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Deformations of Affine Torus Varieties

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1. Introduction

1.1 Based on the paper [6] of Kollár/ Shepherd-Barron, many investigations of the base space of the versal deformation of a two dimensional cyclic quotient singularity were done in the last years: In [2], J. Arndt gave an algorithm for computing the equations of the base space; in [7] and [3], J. Stevens and J.A. Christophersen described the components of the reduced base space in a more qualitative way using continued fractions.

Cyclic quotient singularities are exactly those singularities that appear as two dimensional affine torus varieties:

$X(n,q)$ (quotient of the $\mathbb{Z}/_{n\mathbb{Z}}$-action $\xi \mapsto \left(\begin{smallmatrix}\xi & 0\\ 0 & \xi^q\end{smallmatrix}\right)$ on $\mathbb{C}^2$) can be built as an affine torus variety by the cone $\sigma := \ < (1,0); (-q,n) > \ \subseteq N := \mathbb{Z}^2$.

(For the notion of an affine torus variety: cf. [5].)

On the other hand, in [3] (CDV; §2) Christophersen made the following observation: Deforming such a cyclic quotient singularity, the total spaces over the components of the reduced base space are (after a finite base change) torus varieties, too.

1.2 Having this in mind, the following problem seems to be very interesting: Classify (or describe in combinatorical terms) all possible fibre product diagrammes

$$\begin{array}{ccc} Y & \stackrel{i}{\hookrightarrow} & X \\ \downarrow & \otimes & \downarrow f \\ \{0\} & \hookrightarrow & S \end{array}$$

with:

X, Y = affine torus varieties with closed orbits $0 \in Y$, $0 \in X$, respectively;

$0 \in Y$ is an isolated singularity ($0 \in X$ need not to be isolated);

i = morphism of torus varieties with $i(0) = 0$;

f is a flat map.

In particular, i maps the torus T_Y of Y in the torus T_X of X, and $i : T_Y \hookrightarrow T_X$ is induced by a surjection of lattices:

$$i^* : M \longrightarrow\!\!\!\!\rightarrow M/_L$$

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($M \cong \mathbb{Z}^n$, $T_X = \operatorname{Spec} \mathbb{C}[M]$; $L \subseteq M$ is an m-dimensional sublattice with $M/_L$ torsion free, $T_Y = \operatorname{Spec} \mathbb{C}[M/_L]$).

If X is given by the cone $\sigma \subseteq N \otimes \mathbb{R}$ ($N \cong \mathbb{Z}^n$ is the dual lattice of M), the dual cone $\check{\sigma} \subseteq M \otimes \mathbb{R}$ is squeezed to a cone $\bar{\sigma}^\vee \subseteq M/_L \otimes \mathbb{R}$. (In the dual language: $\bar{\sigma} = \sigma \cap L^\perp$ in $N \otimes \mathbb{R}$). σ and $\bar{\sigma}$ are top dimensional cones (in $N \otimes \mathbb{R}$ and $L^\perp$, respectively) that do not contain any linear subspace.

1.3 In §2, all such deformations were completely described in terms of the cone σ and the sublattice $L \subseteq M$. The corresponding theorem is proved in §3.

In §4, this theorem is used to give an explicit list of all possible pairs $[\sigma, L]$ (in a more or less coarse sense) for the special case that all proper faces $\tau < \sigma$ are simplicial, i.e. $X \setminus \{0\}$ contains only cyclic quotient singularities of arbitrary dimensions.

§5 contains specified computations of the sporadic cases belonging to the previous paragraph. These computations can also be used as examples for the main result, Theorem 2.4.

1.4 In [1], we have proved a formula to compute the vector space T^1 of the affine toric variety $Y_{\bar{\sigma}}$. The spirit of this formula is very closed to the problem of splitting some affine cuts of the cone $\bar{\sigma}$ into a Minkowski sum of two polyhedra.

In a forthcoming paper, we will use the result 2.4 to prove that 1-parameter-deformations, indeed, correspond to special Minkowski sums (such that both summands do not admit any common direction of faces, i.e. the summands are extremal in some sense). The total spaces can be constructed by taking the convex hull of a suitable configuration of both summands.

The next task will be to generalize this point of view to deformations with parameter spaces of higher dimension (more than two Minkowski summands have to be involved). Then, it will be clear, under which conditions two 1-parameter-deformations belong to the same irreducible component of the versal base space.

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2. Relative complete intersections in the category of affine torus varieties

2.1 With the notations of the previous chapter, a deformation diagramme

$$\begin{array}{ccc} Y & \stackrel{i}{\hookrightarrow} & X \\ \downarrow & \otimes & \downarrow f \\ \{0\} & \hookrightarrow & S \end{array}$$

of 1.2 would induce a deformation of the torus T_Y:

$$\begin{array}{ccc} T_Y & \hookrightarrow & T_X \\ \downarrow & \otimes & \downarrow f \\ \{0\} & \hookrightarrow & S \end{array} .$$

T_Y and T_X are smooth. Hence, T_X would split locally in a product $T_X \cong T_Y \times S$, and the germ $(S, 0)$ would have to be smooth also.

2.2 Let a deformation as in 1.2 be given.

Proposition 2.2 *Let $I := \operatorname{Ker}\left(\mathbb{C}[\check{\sigma} \cap M] \to \mathbb{C}[\bar{\sigma}^\vee \cap M/_L]\right)$ be the ideal that defines $Y \subseteq X$. Then, I can be generated by m polynoms of the form $x^r - x^s \in \mathbb{C}[\check{\sigma} \cap M]$ ($r, s \in \check{\sigma} \cap M$; $r - s \in L$). In particular, every deformation $f : X \to S$ of Y described above can be regarded in the following way:*
1) *$Y \subseteq X$ is a relative complete intersection and*
2) *f is a relative deformation of Y inside X.*

Proof. Step 1: On the level of the local rings we obtain the following diagramme:

$$\begin{array}{ccccc} \mathcal{O}_{X,0} & & \xrightarrow{i^*} & \mathcal{O}_{Y,0} & \\ \uparrow & \text{flat} & \otimes & \uparrow & \\ \mathcal{O}_{S,0} & & \longrightarrow & \mathcal{O}_{S,0}/m_{S,0} & = \mathbb{C} \end{array} .$$

Therefore, $I \cdot \mathcal{O}_{X,0} = m_{S,0} \cdot \mathcal{O}_{X,0}$ is generated by m elements $g_1, \ldots, g_m$ ($(S,0)$ is smooth), and by the Nakayama Lemma we can choose these generators among the elements of the form $x^r - x^s$ ($r, s \in \check{\sigma} \cap M$; $r - s \in L$).

Step 2: Let $\tilde{I} := (g_1, \ldots, g_m) \subseteq \mathbb{C}[\check{\sigma} \cap M]$. Then, $\tilde{I} \subseteq I$ are ideals in $\mathbb{C}[\check{\sigma} \cap M]$ meeting the following properties:
(i) $\tilde{I}$, I are homogeneous with respect to the $M/_L$-grading;
(ii) $\tilde{I} = I$ in the local ring $\mathcal{O}_{X,0}$.

Now, we want to show $\tilde{I} = I$. For this let $g \in I$ be an arbitrary $M/_L$-homogeneous element. By (ii) there exists an $h \in \mathbb{C}[\check{\sigma} \cap M]$ with

$$h \cdot g \in \tilde{I} \quad \text{and}$$
$$h \notin m_0 := \bigoplus_{r \in \check{\sigma} \cap M,\, r \neq 0} \mathbb{C} \cdot x^r \quad \text{(i.e. } h \text{ contains a nonvanishing constant term);}$$

by (i) we can assume in addition h to be $M/_L$-homogeneous.

Finally, the closed embedding $i : Y \hookrightarrow X$ sends 0 to 0 (cf. 1.2), and this property implies $L \cap \check{\sigma} = \{0\}$. Hence, the $M/_L$-homogeneous element h has to be a constant. □

Remark. The m vectors $r - s$ that correspond to the generators of I are free generators of the m-dimensional lattice L.

2.3 Now, our task (cf. 1.2) is reduced to describe all relative complete intersections in the category of affine torus varieties. For doing this, we will change our point of view slightly:

Let M, N be n-dimensional dual lattices as before; let $\sigma =< a^1, \dots, a^N >$ be a top dimensional, rational polyhedral cone in $N \otimes \mathbb{R}$ that does not contain any linear subspace. ($a^1, \dots, a^N \in N$ denote the fundamental generators of σ. They are primitive elements of the lattice, i.e. the coordinates of each a^j have the greatest common divisor 1.) For $r_0^1, \dots, r_0^m; r_1^1, \dots, r_1^m \in \check{\sigma} \cap M$ $(1 \leq m \leq n-2)$ we define $L := \sum_{i=1}^{m} \mathbb{Z} \cdot (r_0^i - r_1^i) \subseteq M$. Suppose that L is an m-dimensional sublattice of M with $M/_L$ torsion free and $L \cap \check{\sigma} = \{0\}$. The last condition implies that the "squeezed" cone $\bar{\sigma} := \sigma \cap L^{\perp}$ is top dimensional in $L^{\perp} \subseteq N \otimes \mathbb{R}$; we obtain the affine torus varieties $X := \operatorname{Spec} \mathbb{C}[\check{\sigma} \cap M]$ and $Y := \operatorname{Spec} \mathbb{C}[\bar{\sigma}^{\vee} \cap M/_L]$ together with a map $i : Y \to X$.

Remark. The canonical homomorphism $\mathbb{C}[\check{\sigma} \cap M] \to \mathbb{C}[\bar{\sigma}^{\vee} \cap M/_L]$ need not be surjective (i.e. i would not be a closed embedding) in general. Look at the following example:

$$M := \mathbb{Z}^2, \quad \check{\sigma} :=< (1,0); (0,1) >\subseteq \mathbb{R}^2;$$
$$r_0^1 := (2,0), \quad r_1^1 := (0,3).$$

Then, $L = \mathbb{Z} \cdot (2,-3)$, M splits into $M = L \oplus \mathbb{Z} \cdot (1,-1)$, and the canonical surjection $M \twoheadrightarrow M/_L \cong \mathbb{Z}$ is given by

$$(1,0) \mapsto 3, \quad (0,1) \mapsto 2.$$

Restricting this map to $\check{\sigma} \cap M$, $1 \in \mathbb{Z}$ will not be contained in the image of $\check{\sigma} \cap M \to \bar{\sigma}^{\vee} \cap \mathbb{Z}$ $(= \mathbb{N})$. (The reason for having such cases is that the image of the semigroup $\check{\sigma} \cap M$ need not be saturated.)

Theorem 2.4 *Y has an isolated singularity in $0 \in Y$, i is a closed embedding, and the ideal $I := \operatorname{Ker} \left(\mathbb{C}[\check{\sigma} \cap M] \to \mathbb{C}[\bar{\sigma}^{\vee} \cap M/_L] \right)$ is generated by the elements $x^{r_0^1} - x^{r_1^1}, \dots, x^{r_0^m} - x^{r_1^m} \in \mathbb{C}[\check{\sigma} \cap M]$ (i.e. the situation of 1.2 with $S = \mathbb{C}^m$), iff for all proper faces $\tau < \sigma$ one of the following conditions is fulfilled (with a suitable order of the r^i's and a^j's and with a possible permutation of $r_0^\bullet$ and $r_1^\bullet$):*

(I) $\exists i \in \{1, \dots, m\} : r_0^i \in \tau^{\perp}; r_1^i \notin \tau^{\perp}$, *or*

(II) *$\tau =< a^1, \dots, a^k >$ is smooth (i.e. generated by a part of a $\mathbb{Z}$-basis of N), and there exists an ℓ $(0 \leq \ell \leq k-1)$ with*

- $< a^j, r_0^i >= \delta_{ij} \quad (i = 1, \dots, \ell;\ j = 1, \dots, k)$
- $< a^j, r_1^i >= 0 \quad (1 \leq j \leq i \leq \ell)$
- $< a^j, r_v^i >= 0 \quad (i = \ell+1, \dots, m;\ j = 1, \dots, k;\ v = 0, 1).$

The conditions (I) and (II) of the theorem may look a little bit technical, but they are very strong and easy to handle. As we will see in §4 these two conditions can be used to restrict the choice of possible cones σ to a great extent. Moreover, a coarser (but much nicer) version of the above theorem is sufficient for taking the first big steps to classifying squeezable cones:

2.5 Notations. 1) Let $R := \{(r_0^1)^\perp, \dots, (r_0^m)^\perp; (r_1^1)^\perp, \dots, (r_1^m)^\perp\}$, i.e. we regard the elements r_ν^i as faces of σ (written as "$r_\nu^i \in R$" or "$r_\nu^i \notin R$"). In general, these faces will be of dimension $n-1$. We obtain $\#R \le 2m \le 2n-4$.
2) A top dimensional face $\tau < \sigma$ that does not belong to R will be called a "good face".

Theorem 2.5 *If Y has an isolated singularity in $0 \in Y$, i is a closed embedding, and the ideal $I := \operatorname{Ker}\left(\mathbb{C}[\check{\sigma} \cap M] \to \mathbb{C}[\bar{\sigma}^\vee \cap M/_L]\right)$ is generated by the elements $x^{r_0^1} - x^{r_1^1}, \dots, x^{r_0^m} - x^{r_1^m} \in \mathbb{C}[\check{\sigma} \cap M]$ (i.e. the situation of 1.2 with $S = \mathbb{C}^m$), then, each good (top dimensional) face $\tau < \sigma$ admits the following properties:*
(i) *It is simplicial (even smooth).*
(ii) *With a suitable order of the r^i's and with a possible permutation of $r_0^\bullet$ and $r_1^\bullet$ there holds:*
- *$r_0^1, \dots, r_0^m$ intersect τ in m different $(n-2)$-dimensional faces. ($r_0^1, \dots, r_0^m$ were called the R-neighbours of τ.)*
- *For $i = 1, \dots, m$ the set $(r_0^1 \cap \dots \cap r_0^i) \cup r_1^i$ contains all vertices of τ.*

Remark. If r is an R-neighbour of τ, then either r is an $(n-2)$-dimensional edge of τ, or r is a top dimensional face of σ that is adjacent to τ. In the first case, r can only be an R-neighbour for at most two different good faces of σ.

3. Proof of Theorem 2.4

3.1 With the notations of 2.3 let $\tilde{Y} \subseteq X$ be the zero set of the ideal $(x^{r_0^1} - x^{r_1^1}, \dots, x^{r_0^m} - x^{r_1^m}) \subseteq \mathbb{C}[\check{\sigma} \cap M]$. Then, the map $i : Y \to X$ factorizes through $Y \to \tilde{Y} \stackrel{closed}{\hookrightarrow} X$.

Lemma 3.1 *For an arbitrary proper face $\tau < \sigma$ let $\operatorname{orb} \tau := \operatorname{Spec} \mathbb{C}[\tau^\perp \cap M]$ be the corresponding orbit; denote by $1_\tau \in \operatorname{orb} \tau$ the unit of this torus (defined by sending all monomials to $1 \in \mathbb{C}$). Now, the following conditions are equivalent:*
(1) $1_\tau \in \tilde{Y}$;
(2) $\operatorname{orb} \tau \cap \tilde{Y} \ne \emptyset$;
(3) $\forall i = 1, \dots, m : [r_0^i \in \tau^\perp \Leftrightarrow r_1^i \in \tau^\perp]$.

Proof. (2) $\Longrightarrow$ (3): The existence of a common point of orb τ and $\tilde{Y}$ yields a ring homomorphism $\varphi : \mathbb{C}[\check{\sigma} \cap M] \longrightarrow\!\!\!\!\rightarrow \mathbb{C}$ with the following properties:
(i) $r \notin \tau^\perp \Longleftrightarrow \varphi(x^r) = 0$. ($\varphi$ factorizes through $\mathbb{C}[\check{\sigma} \cap M] \hookrightarrow \mathbb{C}[\tau^\vee \cap M] \longrightarrow\!\!\!\!\rightarrow \mathbb{C}[\tau^\perp \cap M]$. Hence, φ maps x^r to 0 for $r \notin \tau^\perp$ and to a unit otherwise.)
(ii) $\varphi(x^{r_0^i} - x^{r_1^i}) = 0$ for $i = 1, \dots, m$.
Let $i \in \{1, \dots, m\}$. Then, we obtain $\varphi(x^{r_0^i}) = \varphi(x^{r_1^i})$, hence

$$\varphi(x^{r_0^i}) = 0 \quad \text{iff} \quad \varphi(x^{r_1^i}) = 0.$$

By (i), this equivalence implies the condition (3).
(3) $\Longrightarrow$ (1): The ideal of $1_\tau \in X$ is generated (as a $\mathbb{C}$-vectorspace) by
- x^s (for $s \in \check{\sigma} \cap M$; $s \notin \tau^\perp$) and

- $x^s - 1$ (for $s \in (\check{\sigma} \cap \tau^\perp) \cap M$).

Hence, this ideal contains the ideal of $\tilde{Y} \subseteq X$. $\square$

Corollary. *Let $(\tilde{Y}, 0)$ be an isolated singularity. Then, for singular, proper faces $\tau < \sigma$ (i.e. X is not smooth in* orb τ*) the condition (I) of the theorem is fulfilled.*

Proof. Failing (I) is equivalent to the condition (3) of the previous lemma, i.e. $1_\tau \in \tilde{Y}$. Now, $\tau \neq \sigma$ implies $1_\tau \neq 0$, and $1_\tau \in \tilde{Y}$ has to be a smooth point. Hence, in 1_τ the torus variety X would be non-singular too. $\square$

Lemma 3.2 *Let $(\tilde{Y}, 0)$ be an isolated singularity, let $\tau < \sigma$ be a proper face that fails the condition (I) of the theorem. (In particular, $\tau = < a^1, \dots, a^k >$ is a smooth face.) Then, with a suitable order of the r^i's and a^j's and with a possible permutation of $r_0^\bullet$ and $r_1^\bullet$, there exists an ℓ $(0 \leq \ell \leq k)$ with*

- $< a^j, r_0^i > = \delta_{ij} \quad (i = 1, \dots, \ell;\ j = 1, \dots, k)$,
- $< a^j, r_\nu^i > = 0 \quad (i = \ell + 1, \dots, m;\ j = 1, \dots, k;\ \nu = 0, 1)$.

Proof. Step 1: Completing the fundamental generators $a^1, \dots, a^k$ of τ to a $\mathbb{Z}$-basis $a^1, \dots, a^k; b^{k+1}, \dots, b^n$ of the lattice N, we obtain coordinates that split into two blocks: $r \in M$ can be written as $r = (\bar{r}, \bar{\bar{r}})$ $(\bar{r} \in \mathbb{Z}^k;\ \bar{\bar{r}} \in \mathbb{Z}^{n-k})$ with $(\bar{r})_j = < a^j, r >$, and in the same way $(\bar{x}, \bar{\bar{x}}) = (x_1, \dots, x_n)$ denote the coordinates of $\mathbb{C}^k \times (\mathbb{C}^*)^{n-k} \cong \operatorname{Spec}[\tau^\vee \cap M] \overset{open}{\hookrightarrow} X$. (orb $\tau \subseteq \operatorname{Spec} \mathbb{C}[\tau^\vee \cap M]$ is then given by the equation $\bar{x} = 0$, and $1_\tau \in$ orb τ corresponds to the point $(\underline{0}, \underline{1})$.)

Now, $\tilde{Y}$ contains 1_τ, and we can compute the partial derivatives of the equations $g_i := x^{r_0^i} - x^{r_1^i}$ in this point. Let $\ell \in \mathbb{N}$ $(0 \leq \ell \leq m)$ such that (after a permutation of the indices $i \in \{1, \dots, m\}$, if necessary)

$$\begin{aligned} &r_0^i, r_1^i \notin \tau^\perp \quad \text{(i.e. } \bar{r}_0^i, \bar{r}_1^i \neq \underline{0}) \text{ for } i = 1, \dots, \ell; \\ &r_0^i, r_1^i \in \tau^\perp \quad \text{(i.e. } \bar{r}_0^i = \bar{r}_1^i = \underline{0}) \text{ for } i = \ell + 1, \dots, m. \end{aligned}$$

Then, denoting with $\{e_1, \dots, e_k\}$ the canonical basis of $\mathbb{Z}^k$, we obtain

$$\begin{aligned} &\frac{\partial g_i}{\partial x_j}(1_\tau) = \delta_{\bar{r}_0^i, e_j} - \delta_{\bar{r}_1^i, e_j} \quad (i = 1, \dots, m;\ j = 1, \dots, k); \\ &\frac{\partial g_i}{\partial x_j}(1_\tau) = 0 \qquad (i = 1, \dots, \ell;\ j = k + 1, \dots, n); \\ &\frac{\partial g_i}{\partial x_j}(1_\tau) = (r_0^i - r_1^i)_j = < b^j, r_0^i - r_1^i > \quad (i = \ell + 1, \dots, m;\ j = k + 1, \dots, n). \end{aligned}$$

Introducing the following notation

$$\bar{R}_\bullet^i := \begin{cases} \bar{r}_\bullet^i \in \mathbb{Z}^k & \text{if } \bar{r}_\bullet^i \text{ is a unit vector,} \\ \underline{0} \in \mathbb{Z}^k & \text{otherwise,} \end{cases}$$

we can write down our result in a shorter way:

Step 2: After a possible changing of the order of the indices i and j and after a possible permutation of $\bar{R}_0^\bullet$ and $\bar{R}_1^\bullet$, we obtain $\bar{R}_0^i = e_i$ for $i = 1, \ldots, \ell$ $(\leq k)$:
Since 1_τ is a smooth point of $\tilde{Y}$, the Jacobian matrix $\frac{\partial g}{\partial x}(1_\tau)$ has the maximal rank m. Hence, $\operatorname{rank}(\bar{R}_0^i - \bar{R}_1^i)_{i=1,\ldots,\ell} = \ell$, and this implies $\ell \leq k$. Now, by selecting ℓ suitable columns of $(\bar{R}_0^i - \bar{R}_1^i)_{i=1,\ldots,\ell}$ we obtain a new $(\ell \times \ell)$-matrix A with $\det A \neq 0$ and all non-vanishing entries equal to ± 1. It would be enough to prove that there exist ℓ such entries that admit a position of ℓ castles that cannot beat each other in a $\ell \times \ell$ chess game. But this follows easily by induction from the Laplacian Entwicklungssatz: Let

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1\ell} \\ \vdots & \ddots & \vdots \\ a_{\ell 1} & \ldots & a_{\ell\ell} \end{pmatrix};$$

then $\sum_{j=1}^{\ell} (-1)^{j+1} a_{1j} \det A_{1j} = \det A \neq 0$, and there exists an index j with $a_{1j} \neq 0$ and $\det A_{1j} \neq 0$. $\square$

Remark. For τ to meet the condition (II) of the theorem, only the proof of $< a^j, r_1^i >= 0$ $(1 \leq j \leq i \leq \ell)$ is left. This will be done in (3.4).

3.3 Let $Y = \tilde{Y}$, i.e. the ideal $I = \operatorname{Ker}\left(\mathbb{C}[\check{\sigma} \cap M] \to \mathbb{C}[\bar{\sigma}^\vee \cap M/_L]\right)$ is generated by the elements $g_i = x^{r_0^i} - x^{r_1^i}$ $(i = 1, \ldots, m)$. For any index set $\Lambda \subseteq \{1, \ldots, m\}$ we denote by L_Λ the corresponding sublattice $L_\Lambda := \bigoplus_{i \in \Lambda} \mathbb{Z}(r_0^i - r_1^i) \subseteq L$ of M.

Lemma 3.3 1) *Let $\Lambda \subseteq \{1, \ldots, m\}$; let $p \in L \setminus L_\Lambda$, $s \in M$ be elements with $s, s - p \in \check{\sigma}$. Then there exists an index $i \in \{1, \ldots, m\} \setminus \Lambda$ such that $[(s - r_\nu^i) + L_\Lambda] \cap \check{\sigma} \neq \emptyset$ (for some $\nu \in \{0, 1\}$).*
2) *Given a face $\tau =< a^1, \ldots, a^k > < \sigma$, we denote by Λ_τ the index set $\Lambda_\tau := \{i \mid r_0^i, r_1^i \in \tau^\perp\}$. Now, for p, s as before there exist $i \in \{1, \ldots, m\} \setminus \Lambda$ and $\nu \in \{0, 1\}$ such that $< a^j, s - r_\nu^i > \geq 0$ for all $j = 1, \ldots, k$.*

Proof. Step 1: We define a graph with M as the set of vertices as follows: $s, t \in M$ will be connected by an edge iff there exists an index $i \in \{1, \ldots, m\}$ such that either

$$s - r_0^i = t - r_1^i \in \check{\sigma} \qquad \text{or} \qquad s - r_1^i = t - r_0^i \in \check{\sigma}.$$

Now, let elements $p \in L$, $s \in M$ be given such that $s, s-p \in \check{\sigma}$; $x^s - x^{s-p} \in I$ yields an equation

$$(*) \qquad x^s - x^{s-p} = \sum_\mu c_\mu x^{t_\mu}(x^{r_0^{i(\mu)}} - x^{r_1^{i(\mu)}}) \qquad (c_\mu \in \mathbb{C}; t_\mu \in \check{\sigma} \cap M).$$

The exponents $t_\mu + r_0^{i(\mu)}$ and $t_\mu + r_1^{i(\mu)}$ are connected by an edge in the above graph $\mathcal{M}$. Therefore, by deleting all terms that admit exponents not contained in the connected component of $s \in \mathcal{M}$, the equation $(*)$ keeps its form at the right-hand side. Only two different cases can arise:

(i) $x^s - x^{s-p} = \sum_\mu{}' c_\mu x^{t_\mu}(x^{r_0^{i(\mu)}} - x^{r_1^{i(\mu)}})$ or

(ii) $x^s = \sum_\mu{}' c_\mu x^{t_\mu}(x^{r_0^{i(\mu)}} - x^{r_1^{i(\mu)}})$ ("$\sum'$" means a part of "$\sum$").

The latter case would imply $x^s \in I$, i.e. $x^{\bar{s}} = 0$ in the ring $\mathbb{C}[M/_L]$. Hence, only the first case is possible, and we obtain that s and $s-p$ are contained in a common connected component of $\mathcal{M}$.

Step 2: Any path from s to $s-p$ provides a sequence $s = s_0, \ldots, s_N = s-p$ with either

$$\begin{aligned} s_{\mu-1} - r_0^{i(\mu)} &= s_\mu - r_1^{i(\mu)} \in \check{\sigma} \cap M, \quad \text{or} \\ s_{\mu-1} - r_1^{i(\mu)} &= s_\mu - r_0^{i(\mu)} \in \check{\sigma} \cap M \quad (\mu = 1, \ldots, N). \end{aligned}$$

The element $p = s_0 - s_N = \sum_{\mu=1}^N \pm(r_0^{i(\mu)} - r_1^{i(\mu)})$ was presumed not to be contained in the sublattice $L_\Lambda \subseteq L$. Hence, at least one of the indices $i(\mu)$ does not belong to Λ.

Assume μ^* to be the smallest index meeting this property, then we obtain

$$s_0 - s_{\mu^*-1} = \sum_{\mu=1}^{\mu^*-1} \pm(r_0^{i(\mu)} - r_1^{i(\mu)}) \in L_\Lambda$$

and

$$\exists \nu \in \{0,1\}: \quad (s - r_\nu^i) - (s_0 - s_{\mu^*-1}) = s_{\mu^*-1} - r_\nu^i \in \check{\sigma} \cap M \quad (i := i(\mu^*) \notin \Lambda).$$

Finally, the second part of the lemma is a direct consequence from the first part. $\square$

Corollary. *Let $\tau = < a^1, \ldots, a^k > < \sigma$ be a smooth face; denote $\Lambda_\tau := \{i \mid r_0^i, r_1^i \in \tau^\perp\}$ as before. Then, for each $p \in L \setminus L_\Lambda$ there exist $i \in \{1, \ldots, m\} \setminus \Lambda$ and $\nu \in \{0,1\}$ such that*

$$< a^j, r_\nu^i > \leq \max\{< a^j, p >; 0\} \quad \text{for all} \quad j = 1, \ldots, k.$$

Proof. For a given $p \in L$ we construct the following elements of the lattice M:

1) Since τ is a smooth cone (i.e. $\{a^1,\dots,a^k\}$ is a part of a $\mathbb{Z}$-basis of the lattice N), there exists an $s^1 \in M$ such that

$$< a^j, s^1 >= \max\{< a^j, p >; 0\} \quad \text{for all} \quad j = 1,\dots,k.$$

2) Let $s^2 \in [\mathrm{int}\,(\check{\sigma} \cap \tau^\perp)] \cap M$. Then, we obtain

$$< a^j, s^2 >= 0 \quad (j = 1,\dots,k) \quad \text{and} < a^\mu, s^2 >> 0$$

for all remaining fundamental generators a^μ of the cone σ.
3) Now, $s := s^1 + g \cdot s^2$ ($g \in \mathbb{N}$; $g \gg 0$) admits the following properties:

$$s, s-p \in \check{\sigma} \text{ and } < a^j, s >= \max\{< a^j, p >; 0\} \quad (j = 1,\dots,k).$$

Applying the previous lemma to τ, p and s completes the proof. □

Lemma. 3.4 *Let $Y = \tilde{Y}$ have an isolated singularity in 0 (that means a combination of the presumptions of 3.2 and 3.3). Let $\tau =< a^1,\dots,a^k > < \sigma$ be a proper face that fails the conditions (I) of the theorem. Then, additionally to the claim of Lemma (3.2), we can achieve $< a^j, r_1^i >= 0$ for $1 \le j \le i \le \ell$.*

Proof. In 3.2 we have shown

$$\begin{aligned} &< a^j, r_0^i >= \delta_{ij} \quad (i = 1,\dots,\ell\,(\le k);\ j = 1,\dots,k) \quad \text{and} \\ &< a^j, r_\nu^i >= 0 \quad (i = \ell+1,\dots,m;\ j = 1,\dots,k;\ \nu = 0,1); \end{aligned}$$

in particular $\Lambda_\tau = \{\ell+1,\dots,m\}$. Now, we prove the above lemma by induction; let $\ell' \le \ell$ such that

$$< a^j, r_1^i >= 0 \quad \text{for} \quad 1 \le j \le i \le \ell; \quad j \le \ell' - 1.$$

Claim: There exists an index j ($\ell' \le j \le \ell$) such that $< a^j, r_1^i >= 0$ for each $i \ge \ell'$.
Proof of this claim: Let this not be the case, i.e. $< a^j, \sum_{i \ge \ell'} r_1^i > \ge 1$ for each $j = \ell',\dots,\ell$.
Then, we define $p := \sum_{i \ge \ell'} (r_0^i - r_1^i) \in L \setminus L_\Lambda$, which admits the following properties:

1) $< a^j, p >= 0$ for $j = 1,\dots,\ell'-1$ (since $< a^j, r_0^i >=< a^j, r_1^i >= 0$ for $j \le \ell'-1$, $i \ge \ell'$);
2) $< a^j, p > \le 0$ for $j = \ell',\dots\ell$ (since $< a^j, \sum_{i \ge \ell'} r_0^i >= 1$, $< a^j, \sum_{i \ge \ell'} r_1^i > \ge 1$ for $\ell' \le j \le \ell$);
3) $< a^j, p > \le 0$ for $j \ge \ell+1$ (since $< a^j, r_0^i >= 0$ for $j \ge \ell+1$).

Therefore, $\max\{< a^j, p >; 0\} = 0$ for $j = 1,\dots,k$, and we can apply the corollary of 3.3: There exist $i \in \{1,\dots,\ell\}$ and $\nu \in \{0,1\}$ such that $< a^j, r_v^i > \le 0$ for all $j = 1,\dots,k$. By τ failing the condition (I) of the theorem this implies $r_0^i \in \tau^\perp$, and we obtain a contradiction.

Permuting the indices ℓ' and j in both sets $\{a^1,\dots,a^k\}$ and $\{r^1,\dots,r^\ell\}$ the induction hypothesis together with the above claim imply

$$< a^j, r_1^i >= 0 \quad \text{for} \quad 1 \le j \le i \le \ell; \quad j \le \ell'.$$

The statements of Lemma 3.2 also stay valid during this permutation; hence, the lemma is proven. □

Remark. 1) The lemma implies that ℓ has to be smaller than k.
2) The lemmata 3.2 and 3.4 show that meeting the properties (I) or (II) in Theorem 2.4 is a necessary condition (for $Y = \tilde{Y}$ having an isolated singularity). It remains to prove the opposite direction of this theorem.

3.5 Assume that for each proper face $\tau = < a^1, \ldots, a^k > < \sigma$ at least one of the conditions (I) or (II) is fulfilled.

Lemma 3.5 *Let $\tau < \sigma$ be a proper face such that* $\operatorname{orb} \tau \cap \tilde{Y} \neq \emptyset$. *Then,* $\operatorname{Spec} \mathbb{C}[\check{\tau} \cap M] \cap \tilde{Y}$ *is irreducible, smooth of dimension $n-m$, and contains the torus* $\operatorname{Spec} \mathbb{C}[M/_L]$ *as an open, dense subset.*

Proof. $\operatorname{orb} \tau \cap \tilde{Y} \neq \emptyset$ implies that τ fails the condition (I) (cf. Lemma 3.1), hence, condition (II) must be fulfilled. We use the language of the proof of Lemma 3.2: $U_\tau := \operatorname{Spec} \mathbb{C}[\check{\tau} \cap M] \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ is an open, dense subset of X; the equations of $\tilde{Y}$ in U_τ have the following form:

$$
\begin{aligned}
x_1 \cdot \bar{\bar{x}}^{\bar{\bar{r}}^1_0} &= \bar{x}^{\bar{r}^1_1} \cdot \bar{\bar{x}}^{\bar{\bar{r}}^1_1} && \text{(the side on the right hand does not contain } x_1\text{)} \\
x_2 \cdot \bar{\bar{x}}^{\bar{\bar{r}}^2_0} &= \bar{x}^{\bar{r}^2_1} \cdot \bar{\bar{x}}^{\bar{\bar{r}}^2_1} && \text{(the side on the right hand does not contain } x_1, x_2\text{)} \\
&\vdots \\
x_\ell \cdot \bar{\bar{x}}^{\bar{\bar{r}}^\ell_0} &= \bar{x}^{\bar{r}^\ell_1} \cdot \bar{\bar{x}}^{\bar{\bar{r}}^\ell_1} && \text{(the side on the right hand does not contain } x_1, \ldots, x_\ell\text{)} \\
\bar{\bar{x}}^{\bar{\bar{r}}^i_0} &= \bar{\bar{x}}^{\bar{\bar{r}}^i_1} && \text{for } i = \ell+1, \ldots, m.
\end{aligned}
$$

The last $m-\ell$ equalities define a subtorus $(\mathbb{C}^*)^{(n-m)-(k-\ell)} \subseteq (\mathbb{C}^*)^{n-k}$. The variables $x_{\ell+1}, \ldots, x_k$ (belonging to $\bar{x}$) can free range in a space $\mathbb{C}^{k-\ell}$, but the variables $x_1, \ldots, x_\ell$ are completely determined by the first ℓ equations. Hence, $U_\tau \cap \tilde{Y} \cong \mathbb{C}^{k-\ell} \times (\mathbb{C}^*)^{(n-m)-(k-\ell)}$; the torus $\operatorname{Spec} \mathbb{C}[M/_L]$ corresponds to the open, dense subset $(\mathbb{C}^*)^{k-\ell} \times (\mathbb{C}^*)^{(n-m)-(k-\ell)} = (\mathbb{C}^*)^{n-m}$. □

Corollary. *$\tilde{Y} \setminus \{0\}$ is irreducible and smooth of dimension $n-m$. It contains* $\operatorname{Spec} \mathbb{C}[M/_L]$ *as an open, dense subset, and the torus action $T \times X \to X$ induces an action of the small torus*

$$\operatorname{Spec} \mathbb{C}[M/_L] \times (\tilde{Y} \setminus \{0\}) \to (\tilde{Y} \setminus \{0\}).$$

Proof. All these properties can be checked locally, and $\tilde{Y} \setminus \{0\}$ is covered by the open subsets $U_\tau \cap \tilde{Y}$ (taking all proper faces $\tau < \sigma$ such that $\operatorname{orb} \tau \cap \tilde{Y} \neq \emptyset$). □

Therefore, $\tilde{Y}$ itself is irreducible of dimension $n-m$, and $0 \in \tilde{Y}$ is an isolated singularity. As $\tilde{Y}$ is a relative complete intersection in the Cohen-Macaulay variety X, $\tilde{Y}$ has to be Cohen-Macaulay, too. Hence, $\tilde{Y}$ is reduced and normal (non-singular in codimension 1 since $m \leq n-2$).

Spec $\mathbb{C}[M/_L] \subseteq \tilde{Y}$ is an open, dense subset, and the group law of the torus extends to an action on $\tilde{Y}$ (which is the restriction of the action of T_X on X). Therefore, $\tilde{Y}$ must be an affine torus variety, too, i.e. we obtain $Y = \tilde{Y}$. This completes the proof of the theorem.

4. Cones with simplicial faces only

4.1 We use the notations of 2.3. Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone such that for all good faces $\tau < \sigma$ the conditions (i) and (ii) of the Theorem 2.5 are fulfilled. This means a very strong restriction to the form of σ, and we want to determine all abstract cones (i.e. their structure is reduced to the incidence relation between all faces) that can appear in this way.

Remark. Each n-dimensional σ is a cone over a compact, convex polyhedron P_σ of dimension $n-1$. This polyhedron yields the same incidence relations as σ; it can be recovered by intersection of σ with a suitable affine hypersurface in $N_{\mathbb{R}}$. Sometimes (in particular for $n=4$) it is useful to imagine this polyhedron P_σ instead of σ itself. According to this, the fundamental generators of σ were often called vertices also.

4.2 For the rest of this chapter we make the following

Assumption. *Let σ be a cone such that all proper faces are simplicial ones. That means the corresponding toric variety $X = \operatorname{Spec} \mathbb{C}[\check{\sigma} \cap M]$ contains at most cyclic quotient singularities outside $0 \in X$.*

Definition. For top dimensional faces $\tau_1, \tau_2 < \sigma$ we define

$$\begin{aligned} \varrho(\tau_1, \tau_2) :=& (n-1) - \dim(\tau_1 \cap \tau_2) \\ =& \#\{\text{vertices of } \tau_1 \text{ that do not belong to } \tau_2\} \\ =& \#\{\text{vertices of } \tau_2 \text{ that do not belong to } \tau_1\}. \end{aligned}$$

Lemma 4.2 1) *$0 \leq \varrho(\tau_1, \tau_2) \leq n-1$.*
2) *$\varrho(\tau_1, \tau_2) = 0 \Longleftrightarrow \tau_1 = \tau_2$; $\varrho(\tau_1, \tau_2) = 1 \Longleftrightarrow \tau_1$ and τ_2 are adjacent.*
3) *If τ_1 and τ_2 are connected by a path meeting only k $(n-2)$-dimensional faces of σ but no faces of smaller dimension, then $\varrho(\tau_1, \tau_2) \leq k$.*
4) *Let $K = \{\tau_i\}$ be a set of top dimensional faces such that any two faces of K are adjacent to each other. Then, if a face $\tau < \sigma$ is adjacent to two faces of K, τ is adjacent to all faces of K.*
5) *Let $\{\tau_1, \dots, \tau_k\}$ and $\{\varphi_1, \dots, \varphi_\ell\}$ $(k, \ell \geq 2)$ be two sets of top dimensional faces such that $\varrho(\tau_i, \varphi_j) = 1$ for all pairs $[i,j]$. Then there are only two possible cases:*
a) $k = \ell = 2$; $\varrho(\tau_1, \tau_2) = \varrho(\varphi_1, \varphi_2) = 2$, *or*

b) $\varrho(\tau_i, \tau_j) = 1, \quad \varrho(\varphi_i, \varphi_j) = 1$ *for all possible pairs* $[i, j]$.
(In particular, if σ *is not a simplex, (b) implies* $k + \ell \leq n - 1$ *!)*

The proof is easy and will be omitted here.

4.3 We start our investigations by regarding (as a first case) the following class of cones σ:
Assume that there exist (top dimensional) faces τ_L and τ_R such that $\varrho(\tau_L, \tau_R) \geq 3$.
Then the sets of all faces that are adjacent to τ_L and τ_R, respectively, have no common elements. The cone σ must contain at least $2n$ various faces:

Figure 1: Each fat point represents a face; the edges between two of these fat points mean that both faces are adjacent ones.

The faces τ_L and τ_R have the following form:

$$\tau_L =< A, B, C, \tau_L \cap \tau_R, \ldots >; \quad \tau_R =< P, Q, R, \tau_L \cap \tau_R, \ldots >$$

($A, B, C, P, Q, R \in \sigma$ are vertices, and the points "..." can be omitted, iff $\varrho(\tau_L, \tau_R) = 3$).
For any vertex $X \in \tau_\bullet$ we denote by (X) the adjacent face of $\tau_\bullet$ that does not contain X. Using this language, the faces (A), (B), (C) are the only possible adjacencies of τ_L for $\varrho(., \tau_R) \leq 2$ (analog (P), (Q), (R) on the right-hand side).

Therefore, we define the following sets:

$$B_L := \{\tau_L;\ \text{adjacencies of } \tau_L \text{ that contain } A, B \text{ and } C\};$$
$$B_R := \{\tau_R;\ \text{adjacencies of } \tau_R \text{ that contain } P, Q \text{ and } R\}.$$

Claim. *B_L and B_R admit the following properties:*
1) $\#B_L = \#B_R = n - 3$.
2) *There exists a bijection* $\Phi : B_L \setminus \{\tau_L\} \xrightarrow{\sim} B_R \setminus \{\tau_R\}$ *such that*
   (i) $\varrho(\tau, \Phi(\tau)) \geq 2$ *for each* $\tau \in B_L \setminus \{\tau_L\}$, *and*
   (ii) $\varrho(.,.) \geq 3$ *for the remaining pairs of* $B_L \times B_R$.
   (In particular, $\varrho(\tau_L, B_R) \geq 3$ *and* $\varrho(B_L, \tau_R) \geq 3$.*)*

Proof. If $\varrho(\tau_L, \tau_R) \geq 4$, then there exists at most one pair $[\tau_1, \tau_2] \in B_L \times B_R$ such that $\varrho(\tau_1, \tau_2) \leq 2$. For $\varrho(\tau_L, \tau_R) = 3$, we can define Φ as $\Phi : (S) \mapsto (S)$ for vertices $S \in \tau_L \cap \tau_R$: Let $\tau_1 \in B_L$, let $\tau_2 = (S) \in B_R \setminus \{\tau_R\}$ (S is a vertex of $\tau_L \cap \tau_R$). Then, $A, B, C \in \tau_1$, but at most one of these vertices belongs to τ_R. Hence, $\varrho(\tau_1, \tau_2) \leq 2$ implies that S cannot be contained in the face τ_1, too, i.e. $\tau_1 = (S)$. □

Proposition 4.4 *Let σ be a cone such that all proper faces are simplicial ones. Assume that there exist (top dimensional) faces τ_L and τ_R such that $\varrho(\tau_L, \tau_R) \geq 3$. Then, for $2 \leq m \leq n - 3$, σ cannot meet the condition of Theorem 2.5.*

Proof. Otherwise we would be in the situation of 4.1 - 4.3. Then, we distinguish between two cases:

1st case: $B_L \subseteq R = \{r_0^1, \ldots, r_0^m, r_1^1, \ldots, r_1^m\}$. At most $2m - (n-3) \leq m$ elements of R can sit at the right-hand side of our picture in 4.3. Hence, the remaining (at least three) faces have to be good ones that admit all the same set $R \setminus B_L$ of R-neighbours. This fact contradicts to (5) of Lemma 4.2.

2nd case: τ_L, τ_R are good faces. (After renaming we can assume that τ_L is a good face. Because the first case cannot happen even for the right-hand side, there must be another good face $\tau \in B_R$. Since $\varrho(\tau_L, \tau) \geq 3$ we can rename the faces once more to obtain $\tau = \tau_R$.)

Case (2.1): $B_L \setminus \{\tau_L\} \subseteq R$; $B_R \setminus \{\tau_R\} \subseteq R$. Then, each good face at the left-hand side admits at least one R-neighbour at the right-hand side. (Otherwise, there would be a good face τ adjacent to τ_L such that τ and τ_L admit the same set of R-neighbours. Again by Lemma 4.2(5), the faces τ, τ_L and all these R-neighbours would be adjacent to each other, and so would all other left-hand faces admitting at least one left-hand R-neighbour (cf. Lemma 4.2(4)). Hence, there had to exist a good face adjacent to τ_L that admits right-hand R-neighbours only, but this would imply $m \leq 1$.)

Only $m = n-3$ is still possible, and we can assume the following situation:

$(A), (P) \in R$, but $(B), (C), (Q), (R)$ are good faces;

$\varrho((A),(Q)) = \varrho((A),(R)) = \varrho((B),(P)) = \varrho((C),(P)) = 1$.

This implies $P \in (A)$; $P \notin (B), (C)$, hence, $\varrho((A),(B))$, $\varrho((A),(C)) \geq 2$. Therefore, (B) and (C) have $B_L \setminus \{\tau_L\}$ as common set of R-neighbours, i.e. (B) and (C) have to be adjacent.

We can apply Lemma 4.2 again:

$$\varrho(\tau_L,(B)) = \varrho(\tau_L,(C)) = \varrho((B),(C)) = \varrho((B),(P)) = \varrho((C),(P)) = 1$$

would imply $\varrho(\tau_L,(P)) = 1$ but this means a contradiction.

Case (2.2): There are good faces $\tau_L, \tau \in B_L$; $\tau_R \in B_R$. From τ_L and τ having the same R-neighbours we can derive the existence of a left-hand good face ψ that admits the same R-neighbours as τ_R (cf. the beginning of case (2.1)). In particular, $m = 2$ and these two right-hand R-neighbours have to be outside B_R. Hence, B_R consists of good faces only. We have found $n-2$ faces (B_R and ψ) admitting the same two R-neighbours, which is a contradiction because of Lemma 4.2. □

4.5 As a next step we will investigate the case $2 \leq m = n-2$. In this situation, the property (ii) (cf. Theorem 2.5) of good faces implies

(ii)' There exist not only $m = n-2$ but $m+1 = n-1$ R-neighbours.

(ii)" If $\#R = 2m = 2n-4$, then there is an element $s \in R$ (different from the $n-1$ R-neighbours) that intersects the good face τ in an $(n-3)$-dimensional edge, i.e. $\varrho(\tau, s) \leq 2$.

Proposition 4.5 *Let σ be a cone such that all proper faces are simplicial ones. Assume that there exist (top dimensional) faces τ_L and τ_R such that $\varrho(\tau_L, \tau_R) \geq 3$. Then, for $2 \leq m = n-2$, σ cannot meet the condition of Theorem 2.5.*

Proof. As in the proof of the last proposition, we assume that σ would meet this condition. Obviously, τ_L and τ_R cannot be good faces simultaneously (each of them had to have $n-1$ neighbours from a set with at most $2n-4$ elements). Hence, by the same arguments as in the proof of the previous proposition, we may assume $B_L \subseteq R$.

The right-hand side contains at most $(2n-4)-(n-3) = n-1$ faces of R, i.e. at least one face τ is left to be good. On the other hand, more than or equal to two good faces at the right-hand side would admit the same set $R \setminus B_L$ of R-neighbours, and this would be a contradiction because of Lemma 4.2. Hence, the right-hand side consists exactly of τ and its $n-1$ R-neighbours. This fact implies $\tau = \tau_R$, and we obtain the contradiction by missing the announced element $s \in R$ for the good face τ_R (cf. (ii)" at the beginning of 4.5). □

4.6 In the case $m = 1$ there are two positive examples. The set $R = \{r_0, r_1\}$ consists of two top dimensional faces that have no common vertex (see the next figures).

Proposition 4.6 *Let σ be a cone such that all proper faces are simplicial ones. Assume that there exist (top dimensional) faces τ_L and τ_R such that $\varrho(\tau_L, \tau_R) \geq 3$. Then, for $1 = m \leq n-2$, the above two examples are the only ones such that σ can meet the condition of Theorem 2.5.*

Proof. Each face of σ has to be adjacent either to r_0 or r_1. In particular, σ consists of exactly $2n$ faces, and r_0, r_1 are two of them. By Lemma 4.2(1) we obtain $n \geq 4$.

Claim: τ_R is the only face of σ such that $\varrho(\tau_L, .) \geq 3$. (Let $\tau < \sigma$ such that $\varrho(\tau_L, \tau) \geq 3$. Then, the set of all top dimensional faces of σ can be written as a disjoint union in two different ways:

$$\begin{aligned}\{\tau_L \text{ and its neighbours}\} \amalg \{\tau_R \text{ and its neighbours}\} &= \{\text{faces of } \sigma\} = \\ &= \{\tau_L \text{ and its neighbours}\} \amalg \{\tau \text{ and its neighbours}\}.\end{aligned}$$

Hence, $\{\tau_L$ and its neighbours$\} = \{\tau$ and its neighbours$\}$, and this implies $\tau_L = \tau$.)

In particular, the set B_R consists of the single element τ_R only. That means $n = 4$, and we obtain the following picture:

The two examples described above arise from the following two cases:

(I) The sets $\{(A), (B), (C)\}$ and $\{(P), (Q), (R)\}$ both do not contain any pair of adjacent faces.

(II) $\varrho((A), (B)) = 1$. $\square$

Lemma 4.7 *Let σ be an n-dimensional cone ($n \geq 4$) such that all proper faces are simplicial ones. Then, the following three statements are equivalent:*

(1) $\varrho(.,.) \leq 2$;

(2) *σ admits at most $n+1$ vertices;*

(3) *$\check{\sigma}$ is equal to the cone over $\Delta^k \times \Delta^{n-k-1}$ for a suitable $k = 0, \ldots, \left[\frac{n-1}{2}\right]$. ($\Delta^i$ denotes the i-dimensional, compact simplex.)*

Remark. The equivalence of (2) and (3) is, of course, well known (cf. [4]).

Proof of the Lemma: Assume $\varrho(.,.) \leq 2$. We will deal with the compact polyhedron P_σ instead of σ (cf. 4.1) and have to introduce some notations: Choose a top dimensional face I that will be regarded as basic face of P_σ. The vertices of I will be represented by the integers from 1 to $n-1$: $I = < 1, \ldots, n-1 >$. Let $A = \{a^1, \ldots, a^N\}$ be the set of the remaining vertices (not contained in I) of the polyhedron P_σ.

Then, the faces of P_σ split into only two different types:

Type 1: $[i, a(i)] := < 1, \ldots, \hat{i}, \ldots, n-1;\ a(i) \in A >$, and $a : \{1, \ldots, n-1\} \to A$ becomes a map which is constant iff σ is equal to a simplex.

Type 2: $[i, j; a, b] := < 1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, n-1;\ a, b \in A >$, but the elements $a, b \in A$ are not uniquely determined by giving the pair $[i, j]$. It is even possible that there does not exist any face of this type for given i, j.

For each pair $i, j \in I$ we define a non-oriented graph $G(i, j)$ with its set of vertices contained in A as follows:

$$\overline{ab} \quad \text{is an edge in} \quad G(i,j), \ \text{ iff } \quad [i,j;a,b] \quad \text{is a face of} \quad \sigma \qquad (a, b \in A).$$

We obtain the following properties:

(i) If $a(i) = a(j)$, then there is no face of the form $[i, j; a(i), .]$.

(ii) If $a(i) \neq a(j)$, then there exists exactly one element $b \in A$ such that $[i, j; a(i), b]$ is a face of σ.

(iii) Let $a \neq a(i), a(j)$. Then, the number of faces $[i, j; a, .]$ of σ is either 0 or 2.

Consequences for the graph $G(i, j)$: $G(i, j)$ is the disjoint union of

a) a path from $a(i)$ to $a(j)$ that does not contain any loop (in particular, it is empty if $a(i) = a(j)$) and

b) possibly some loops that do not contain $a(i)$, $a(j)$.

Furthermore we obtain the additional properties:

(iv) Let $i, j, k \in I$; $a, b \in A$. Then, the number of σ-faces among $[i, j; a, b]$, $[j, k; a, b]$ and $[i, k; a, b]$ is either 0 or 2. [That means that the edge $\overline{ab}$ appears in either 0 or 2 of the graphs $G(i, j)$, $G(j, k)$, or $G(i, k)$.]

(v) Let $i, j, k \in I$ be three different elements. Then, for each face $[i, j; a, b]$, either $a = a(k)$ or $b = a(k)$ has to be true. [The graph $G(i, j)$ is star shaped with $a(k)$ as its centre.]

(vi) Let $i, j, k \in I$ be three different elements. If $[i, j; a, b]$ and $[j, k; c, d]$ are faces of σ, then the sets $\{a, b\}$ and $\{c, d\}$ have at least one common element. [Any two edges of $G(i, j)$ and $G(j, k)$, respectively, have at least one common vertex.]

(vii) Let $i, j, k, \ell \in I$ be four different elements. If $[i, j; a, b]$ and $[k, \ell; c, d]$ are faces of σ, then the sets $\{a, b\}$ and $\{c, d\}$ are equal. [If $G(i, j)$, $G(k, \ell)$ are not empty, both graphs coincide and consist of a single edge only.]

(viii) If σ is not a simplex, then for each element $a \in A$ there exists a face of the form $[., .; a, .]$. [Each element of the set A appears as a vertex in some graph $G(i, j)$.]

The proofs of (i) - (viii) are easy; they result from three basic ideas: Regard a suitable $(n-2)$-dimensional face of σ (i.e. a $(n-3)$-dimensional face of P_σ) and use the fact that it is always the intersection of exactly two top dimensional faces. This leads to (i) - (iv).

(v) - (vii): If the claim would not be true, the corresponding faces would differ in more than two vertices (regard $[k, a(k)]$ in (v)). In (viii), the dual statement is trivial: Let $P \in \sigma$ be a vertex that is not contained in a top dimensional face F. Then, there exists another top dimensional face $\tilde{F}$ not containing P such that F and $\tilde{F}$ have contact of codimension 1.

Now, we can use the properties (i) - (iii) and (v) to obtain the following description of the graphs $G(i, j)$:

1) $a(i) = a(j)$ iff $G(i, j) = \emptyset$.

2) If $a(i) \neq a(j)$, then there are two possibilities:

a) $G(i, j) = \text{edge } \overline{a(i)a(j)}\}$ and $a(k) \in \{a(i), a(j)\}$ for each $k \neq i, j$, or

b) $G(i, j) = \text{edges } \overline{a(i)b}, \overline{a(j)b}\}$ and $a(k) = b$ for each $k \neq i, j$.

In particular, if σ is not a simplex, $N = \#A$ can be equal to 2 or 3 only; we will consider these two cases separately:

Case $N = 3$: Each non-empty graph $G(i, j)$ has to be of the form that is described in (b). Hence, the map $a : \{1, \ldots, n-1\} \to A$ is surjective, and we can assume the following situation:

$$
\begin{aligned}
&n=4; \quad A=\{a,b,c\};\\
&a(1)=a, \quad a(2)=b, \quad a(3)=c;\\
&G(1,2)=\{\overline{ac},\overline{bc}\}, \quad G(1,3)=\{\overline{ab},\overline{bc}\}, \quad G(2,3)=\{\overline{ab},\overline{ac}\}.
\end{aligned}
$$

On the other hand, such a polyhedron P_σ cannot exist by the Euler polyhedron formulae (notice the equation $2\cdot\#(\text{edges})=3\cdot(\text{facets})$ because of P_σ having simplicial faces only).

Case $N=2$: We change our notation slightly: Let $A=\{A_0,B_0\}$ and $I=\{A_1,\dots,A_k; B_1,\dots,B_{n-k-1}\}$ $\left(1\le k\le\left[\frac{n-1}{2}\right]\right)$ such that $a(A_i)=A_0$ and $a(B_j)=B_0$.

Then, we obtain the following graphs:

$$
\begin{aligned}
G(A_i,A_j)&=G(B_i,B_j)=\emptyset;\\
G(A_i,B_j)&=\{\text{edge}\quad\overline{A_0B_0}\}.
\end{aligned}
$$

In particular, P_σ consists of the top dimensional faces

$$S_{ij}:=<A_0,\dots,\hat{A}_i,\dots,A_k;B_0,\dots,\hat{B}_j,\dots,B_{n-k-1}>$$

$(i=0,\dots,k;\ j=0,\dots,n-k-1)$ only, i.e. P_σ is the polyhedron that is dual to $\Delta^k\times\Delta^{n-k-1}$. □

4.8 We are looking for cones σ that are able to meet the condition of Theorem 2.5. By the previous lemma, only the investigation of the polyhedra $P_\sigma=(\Delta^k\times\Delta^{n-k-1})^\vee$ $(k=0,\dots,\left[\frac{n-1}{2}\right])$ is left. (The case $n=3$ (and $m=1$) is integrated, since only triangles and quadrangles are candidates for squeezing.)

Cones over $(\Delta^k\times\Delta^{n-k-1})^\vee$ have the following form:

$$
\begin{aligned}
\sigma&=<a^0,\dots,a^k;b^0,\dots,b^{n-k-1}>;\\
(\nu,\mu)&:=<a^0,\dots,\hat{a}^\nu,\dots,a^k;b^0,\dots,\hat{b}^\mu,\dots,b^{n-k-1}>
\end{aligned}
$$

are the top dimensional faces of σ $(\nu=0,\dots,k;\ \mu=0,\dots,n-k-1)$. These faces are illustrated as vertices of a lattice:

Then, $\varrho((\nu,\mu);(\tilde{\nu},\tilde{\mu}))=1$ means that the points (ν,μ) and $(\tilde{\nu},\tilde{\mu})$ sit in a common line (either row or column).

Lemma 4.8 *Let σ meet the condition of Theorem 2.5. If r denotes the cardinality of $R=\{(r_0^1)^\perp,\dots,(r_0^m)^\perp;(r_1^1)^\perp,\dots,(r_1^m)^\perp\}$ (cf. 2.5), then the following inequality is true:*

$$(n+m+1)r \geq (k+1)(n-k)m + \max\left\{r, \frac{r^2}{n+1}\right\} + \max\left\{r, \frac{r^2}{n-k}\right\}.$$

(If R does not contain top dimensional faces only, then correct the definition of r as follows: replace all faces of dimension less or equal than $n-2$ by one of the enclosed $(n-1)$-dimensional ones.)

Proof. We may assume that R consists of top dimensional faces only. (Otherwise, the changes suggested above have to be done, but the conditions (i) and (ii) of 2.5 still keep true.)

Now, we look at the "face lattice" drawn above and count the number of R-faces sitting in each row and column, respectively:

$$\begin{aligned} s_\nu &:= \#(R\cap \nu^{th}\text{ line}) && (\nu=0,\dots,k);\\ t_\mu &:= \#(R\cap \mu^{th}\text{ column}) && (\mu=0,\dots,n-k-1). \end{aligned}$$

These two numbers admit the following properties:

1) $\sum\limits_{\nu=0}^{k} s_\nu = r;\quad \sum\limits_{\mu=0}^{n-k-1} t_\mu = r.$
2) If $(\nu,\mu)\notin R$, then $s_\nu+t_\mu\geq m$ (by (ii) of 2.5).
3) $\sum\limits_{(\nu,\mu)\in R} s_\nu = \sum\limits_{\nu=0}^{k}(\#\{(\nu,.)/(\nu,.)\in R\}\cdot s_\nu) = \sum\limits_{\nu=0}^{k} s_\nu^2;\quad \sum\limits_{(\nu,\mu)\in R} t_\mu = \sum\limits_{\mu=0}^{n-k-1} t_\mu^2.$

Hence, we obtain

$$\begin{aligned}(n+1)r &= (n-k)\sum_{\nu=0}^{k} s_\nu \;+\; (k+1)\sum_{\mu=0}^{n-k-1} t_\mu = \sum_{(\nu,\mu)}(s_\nu+t_\mu) = \\ &= \sum_{(\nu,\mu)\notin R}(s_\nu+t_\mu) \;+\; \sum_{(\nu,\mu)\in R}(s_\nu+t_\mu)\\ &\geq [(k+1)(n-k)-r]m + \sum_{\nu=0}^{k} s_\nu^2 + \sum_{\mu=0}^{n-k-1} t_\mu^2.\end{aligned}$$

Finally, $\sum\limits_\nu s_\nu^2$ (and $\sum\limits_\mu t_\mu^2$) can be estimated in two different ways:

a) $s_\nu^2\geq s_\nu$ provides $\sum\limits_\nu s_\nu^2 \geq \sum\limits_\nu s_\nu = r$, and
b) $\sqrt{\frac{1}{k+1}\sum\limits_\nu s_\nu^2} \geq \frac{1}{k+1}\sum\limits_\nu s_\nu$ implies $\sum\limits_\nu s_\nu^2 \geq \frac{r^2}{k+1}$. □

Remarks. 1) The inequality of the lemma could be improved by finding a better estimation for $\left(\sum_\nu s_\nu^2 + \sum_\mu t_\mu^2\right)$.
2) If (n,k,m,r) satisfies not only the inequality but also the sharp version $(n+m+1)r = (k+1)(n-k)m + \frac{r^2}{k+1} + \frac{r^2}{n-k}$, then the values of $s_0,\dots,s_k$ and $t_0,\dots,t_{n-k-1}$ have to be equal, respectively. In particular, r has to be divisible by $k+1$ and $n-k$.
3) In the case $m = n-2$ we are able to use (ii)' of 4.5 instead of (ii) of 2.5. In particular, m can be replaced by $m+1$, and we obtain the following inequality:

$$2nr \ge (k+1)(n-k)(n-1) + \max\left\{r, \frac{r^2}{k+1}\right\} + \max\left\{r, \frac{r^2}{n-k}\right\}.$$

Corollary. *Only few possiblities for σ remain:*
a) $k \le 1$, *or*
b) $k = 2$; $n = 5,6,7,8$, *or*
c) $k = 3$; $n = 7$.

Proof. Denote $q := (k+1)(n-k)$. Then, $k \ge 2$ implies $q > r$, and we obtain the following inequalities:

$$\begin{aligned}
(n+1)r &\ge (q-r)m + \frac{r^2}{k+1} + \frac{r^2}{n-k} \ge (q-r)\frac{r}{2} + \frac{r^2}{k+1} + \frac{r^2}{n-k},\\
2q(n+1)r &\ge q(q-r)r + 2((n-k)+(k+1))r^2,\\
2q(n+1) &\ge q(q-r) + 2(n+1)r,\\
2(n+1)(q-r) &\ge q(q-r), \quad \text{and this means}\\
2(n+1) &\ge q.
\end{aligned}$$

Therefore, the inequalities $q \ge 3(n-2)$ (for $k \ge 2$) and $q \ge 4(n-3)$ (for $k \ge 3$) provide the restrictions $n \le 8$ and $n \le 7$, respectively. $\square$

4.9 The investigation of the remaining cases has to be more careful now. We still assume that R consists of top dimensional faces only and recall the essential part of Theorem 2.5 (applied to the special case $P_\sigma = (\Delta^k \times \Delta^{n-k-1})^\vee$):
1) R consists of m pairs of faces of σ (but these pairs need not to be disjoint ones).
2) Given a good face $\tau < \sigma$ (i.e. $\tau \notin R$) it is possible to introduce an order of both, the set of pairs in R on the one hand, and the elements of each pair on the other hand: R consists of $(r_0^1, r_1^1), \dots, (r_0^m, r_1^m)$. These orders (that depend on the choice of τ) admit the following properties:
   a) The faces $r_0^1, \dots, r_0^m$ correspond to m distinct fat points sitting in the same line (row or column) as τ (cf. the figure at the beginning of 4.8).
   b) For $i = 1, \dots, m$, the line that contains r_0^i but not τ does not pass through any of the fat points $r_1^i, r_1^{i+1}, \dots, r_1^m$.

Case $k \geq 2$: 13 quadruples (n,k,m,r) (such that $k \leq \left[\frac{n-1}{2}\right]$, $1 \leq m \leq n-2$ and $m+1 \leq r \leq 2m$) are left by the inequality of the previous lemma ((2) and (3) of the added remark included):

$$\begin{array}{ccc}
\begin{array}{l}(5,2,1,2);\quad (5,2,2,3);\\ (5,2,2,4);\quad (5,2,3,6)\end{array} & \begin{array}{ccc}\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet\end{array} & (P_\sigma = (\Delta^2 \times \Delta^2)^\vee)\\[2ex]
\begin{array}{l}(6,2,1,2);\quad (6,2,2,4);\\ (6,2,3,6)\end{array} & \begin{array}{cccc}\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet\end{array} & (P_\sigma = (\Delta^2 \times \Delta^3)^\vee)\\[2ex]
\begin{array}{l}(7,2,2,4);\quad (7,2,3,6);\\ (7,2,4,8)\end{array} & \begin{array}{ccccc}\bullet&\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet&\bullet\end{array} & (P_\sigma = (\Delta^2 \times \Delta^4)^\vee)\\[2ex]
(8,2,3,6) & \begin{array}{cccccc}\bullet&\bullet&\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet\end{array} & (P_\sigma = (\Delta^2 \times \Delta^5)^\vee)\\[2ex]
(7,3,1,2);\quad (7,3,2,4) & \begin{array}{cccc}\bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet\\ \bullet&\bullet&\bullet&\bullet\end{array} & (P_\sigma = (\Delta^3 \times \Delta^3)^\vee).
\end{array}$$

After checking the above conditions (1), (2a) and (2b) in each of these 13 cases, the only quadruple that survives is $(5,2,2,3)$. Hence, we are able to continue our positive list started in 4.6:

Case $k=1$: $P_\sigma = (\Delta^1 \times \Delta^{n-2})^\vee$ admits the following lattice of top dimensional faces:

$$\underbrace{\begin{array}{cccccc}\bullet&\bullet&\bullet&\dots&\bullet\\ \bullet&\bullet&\bullet&\dots&\bullet\end{array}}_{n-1\ columns}$$

Applying the conditions (1), (2a) and (2b) to this special situation, we obtain:

- There exist good faces in each of the two lines.
- Each line contains exactly one element of each R-pair. We number these pairs and denote their elements by r^i and s^i (sitting in the upper or lower line, respectively).
- $r = 2m$. (If this would not be the case, we would be able to assume $r^1 = r^2$. Then, the condition (2a) would imply that there had to be an arrangement of faces of the form $\left(\begin{smallmatrix} x & r_0^1=r_0^2 & x \\ r_1^1 & ? & r_1^2 \end{smallmatrix}\right)$ ("x" denotes a good face), but this contradicts (2b).)
- If we would draw edges between both elements of each R-pair on the one hand and between any two elements of R that are contained in the same column on the other hand, then no loops are allowed to arise. That means, constellations like the following two are forbidden:

These conditions lead to the following class of squeezable cones:

4.10 We still regard the polyhedron $P_\sigma = (\Delta^k \times \Delta^{n-k-1})^\vee$. But now, we want to omit the restriction that R is allowed to contain top dimensional faces only.

If $K \in R$ is an $(n-2)$-dimensional face of a squeezable cone σ (and we also will assume that it is the only member of R being not of dimension $n-1$), then, in 4.8 and 4.9 K has been replaced by one of the two top dimensional faces containing this edge: R changes to R', and $(\sigma; R')$ remains squeezable (at least in the sense of 2.5).

Let $r^i \in R'$ be the face that arises from $K \in R$. Then, (σ, R') must belong to one of the two classes (III) or $(\mathrm{IV})_{p_1,\dots,p_\ell}$, and the following additional properties have to be satisfied:

- Replacing r^i by the other top dimensional face that contains K, the result still has to belong to the classes (III) or (IV), respectively.
- There exists almost one good face τ of $(\sigma; R')$ that requires r^i to be its R-neighbour of the i^{th} pair. Then, K has to be the edge connecting both faces, τ and r^i.
- Let α be a good face such that K has to play the role of r_1^i in $(\sigma; R)$. Then, in $(\sigma; R')$ not only r^i but also τ satisfies the condition for r_1^i described in (2b) of 4.9.

If R contains more than one $(n-2)$-dimensional face we can proceed inductively: Step by step (in an arbitrary order) these faces can be changed into top dimensional ones - the whole system always keeps squeezable.

Having all this in mind, the following two additional classes of squeezable cones will be obtained:

$(IV)^1_{p,q}$ *The double simplex (with a single edge)*

$P^\vee_\sigma = \left({x \atop r^1_1} \,|\, B_p \,|\, B_q \right)$, and $r^1_0 \in R$ equals the $(n-2)$-dimensional edge between the two upper good faces of B_p and B_q, respectively. ($p, q \geq 0$; $p + q = m - 1 = n - 4$. We also assume $p \leq q$.)

$(IV)^2_{p,q}$ *The double simplex (with two edges)*

$P^\vee_\sigma = \left({x \atop r^1_1} \,|\, B_p \,|\, B_q \,|\, {r^m_0 \atop x} \right)$, and $r^1_0 \in R$ ($r^m_1 \in R$) equals the $(n-2)$-dimensional edge between the two upper (lower) good faces of B_p and B_q, respectively. ($p, q \geq 0$; $p + q = m - 2 = n - 5$. We also assume $p \leq q$.)

Remark. If R contains an edge ℓ of dimension less than $n-2$, then the condition of Theorem 2.5 cannot be satisfied: Otherwise we could replace ℓ by an enclosed edge of dimension $n-2$, and the whole set up would "factorize" through $(\mathrm{IV})^1_{p,q}$. But, in this class, the edge r^1 is needed to be the R-neighbour of two good faces, i.e. it can not be replaced by a smaller one.

4.11 In the last two sections we have regarded the polyhedra $P_\sigma = (\Delta^k \times \Delta^{n-k-1})^\vee$ with $k \geq 1$. Therefore, only one class of cones remains to be mentioned:

(V) *The simplex*

$$\sigma < a^1, \dots, a^n >; \quad 1 \leq m \leq n-2.$$

Remark. As in the previous cases it remains to classify the possible constellations of R-elements in this class also. But, because of the absence of sufficient good faces, to carry out this task seems to be very stupid and will be omitted here.

4.12 In this chapter we have proved the following statement: let $i : Y_{\bar{\sigma}} \hookrightarrow X_\sigma$ be a closed embedding of two affine torus varieties that is a relative complete intersection (cf. chapter 2). If, moreover, $Y_{\bar{\sigma}} \setminus \{0\}$ is smooth and $X_\sigma \setminus \{0\}$ admits at most cyclic quotient singularities (i.e. the cone σ consists of simplicial faces only), then for σ together with the equations $x^{r^i_0} - x^{r^i_1}$ ($i = 1, \dots, m$; $r^i_\nu \in \check{\sigma} \cap M$) there are only the possibilities (I) - (V) described above.

In the next chapter we will complete our work for the classes (I) - (III): all squeezable cones will be classified not only as abstract cones, but up to automorphisms of the lattice N. Moreover, we will characterize those (3-dimensional) isolated toric singularities that can appear in this way. The classes (IV) and (V) consist of the simplest cones one can think about - but there are many different cases for the arrangement of the R-neighbours belonging to each good face. We omit further investigations here, but it seems to be quite interesting to keep these cases in mind: for instance, the whole series of cones over scrolls — given by the determinantic equations

$$\det \begin{pmatrix} x_{10} & x_{11} & \dots & x_{1\,k_1-1} & x_{20} & x_{21} & \dots & x_{2\,k_2-1} & \dots & x_{\ell\,k_\ell-1} \\ x_{11} & x_{12} & \dots & x_{1\,k_1} & x_{21} & x_{22} & \dots & x_{2\,k_2} & \dots & x_{\ell\,k_\ell} \end{pmatrix} = 0$$

— is hidden in class (IV).

5. The hexagon singularities

The classes (I) - (III) of squeezable cones (cf. 4.6 and 4.9) provide 3-dimensional toric singularities that are given by cones over a hexagon. Now, by using Theorem 2.4 we will describe these cases in detail:

5.1 Class (I): The octahedron (see Figure 5.1)

Figure 5.1

$< A, B, X >$ is a good face. In particular, it is smooth, and we can assume

$$A = (1,0,0,0); \quad B = (1,1,0,0); \quad X = (1,0,1,0).$$

Hence, $< A, r_0 > = < X, r_0 > = 0$ and $< B, r_0 > = 1$ imply $r_0 = [0,1,0,*]$, and we can choose our coordinates of N in such a way that $r_0 = [0,1,0,0]$.

Now, by the properties of good faces or simply by incidence relations only, we obtain the following equations:

$$< a, r_0 > = < B, r_1 > = < b, r_1 > = < Y, r_1 > = 0;$$
$$< b, r_0 > = < Y, r_0 > = < A, r_1 > = < a, r_1 > = < X, r_1 > = 1.$$

In particular, that means $r_1 = [1,-1,0,*]$. We can use the remaining feasibilities of choosing the coordinates well to kill the last component of r_1; then, we obtain the result

$$\begin{array}{c} A = (1,0,0,0) \\ a = (1,0,a_3,a_4) \end{array} \quad X = (1,0,1,0); \quad r_0 = [0,1,0,0]$$
$$\begin{array}{c} B = (1,1,0,0) \\ b = (1,1,b_3,b_4) \end{array} \quad Y = (1,1,Y_3,Y_4); \quad r_1 = [1,-1,0.0].$$

On the other hand, for P_σ to be a convex body of the form drawn above, some 4×4 determinants have to be positive. This provides conditions for our parameters $a_3, a_4, b_3, b_4, Y_3, Y_4 \in \mathbb{Z}$:

$$(5.1.1)\qquad \begin{aligned} & a_4,\ b_4,\ Y_4\ >\ 0, \\ & \begin{vmatrix} a_3 & a_4 \\ Y_3 & Y_4 \end{vmatrix},\ \begin{vmatrix} b_3 & b_4 \\ Y_3 & Y_4 \end{vmatrix} > 0, \\ & \begin{vmatrix} b_3 - Y_3 & b_4 - Y_4 \\ a_3 & a_4 \end{vmatrix},\ \begin{vmatrix} b_3 - Y_3 & b_4 - Y_4 \\ a_3 - 1 & a_4 \end{vmatrix},\ \begin{vmatrix} b_3 & b_4 \\ a_3 - 1 & a_4 \end{vmatrix} > 0. \end{aligned}$$

The smoothness of the good faces (only $< A, B, X >$ was already used) yields

$$(5.1.2)\qquad \begin{aligned} & \gcd(a_3, a_4) = \gcd(a_3 - 1, a_4) = \gcd(b_3, b_4) = 1; \\ & \gcd(Y_3, Y_4) = \gcd(b_3 - Y_3, b_4 - Y_4) = 1. \end{aligned}$$

Remark. 1) The vertices of σ together sit inside the affine hyperplane $[a_1 = 1]$ of $N \otimes \mathbb{R}$. That means that the corresponding toric variety X_σ is Gorenstein.
2) X_σ has an isolated singularity in 0, iff $a_4 = \begin{vmatrix} b_3 & b_4 \\ Y_3 & Y_4 \end{vmatrix} = 1$. (Then, this even implies all gcd-conditions.)

Now, we are ready to determine the special fibre $Y_{\bar\sigma}$ of the map $(x^{r_0} - x^{r_1}) : X_\sigma \to \mathbb{C}^1$. We compute the "squeezed" cone $\bar\sigma = \sigma \cap [r_0 - r_1]^\perp :\ r_0 - r_1 = [-1, 2, 0, 0]$, i.e. the linear subspace $[r_0 - r_1]^\perp \subseteq N$ is given by the equation $a_1 = 2a_2$. The points of intersection of this subspace with the six edges $\overline{AB}$, $\overline{BX}$, $\overline{Xb}$, $\overline{ba}$, $\overline{aY}$ and $\overline{YA}$, respectively, are

$$\begin{aligned} & (2,1,0,0);\ (2,1,1,0);\ (2,1;b_3+1,b_4); \\ & (2,1,a_3+b_3,a_4+b_4);\ (2,1,a_3+Y_3,a_4+Y_4);\ (2,1,Y_3,Y_4). \end{aligned}$$

Taking $(2,1,0,0)$; $(0,0,1,0)$; $(0,0,0,1)$ as a $\mathbb{Z}$-basis of $[r_0 - r_1]^\perp$, we can omit the first component "2" in each point. Then, $\overline{\sigma}$ can be seen to be the cone over the hexagon

$$H_I := < (0,0); (1,0); (b_3+1, b_4); (a_3+b_3, a_4+b_4); (a_3+Y_3, a_4+Y_4); (Y_3, Y_4) >$$

that sits in the affine hyperplane $[\bar{a}_1 = 1]$ ($\bar{a}_1$, $\bar{a}_2$, $\bar{a}_3$ denote the coordinates of $[r_0 - r_1]^\perp$).

Remark. 3) $Y_{\bar{\sigma}}$ is Gorenstein, too.
The conditions of (5.1.1) express exactly the fact that H_I is a convex, non-degenerate hexagon of the form drawn above.
5) The conditions of (5.1.2) express that the (oriented) edges of H_I consist of primitive vectors only. But this fact is equivalent to $(Y_{\bar{\sigma}}, 0)$ having at most an isolated singularity.
6) $P_0 + P_2 + P_4 = P_1 + P_3 + P_5$, i.e. the centres of both triangles, $\Delta P_0 P_2 P_4$ and $\Delta P_1 P_3 P_5$, coincide. This condition is not only necessary, but also sufficient (up to $\mathbb{Z}$-isomorphisms) for H_I to have those special vertices as mentioned above.

Therefore, it is exactly the isolated Gorenstein hexagon singularities Y with $P_0+P_2+P_4 = P_1+P_3+P_5$ that admit toric deformations of type (I). The corresponding total space X_σ is uniquely determined by Y.

The view of the dual situation illustrates the notion of squeezing:
The hexagon results by pressing the cube into dimension 2.

5.2 Class (II): The tent (see Figure 5.2)
We proceed as in the last section; for shortness we will only list the results in the order of appearance:

(i) The good face $< A, B, X >$ together with a good choice of coordinates yields $A = (1,0,0,0)$; $B = (1,1,0,0)$; $X = (s,0,1,0)$ and $r_0 = [0,1,0,0]$. ($s \in \mathbb{Z}$ is a parameter which keeps for us the possibility to change the first component of X if it is necessary.)

(ii) The equations

$$< a, r_0 > = < B, r_1 > = < b, r_1 > = < y, r_1 > = 0;$$
$$< b, r_0 > = < A, r_1 > = < a, r_1 > = 1$$

Figure 5.2

provide $r_1 = [1, -1, 0, 0]$ (the last two components vanish after improving the coordinates again - the parameter s of (i) is fixed now). We obtain the following result:

$$\begin{array}{c} A = (1,0,0,0) \\ a = (1,0,a_2,a_4) \end{array} \quad X = (s,0,1,0); \quad r_0 = [0,1,0,0]$$

$$\begin{array}{c} B = (1,1,0,0) \\ b = (1,1,b_3,b_4) \end{array} \quad Y = (t,t,Y_3,Y_4); \quad r_1 = [1,-1,0,0].$$

(iii) Conditions for the parameters $s, t, a_3, a_4, b_3, b_4, Y_3, Y_4 \in \mathbb{Z}$:

(5.2.1)
$$\begin{array}{l} s,\ t,\ a_4,\ b_4,\ Y_4\ >\ 0; \\ \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix}, \begin{vmatrix} b_3 & b_4 \\ Y_3 & Y_4 \end{vmatrix}, \begin{vmatrix} a_3 & a_4 \\ Y_3 & Y_4 \end{vmatrix} > 0\,; \\ s\begin{vmatrix} a_3 & a_4 \\ Y_3 & Y_4 \end{vmatrix} > Y_4;\ t\begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} > \begin{vmatrix} a_3 & a_4 \\ Y_3 & Y_4 \end{vmatrix};\ tb_4 > Y_4. \end{array}$$

(5.2.2)
$$\begin{array}{l} \gcd(a_3, a_4) = \gcd(b_3, b_4) = \gcd(Y_3, Y_4) = 1; \\ \gcd(sa_3 - 1, a_4) = \gcd(Y_3 - tb_3, Y_4 - tb_4) = 1. \end{array}$$

Remark. 1) X_σ is Gorenstein, iff X_σ is $\mathbb{Q}$-Gorenstein, iff $s = t = 1$.
2) X_σ has an isolated singularity in 0, iff $a_4 = \begin{vmatrix} b_3 & b_4 \\ Y_3 & Y_4 \end{vmatrix} = 1$. (Then, this even implies all gcd-conditions.)

(iv) The 3-dimensional cone $\bar{\sigma} = \sigma \cap [r_0 - r_1]^\perp$ is spanned by the vertices

$$(1,0,0);\ (s,1,0);\ (1,a_3,a_4);\ (1,a_3+b_3,a_4+b_4);\ (1,b_3+b_4);\ (t,Y_3,Y_4),$$

which arise as points of intersection of $[r_0 - r_1]^\perp$ with the edges $\overline{AB}$, $\overline{BX}$, $\overline{Ba}$, $\overline{ab}$, $\overline{bA}$ and $\overline{YA}$, respectively. That means, $\bar{\sigma}$ is the cone over the hexagon

$$H_{II} :=< (0,0); \frac{1}{s}(1,0); (a_3, a_4); (a_3 + b_3, a_4 + b_4); (b_3, b_4); \frac{1}{t}(Y_3, Y_4) >$$

that sits in the affine hyperplane $[\bar{a}_1 = 1]$.

Attention. The points $\frac{1}{s}(1,0)$ and $\frac{1}{t}(Y_3, Y_4)$ belong to the lattice of $[r_0 - r_1]^\perp$, iff $s = t = 1$!

Remark. 3) $Y_{\bar{\sigma}}$ is Gorenstein, iff it is $\mathbb{Q}$-Gorenstein, iff $s = t = 1$.
4) The conditions of (5.2.1) express exactly the fact that H_{II} is a convex, non-degenerate hexagon of the form drawn above.
5) The conditions of (5.2.2) express that $(Y_{\bar{\sigma}}, 0)$ has, at most, an isolated singularity.
6) $P_0 + P_3 = P_2 + P_4$, i.e. the vertices P_0, P_2, P_3, P_4 form a parallelogram inside H_{II}. This condition is not only necessary, but also sufficient (up to $\mathbb{Z}$-isomorpisms) for H_{II} to have those special vertices as mentioned above.

Therefore, the isolated singularities Y that admit toric deformations of type (II) are exactly those hexagon singularities such that P_0, P_2, P_3, P_4 form an (isolated) Gorenstein parallelogram singularity. The corresponding total space X_σ is uniquely determined by Y.

5.3 Class (III): The super triangle (see Figure 5.3)
The cone $\sigma =< a^0, a^1, a^2; b^0, b^1, b^2 >$ admits 9 top dimensional faces

$$(\nu, \mu) :=< a^0, \dots, \widehat{a^\nu}, \dots a^2; b^0, \dots, \widehat{b^\mu}, \dots, b^2 >;$$

each face is represented by a fat point in the (3×3)-lattice drawn in Figure 5.3.

In particular,

$$\begin{aligned} r_0^1 = (0,0) &=< a^1, a^2, b^1, b^2 >, \\ r := r_1^1 = r_0^2 = (1,1) &=< a^0, a^2, b^0, b^2 >, \quad \text{and} \\ r_1^2 = (2,2) &=< a^0, a^1, b^0, b^1 > . \end{aligned}$$

Figure 5.3

Now, we can proceed as usual (kill the last components of r_0^1, r and r_1^2, respectively, by choosing the coordinates of N well):

(i) The property for $(0,2) =< a^1, a^2, b^0, b^1 >$ to be a good face provides

$$a^1 = (1,1,0,0,0);\ \ a^2 = (1,0,1,0,0);\ \ b^0 = (1,0,0,0,0);\ \ b^1 = (1,1,0,1,0)$$

and

$$r_0^1 = [1,-1,-1,0,0]; \quad r_1^2 = [0,0,1,0,0].$$

(ii) Again, there are many equations:

$$< b^2, r_0^1 > =< a^0, r >=< a^2, r >=< b^0, r >=< b^2, r >=< a^0, r_1^2 >= 0;$$
$$< a^0, r_0^1 > =< a^1, r >=< b^1, r >=< b^2, r_1^2 >= 1.$$

Hence, we obtain the following result:

$$a^0 = (1,0,0;a_4,a_5),\ \ a^1 = (1,1,0;0,0),\ \ a^2 = (1,0,1;0,0),$$
$$b^0 = (1,0,0;0,0),\ \ b^1 = (1,1,0;1,0),\ \ b^2 = (1,0,1;b_4,b_5),$$
$$r_0^1 = [1,-1,-1,0,0],\ \ r = [0,1,0,0,0],\ \ r_1^2 = [0,0,1,0,0].$$

(iii) Conditions for the parameters a_4, a_5, b_4, b_5:

(5.3.1) $$a_5,\ b_5,\ \begin{vmatrix} a_4 & a_5 \\ b_4 & b_5 \end{vmatrix} > 0.$$

(5.3.2) $$\gcd(a_4, a_5) = \gcd(b_4, b_5) = 1.$$

Remark. 1) X_σ is Gorenstein.
2) X_σ has an isolated singularity in 0, iff $a_4 = b_4 + 1$ and $a_5 = b_5 = 1$.

(iv) The linear subspace $L^\perp = [r_0^1 - r]^\perp \cap [r - r_1^2]^\perp \subseteq N$ is given by the equations $a_1 = 3a_3$; $a_2 = a_3$. To determine $\bar{\sigma} = \sigma \cap L^\perp$ it is sufficient to compute the intersection with each of the 3-dimensional faces of σ only. Then, exactly the six faces of the form $< .^0, .^1, .^2 >$ give a contribution, and we obtain $\bar{\sigma}$ to be the cone over

$$H_{III} :=< (0,0); (1,0); (a_4+1, a_5); (a_4+b_4+1, a_5, b_5); (a_4+b_4, a_5+b_5); (b_4, b_5) >$$

that sits in the affine hyperplane $[\bar{a}_1 = 1] \subseteq L^\perp$.

Remark. 3) $Y_{\bar{\sigma}}$ is Gorenstein.
4) The conditions of (5.3.1) mean that H_{III} is a convex, non-degenerate hexagon of the form drawn above.
5) The condition of (5.3.2) expresses that $(Y_{\bar{\sigma}}, 0)$ has at most an isolated singularity.
6) $P_0 + P_3 = P_1 + P_4 = P_2 + P_5$, i.e. opposite edges are parallel ones which have the same length too. This condition is not only necessary, but also sufficient (up to $\mathbb{Z}$-isomorphism) for H_{III} to have those special vertices as mentioned above.

Therefore, the isolated singularities Y that admit toric deformations of type (III) are exactly those Gorenstein hexagon singularities such that the hexagon consists of three parallelograms as shown in the picture above. The corresponding total space X_σ is uniquely determined by Y.

5.4 Which isolated hexagon singularities admit toric deformations of more than one type?

Obviously, the type (II) is not compatible with any of the two remaining types. On the other hand, the hexagons H belonging to both classes, (I) and (III) are easy to describe: they consist of six copies of a fixed triangle, and these copies are arranged in such a way that any two adjacent ones form a parallelogram (see the figure above).

Remark. The total spaces of both toric deformations have isolated singularities, iff $a_5 = 1$. If this is the case, then the coordinates can be improved once more to obtain $a_4 = 1$ also. In particular, the corresponding toric variety $Y_{\bar{\sigma}}$ is uniquely determined; the two total spaces are isomorphic to the cones over $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2 \times \mathbb{P}^2$, respectively.

$Y_{\bar{\sigma}}$ has embedding dimension 7 and can be given by 9 equations:

$$x_1 y_1 = x_2 y_2 = x_3 y_3 = z^2$$

$$x_1 z = x_2 x_3;\ x_2 z = x_1 y_3;\ x_3 z = x_1 y_2;$$

$$y_1 z = y_2 y_3;\ y_2 z = x_3 y_1;\ y_3 z = x_2 y_1.$$

In fact, as J. Stevens pointed out to me, this singularity is the cone over the Del Pezzo surface. Moreover, by explicit computations with the computer programme "Macaulay", Duco van Straten has shown that the above two total spaces represent, indeed, exactly the two components of the reduced versal base space of $Y_{\bar{\sigma}}$.

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