The Tree Stabbing Number is not Monotone

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Abstract

Let $P \subseteq \mathbb{R}^2$ be a set of points and T be a spanning tree of P. The *stabbing number* of T is the maximum number of intersections any line in the plane determines with the edges of T. The *tree stabbing number* of P is the minimum stabbing number of any spanning tree of P. We prove that the tree stabbing number is not a monotone parameter, i.e., there exist point sets $P \subsetneq P'$ such that $\text{TREE-STAB}(P) > \text{TREE-STAB}(P')$, answering a question by Eppstein [4, Open Problem 17.5].

1 Introduction

Let $P \subseteq \mathbb{R}^2$ be a set of points in general position, i.e., no three points lie on a common line. A *geometric graph* $G = (P, E)$ is a graph equipped with a drawing where edges are realized as straight-line segments. The *stabbing number* of G is the maximum number of proper intersections that any line in the plane determines with the edges of G. Let $\mathcal{G}$ be a graph class (e.g., trees, paths, triangulations, perfect matchings etc.). The $\mathcal{G}$*-stabbing number* of P is the minimum stabbing number of any geometric graph $G = (P, E)$ belonging to $\mathcal{G}$ (as a function of P).

Stabbing numbers are a classic topic in computational geometry and received a lot of attention both from an algorithmic as well as from a combinatorial perspective. We mainly focus on the stabbing number of spanning trees (see, e.g., [11] for more information), which has numerous applications. For instance, Welzl [10] used spanning trees with low stabbing number to efficiently answer triangle range searching queries, Agarwal [1] used them in the context of ray shooting (also see [2, 3] for more examples). Furthermore, Fekete, Lübbecke and Meijer [5] proved $\mathcal{NP}$-hardness of stabbing numbers for several graph classes, namely for spanning trees, triangulations and matchings, though for paths this question remains open.

It is natural to ask whether stabbing numbers are monotone, i.e., does it hold for any pointset $P \subseteq \mathbb{R}^2$ that the $\mathcal{G}$-stabbing number of P is not smaller than the $\mathcal{G}$-stabbing number of any proper subset $P' \subsetneq P$. Recently, Eppstein [4] gave a detailed analysis of several parameters that are monotone and depend only on the point set's order type. Clearly, stabbing numbers depend only on the order type. Eppstein observed that the path stabbing number is monotone [4, Observation 17.4] and asked whether this is also the case for the tree stabbing number [4, Open Problem 17.5]. We prove that neither the tree stabbing number (Corollary 3.4) nor the triangulation stabbing number (Corollary 4.2) nor the matching stabbing number (Corollary 5.2) are monotone. A more detailed analysis can also be found in the second author's Master thesis [9]. Each of the following sections is dedicated to one graph class.

* Partially supported by ERC StG 757609

36th European Workshop on Computational Geometry, Würzburg, Germany, March 16–18, 2020.

2 Path Stabbing Number

For completeness we repeat the main argument that the path stabbing number, denoted by PATH-STAB($\cdot$), is monotone, which can be found in [4, Observation 17.4] for example.

▶ **Lemma 2.1.** *Let G be a geometric graph. The following two operations do not increase the stabbing number of G:*

1. *Removing a vertex of degree 1.*
2. *Replacing a vertex v of degree 2 with the segment connecting its two neighbours w_1, w_2.*

Proof. Clearly, the first operation cannot increase the stabbing number, since it does not add any new segments.

For the second part, let G' be the geometric graph obtained from G by performing operation 2 and let ℓ be an arbitrary line. If ℓ has strictly less than STABBING-NUMBER(G) intersections in G, it has at most STABBING-NUMBER(G) intersections in G', since we added only one segment. Otherwise, if ℓ has STABBING-NUMBER(G) intersections in G, it clearly does not pass through any vertex of G and if ℓ intersects the newly inserted segment $\overline{w_1 w_2}$ it must have also intersected either $\overline{w_1 v}$ or $\overline{v w_2}$. ◀

▶ **Corollary 2.2.** PATH-STAB*($\cdot$) is monotone.*

3 Tree Stabbing Number

We construct point sets $P_1 \subsetneq P_2$ of size n and $n+1$ such that TREE-STAB(P_1) $>$ TREE-STAB(P_2). The point $p \in P_2 \setminus P_1$ we want to remove, must, of course, have degree at least 3 in any spanning tree of minimum stabbing of P_2, since otherwise the arguments of Lemma 2.1 apply.

Our construction, which is depicted in Figure 1 (a), is as follows. Start with a unit circle around the origin O and place 3 evenly distributed points x_1, x_2, x_3 on this circle (in counterclockwise order). Next, add an "arm" consisting of 2 points y_i, z_i ($i = 1, 2, 3$) at each of the x_i (outside the circle) such that the points O, x_i, y_i, z_i form a convex chain for $i = 1, 2, 3$ (which are all three oriented the same way). These arms need to be flat enough, i.e., the line supporting the segment $\overline{x_i y_i}$ must intersect the interior of the segment $\overline{O x_{i+2}}$ (indices are taken modulo 3), but also curved enough, i.e., the line supporting the segment $\overline{y_i z_i}$ must have the remaining 8 points on the same side. In particular, there are lines intersecting the segments $\overline{x_i y_i}$, $\overline{y_i z_i}$ and also $\overline{O x_{i+2}}$ on the one hand and $\overline{y_{i+2} z_{i+2}}$ on the other hand (the red lines in Figure 1 (a)). If there is no danger of confusion, we might omit that indices are taken modulo 3 (as in the previous sentence).

Define the two point sets P_1, P_2 (which are both in general position) to be

$$P_1 = \{x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3\}, \qquad P_2 = P_1 \cup \{O\}.$$

▶ **Lemma 3.1.** *It holds that* TREE-STAB(P_1) $= 4$ *and* TREE-STAB(P_2) ≤ 3.

Proof. This result was obtained by a computer-aided brute-force search (the source code is available on github [8]). In order to compute the stabbing number of a given geometric graph spanning some point set, it is enough to consider a *representative set* H_P of lines. For any line ℓ that partitions the point set into two non-empty subsets, there is a line in the representative set inducing the same partitioning. For an n-point set in general position, the size of a representative set is $\binom{n}{2}$ (see the full version of this paper [7]). Hence, we

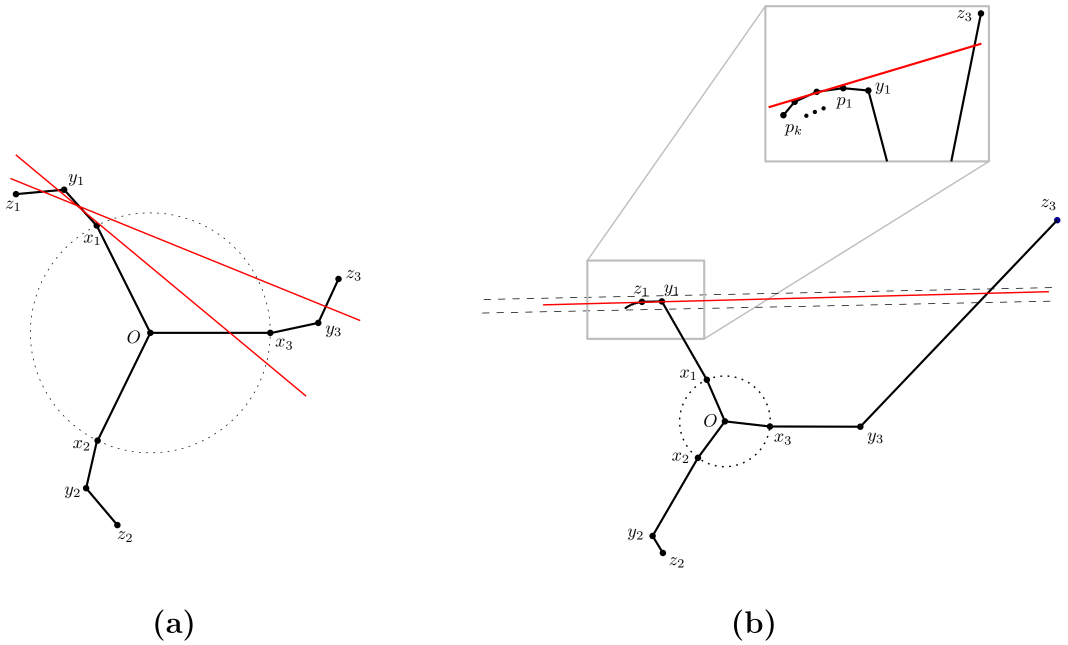

Figure 1 Illustration of a set of (a) 9 points and (b) n points such that removing the point O increases the tree stabbing number.

have $|H_{P_1}| = 36$ and $|H_{P_2}| = 45$. The sets H_{P_1} and H_{P_2} were also obtained by computer assistance. Any pair of points induces four distinct representative lines, computing these and removing duplicates yields H_{P_1} and H_{P_2} (as in [6] for example).

Now, it is enough to compute – for all $9^7 = 4782969$ possible spanning trees on P_1 – their intersections with the lines in H_{P_1}, yielding TREE-STAB$(P_1) = 4$.

On the other hand, for P_2 the spanning tree depicted in Figure 1 has stabbing number 3 (again by computing all intersections with lines in H_{P_2}) implying TREE-STAB$(P_2) \leq 3$. ◀

Next, we generalize this construction to arbitrarily large point sets. We simply replace one of the z_i (say z_1) by a convex chain C consisting of k points $p_1, \ldots, p_k$ (see Figure 1 (b)). Denote the convex chains x_1y_1C, $x_2y_2z_2$ and $x_3y_3z_3$ by C_1, C_2 and C_3.

Our goal will be to remove all but two points of $C \cup \{y_1\}$ to get back to our 9-point setting. Of course, it is crucial to keep the relative position of the points as it is in the 9-point set. Thus, place the points $p_1, \ldots, p_k$ such that:

1. $O, x_1, y_1, p_1, \ldots, p_k$ forms a convex chain.
2. close enough to y_1, so that the order type of the resulting point set is the same no matter which $k-1$ of the points in $C \cup \{y_1\}$ we remove. In particular, no line through any two points not belonging to $y_1, p_1, \ldots, p_k$ may separate these points.
3. for any two segments formed by any triple of points in C_1 (consecutively along the convex chain) there is a line intersecting these two segments and also $\overline{y_3z_3}$. To achieve this, C needs to be sufficiently flat and z_3 needs to be pushed further away.

Note that Lemma 3.1 has been verified to still hold after the modification of pushing z_3 further out. Before proving that this construction fulfills the desired properties, we need one more preliminary lemma (see Figure 2).

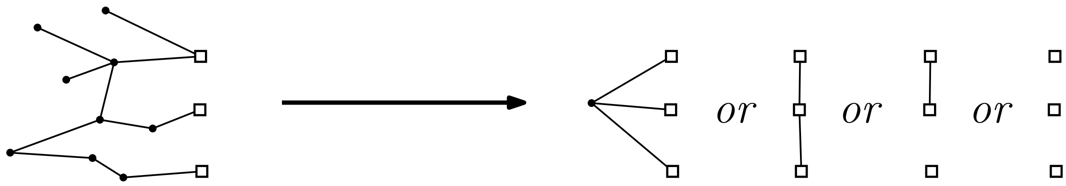

Figure 2 Illustration of Lemma 3.2. Special vertices are depicted as squares. Other vertices of degree 1 or 2 are successively removed.

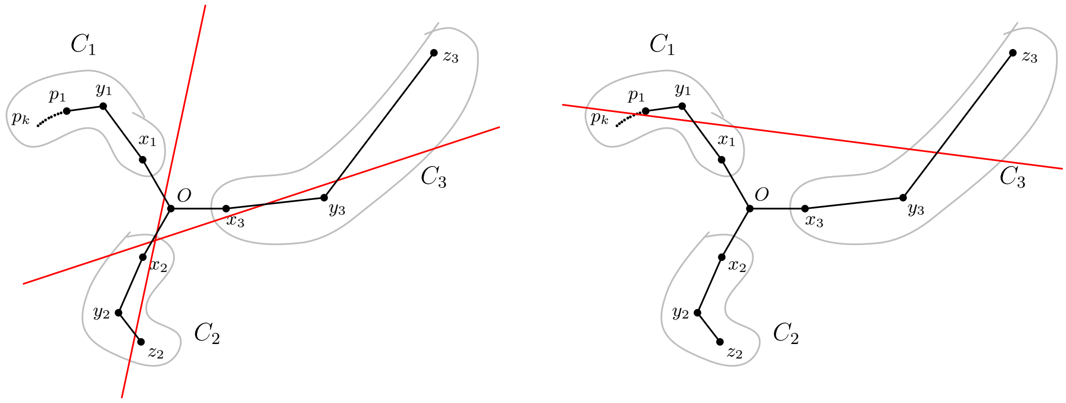

Figure 3 There is no line that intersects more than 3 segments in this spanning tree.

▶ **Lemma 3.2.** *Let $G = (V, E)$ be a forest with c connected components and $|V| \geq 4$. Mark three of the vertices as special (call them v_1, v_2, v_3) and iteratively remove/replace vertices of degree 1 and 2 (as in Lemma 2.1) until no non-special vertex of degree ≤ 2 remains. Then the resulting graph is a forest and consists of the three special vertices and at most one non-special vertex.*

The proof is straightforward and can be found in the full version of this paper [7]. Now, we are prepared to prove our main lemma.

▶ **Lemma 3.3.** *For any integer $n \geq 9$, there exist (planar) point sets $P'_1 \subsetneq P'_2$ of size $|P'_1| = n$ and $|P'_2| = n + 1$ such that* TREE-STAB*$(P'_1) >$* TREE-STAB*(P'_2).*

Proof. Let $k = n - 8$ and define P'_1 and P'_2 as above (Figure 1 (b)), replacing z_1 by $p_1, \ldots, p_k$:

$$P'_1 = \{x_1, y_1, p_1, \ldots, p_k, x_2, y_2, z_2, x_3, y_3, z_3\}, \qquad P'_2 = P'_1 \cup \{O\}.$$

On the one hand, it is straightforward to see that the spanning tree depicted in Figure 1 (b) has stabbing number 3 (see Figure 3 for an illustration) and hence TREE-STAB$(P'_2) \leq 3$.

On the other hand, we show TREE-STAB$(P'_1) \geq 4$ next. Assume for the sake of contradiction that there is a spanning tree T of P'_1 with stabbing number at most 3. Our goal will be to carefully remove points from P_1 such that the stabbing number of T cannot increase until there are only 9 points left in exactly the same relative position as in Lemma 3.1. Clearly, this would be a contradiction.

Consider the set of edges of T with at least one endpoint among the points in C_1. There are at most 3 edges having only one endpoint in C_1 (we call them *bridges*). If there would be more than 3 bridges, there is a line that intersects at least 4 line segments, namely a line that separates C_1 from the rest. Because of the same reason, not all three bridges can go to the same other component (C_2 or C_3).

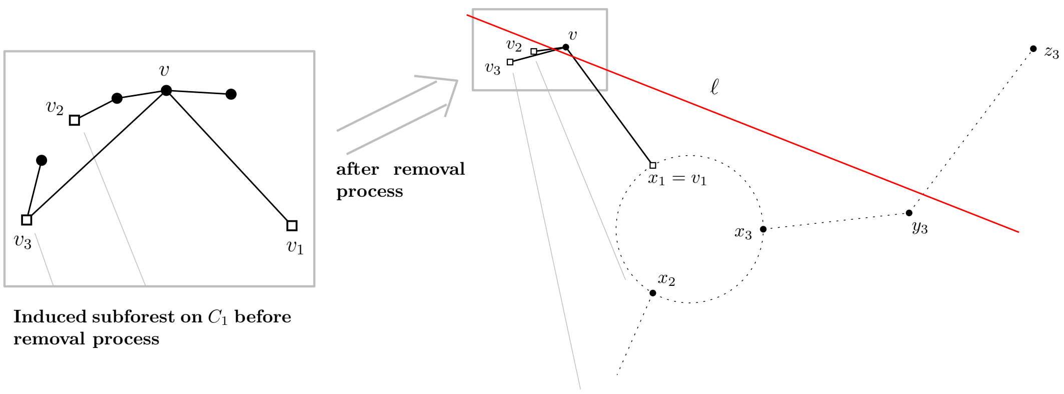

Figure 4 Illustration of Case 2. If a non-special vertex v survives the removal process, the red line has too many intersections.

There are at most 3 points in C_1 that are incident to a bridge and if they are distinct, one of them needs to be x_1, otherwise the line separating x_1 from the rest of C_1 has 4 intersections. Pick three vertices v_1, v_2, v_3 in C_1 such that x_1 and any point incident to a bridge is among them and mark them as special.

Next, we apply Lemma 3.2 to the subforest induced by C_1:

Case 1: No non-special vertex in C_1 survives the removal process.

Then 9 points with the same order type as in Lemma 3.1 and a spanning tree with stabbing number 3 remain, which is a contradiction to Lemma 3.1.

Case 2: One non-special vertex v in C_1 survives the removal process.

Then v is incident to all special vertices v_1, v_2, v_3. If v is the last vertex along C_1, there is obviously a line having more than three intersections. Otherwise, by construction, there is a line ℓ that separates v from v_1, v_2, v_3 and at the same time z_3 from the rest of the point set (see Figure 4). In particular, ℓ has only z_3 and v on one side and all other points on the other. z_3 cannot be adjacent to v, since v is not incident to a bridge and therefore contributes another intersection to ℓ. This is a contradiction to the assumption that T was a spanning tree of stabbing number 3. ◀

▶ **Corollary 3.4.** TREE-STAB$(\cdot)$ *is not monotone.*

4 Triangulation Stabbing Number

We denote the triangulation stabbing number by TRI-STAB$(\cdot)$. Proving non-monotonicity of TRI-STAB$(\cdot)$ is much simpler, only exploiting the additional structure enforced by triangulations. Consider two symmetric convex chains $C_1 = \{p_1, \ldots, p_n\}$ and $C_2 = \{p'_1, \ldots, p'_n\}$ (sufficiently flat) each consisting of n points and facing each other as depicted in Figure 5 (a). These points constitute the point set P. P' consists of the same $2n$ points and two more (slightly perturbed) points added on the line segment connecting the two middle points of C_1 and C_2 (as in Figure 5 (b)). Then the following holds:

▶ **Lemma 4.1.** TRI-STAB*(P)* $\geq 2n - 1$ *and* TRI-STAB*(P')* $\leq n + 4 \log n + 3$.

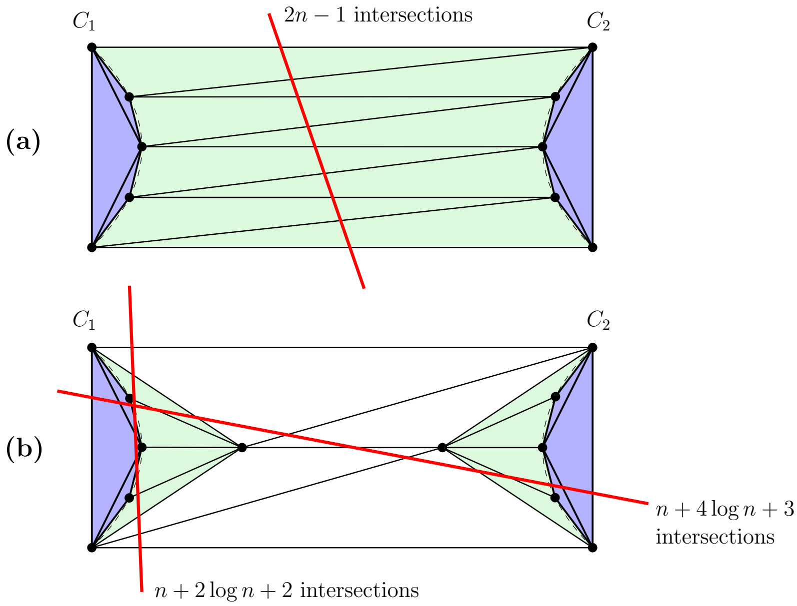

Figure 5 Two symmetric chains in (a) might have a larger triangulation stabbing number compared to the same point set with additional points inbetween (b).

The proof of Lemma 4.1 is straightforward and can be found in the full version of this paper [7].

▶ **Corollary 4.2.** TRI-STAB$(\cdot)$ *is not monotone.*

5 Matching Stabbing Number

First note that the point sets in the case of matchings have to be of even size and all matchings are perfect. Again, we only illustrate the construction, which simply exploits the structure of matchings (again, the proof can be found in the full version [7]).

Take k points $p_1, \ldots, p_k$ in convex position and one point x inside such that any segment $\overline{xp_i}$ is intersected by some $\overline{p_j p_k}$. Next, double all points within a small enough ε-radius (preserving general position) and for a point p name the partner point p' (see Figure 6).

Define the point sets P_1 and P_2 to be:

$$P_2 = \{x, x', p_1, \ldots, p_k, p'_1, \ldots, p'_k, \}, \qquad P_1 = P_2 \setminus \{x', p'_1\}.$$

▶ **Lemma 5.1.** *It holds that* MAT-STAB$(P_1) \geq 3$ *and* MAT-STAB$(P_2) \leq 2$.

▶ **Corollary 5.2.** *The matching stabbing number,* MAT-STAB$(\cdot)$*, is not monotone.*

6 Conclusion

Our proof of Lemma 3.1 relies on computer assistance and of course it would be interesting to turn this into a pen-and-paper proof.

Furthermore, it is easy to generalize stabbing numbers to the context of range spaces $(X, \mathcal{R})$, where X is a set and $\mathcal{R}$ a set of subsets of X, called *ranges*. A spanning path then

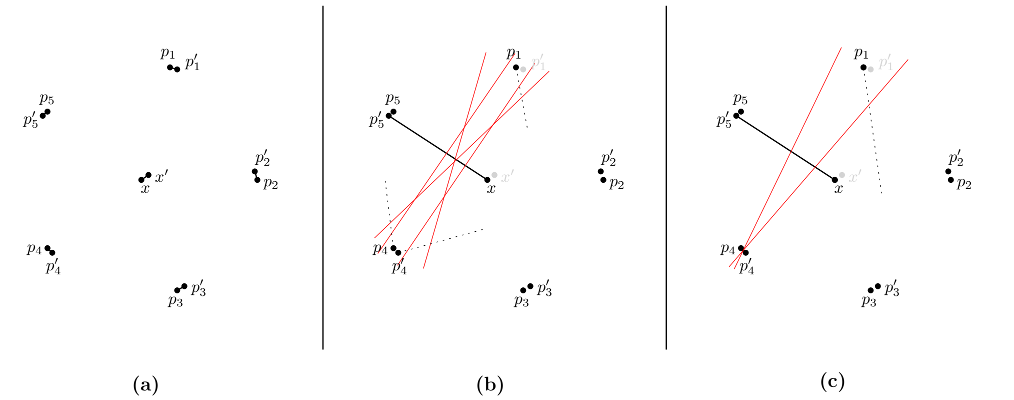

Figure 6 A point set with matching stabbing number 2 in (a) and removing p_1 and x' results in a point set with larger matching stabbing number, illustrated in (b) and (c).

corresponds to a permutation of X and a set $A \subseteq X$ is *stabbed* by a range $r \in \mathcal{R}$ if there are $x, y \in A$ such that $x \in r$ and $y \notin r$. It is straightforward to prove Corollary 2.2 in this context, but we don't know how to apply this for other graph classes.

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