\mathrm{O}(n+2)/\mathrm{O}(n)$ classifies n-dimensional vector bundles V with a trivialization of $V \oplus \mathbb{R}^2$. Explain that. So what does $\Omega^{n+2}(\mathrm{O}(n+2)/\mathrm{O}(n))$ classify? Use your answer and the equation $e^2 = p_{n/2}$ to sketch a geometric definition of an $(n-3)$-cocycle on

$$\Omega^{n+2}(\mathrm{O}(n+2)/\mathrm{O}(n))$$

whose class generates $H^{n-3}(...;\mathbb{Q})$. See also problem 1g.

1i. Assuming hypo A, repeat previous problem with $\Omega^{n+2}(\mathrm{TOP}(n+2)/\mathrm{TOP}(n))$ to construct an interesting rational $(n-3)$-cocycle on $\Omega^{n+2}(\mathrm{TOP}(n+2)/\mathrm{TOP}(n))$. How does this imply hypo B ?

VI. Exercises 2

2a. Show that a function $g\colon \mathbb{R}^p \to \mathbb{R}$ is polynomial of degree $\leq n$ iff, for every v and $w^1, \dots, w^{n+1}$ in $\mathbb{R}^p$ we have

$$g(v) = \sum_{\substack{S\subset\{1,\dots,n+1\}\\ S\neq\emptyset}} (-1)^{|S|-1} g(v + \Sigma_{i\in S}\, w^i) \ .$$

2b. "Discuss" the analogy between the Taylor tower of a functor F in orthogonal calculus and the Postnikov tower of a CW-space X. Do not assume that X is simply connected.

2c. (i) Let $\mathcal{A} = \mathrm{fun}(\mathcal{V}, \mathcal{Z})$ and $\mathcal{B} = \mathrm{fun}(\mathcal{U}, \mathcal{Z})$ where $\mathcal{U}$ is a subcategory of a small category $\mathcal{V}$, and $\mathcal{Z}$ is the category of sets. Then we have $\mathrm{res}\colon \mathcal{A} \to \mathcal{B}$, the restriction functor. Show that it has a right adjoint $\mathrm{ind}\colon \mathcal{B} \to \mathcal{A}$ and that this is given by

$$\mathrm{ind}(f)(x) = \mathrm{mor}_{\mathcal{B}}(\mathrm{mor}_{\mathcal{V}}(x, —), f)$$

for $f\colon \mathcal{U} \to \mathcal{Z}$ in $\mathcal{B}$ and x an object in $\mathcal{V}$.
(ii) If enriched categories and monoidal categories mean something to you, explain in detail what a category enriched over based spaces is. (*Based spaces* is a monoidal category with the smash product.) Here is an example. Define $\mathcal{J}_1 \supset \mathcal{J}$ as follows. The objects are finite dimensional real vector spaces $V, W, \dots$ with inner product. Let $\mathrm{mor}(V, W)$ be the space of morphisms from V to W in $\mathcal{J}$. The space of morphisms $\mathrm{mor}_1(V, W)$ in $\mathcal{J}_1$ is the Thom space of a vector bundle on $\mathrm{mor}(V, W)$ whose fiber over $g \in \mathrm{mor}(V, W)$ is the cokernel of g (alias orthogonal complement of image of g.) Composition in $\mathcal{J}_1$ ... details left to you. Explain how $\mathcal{J}_1$ is a category enriched over based spaces. In particular, composition of morphisms in $\mathcal{J}_1$ has something to do with smash products.
(iii) Show that there is a homotopy cofiber sequence

$$\mathrm{mor}_{\mathcal{J}}(\mathbb{R} \oplus V, W) \to \mathrm{mor}_{\mathcal{J}}(V, W) \to \mathrm{mor}_{\mathcal{J}_1}(V, W) \ .$$

(iv) Combine all the above to show that the restriction functor

$$\mathrm{res} : \mathrm{fun}(\mathcal{J}_1\,, \mathrm{Spaces}_*) \longrightarrow \mathrm{fun}(\mathcal{J}\,, \mathrm{Spaces}_*)$$

has a right adjoint ind, and what is more, for an object E in $\mathrm{fun}(\mathcal{J}\,, \mathrm{Spaces}_*)$ and V in $\mathcal{J}$ or $\mathcal{J}_1$ we have

$$\mathrm{ind}(E)(V) \cong \mathrm{hofiber}[\, E(V) \to E(V \oplus \mathbb{R})\,] \ .$$

Here fun refers to *continuous* functors at the very least, if not *enriched* functors.
(v) Deduce that for E in $\mathrm{fun}(\mathcal{J}\,, \mathrm{Spaces}_*)$, the rule

$$V \mapsto \Omega^V \,\mathrm{hofiber}[\, E(V) \to E(V \oplus \mathbb{R})\,]$$

is again a *functor* from $\mathcal{J}$ to spaces. This can be iterated; therefore E determines a sequence of spectra. Explain.

2d. For V and W in $\mathcal{J}$, we have the Hilbert-Schmidt inner product on $\hom_{\mathbb{R}}(V, W)$ given by $\langle f, g\rangle = \mathrm{trace}(g^* f)$. Use this to define the unit sphere $S(\hom_{\mathbb{R}}(V, W))$ and show that

$$S(\hom_{\mathbb{R}}(V, W)) \cong \operatorname*{hocolim}_{0\neq U\subset V} \mathrm{mor}_{\mathcal{J}}(U, W)$$

where U runs through the nonzero linear subspaces of V.
[*Hint: What does it mean to say that* $f \in \hom_{\mathbb{R}}(V, W)$ *has norm 1 ?*]
How does this help you to understand the formula (see talk notes) given for the term in degree kn of the spectrum $\Theta^{(k)}E$, where E is a continuous functor from $\mathcal{J}$ to based spaces?

VII. Exercises 3

3a. Prove that the comparison map (also known as jet prolongation)

$$\mathbf{imm}(M,N) \to \mathbf{fimm}(M,N)$$

is a homotopy equivalence when $M = \mathbb{R}^d$ and $d \le \dim(N)$. (*This is the induction beginning in a standard proof of the main theorem of immersion theory. You should not* use *the main theorem of immersion theory to prove it. You may use simplicial set models for the two spaces, or if not, you may use that* $\mathbf{imm}(M,N)$ *has the homotopy type of a CW-space. Or you may use Whitehead's theorem or similar.*)

3b. Let M be a smooth submanifold of a smooth N, of codimension > 0, with normal bundle $V \to M$. It is sometimes claimed that $\mathbf{imm}(M,N) \simeq \mathbf{imm}(V,N)$, so that immersion theory reduces to the codimension zero setting. Show by example that the claim is wrong. Come up with a corrected statement which still permits a reduction of immersion theory to the codimension zero setting.

3c. Let $j\colon S^1 \to \mathbb{R}^2$ be the standard inclusion. Let $f\colon S^1 \to S^1$ be any orientation–reversing diffeomorphism (for example, $z \mapsto z^{-1}$ in complex number notation). Using the main theorem of immersion theory, show that the immersions j and jf are not regularly homotopic, i.e., not in the same path component of $\mathbf{imm}(S^1,\mathbb{R}^2)$. More generally, use the main theorem to make a bijection from $\pi_0\mathbf{imm}(S^1,\mathbb{R}^2)$ to $\mathbb{Z}$. Describe the geometric meaning of this bijection. Draw representing immersions for each $z \in \mathbb{Z}$.

3d. Let $j\colon S^2 \to \mathbb{R}^3$ be the standard inclusion. Let $f\colon S^2 \to S^2$ be any orientation–reversing diffeomorphism (for example, reflection at the equator). Using the main theorem of immersion theory, show that the immersions j and jf are regularly homotopic. How big is $\pi_0\mathbf{imm}(S^2,\mathbb{R}^3)$?

3e. Let $j\colon S^{n-1} \to \mathbb{R}^n$ be the standard inclusion. Let $f\colon S^{n-1} \to S^{n-1}$ be any orientation–reversing diffeomorphism (for example, reflection at the equator). *Suppose* that the immersions j and jf are regularly homotopic. What can you say about n ?

3f. Show that in $\mathcal{D}_1$, each morphism space is homotopy equivalent to a finite discrete set. Make this explicit and describe the discrete category obtained from $\mathcal{D}_1$ by replacing each morphism space $\mathrm{mor}(V,W)$ by its set of connected components $\pi_0\mathrm{mor}(V,W)$.

3g. Show that in $\mathcal{D}_2$, each morphism space $\mathrm{mor}(V,W)$ is aspherical (i.e., the homotopy groups $\pi_n\mathrm{mor}(V,W)$ for $n > 1$ and any choice of base point are trivial).

3h. Let F be a contravariant (continous/enriched) functor from $\mathcal{D}_1$ to spaces and let $F^!$ be the a contravariant (continous/enriched) functor on $\mathcal{M}an$ defined by

$$F^!(M) := \mathrm{honat}_{\mathcal{D}_1}(\mathbf{emb}(-,M),F)\ .$$

Show that $F^!$ has the following sheaf-like property: given disjoint closed subsets A_0 and A_1 of some M in $\mathcal{M}an$, the square

$$\begin{tikzcd}
F^!(M) \arrow[r] \arrow[d] & F^!(M \smallsetminus A_0) \arrow[d] \\
F^!(M \smallsetminus A_1) \arrow[r] & F^!(M \smallsetminus (A_0 \cup A_1))
\end{tikzcd}$$

is a homotopy pullback square.
Similarly: Let F be a contravariant (continous/enriched) functor from $\mathcal{D}_k$ to spaces and let $F^!$ be the contravariant (continous/enriched) functor on $\mathcal{M}an$ defined by

$$F^!(M) := \mathrm{honat}_{\mathcal{D}_k}(\mathbf{emb}(-,M),F)\ .$$

Show that $F^!$ has the following sheaf-like property: given pairwise disjoint closed subsets $A_0, A_1, \dots, A_k$ of some M in $\mathcal{M}an$, the $(k+1)$-dimensional cube of spaces

$$R \mapsto F^!(M \smallsetminus \cup_{i\in R} A_i) \qquad\qquad (R \subset \{0,1,\dots,k\}\,)$$

is a homotopy cartesian cube.

3i. Let S be a finite nonempty set of cardinality $k \geq 2$. Let N be a smooth manifold of dimension n. Show that the diagram of spaces

$$R \mapsto \mathbf{emb}(R,N) \qquad\qquad (R \subset S)$$

(a k-dimensional cube) is so-and-so cartesian. This means that the forgetful or inclusion map

$$\mathbf{emb}(S,N) \to \underset{\substack{R\subset S\\ R\neq S}}{\mathrm{holim}}\ \mathbf{emb}(R,N)$$

is so-and-so connected.
Hint: Try $k = 2$ first. In the general case, you may use without proof a (higher Blakers-Massey) theorem of the form *if a q-cube of spaces is strongly cocartesian and the maps from the initial term to the nearest neighbors are $k_1, k_2, \dots, k_q$-connected, then the comparison map from the initial term of the cube to the homotopy limit of the other terms is k-connected, where $k = 1 - q + \sum k_s$* . This is theorem 2.3 in Goodwillie's *Calculus II*. Goodwillie also refers to Ellis-Steiner.

3j. Notation as in 3h. Fix $d \leq n$ and for $R \subset S$ let $V_R = S \times \mathbb{R}^d$. Show that the cube

$$R \mapsto \mathbf{emb}(V_R,N) \qquad\qquad (R \subset S)$$

is so-and-so-cartesian (same number as in 3h).

Math. Institut, Universität Münster, 48147 Münster, Germany
E-mail address: `m.weiss@uni-muenster.de`