# Computational Complexity of the $\alpha$-Ham-Sandwich Problem*

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## Abstract

A variant of the Ham-Sandwich Theorem by Bárány, Hubard, and Jerónimo [DCG 2008] states that given any $d$ measurable sets in $\mathbb{R}^d$ that are convex and *well-separated*, and any given $\alpha_1, \ldots, \alpha_d \in [0,1]$, there is a unique oriented hyperplane that cuts off a respective fraction $\alpha_1, \ldots, \alpha_d$ from each set. Steiger and Zhao [DCG 2010] proved a discrete analogue, which we call the *$\alpha$-Ham-Sandwich theorem.* They gave an algorithm to find the hyperplane in time $O(n(\log n)^{d-3})$, where $n$ is the total number of input points. The computational complexity of this search problem in high dimensions is open, unlike that of the Ham-Sandwich problem, which is now known to be PPA-complete (Filos-Ratsikas and Goldberg [STOC 2019]).

Recently, Fearley, Gordon, Mehta, and Savani [ICALP 2019] introduced a new sub-class of CLS (Continuous Local Search) called *Unique End-of-Potential Line* (UEOPL). This class captures problems in CLS that have unique solutions. We show that for the $\alpha$-Ham-Sandwich theorem, the search problem of finding the dividing hyperplane lies in UEOPL. This gives the first non-trivial containment of the problem in a complexity class and places it in the company of several classic search problems.

## 1 Introduction and preliminaries

The classic Ham-Sandwich theorem [7, 8, 12] states that for any $d$ measurable sets in $\mathbb{R}^d$, there is a hyperplane that bisects them simultaneously. Bárány et al. [2] proved a variant of this classic theorem that aims at dividing sets into arbitrary given ratios instead of simply bisecting them. The sets $S_1, \ldots, S_d \subset \mathbb{R}^d$ are *well-separated* if every selection of the sets can be strictly separated from the others by a hyperplane. If the sets are well-separated and convex, then for any given choice $\alpha_1, \ldots, \alpha_d \in [0,1]$, there is a unique oriented hyperplane that divides $S_1, \ldots, S_d$ in the ratios $\alpha_1, \ldots, \alpha_d$, respectively.

Steiger and Zhao [11] gave a discrete version of [2] and called their result the *Generalized Ham-Sandwich Theorem*, yet it is not a strict generalization of the classic Ham-Sandwich Theorem. Their result requires that the point sets obey well-separation and weak general position, while the classic theorem always holds without these assumptions. Therefore, we call this result the *$\alpha$-Ham-Sandwich theorem*, for a clearer distinction. Formally, given $d$ finite point sets $P_1, \ldots, P_d \subset \mathbb{R}^d$ and any set of positive integers $\{\alpha_1, \ldots, \alpha_d\}$ satisfying $1 \le \alpha_i \le |P_i|$, for all $i \in [d]$, where $[d]$ denotes the set $\{1, \ldots, d\}$, an *$(\alpha_1, \ldots, \alpha_d)$-cut* is an oriented hyperplane $H$ that contains one point from each set and satisfies $|H^+ \cap P_i| = \alpha_i$ for all $i \in [d]$, where $H^+$ is the closed positive half-space bounded by $H$.

▶ **Theorem 1.1** ($\alpha$-Ham-Sandwich Theorem [11]). *Let $P_1, \ldots, P_d$ be finite, well-separated point sets in $\mathbb{R}^d$. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a vector, where $\alpha_i \in [|P_i|]$ for all $i \in [d]$.*

***1.*** *If an $\alpha$-cut exists, then it is unique.*

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This is an extended abstract of a presentation given at EuroCG'20. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

*2. If $P$ is in a sufficiently general position, then a cut exists for each choice of $\alpha$.*
This statement does not necessarily hold if the sets are not well-separated, see Figure 1.

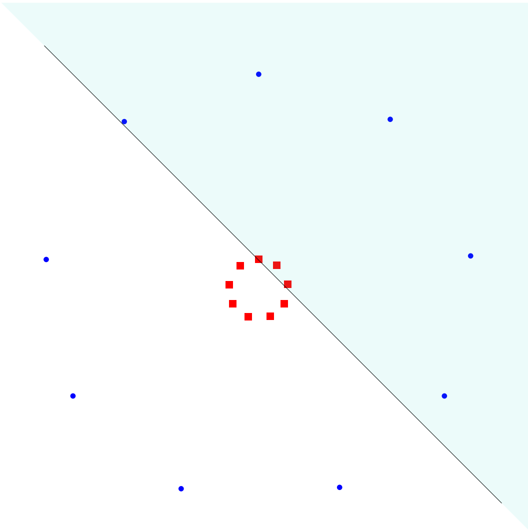

**Figure 1** The red (square) and the blue (round) point sets are not well-separated. There is no halfplane that contains exactly three red and three blue points.

We call the associated computational search problem of finding the dividing hyperplane Alpha-HS. Set $n = \sum_{i\in[d]} |P_i|$. Steiger and Zhao gave an algorithm that computes the dividing hyperplane in $O\left(n(\log n)^{d-3}\right)$ time, which is exponential in $d$. Later, Bereg [3] improved this algorithm to achieve a running time of $n2^{O(d)}$, which is linear in $n$ but still exponential in $d$. No polynomial algorithms are known for Alpha-HS if $d$ is not fixed. Despite their superficial similarity, it is not immediately apparent whether the classic Ham-Sandwich theorem problem and Alpha-HS are comparable in terms of their complexity. Due to the additional requirements on an input for Alpha-HS, an instance of the Ham-sandwich problem may not be reducible to Alpha-HS in general.

Alpha-HS is a total search problem and is modeled by the complexity class TFNP (Total Function Nondeterministic Polynomial) of NP-search problems that always admit a solution. A noteworthy sub-class is CLS (continuous local search), that was introduced by Daskalakis and Papadimitriou [4]. It models optimization problems that can be solved by local search over a continuous domain using a continuous potential function. Recently there have been increasing efforts towards mapping the complexity landscape of existence theorems in high-dimensional discrete geometry in such classes. It was shown in [6] that the search problem for the Ham-Sandwich theorem is complete for PPA. Finding a solution to the Colorful Carathéodory problem [1] was shown to lie in the intersection PPAD $\cap$ PLS [9, 10]. Here, PPAD $\subseteq$ PPA, CLS $\subseteq$ PLS $\cap$ PPAD are other sub-classes of TFNP.

Recently, Fearley et al. [5] defined a sub-class of CLS by the name *Unique End of Potential Line* that represents problems in CLS with unique solutions. They define it through a canonical complete problem UniqueEOPL:

▶ **Definition 1.2 (from [5]).** Let $n, m$ be positive integers. The input consists of
- a pair of Boolean circuits $\mathbb{S}, \mathbb{P} : \{0,1\}^n \to \{0,1\}^n$ such that $\mathbb{P}(0^n) = 0^n \neq \mathbb{S}(0^n)$, and
- a Boolean circuit $\mathbb{V} : \{0,1\}^n \to \{0,1,\dots,2^m-1\}$ such that $\mathbb{V}(0^n) = 0$,

each circuit having poly$(n, m)$ size. The UniqueEOPL problem is to report one of the following:

**(U1).** A point $v \in \{0,1\}^n$ such that $\mathbb{P}(\mathbb{S}(v)) \neq v$.

**(UV1).** A point $v \in \{0,1\}^n$ such that $\mathbb{S}(v) \neq v$, $\mathbb{P}(\mathbb{S}(v)) = v$, and $\mathbb{V}(\mathbb{S}(v)) - \mathbb{V}(v) \leq 0$.
**(UV2).** A point $v \in \{0,1\}^n$ such that $\mathbb{S}(\mathbb{P}(v)) \neq v \neq 0^n$.
**(UV3).** Two points $v, u \in \{0,1\}^n$ such that $v \neq u$, $\mathbb{S}(v) \neq v$, $\mathbb{S}(u) \neq u$, and either $\mathbb{V}(v) = \mathbb{V}(u)$ or $\mathbb{V}(v) < \mathbb{V}(u) < \mathbb{V}(\mathbb{S}(v))$.

The problem defines a graph $G$ with up to $2^n$ vertices. Informally, $\mathbb{S}(\cdot), \mathbb{P}(\cdot), \mathbb{V}(\cdot)$ represent the *successor*, *predecessor* and *potential* functions that act on the vertices. There is an edge $(u, v) \in G$ if and only if $\mathbb{S}(u) = v$, $\mathbb{P}(v) = u$ and $\mathbb{V}(u) < \mathbb{V}(v)$. Thus, $G$ is a directed path (line) along which the potential strictly increases. $\mathbb{S}(\mathbb{P}(x)) \neq x$ represents a start of a line, $\mathbb{P}(\mathbb{S}(x)) \neq x$ represents the end, $\mathbb{P}(\mathbb{S}(x)) = x$ otherwise, and $0^n$ is a given starting vertex.

**(U1)** is a solution representing the end of a path. **(UV1)**, **(UV2)** and **(UV3)** are violations. **(UV1)** gives a violation of our assumption that $\mathbb{V}$ increases strictly along the path. **(UV2)** gives a start of a path that is not $0^n$. **(UV3)** shows that $G$ has more than one path. If there are no violations, $G$ is a single path starting at $0^n$ and ending at **(U1)**. UniqueEOPL is formulated in the non-promise setting, placing it in TFNP. UEOPL contains three classical problems [5], including finding the fixed point of a contraction map.

A notion of *promise-preserving* reductions is also defined in [5]. A reduction from problem $X$ to $Y$ is said to be promise-preserving, if whenever it is promised that $X$ has no violations, then the reduced instance of $Y$ is free of violations. Such a reduction would imply that whenever the original problem is free of violations, then the reduced instance always has a single line that ends at a valid solution.

**Contributions.** We formalize the search problem for Alpha-HS in a non-promise setting:

▶ **Definition 1.3** (Alpha-HS)**.** Given $d$ finite point sets $P = P_1 \cup \cdots \cup P_d \subset \mathbb{R}^d$ each interpreted as a different color, and a vector $(\alpha_1, \ldots, \alpha_d)$ of positive integers such that $\alpha_i \leq |P_i|$ for $i \in [d]$, the Alpha-HS problem is to find one of the following:

**(G1).** A $(\alpha_1, \ldots, \alpha_d)$-cut.

**(GV1).** A subset of $P$ of size $d+1$ and at least $d-1$ colors that lies on a hyperplane.
**(GV2).** A disjoint pair of sets $I, J \subset [d]$ such that $\text{conv}(\{\cup_{i \in I} P_i\}) \cap \text{conv}(\{\cup_{j \in J} P_j\}) \neq \emptyset$.

**(G1)** corresponds to a solution representing a valid cut, while **(GV1)** and **(GV2)** refer to violations of weak general position and well-separation, respectively. From Theorem 1.1 we see that **(G1)** is guaranteed if no violations are presented, so that Alpha-HS is a total search problem. We give the first non-trivial complexity-theoretic upper bound for Alpha-HS:

▶ **Theorem 1.4.** *There is a* poly$(n, d)$*-time promise-preserving reduction from* Alpha-HS *to* UniqueEOPL*, so that* Alpha-HS $\in$ UEOPL $\subseteq$ CLS*.*

It is not surprising to discover that Alpha-HS $\in$ PPAD, since the proof of the continuous version [2] was based on Brouwer's Fixed Point Theorem. The observation that it also lies in PLS is new and noteworthy, putting Alpha-HS into the reach of local search algorithms. See Figure 2 for a pictorial view.

## 2 Alpha-HS is in UEOPL

For space reasons, we cannot provide much technical detail. Instead, we give a broad overview and some difficulties we encountered. We call a hyperplane *colorful* if it passes through exactly $d$ colorful points $p_1, \ldots, p_d \subset P$. Otherwise, we call the hyperplane *non-colorful*. We

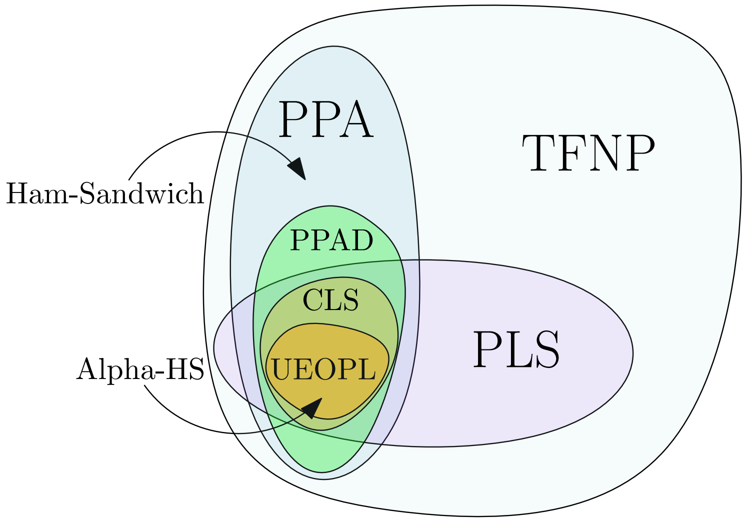


**Figure 2** The hierarchy of complexity classes.

follow the notation of [11] to define the orientation of hyperplanes. If a hyperplane is colorful, the orientation is determined by the $d$ colorful points. If a hyperplane is non-colorful, we design a deterministic way to pick a point in the intersection of the convex of the missing color with the hyperplane to define the orientation (see Figure 3). The *$\alpha$-vector* of any oriented hyperplane $H$ is a $d$-tuple $(\alpha_1, \dots, \alpha_d)$ of integers where $\alpha_i$ is the number of points of $P_i$ in the closed halfspace $H^+$ for $i \in [d]$.

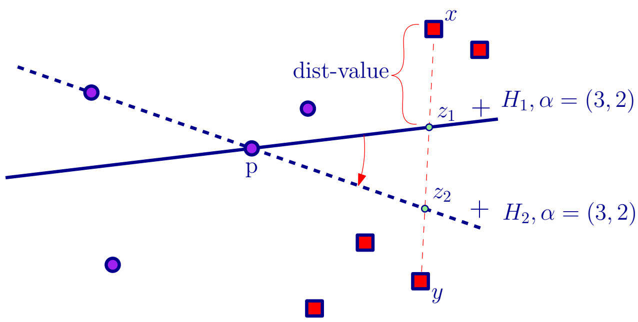


**Figure 3** Purple (disk) is the first color and red (square) is the second color. $H_2$ is a hyperplane that rotates from $H_1$ at the anchor $p$. $x, y$ are the highest ranked points of red color on each side of $H_1, H_2$ under a given order. The orientations of $H_1, H_2$ are determined by $p$ and $z_1, z_2$ respectively.

Our intuition is based on rotating a colorful hyperplane $H$ to another colorful hyperplane $H'$ through a sequence of local changes of the points on the hyperplanes such that the $\alpha$-vector of $H'$ increases in some coordinate by one from that of $H$. The hyperplane rotates about an *anchor*, which is a colorful $(d-1)$-tuple of $P$ that spans a $(d-2)$-flat. Whenever the non-colorful hyperplane hits a new point of a repeated color, the point in the anchor of the same color is swapped with it and continues the rotation until a point of the missing color is hit (see Figure 4). Roughly speaking, the colorful hyperplanes represent the vertices of the UniqueEOPL instance and the rotations determine the edges. We first describe our approach assuming that both well-separation and sufficient general position hold. We then describe how to handle the cases when these assumptions are violated.

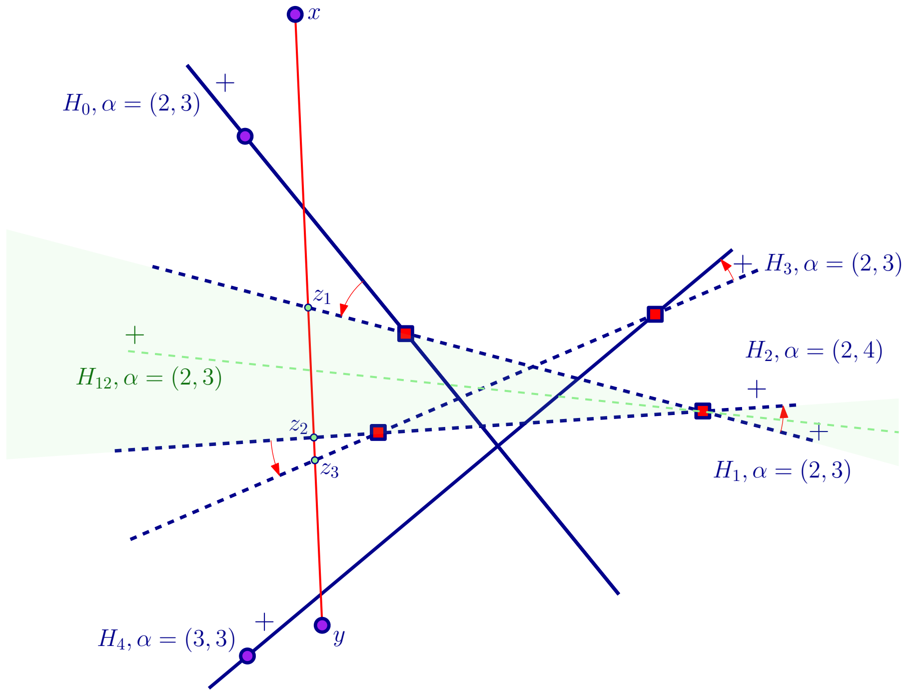


**Figure 4** An example showing a sequence of rotations from $H_0$ to $H_4$ through $H_1, H_2, H_3$. Purple (disk) is the first color and red (square) is the second color. This sequence represents a path between two vertices in the UniqueEOPL graph that is generated in the reduction. The shaded region represents a rotation and $H_{12}$ is its angular bisector. The segment $xy$ is used to define the orientations of $H_1, H_2, H_3, H_{12}$.

**Canonical path.** Each colorful hyperplane $H$ is incident to a colorful set of $d$ points. This set of points defines $d$ possible anchors, and each anchor can be used to rotate $H$ in a different fashion. To define a unique sequence of rotations, we pick a specific order as follows: first, we assume that the colorful hyperplane $H$ whose $\alpha$-vector is $(1, \ldots, 1)$ is given (we show later how this assumption can be removed). We start at $H$ and pick the anchor that excludes the first color, then apply a sequence of rotations until we hit another colorful hyperplane with $\alpha$-vector $(2, 1, \ldots, 1)$. Similarly, we move to a colorful hyperplane with $\alpha$-vector $(3, 1, \ldots, 1)$ and so on until we reach $(\alpha_1, 1, \ldots, 1)$. Then, we repeat this for the other colors in order to reach $(\alpha_1, \alpha_2, 1, \ldots, 1)$ and so on until we reach the target $\alpha$-vector. This pattern of $\alpha$-vectors helps in defining a potential function that strictly increases along the path. We can encode this sequence of rotations as a unique path in the UniqueEOPL instance, and we call it *canonical path*.

**Distance parameter and potential function.** The $\alpha$-vector is not sufficient to define the potential function, since the sequence of rotations between two colorful hyperplanes may have the same $\alpha$-vector. For instance, the angular bisectors of the rotations in $H_0, \ldots, H_3$ in Figure 4 all have the same $\alpha$-vector. Hence, we need an additional measurement in order to determine the direction of rotation that increases the $\alpha$-vector. Similar to how we define the orientation for a non-colorful hyperplane $H$, we deterministically select a directed segment $xy$ that intersects $H$. We define a distance parameter called *dist-value* of $H$ to be the distance from $x$ to the intersection point (see Figure 3). We define a potential value for each vertex on the canonical path in UniqueEOPL using the sum of weighed components of $\alpha$-vector

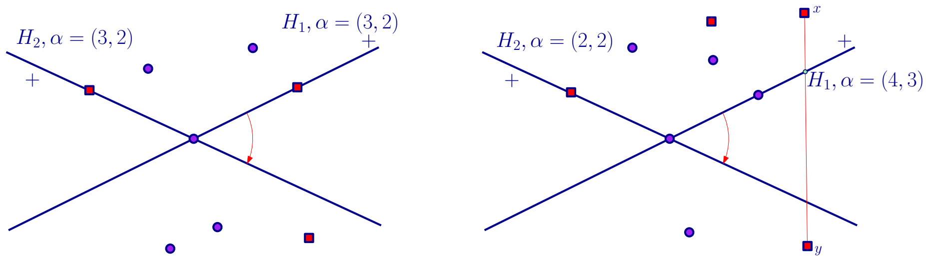


**Figure 5** The examples show two sets of points that are not well-separated. Purple (circle) represents the first color and red (square) represents the second color. In both examples the rotation procedure does not increase the $\alpha$-vector. Both examples show that the orientation of the hyperplane may be flipped after the rotation, so the resulting $\alpha$-vector can go wrong.

and dist-value for the tie-breaker.

We do not need to know the vertex with $\alpha$-vector $(1, \ldots, 1)$ in advance. We split the problem into two sub-problems: in the first we start with a copy of $G$ and any arbitrary vertex. We reverse the direction determined by the potential and construct a Alpha-HS instance for which the vertex with $\alpha$-vector $(1, \ldots, 1)$ is the solution. In the second, we use this vertex as the input to the main Alpha-HS instance. If the input is free of violations, then both sub-problems give valid solutions and together they answer the original question.

**Handling violations.** We show that if there are no violations, then the reduced instance of UniqueEOPL only gives a **(U1)** solution, which readily translates to a **(G1)** solution, so our reduction is promise-preserving, and this can be done in poly$(n, d)$ time.

If $P$ violates well-separation or weak general position, there may be multiple solutions for the same $\alpha$-cut (see Figure 5, left), and no solutions for other cuts. Many nice properties of rotations are destroyed because the orientation of the rotating hyperplane may flip. For instance, the $\alpha$-vector may fail to increment (see Figure 5, right). From the point of view of the canonical path we create, the path may be split into several pieces, which fails the assumption of the unique line. The vertex that corresponds to the target $\alpha$-vector may not exist.

We design our reduction in such a way that any violations on the canonical path can be captured from the violations of the UniqueEOPL instance. After we obtain a violation solution from the reduced instance, we can process it to generate a certificate that witnesses a violation of Alpha-HS. When weak general position fails, then the hyperplanes may have additional points of $P$. These give rise to many different $d$-tuples (each corresponding to some vertex in the UniqueEOPL graph $G$) that represent the same hyperplane. We join these vertices to form a cycle in $G$. For some other case, we show that when two hyperplanes have the same $\alpha$-vector (and dist-value for non-colorful), we can compute a witness for the violation of well-separation. To summarize, we show how to compute a

- **(GV1)** solution from a **(UV1)** solution.
- **(GV1)** or **(GV2)** solution, given a **(UV2)** or **(UV3)** solution.
- **(GV1)** or **(GV2)** solution, that occurs with a **(U1)** solution with the incorrect $\alpha$-vector.

We show that converting these solutions always takes poly$(n, d)$ time. See Figure 6 for an example.

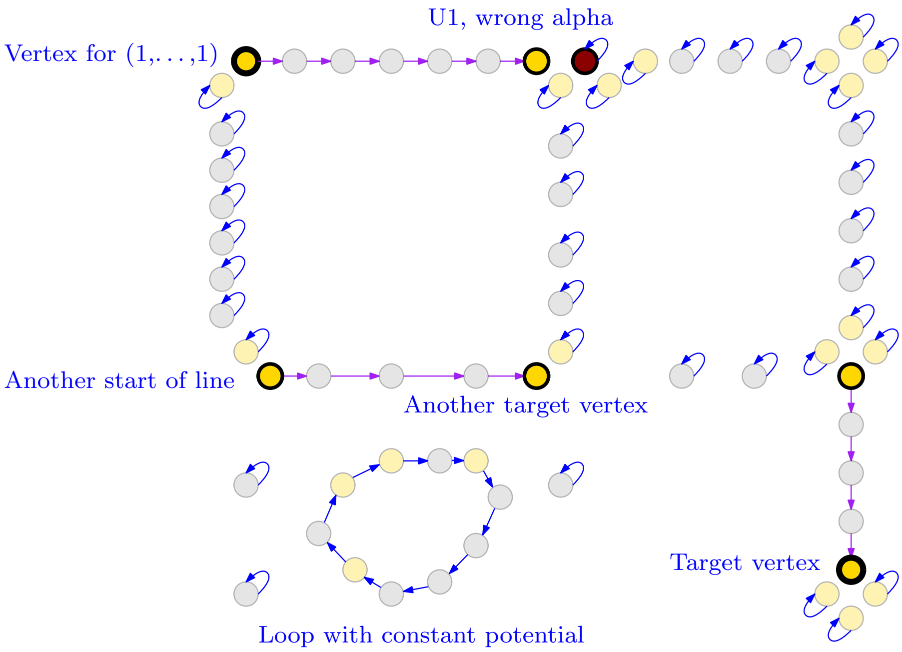


**Figure 6** A subgraph with multiple violations. The vertices that are not on the canonical path are isolated by self-loops. Some vertex that witnesses a violation splits the canonical path into two. Since the orientations are not consistent, there may exist multiple paths that contain vertices with the same $\alpha$-vector.

## 3 Conclusion and future work

We gave an upper bound on the complexity of ALPHA-HS. The next question is determining if the problem is hard for UEOPL. One challenge is that UNIQUEEOPL is formulated as Boolean circuits, whereas ALPHA-HS is purely geometric. Emulating circuits using purely geometric arguments is highly non-trivial. It could be worthwhile to investigate if the techniques used in [6] can prove useful in answering this question.

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