André-Quillen cohomology of monoid algebras

Klaus Altmann Arne B. Sletsjøe

Abstract

We compute the André-Quillen (or Harrison) cohomology of an affine toric variety. The best results are obtained either in the general case for the first three cohomology groups, or in the case of isolated singularities for all cohomology groups, respectively.

1 Introduction

(1.1) Let k be a field of characteristic 0. For any finitely generated k-algebra A the so-called cotangent complex yielding the André-Quillen cohomology $T^n_A = T^n(A, A; k)$ $(n \geq 0)$ may be defined. The first three of these A-modules are important for the deformation theory of A or its geometric equivalent Spec$A$: T^1_A equals the set of infinitesimal deformations, T^0_A describes their automorphisms, and T^2_A contains the obstructions for lifting infinitesimal deformations to larger base spaces. Apart from occurring in long exact sequences no meaning of the higher cohomology groups seems to be known when studying the deformation theory of closed subsets of Spec$A$. A very readable reference for the definition of André-Quillen cohomology and its relations to Hochschild and Harrison cohomology is Loday's book [Lo]. For applications in deformation theory see for instance [Ld], [Pa], or the summary of the properties one has to know (without proofs) in the first section of [BC].

(1.2) For smooth k-algebras A, the higher T^n_A (i.e. $n \geq 1$) vanish. For complete intersections the situation is still easy; T^0_A and T^1_A are well understood, and the remaining cohomology groups vanish. As far as we know, only few further examples exist where the cotangent complex or at least the André-Quillen cohomology groups are known. Palamodov has told us that he has computed (unpublished) the cotangent complex of an embedded point on a line; it turned out that the Poincaré series $T(s) := \sum_{n\geq 0}(\dim_k T^n) \cdot s^n$ of this singularity is a rational function. It would be interesting to know whether this is always the case for isolated singularities.
The result of the present paper is a spectral sequence converging to the Harrison (or André-Quillen) cohomology for affine toric varieties. In the case of an isolated singularity, this spectral sequence degenerates; this leads to a down to earth description of the modules T^n_A. Moreover, in the general case, the information is still sufficient to determine T^0_A, T^1_A, and T^2_A. That is, we use the methods of [Sl] to obtain straight formulas generalizing part of the results of [Al 1].

(1.3) The paper is organized as follows: We begin in §2 with fixing notation and recalling those facts of [Sl] that will be used in the following. The sections §3 and §4 contain the main theorems of the paper: First, we state our spectral sequence; then, in §4, we calculate some of the E_1-terms and show the vanishing of others. Finally, in §5, we present the resulting T^n_A-formulas promised before.

(1.4) *Acknowledgements:* The first author became interested in describing the algebra cohomology of monoid algebras during his visit at the Senter for høyere studier (Academy of Sciences) in Oslo. He liked the warm and stimulating atmosphere very much, and would like to thank those who made it possible for him to stay there. Moreover, we are grateful to R.-O. Buchweitz, J.A. Christophersen, and A. Laudal for many helpful discussions.

2 Inhomogeneous Harrison cohomology

(2.1) Let M, N be mutually dual, finitely generated, free Abelian groups; we denote by $M_{\mathbb{R}}$, $N_{\mathbb{R}}$ the associated real vector spaces obtained via base change with $\mathbb{R}$. Assume we are given a rational, polyhedral cone $\sigma = \langle a^1, \dots, a^m \rangle \subseteq N_{\mathbb{R}}$ with apex in 0 and with $a^1, \dots, a^m \in N$ denoting its *primitive* fundamental generators (i.e. none of the a^i is a proper multiple of an element of N). We define the dual cone $\sigma^\vee := \{r \in M_{\mathbb{R}} \mid \langle \sigma, r \rangle \geq 0\} \subseteq M_{\mathbb{R}}$ and denote by $\Lambda := \sigma^\vee \cap M$ the resulting monoid of lattice points.
The corresponding monoid algebra $A := k[\Lambda]$ will be the object of the upcoming investigations. It is the ring of regular functions on the toric variety $Y_\sigma = \operatorname{Spec} A$ associated to σ. The ring A itself as well as most of its important modules (such as T^n_A) admit an M-(multi)grading. It is this grading which will make computations possible. For general facts concerning toric varieties see for instance [Fu] or [Od].

(2.2) The following definitions are taken from [Sl], §2. We note that the original requirement that the monoids involved have no non-trivial subgroups is unnecessary for our purposes.

Definition: *$L \subseteq \Lambda$ is said to be monoid-like if for all elements $\lambda_1, \lambda_2 \in L$ the relation $\lambda_1 - \lambda_2 \in \Lambda_+ := \Lambda \setminus \{0\}$ implies $\lambda_1 - \lambda_2 \in L$.*
Moreover, a subset $L_0 \subseteq L$ of a monoid-like set is called full if $(L_0 + \Lambda) \cap L = L_0$.

For any subset $P \subseteq \Lambda$ and $n \geq 1$ we introduce $S_n(P) := \{(\lambda_1, \dots, \lambda_n) \in P^n \mid \sum_v \lambda_v \in P\}$. If $L_0 \subset L$ are as in the previous definition, then this gives rise to the following set:

$$C^n(L \setminus L_0, L; k) := \{\varphi : S_n(L) \to k \mid \varphi \text{ is shuffle invariant and vanishes on } S_n(L \setminus L_0)\}.$$

(φ is said to be shuffle invariant if it vanishes on $\big(\sum_\pi \operatorname{sgn}(\pi)\, \pi\big)(\underline{\lambda})$ where π runs through all shuffles of the set $\{1, \dots, n\}$.) These k-vector spaces turn into a complex with the differential $\delta^n : C^{n-1}(L \setminus L_0, L; k) \to C^n(L \setminus L_0, L; k)$ defined via

$$(\delta^n \varphi)(\lambda_1, \dots, \lambda_n) := \varphi(\lambda_2, \dots, \lambda_n) + \sum_{v=1}^{n-1} (-1)^v \varphi(\lambda_1, \dots, \lambda_v + \lambda_{v+1}, \dots, \lambda_n) + (-1)^n \varphi(\lambda_1, \dots, \lambda_{n-1}).$$

Definition: *The k-vector space $HA^n(L \setminus L_0, L; k) := H^n\big(C^\bullet(L \setminus L_0, L; k)\big)$ is called the n-th inhomogeneous Harrison cohomology of the pair (L, L_0).*

(2.3) **Theorem** ([Sl]): *Let $R \in M$. Then, defining $\Lambda_+ := \Lambda \setminus \{0\}$, the homogeneous part of T^n_A in degree $-R$ equals*

$$T^n_A(-R) = HA^{n+1}(\Lambda_+ \setminus (R + \Lambda), \Lambda_+; k) \quad \textit{for } n \geq 0.$$

The proof of the theorem is spread throughout the first two sections of [Sl]: First, in (1.13), (1.14), the calculation of T^n_A has been reduced down to the monoid level. Then, Proposition (2.9) shows that the homogeneous pieces $T^n_A(-R)$ equal the so-called graded Harrison cohomology groups $\operatorname{Harr}^{n+1,-R}(\Lambda, k[\Lambda])$, and, finally, Theorem (2.10) states the above result.

3 The spectral sequence

(3.1) With the notation of (2.1) we define for any face $\tau \leq \sigma$ and any degree $R \in M$ the monoid-like set

$$K_\tau^R := \Lambda_+ \cap \left(R - \operatorname{int} \tau^\vee\right).$$

These sets admit the following elementary properties:

(i) $K_0^R = \Lambda_+$, and $K_{(i)}^R := K_{a^i}^R = \{r \in \Lambda_+ \mid \langle a^i, r\rangle < \langle a^i, R\rangle\}$ with $i = 1, \dots, m$.

(ii) For $\tau \neq 0$ the equality $K_\tau^R = \bigcap_{a^i \in \tau} K_{(i)}^R$ holds. Moreover, if σ is a top-dimensional cone, $K_\sigma^R = \Lambda_+ \cap \left(R - \operatorname{int} \sigma^\vee\right)$ is a (diamond shaped) finite set.

(iii) $\Lambda_+ \setminus (R + \Lambda) = \bigcup_{i=1}^m K_{(i)}^R$.

(3.2) Let us fix an element $R \in M$. The dependence of the sets K_τ^R on τ is a contravariant functor. This gives rise to the complexes $C^q(K_\bullet^R; k)$ $(q \geq 1)$ defined as

$$C^q(K_p^R; k) := \oplus_{\tau \leq \sigma,\, \dim \tau = p}\, C^q(K_\tau^R; k) \qquad (0 \leq p \leq \dim \sigma)$$

with $C^q(K_\tau^R; k) := C^q(\emptyset, K_\tau^R; k)$ and the obvious differentials $d^p : C^q(K_{p-1}^R; k) \to C^q(K_p^R; k)$. (One has to use the maps $C^q(K_\tau^R; k) \to C^q(K_{\tau'}^R; k)$ for any pair $\tau \leq \tau'$ of $(p-1)$- and p-dimensional faces, respectively. The only problem might be the sign; it arises from comparison of the (pre-fixed) orientations of τ and τ'.) Our complex begins as

$$0 \to C^q(\Lambda_+; k) \to \oplus_{i=1}^m C^q(K_{(i)}^R; k) \to \oplus_{\langle a^i, a^j\rangle \leq \sigma} C^q(K_{(ij)}^R; k) \to \oplus_{\dim \tau = 3}\, C^q(K_\tau^R; k) \to \cdots.$$

Lemma: *The canonical k-linear map $C^q(\Lambda_+ \setminus (R+\Lambda), \Lambda_+;\, k) \to C^q(K_\bullet^R; k)$ is a quasiisomorphism, i.e. a resolution of the first vector space.*

Proof: For an $r \in \Lambda_+ \subseteq M$ we define the k-vector space

$$V^q(r) := \left\{\varphi : \{\underline{\lambda} \in \Lambda_+^q \mid \textstyle\sum_v \lambda_v = r\} \to k \;\middle|\; \varphi \text{ is shuffle invariant}\right\}.$$

Then, our complex splits into a direct product over $r \in \Lambda_+$. Its homogeneous factors equal

$$0 \to V^q(r) \to V^q(r)^{\{i \,\mid\, r \in K_{(i)}^R\}} \to V^q(r)^{\{\tau \leq \sigma \,\mid\, \dim \tau = 2;\, r \in K_\tau^R\}} \to V^q(r)^{\{\tau \leq \sigma \,\mid\, \dim \tau = 3;\, r \in K_\tau^R\}} \to \cdots.$$

On the other hand, denoting by $H_{r,R}^+$ the halfspace $H_{r,R}^+ := \{a \in N_{I\!R} \mid \langle a, r\rangle < \langle a, R\rangle\} \subseteq N_{I\!R}$, the relation $r \in K_\tau^R$ is equivalent to $\tau \setminus \{0\} \subseteq H_{r,R}^+$. Hence, the complex for computing the reduced cohomology of the topological space $\bigcup_{\tau \setminus \{0\} \subseteq H_{r,R}^+} (\tau \setminus \{0\}) \subseteq \sigma$ equals

$$0 \to k \to k^{\{i \,\mid\, r \in K_{(i)}^R\}} \to k^{\{\tau \leq \sigma \,\mid\, \dim \tau = 2;\, r \in K_\tau^R\}} \to k^{\{\tau \leq \sigma \,\mid\, \dim \tau = 3;\, r \in K_\tau^R\}} \to \cdots$$

if $\sigma \cap H_{r,R}^+ \neq \emptyset$, i.e. $r \in \bigcup_i K_{(i)}^R$; it is trivial otherwise. Since $\bigcup_{\tau \setminus \{0\} \subseteq H_{r,R}^+} (\tau \setminus \{0\})$ is contractible, this complex is always exact. Thus, $C^q(K_\bullet^R; k) = \prod_{r \in \Lambda_+} V^q(r)^{\{\tau \leq \sigma \,\mid\, \dim \tau = \bullet;\, r \in K_\tau^R\}}$ has $\prod_{r \in \Lambda_+ \setminus (\cup_i K_{(i)}^R)} V^q(r) = C^q(\Lambda_+ \setminus (R + \Lambda), \Lambda_+;\, k)$ as cohomology in 0, and it is exact elsewhere. $\square$

(3.3) Combining the differentials d^p from (3.2) and δ^q from (2.2), we obtain a double complex $C^\bullet(K_\bullet^R; k)$ $(0 \leq p \leq \dim \sigma;\ q \geq 1)$.

Theorem: *The André-Quillen cohomology of $A = k[\Lambda]$ equals the cohomology of the total complex, that is*

$$T^n_A(-R) = H^{n+1}\big(\mathrm{tot}^\bullet\big[C^\bullet(K^R_\bullet; k)\big]\big) \quad \textit{for } n \geq 0.$$

Moreover, given an element $s \in \Lambda$, the multiplication $[\cdot x^s] : T^n_A(-R) \to T^n_A(-R+s)$ is given by the homomorphism $C^\bullet(K^R_\bullet; k) \to C^\bullet(K^{R-s}_\bullet; k)$ assigned to the inclusions $K^{R-s}_\tau \subseteq K^R_\tau$.

Proof: The first part is a straightforward consequence of Theorem (2.3) and the previous lemma – just use the corresponding spectral sequence of the double complex. For the A-module structure of T^n_A, one easily observes from the proofs in [Sl] that the multiplication with x^s arises from the complex homomorphism $C^\bullet(\Lambda_+ \setminus (R+\Lambda),\, \Lambda_+;\, k) \to C^\bullet(\Lambda_+ \setminus ([R-s]+\Lambda),\, \Lambda_+;\, k)$ provided by the inclusion $R + \Lambda \subseteq [R-s] + \Lambda$. $\square$

(3.4) **Corollary:** *There is a spectral sequence $E^{p,q}_1 = \oplus_{\dim \tau = p} HA^q(K^R_\tau; k) \Longrightarrow T^{p+q-1}_A(-R)$.*

Proof: This is the other spectral sequence associated to the double complex $C^\bullet(K^R_\bullet; k)$. $\square$

4 The E_1-level

(4.1) **Definition:** Let $K \subseteq M$ be an arbitrary subset of the lattice M. A function $f : K \to k$ is called *quasilinear* if $f(r) + f(s) = f(r+s)$ for any r and s with $r, s, r+s \in K$. The vector space of quasilinear functions is denoted by $\overline{\mathrm{Hom}}\,(K, k)$.

Recalling the differential $\delta^2 : C^1(K^R_\tau) \to C^2(K^R_\tau)$ from (2.2) shows that the $E^{\bullet,1}_1$-summands $HA^1(K^R_\tau; k)$ equal $\overline{\mathrm{Hom}}\,(K^R_\tau, k)$.

(4.2) The orbits of the torus acting on Y_σ are parametrized by the faces of σ; the singular locus of Y_σ is the disjoint union of some of these orbits. We call a face $\tau \leq \sigma$ smooth if our toric variety is smooth along $\mathrm{orb}(\tau)$. It is one of the basic facts that smooth faces are characterized by being generated from a part of a $\mathbb{Z}$-basis of N. In particular, 0 and the one-dimensional faces are always smooth.

Proposition: *If $\tau \leq \sigma$ is a smooth face, then the injections $\mathrm{Hom}_k(\mathrm{span}_k K^R_\tau,\, k) \hookrightarrow \overline{\mathrm{Hom}}\,(K^R_\tau, k)$ are even isomorphisms. Moreover, $\mathrm{span}_k K^R_\tau = \bigcap_{a^i \in \tau} \mathrm{span}_k K^R_{(i)}$, and the latter vector spaces equal $\mathrm{span}_k K^R_{(i)} = M_k$, $(a^i)^\perp$, or 0 if $\langle a^i, R\rangle \geq 2$, $= 1$, or ≤ 0, respectively.*

Proof: Let $f : K^R_\tau \to k$ be quasilinear; it suffices to extend f to a $\mathbb{Z}$-linear map defined on $\mathrm{span}_{\mathbb{Z}} K^R_\tau \subseteq M$. If R was non-positive on any of the generators of τ, then K^R_τ would be empty anyway. Hence, if (w.l.o.g.) $\tau = \langle a^1, \dots, a^k\rangle$, we may assume that $\langle a^i, R\rangle \geq 2$ for $i = 1, \dots, l$ and $\langle a^j, R\rangle = 1$ for $j = l+1, \dots, k$.
K^R_τ contains the easy part $\tau^\perp \cap \Lambda_+$; it is no problem at all to extend $f_{|\tau^\perp \cap \Lambda_+}$ to a $\mathbb{Z}$-linear function defined on $\tau^\perp \cap M$. In general, we have to show that for elements $s^v \in K^R_\tau$ the value $\sum_v f(s^v)$ only depends on $s := \sum_v s^v$, not on the summands themselves. (Then, $f(s)$ may be defined as this value.)
By smoothness of τ there exist elements $r^1, \dots, r^l \in K^R_\tau$ such that $\langle a^i, r^j\rangle = \delta_{ij}$ for $1 \leq i \leq k$ and $1 \leq j \leq l$. Hence, quasilinearity of f implies

$$f(s^v) = \sum_{i=1}^{l} \langle a^i, s^v\rangle f(r^i) + f(p^v) \quad \text{with } p^v := s^v - \textstyle\sum_i \langle a^i, s^v\rangle r^i \in \tau^\perp \cap M\,.$$

Summing up yields

$$\sum_v f(s^v) = \sum_{i=1}^{l} \langle a^i, s\rangle f(r^i) + \sum_v f(p^v) = \sum_{i=1}^{l} \langle a^i, s\rangle f(r^i) + f\big(s - \sum_i \langle a^i, s\rangle r^i\big)\,.$$

Finally, the second claim follows by $\bigcap_{a^i\in\tau} \operatorname{span}_k K^R_{(i)} = \bigcap_{j=l+1}^{k} (a^j)^\perp = \operatorname{span}_k\big(\tau^\perp; r^1,\dots,r^l\big) = \operatorname{span}_k K^R_\tau$. $\square$

(4.3) We turn to the remaining part of the first level and show the vanishing of $E_1^{p,\geq 2}$ if Y_σ is smooth in codimension p:

Theorem: *If $\tau \leq \sigma$ is a smooth face, then $HA^q(K^R_\tau; k) = 0$ for $q \geq 2$.*

Proof: We proceed by induction on $\dim\tau$, i.e. we may assume that the vanishing holds for all proper faces of τ. Let $r(\tau)$ be an arbitrary element of $\operatorname{int}(\sigma^\vee \cap \tau^\perp) \cap M$, i.e. $\tau = \sigma \cap [r(\tau)]^\perp$. Then, via $R_g := R - g\cdot r(\tau)$ with $g \in \mathbb{Z}$, one obtains an infinite (if $\tau \neq \sigma$) series of degrees admitting the following two properties:

(i) $K^{R_g}_\tau = K^R_\tau$ for any $g\in\mathbb{Z}$ (since $R_g = R$ on τ), and

(ii) $K^{R_g}_{\tau'} \neq \emptyset$ implies $\tau' \leq \tau$ for any face $\tau' \leq \sigma$ and $g \gg 0$ (since $\langle a^j, R_g\rangle \leq 0$ if $a^j \notin \tau$).

In particular, in degree $-R_g$ with $g\gg 0$, the first level of our spectral sequence is shaped as follows:

- For $p < \dim\tau$ only $HA^q(K^R_{\tau'}; k)$ with $\tau' \leq \tau$ appear as summands of $E_1^{p,q}$. By the induction hypothesis they even vanish for $q \geq 2$,
- for $p = \dim\tau$ it follows that $E_1^{p,q} = HA^q(K^R_\tau; k)$, and
- all vector spaces $E_1^{p,q}$ vanish beyond the $[p = \dim\tau]$–line.

Hence, the differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are trivial for $r\geq 1$, $q \geq 2$, and we obtain

$$T_A^{q+\dim\tau-1}(-R_g) = HA^q(K^R_\tau; k) \quad \text{for } g \gg 0\,.$$

Moreover, under this identification, the multiplication

$$[\cdot x^{r(\tau)}] : T_A^{q+\dim\tau-1}(-R_g) \to T_A^{q+\dim\tau-1}(-R_{g+1})$$

is just the identity map. On the other hand, we may restrict $T^n_\sigma := T^n_A$ onto the *smooth*, open subset $Y_\tau := \operatorname{Spec} k[\tau^\vee\cap M] \subseteq Y_\sigma$. Since $k[\tau^\vee\cap M]$ equals the localization of $k[\sigma^\vee \cap M]$ by the element $x^{r(\tau)}$, we obtain

$$T^n_\sigma \otimes_{k[\sigma^\vee\cap M]} k[\sigma^\vee\cap M]_{x^{r(\tau)}} = T^n_\tau = 0 \quad \text{for } n \geq 1.$$

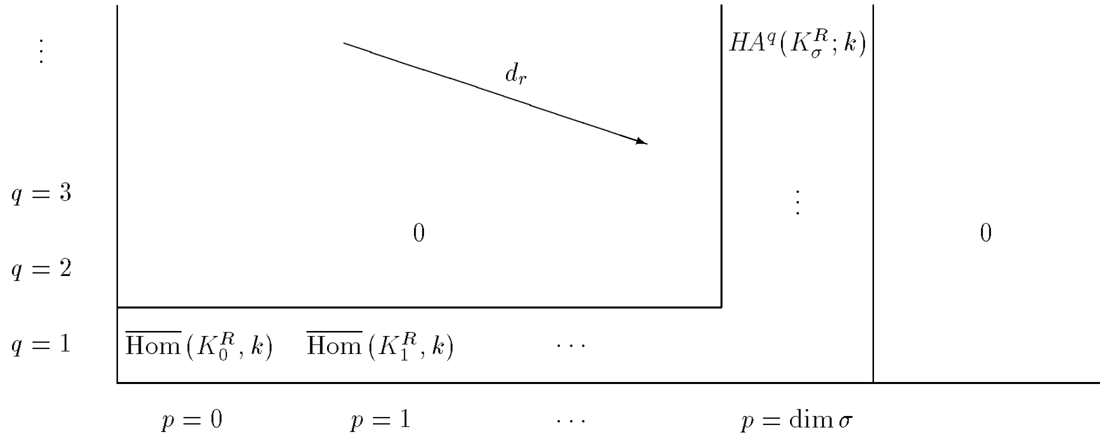
In particular, any element of $T_A^{q+\dim\tau-1}(-R_g) \subseteq T_A^{q+\dim\tau-1}$ will be killed by some power of $x^{r(\tau)}$; but this means $HA^q(K^R_\tau; k) = 0$. $\square$

(4.4) **Corollary:** *$E_1^{0,q} = E_1^{1,q} = 0$ for $q \geq 2$.*

5 Applications

(5.1) The main ingredient for describing the André-Quillen cohomology of Y_σ will be the complex $\overline{\mathrm{Hom}}\,(K_\bullet^R, k) = (E_1^{\bullet,1},\, d_1)$ which is built from the vector spaces $\overline{\mathrm{Hom}}\,(K_\tau^R, k)$ as $C^q(K_\bullet^R; k)$ was from $C^q(K_\tau^R; k)$ in (3.2). It contains $(\mathrm{span}_k K_\bullet^R)^*$ as a subcomplex.

(5.2) First, we discuss the case of an *isolated singularity* Y_σ. Here, our spectral sequence degenerates completely.



Theorem: *Let Y_σ be an isolated singularity. Then, the André-Quillen cohomology in degree $-R$ equals*

$$T_A^n(-R) = \begin{cases} H^n\big(\overline{\mathrm{Hom}}\,(K_\bullet^R, k)\big) = H^n\big((\mathrm{span}_k K_\bullet^R)^*\big) & \textit{for } 0 \le n \le \dim\sigma - 1 \\ H^{\dim\sigma}\big(\overline{\mathrm{Hom}}\,(K_\bullet^R, k)\big) & \textit{for } n = \dim\sigma \\ HA^{n-\dim\sigma+1}(K_\sigma^R; k) & \textit{for } n \ge \dim\sigma + 1\,. \end{cases}$$

Proof: The first level of the spectral sequence is non-trivial only in $E_1^{p,1}$ with $0 \le p \le \dim\sigma$ and $E_1^{\dim\sigma,q}$ with $q \ge 1$, respectively. Moreover, the complexes $\overline{\mathrm{Hom}}\,(K_\bullet^R, k)$ and $(\mathrm{span}_k K_\bullet^R)^*$ are equal up to position $p = \dim\sigma - 1$. □

(5.3) In the *general case*, we still have enough information to determine the deformation relevant modules T_A^0, T_A^1, and T_A^2. Under the additional hypothesis of smoothness in codimension two, these results have already been obtained in [Al 1] with a different proof.

Theorem: *Let σ be an arbitrary rational, polyhedral cone with apex in 0. Then, for every $R \in M$,*

$$T_A^n(-R) = H^n\big(\overline{\mathrm{Hom}}\,(K_\bullet^R, k)\big) \qquad \textit{for } n = 0, 1, 2.$$

Moreover, for $n = 0, 1$, this vector space equals $H^n\big((\mathrm{span}_k K_\bullet^R)^*\big)$, too.

Proof: This is a direct consequence of Corollary (4.4). □

(5.4) Since the complex $(\mathrm{span}_k K_\bullet^R)^*$ is much easier to handle than $\overline{\mathrm{Hom}}\,(K_\bullet^R, k)$, it pays to look for sufficient conditions for $T_A^2 = H^2\big((\mathrm{span}_k K_\bullet^R)^*\big)$ to hold.

Proposition: *If Y_σ is Gorenstein in codimension two, i.e. for every two-dimensional face $\langle a^i, a^j\rangle \leq \sigma$ there is an element $r(i,j) \in M$ such that $\langle a^i, r(i,j)\rangle = \langle a^j, r(i,j)\rangle = 1$, then*

$$T^2_A = H^2\big((\operatorname{span}_k K^R_\bullet)^*\big).$$

Proof: For edges $\langle a^i, a^j\rangle \leq \sigma$ we have to show that $(\operatorname{span}_k K^R_{(ij)})^* \hookrightarrow \overline{\operatorname{Hom}}\,(K^R_{(ij)}, k)$ is an isomorphism. We will adapt the proof of (4.2).
It may be assumed that $\langle a^i, R\rangle; \langle a^j, R\rangle \geq 1$ and $r(i,j) \in K^R_{(ij)}$. Let $d := |\det(a^i, a^j)| \in \mathbb{Z}$; it is the smallest positive value of a^i possible on elements of $\Lambda \cap (a^j)^\perp$. We choose an $r^i \in \Lambda \cap (a^j)^\perp$ with $\langle a^i, r^i\rangle = d$; together with $r(i,j)$ it will play the same role as $r^1, \ldots, r^l$ did in (4.2).
Case 1: $\langle a^i, R\rangle > d$ and $\langle a^i, R\rangle \geq \langle a^j, R\rangle$ (in particular, $r^i \in K^R_{(ij)}$): Then, elements s^v or s (cf. (4.2)) may be represented as

$$s = \langle a^j, s\rangle \cdot r(i,j) + \frac{\langle a^i, R\rangle - \langle a^j, R\rangle}{d}\, r^i + \big[(a^i, a^j)^\perp - \text{elements}\big]\,.$$

The difference $\langle a^i, R\rangle - \langle a^j, R\rangle$ is always divisible by d, i.e. the coefficients are integers.
Case 2: $\langle a^i, R\rangle; \langle a^j, R\rangle \leq d$: This implies $K^R_{(ij)} \subseteq (a^i, a^j)^\perp + \mathbb{Z}\cdot r(i,j) = (a^i - a^j)^\perp$. In particular, we may use the representation $s = \langle a^i, s\rangle \cdot r(i,j) + \big[(a^i, a^j)^\perp - \text{elements}\big]\,.$ □

Corollary: *Let Y_σ be a three-dimensional, toric Gorenstein singularity, i.e. $\sigma = \langle a^1, \ldots, a^m\rangle$ with $a^{m+1} := a^1$ being the cone over a lattice polygon embedded in height one. Then*

$$T^2_A(-R)^* = \bigcap\nolimits_{i=1}^m \big(\operatorname{span}_k K^R_{(i,i+1)}\big) \Big/ \operatorname{span}_k\big(\bigcap\nolimits_{i=1}^m K^R_{(i,i+1)}\big)\,.$$

(5.5) Finally, we would like to mention an alternative to the complexes $(\operatorname{span}_k K^R_\bullet)^*$ and $\overline{\operatorname{Hom}}\,(K^R_\bullet, k)$. Let $E \subseteq \Lambda_+$ be the (finite) set of non-splittable elements in Λ_+; it is the minimal generator set of the monoid $\Lambda = \sigma^\vee \cap M$ and gives rise to a canonical surjection $\pi : \mathbb{Z}^E \to M$. The relations among E-elements are gathered in the $\mathbb{Z}$-module $L(E) := \ker \pi$. Every $q \in L(E)$ splits into a difference $q = q^+ - q^-$ with $q^+, q^- \in \mathbb{N}^E$ and $\sum_v q^+_v\, q^-_v = 0$. We denote by $\bar{q} \in \Lambda$ the image $\bar{q} := \pi(q^+) = \pi(q^-)$.

Definition: With $E^R_\tau := E \cap K^R_\tau$, we define $L(E^R_\tau) := L(E) \cap \mathbb{Z}^{E^R_\tau}$ and $\overline{L}(E^R_\tau) \subseteq L(E^R_\tau)$ to be the submodule generated by the relations $q \in L(E)$ such that $\bar{q} \in K^R_\tau$. (Notice that with $E^R_0 = E$ the notation differs slightly from that in [Al 1].)

As usual (cf. (3.2)), we may construct complexes from these finitely generated, free abelian groups. They fit into the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\operatorname{span}_k K^R_\bullet)^* & \longrightarrow & k^{E^R_\bullet} & \to & \operatorname{Hom}_{\mathbb{Z}}(L(E^R_\bullet), k) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow\wr & & \downarrow & & \\
0 & \longrightarrow & \overline{\operatorname{Hom}}\,(K^R_\bullet, k) & \longrightarrow & k^{E^R_\bullet} & \to & \operatorname{Hom}_{\mathbb{Z}}(\overline{L}(E^R_\bullet), k) & \longrightarrow & 0
\end{array}$$

Proposition: *The k-duals of $T^n_A(-R)$ equal*

$$T^1_A(-R)^* = L_k\big(\bigcup\nolimits_i E^R_{(i)}\big) / \textstyle\sum_i L_k(E^R_{(i)})$$

and

$$T^2_A(-R)^* = \ker\big(\oplus_i\, L_k(E^R_{(i)}) \to L(E)\big) \Big/ \operatorname{im}\big(\oplus_{ij}\, \overline{L}_k(E^R_{(ij)}) \to \oplus_i L_k(E^R_{(i)})\big)\,.$$

Proof: As in the proof of Lemma (3.2), one obtains that the complex $k^{E^R_\bullet}$ has no cohomology except $H^0 = k^{E\setminus\cup_i E^R_{(i)}}$. Hence, the long exact sequence for the second row in the above diagram yields isomorphisms

$$H^{p-1}\big(\mathrm{Hom}_{\mathbb{Z}}(\overline{L}(E^R_\bullet), k)\big) \xrightarrow{\sim} H^p\big(\overline{\mathrm{Hom}}\,(K^R_\bullet, k)\big) \quad \text{for } p \geq 2.$$

Taking a closer look at the first terms shows that the same result is true for $p = 1$ if E^R_0 is replaced by $\bigcup_i E^R_{(i)}$. Finally, we know that $\overline{L}(E^R_\tau) = L(E^R_\tau)$ if $\dim \tau \leq 1$. $\square$

(5.6) **Remark:** The vector space $T^1_A(-R)$ has also a convex-geometric interpretation; it is related to the set of Minkowski summands of the cross cut $\sigma \cap [R = 1]$. For details, we refer to [Al 2].

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Klaus Altmann
Institut für reine Mathematik der
Humboldt-Universität zu Berlin
Ziegelstr. 13A
D-10099 Berlin, Germany
E-mail: altmann@mathematik.hu-berlin.de

Arne B. Sletsjøe
Department of Mathematics
University of Oslo
Pb. 1053 Blindern
N-0316 Oslo, Norway
E-mail: arnebs@math.uio.no