Variational methods for rate- and state-dependent friction problems

Elias Pipping[1] Oliver Sander[2] Ralf Kornhuber[1]

[1]Free University Berlin: Institute for Mathematics

[2]RWTH Aachen University: Institute for Geometry and Practical Mathematics

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Experimental background

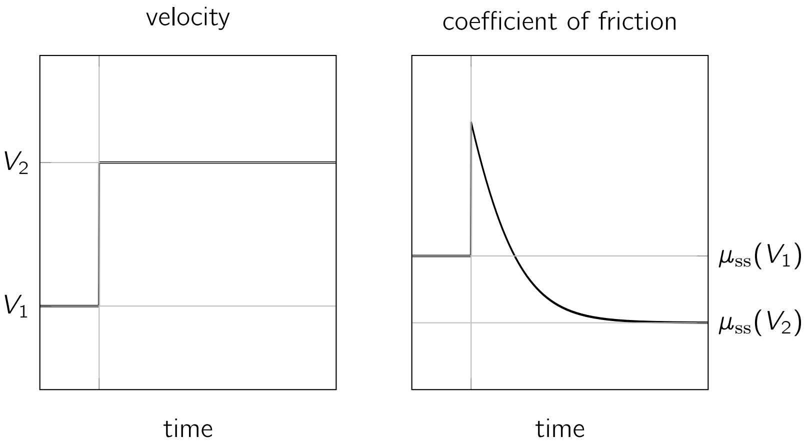

Figure : System response to jump in velocity (after steady-state sliding)

Phenomenological Law

Abstract setting

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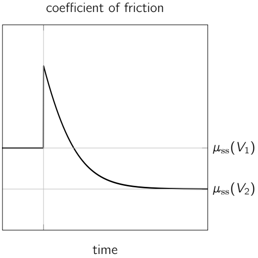

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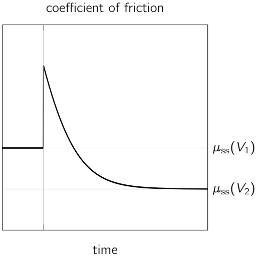

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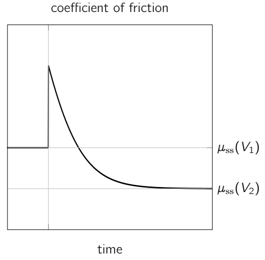

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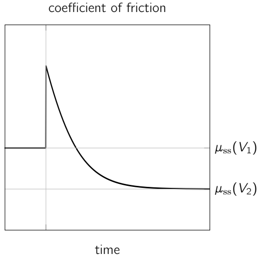

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Interpretation:

- time scale: L/V, regularisation from $\mu = \mu(V)$.

Opinions

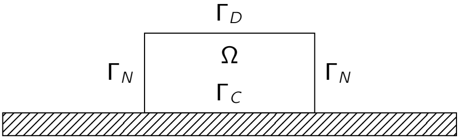
What geoscientists think

- Widely applicable (e.g. wood/rock, pulverised fault gouge); can even be used for *quantitative* reproduction of data.
- Cumbersome.

What mathematicians think

- $\mu(V, \theta)$ monotone in V ($\rightsquigarrow$ convex energies, etc.)
- $\dot{\theta}(V, \theta)$ gradient flow for fixed V.
- aside: needs slight modifications to be meaningful ($\mu \geq 0$): $\mu \rightsquigarrow \mu_{\mathrm{s}}$.

A problem involving RSD friction



With prescribed $\mathbf{u}(0)$, $\dot{\mathbf{u}}(0)$, and $\theta(0)$.

$$
\begin{aligned}
\boldsymbol{\sigma}(\mathbf{u}) &= \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}) && \text{in } \Omega && \text{(linear elasticity)} \\
\operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{b} &= \rho \ddot{\mathbf{u}} && \text{in } \Omega && \text{(momentum balance)} \\
\dot{\mathbf{u}}_n &= 0 && \text{on } \Gamma_C && \text{(bilateral contact)}^1\text{, i.e. } \dot{\mathbf{u}} = \dot{\mathbf{u}}_t \\
\boldsymbol{\sigma}_t = -\lambda \dot{\mathbf{u}}, \quad \lambda &= \frac{|\boldsymbol{\sigma}_t|}{|\dot{\mathbf{u}}|} = \frac{|s_n| \mu_{\mathrm{s}}(|\dot{\mathbf{u}}|, \theta)}{|\dot{\mathbf{u}}|} && \text{on } \Gamma_C && \text{with } \lambda = 0 \text{ for } \dot{\mathbf{u}} = 0 \\
&\dots && \text{on } \Gamma_{N,D} && \\
\dot{\theta} &= \dot{\theta}(|\dot{\mathbf{u}}|, \theta) && \text{on } \Gamma_C && \text{(family of ODEs)}
\end{aligned}
$$

with $s_n \approx \sigma_n$, constant in time[1].

[1] Inherited from the RSD friction model

Weak formulation

We get

$$\int_\Omega \rho\ddot{\mathbf{u}}(\mathbf{v}-\dot{\mathbf{u}}) + \int_\Omega \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u})\colon \boldsymbol{\varepsilon}(\mathbf{v}-\dot{\mathbf{u}}) + \int_{\Gamma_C} \phi(\mathbf{v},\theta) \geq \int_{\Gamma_C} \phi(\dot{\mathbf{u}},\theta) + \ell(\mathbf{v}-\dot{\mathbf{u}})$$

for every $\mathbf{v} \in \mathcal{H}$ with

$$\mathcal{H} = \{\mathbf{v} \in H^1(\Omega)^d \colon \mathbf{v} = 0 \text{ on } \Gamma_D,\, \mathbf{v}_n = 0 \text{ on } \Gamma_C\}$$

or briefly

$$0 \in M\ddot{\mathbf{u}} + A\mathbf{u} + \partial\Phi(\dot{\mathbf{u}},\theta) - \ell \subset \mathcal{H}^*$$

Time discretisation

Turn

$$0 \in M\ddot{\mathbf{u}} + A\mathbf{u} + \partial\Phi(\dot{\mathbf{u}}, \theta) - \ell, \qquad \dot{\theta} = \dot{\theta}(|\dot{\mathbf{u}}|, \theta)$$

into

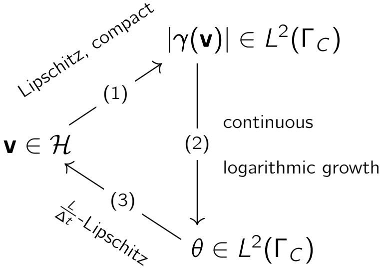
$$0 \in M\ddot{\mathbf{u}}_n + A\mathbf{u}_n + \partial\Phi(\dot{\mathbf{u}}_n, \theta_n) - \ell_n, \qquad \dot{\theta} = \dot{\theta}(|\dot{\mathbf{u}}_n|, \theta)$$

and then (e.g. using the Newmark-β-method)

$$0 \in \frac{2}{\Delta t} M\dot{\mathbf{u}}_n + \frac{\Delta t}{2} A\dot{\mathbf{u}}_n + \partial\Phi(\dot{\mathbf{u}}_n, \theta_n) - \ell_n + \dots \qquad \theta_n = \theta_n(|\dot{\mathbf{u}}_n|, \dots)$$

$\rightsquigarrow$ convex minimisation problem + a step on each ODE

The big picture



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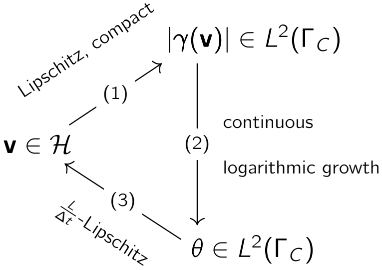

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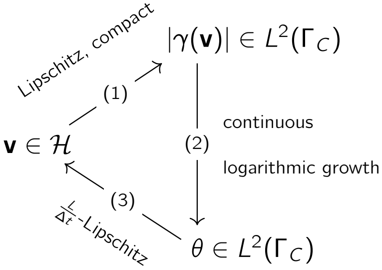

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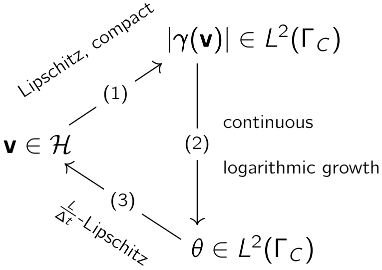

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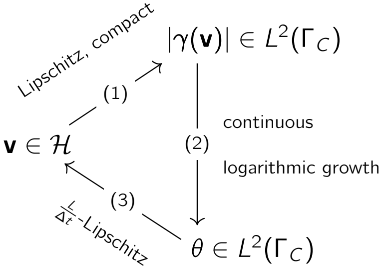

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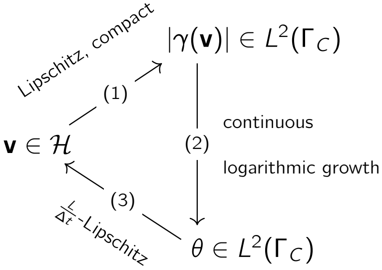

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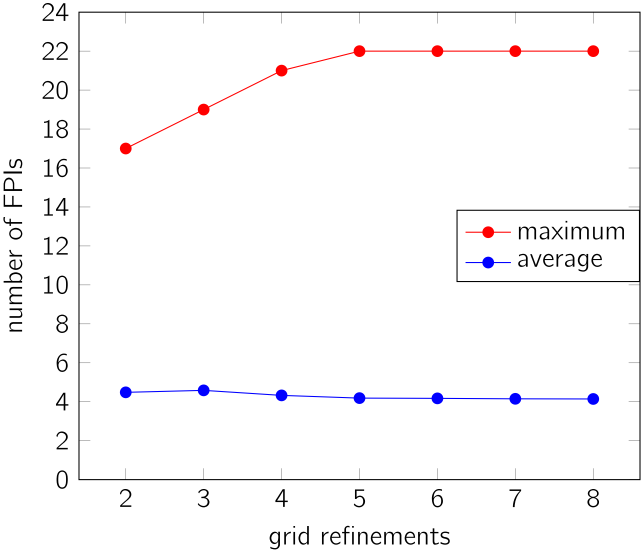
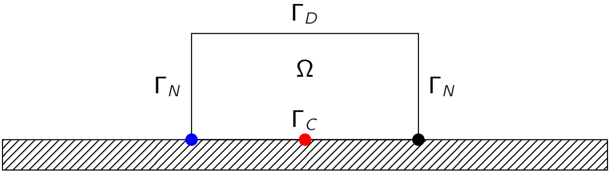
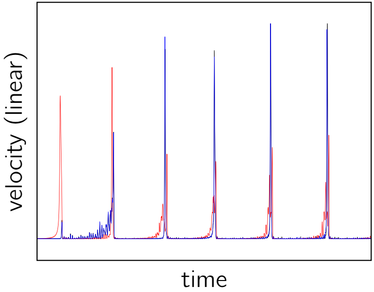
- **Q**: Does it converge to a fixed point?
- **Q**: What about the time-continuous case?

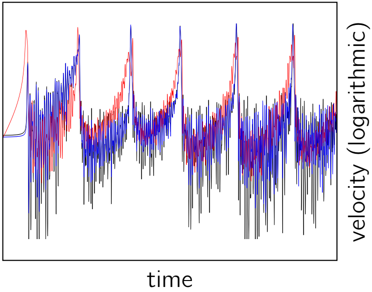
Numerical simulation of a sample problem



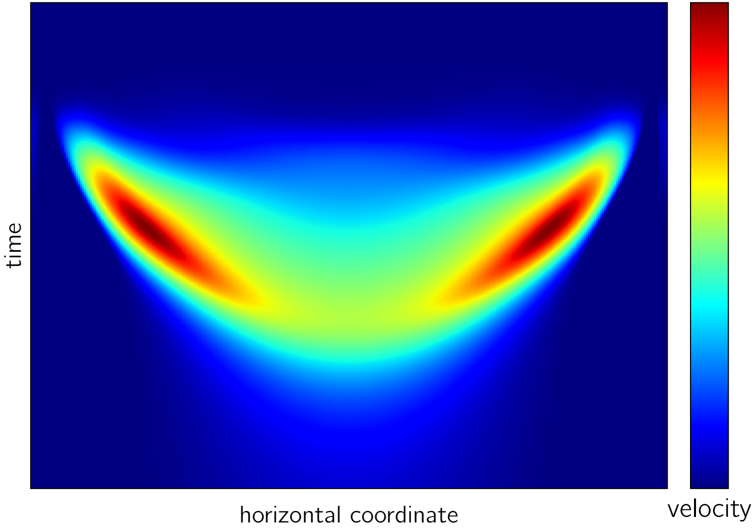
Snapshots of the same problem (1/2)







Snapshots of the same problem (2/2)



Further reading

- P. Bastian et al. "A Generic Grid Interface for Parallel and Adaptive Scientific Computing Part II: Implementation and Tests in DUNE". In: *Computing* 82.2–3 (2 2008), pp. 121–138. ISSN: 0010-485X. DOI: 10.1007/s00607-008-0004-9.
- C. Gräser and R. Kornhuber. "Multigrid Methods for Obstacle Problems". In: *J. Comp. Math.* 27.1 (2009), pp. 1–44.
- E. Pipping, O. Sander and R. Kornhuber. "Variational Formulation of Rate- and State-dependent Friction Problems". Preprint; to appear in ZAMM. URL: ftp://ftp.math.fu-berlin.de/pub/math/publ/pre/2013/Ab-A-13-03.html.