The chain property for the associated primes of $\mathcal{A}$-graded ideals

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Abstract

We investigate how the chain property for the associated primes of monomial degenerations of toric (or lattice) ideals can be generalized to arbitrary $\mathcal{A}$-graded monomial ideals. The generalization works in dimension $d = 2$, but it fails for $d \geq 3$.
Moreover, for a certain class of binomial ideals (including the $\mathcal{A}$-graded ones) we present an explicit cellular primary decomposition.

1 Introduction

(1.1) Challenged by the question of Arnold for the ideals with the easiest Hilbert function, Sturmfels has invented in [St1] and §10 of [St2] the notion of $\mathcal{A}$-graded ideals. For a given linear map $\mathcal{A} : \mathbb{Z}^n \to \mathbb{Z}^d$ with $(\ker \mathcal{A}) \cap \mathbb{Z}^n_{\geq 0} = 0$ an ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$ is called $\mathcal{A}$-graded if it is $\mathbb{Z}^d$-homogeneous via $\mathcal{A}$ and, moreover, if it has the Hilbert function

$$\dim_{\mathbb{C}} \left(\mathbb{C}[\underline{x}]/I\right)_q = \begin{cases} 1 & \text{if } q \in \mathcal{A}(\mathbb{Z}^n_{\geq 0}) \\ 0 & \text{otherwise.} \end{cases}$$

Examples are the so-called toric ideal $J_{\mathcal{A}} := \left(x^a - x^b \,\middle|\, a, b \in \mathbb{Z}^n_{\geq 0},\, a - b \in \ker \mathcal{A}\right)$ and all its Gröbner degenerations. Indeed, these ideals form an irreducible, the "coherent" component in the parameter space of all $\mathcal{A}$-graded ideals. See [PS2], [PS1], and [St1] for different treatments of this interesting subject.
The importance of $\mathcal{A}$-graded ideals seems to be two-fold. First, they give insight into a small layer of the deformation space of monomial ideals. Second, via taking radicals and using the Stanley-Reisner construction, the monomial $\mathcal{A}$-graded ideals provide triangulations of the convex cone $\mathcal{A}(\mathbb{R}^n_{\geq 0})$. Hence, the set of those triangulations may be studied by algebraic tools, cf. [MT].

(1.2) Sturmfels has investigated intrinsic properties of monomial ideals that are satisfied for coherent ideals, but fail for general $\mathcal{A}$-graded ones. First, there is a combinatorial property obtained by observing the vertices of the fibers of $\mathcal{A}$ restricted to $\mathbb{Z}^n_{\geq 0}$. The second property involves more algebraic concepts such as the degree of the generators of the ideal, cf. [St1], [St2].
In this context, Hoşten and Thomas have addressed another point. In [HT] they observe that monomial degenerations of toric ideals admit a very special primary decomposition; the associated prime ideals occur in chains:

Definition: The ideal I fulfils the *chain property* for its associated primes if, for any associated, non-minimal prime P of I, there is another one $P' \subseteq P$ with $\mathrm{ht}(P') = \mathrm{ht}(P) - 1$.

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The subject of the present paper is to show that this property is of the same type as the first two mentioned above, i.e., not true for non-coherent $\mathcal{A}$-graded monomial ideals, in general. In detail, we will prove the following

Theorem (3.4)/(4.1): *The chain property holds for $\mathcal{A}$-graded monomial ideals of dimension $d \leq 2$. However, there are counter examples for $d = 3$.*

(1.3) The main tool for proving the previous theorem is the explicit knowledge of the primary decomposition of monomial ideals from [STV]. However, instead of quoting the result, we start our paper in §2 with the presentation of a generalization to a certain class of binomial ideals, cf. Theorem (2.7). This result implies the

Theorem: *$\mathcal{A}$-graded ideals admit a cellular primary decomposition.*

The fact that primary decomposition does not leave the category of binomial ideals at all follows already from [ESt].

2 Primary decomposition of saturated binomial ideals

(2.1) Let $I \subseteq \mathbb{C}[\underline{x}] := \mathbb{C}[x_1, \ldots, x_n]$ be a binomial ideal. By $T(I) \subseteq \mathbb{Z}^n_{\geq 0}$ we denote the set

$$T(I) := \{a \in \mathbb{Z}^n_{\geq 0} \,|\, x^a \notin I\}$$

of the non-monomials in I. The set $T := T(I)$ has the property

(i) If $a, b \in \mathbb{Z}^n_{\geq 0}$ with $a \geq b$ (i.e., $a - b \in \mathbb{Z}^n_{\geq 0}$) and $a \in T$, then $b \in T$.

Every T fulfilling this property occurs as $T(I)$ for some binomial (even monomial) ideal. For the upcoming definition, we also need the following notation. If $\ell \subseteq [n] := \{1, \ldots, n\}$ is an arbitrary subset, then we write

$$\mathbb{Z}^\ell := \{a \in \mathbb{Z}^n \,|\, \operatorname{supp} a \subseteq \ell\} \qquad \text{and} \qquad \mathbb{Z}^\ell_{\geq 0} := \mathbb{Z}^n_{\geq 0} \cap \mathbb{Z}^\ell.$$

(2.2) **Definition:** Let $T \subseteq \mathbb{Z}^n_{\geq 0}$ with property (i). A set $(r, \ell) := r + \mathbb{Z}^\ell_{\geq 0}$ with $r \in \mathbb{Z}^n_{\geq 0}$ and $\ell \subseteq [n]$ disjoint to $\operatorname{supp} r$ is called a *standard pair* if it is maximal (with respect to inclusion) for the property $r + \mathbb{Z}^\ell_{\geq 0} \subseteq T$.

Denoting by $B := B(T)$ the set of standard pairs, we obviously have $T = \cup_{(r,\ell)\in B}(r + \mathbb{Z}^\ell_{\geq 0})$. The following result is well known, however, we include a short proof here for the reader's convenience.

Proposition: *The set $B(T)$ is finite.*

Proof: Otherwise, let $\ell \subseteq [n]$ be a subset such that there are infinitely many r^i with $(r^i, \ell) \in B$. Then, we may choose a subsequence of (r^i) that is increasing via the partial order provided by $\mathbb{Z}^{[n]\setminus\ell}_{\geq 0} \subseteq \mathbb{Z}^n_{\geq 0}$: If there is one value occurring infinitely often as the entry of the r^i in one of the $[n] \setminus \ell$ coordinates, then this follows via induction by $\#([n] \setminus \ell)$. If not, then the claim is trivial, anyway.

Assuming that the r^i are increasing, then there is at least one coordinate (e.g. the first one) that becomes arbitrary large. Since $\big(r^1 + (r^i_1 - r^1_1)e^1\big) \leq r^i$, the property (i) implies $\big(r^1 + (r^i_1 - r^1_1)e^1\big) + \mathbb{Z}^\ell_{\geq 0} \subseteq T$; hence $r^1 + \mathbb{Z}^{\ell\cup\{1\}}_{\geq 0} \subseteq T$. In particular, the set $r^1 + \mathbb{Z}^\ell_{\geq 0}$ was not minimal, i.e., $(r^1, \ell) \notin B$. $\square$

(2.3) Let $T \subseteq \mathbb{Z}^n_{\geq 0}$ with property (i) of (2.1). Then we define for any subset $\ell \subseteq [n]$

$$T(\ell) := \bigcup_{(r^i,\ell)\in B} \big(r^i + \mathbb{Z}^\ell_{\geq 0}\big) \subseteq T$$

and its closure via the partial order "$\geq$"

$$\overline{T(\ell)} := \bigcup_{(r^i,\ell)\in B,\, r\leq r^i} \big(r + \mathbb{Z}^\ell_{\geq 0}\big) \subseteq T\,.$$

Remark: $T(\ell) = \{r \in \mathbb{Z}^n_{\geq 0} \,|\, r + \mathbb{Z}^\ell_{\geq 0} \subseteq T$ and ℓ is maximal with this property$\}$.

(2.4) **Definition:** Let $I \subseteq \mathbb{C}[\underline{x}]$ be a binomial ideal. Define $K(I) \subseteq \mathbb{Z}^n$ as the abelian subgroup generated as

$$K(I) := \big\langle a - b \,\big|\, a, b \in T(I) \text{ with } \lambda_a x^a - \lambda_b x^b \in I \text{ for some } \lambda_a, \lambda_b \neq 0\big\rangle.$$

The binomial ideal I is called *saturated* if for any $a, b \in T(I)$ with $a-b \in K(I)$ there are coefficients $\lambda_a, \lambda_b \neq 0$ such that $\lambda_a x^a - \lambda_b x^b \in I$.

Remark: The abelian subgroup $K(I) \subseteq \mathbb{Z}^n$ is minimal for I to be $\mathbb{Z}^n/_{K(I)}$-homogeneous. Moreover, using this language, the saturation property means that the associated Hilbert function of the graded ring $\mathbb{C}[\underline{x}]/_I$ always yields 0 or 1.

Examples:

1) Monomial ideals: Here is $K(I) = 0$.
2) $\mathcal{A}$-graded ideals with $\mathcal{A} : \mathbb{Z}^n \to \mathbb{Z}^d$: We have $K(I) = \big\langle (a-b) \in \ker \mathcal{A} \,|\, a, b \in T(I)\big\rangle \subseteq \ker \mathcal{A}$. In particular, the map $\mathbb{Z}^n/_{K(I)} \twoheadrightarrow \mathbb{Z}^n/_{\ker \mathcal{A}} \hookrightarrow \mathbb{Z}^d$ shows that the $\mathbb{Z}^d$-grading is in general weaker than the $\mathbb{Z}^n/_{K(I)}$-grading.
3) The ideal $I := (x^2 - xy)$ is *not* saturated: While $K(I) = (1, -1) \cdot \mathbb{Z} \subseteq \mathbb{Z}^2$, we have $x - y \notin I$.

As the ideals $(x^2 - xy)$ and $(x - y)$ admit the same invariants K and T, it follows that the saturation property does not depend on them alone. Nevertheless, the saturation of an ideal I implies a special feature of the sets $K(I)$ and $T(I)$:

Proposition: *Let $I \subseteq \mathbb{C}[\underline{x}]$ be a binomial ideal. If I is saturated, then $T = T(I)$ and the fibers $T_q := \{a \in T \,|\, a \mapsto q\}$ with $q \in \mathbb{Z}^n/_{K(I)}$ fulfil, in addition to (2.1)(i), the following property:*

(ii) *If $g \in \mathbb{Z}^n_{\geq 0}$ and $q \in \mathbb{Z}^n/_{K(I)}$, then $(g + T_q) \cap T = \emptyset$ or $g + T_q \subseteq T$.*

Proof: If $a, b \in T_q$, then there is an element $\lambda_a x^a - \lambda_b x^b \in I$, hence, $\lambda_a x^{a+g} - \lambda_b x^{b+g} \in I$. Since $\lambda_a, \lambda_b \neq 0$, it follows that the latter monomials are either both contained in I or both not. □

Open problem: Do there exist $\mathcal{A}$-graded ideals I containing at least one monomial, but still satisfying $K(I) = \ker \mathcal{A}$ or at least $K(I)_{\mathbb{Q}} = \ker \mathcal{A}_{\mathbb{Q}}$?

(2.5) **Lemma:** *Assume that $T \subseteq \mathbb{Z}^n_{\geq 0}$ satisfies, for some subgroup $K \subseteq \mathbb{Z}^n$, the properties (i) and (ii) of (2.1) and (2.4), respectively.*

(a) *If $a, b \in T_p$ and $g, h \in T_q$, then $a + g \in T$ iff $b + h \in T$.*

(b) *If* $T(\ell)_q \neq \emptyset$ *, then* $T(\ell)_q = T_q$.

(c) *If* $\overline{T(\ell)}_q \neq \emptyset$ *, then* $\overline{T(\ell)}_q = T_q$.

In particular, both sets $T(\ell) \subseteq T$ *and* $\overline{T(\ell)} \subseteq T$ *are unions of selected whole fibers* T_q.

Proof: Part (a) uses property (ii) twice to compare $(a+g)$, $(a+h)$, and $(b+h)$ successively.
For (b) let $a \in T(\ell)_q$, $b \in T_q$. We will use Remark (2.3) several times: First, it follows that $a + \mathbb{Z}^\ell_{\geq 0} \subseteq T$. Then, with g browsing through $\mathbb{Z}^\ell_{\geq 0}$, property (ii) implies that also $b + \mathbb{Z}^\ell_{\geq 0} \subseteq T$. Moreover, the same argument applied in the opposite direction shows that ℓ is maximal with this property, i.e., $b \in T(\ell)_q$.
Finally, let $a \in \overline{T(\ell)}_q$, $b \in T_q$. In particular, there is an element $g \in \mathbb{Z}^n_{\geq 0}$ such that $(a+g) \in T(\ell)_{q+p}$ with p being the image of g via $\mathbb{Z}^n \to \mathbb{Z}^n/K$. Applying the parts (a) and (b) successively, this means that $(b+g) \in T_{q+p} = T(\ell)_{q+p}$, hence $b \in \overline{T(\ell)}_q$. □

A consequence of the previous lemma is that, for saturated ideals I, the graded pieces I_q are generated, as vector spaces, by the monomials x^a with $a \in (\mathbb{Z}^n_{\geq 0})_q \setminus T_q$ and the binomials $\lambda_a x^a - \lambda_b x^b$ with $a, b \in T_q$ and $\lambda_\bullet$ as in (2.4).
Moreover, if abstract K, T are given as in the lemma, then we may use these generators with $\lambda_\bullet := 1$ to construct *some* saturated ideal I with $T(I) = T$ and $K(I) = \langle a - b \,|\, a, b \in T;\ a - b \in K \rangle \subseteq K$.

(2.6) Let $I \subseteq \mathbb{C}[\underline{x}]$ be a binomial ideal such that $T = T(I)$ and $K = K(I)$ meet the conditions (i) and (ii) of (2.1) and (2.4), respectively. Then, for any $\ell \subseteq [n]$ such that some $(\bullet, \ell)$ occurs as a standard pair, we define the ideal

$$I^{(\ell)} := I + \left(x^a \mid a \notin \overline{T(\ell)}\right) \subseteq \mathbb{C}[\underline{x}].$$

If there was no standard pair containing ℓ, then $T(\ell) = \overline{T(\ell)} = \emptyset$, and the above definition would yield $I^{(\ell)} = (1)$, anyway.

Remark: The ideals $I^{(\ell)}$ are still binomial with $T(I^{(\ell)}) = \overline{T(\ell)}$.

Proof: Indeed, if $a \notin T(I^{(\ell)})$, i.e. $x^a \in I^{(\ell)}$, then the remark at the end of (2.5) shows that either $a \notin \overline{T(\ell)}$, or that a is connected with some $b \in T \setminus \overline{T(\ell)}$ via an element of $K(I)$. However, by Lemma (2.5), the latter would imply $a \notin \overline{T(\ell)}$, too. □

(2.7) Now, let us assume that the ideal I is saturated. Then, we have gathered everything we need for the description of its cellular primary decomposition:

Theorem: *Let* $I \subseteq \mathbb{C}[\underline{x}]$ *be a saturated binomial ideal in the sense of (2.4). Then*

(1) $I = \bigcap_\ell I^{(\ell)}$ *is a primary decomposition, and*

(2) the ideals $I^{(\ell)}$ *are cellular in the sense of §6 of [ESt].*

Proof: *Step 1: Being primary may be checked by means of the homogeneous elements only:*
If $J \subseteq R$ is an ideal, then J is primary if and only if the multiplication maps $(\cdot r) : R/J \to R/J$ are either injective or nilpotent. On the other hand, if J is homogeneous in a graded ring, then these two properties of linear maps $\psi : R/J \to R/J$ may be checked by using homogeneous arguments only. Moreover, the sum of injective, homogeneous maps of different degrees remains injective, the sum of nilpotent maps remains nilpotent, and the sum of an injective and a nilpotent map is injective.

Step 2: The ideals $I^{(\ell)}$ *are primary:*

For $q \in \mathbb{Z}^n/K$ denote by $F_q := \{a \in \mathbb{Z}^n_{\geq 0} \mid a \mapsto q\}$ the whole fiber of q; in particular, $T_q \subseteq F_q$. If $I^{(\ell)}$ was not primary, then there would be elements $s \in \mathbb{C}[\underline{x}]_p$ and $t \in \mathbb{C}[\underline{x}]_q$ such that $st \in I^{(\ell)}$, $s \notin I^{(\ell)}$, and $t^N \notin I^{(\ell)}$ for every $N \geq 1$. Moreover, if we replace s, t by different representatives of their equivalent classes in $\mathbb{C}[\underline{x}]/I^{(\ell)}$, then the previous property does not change.
For any degree $q \in \mathbb{Z}^n/K$ we know that $I_q \subseteq I_q^{(\ell)} \subseteq \mathbb{C}[\underline{x}]_q$ and $\dim_{\mathbb{C}} \mathbb{C}[\underline{x}]_q/I_q \leq 1$. Applied to our special situation this means that $I_p = I_p^{(\ell)} \subset \mathbb{C}[\underline{x}]_p$, $I_{Nq} = I_{Nq}^{(\ell)} \subset \mathbb{C}[\underline{x}]_{Nq}$, and s, t may be assumed to be monomials x^a and x^b, respectively. Moreover, the product $st = x^{a+b}$ is either contained in I_{p+q}, or we have that $I_{p+q}^{(\ell)} = \mathbb{C}[\underline{x}]_{p+q}$. Translated into the language of exponents, this means:

$$a \in T_p = \overline{T(\ell)}_p \;\text{ and }\; Nb \in T_{Nq} = \overline{T(\ell)}_{Nq}\; \forall N \geq 1\,, \;\text{ but }\; a + b \notin T \;\text{ or }\; T_{p+q} \neq \overline{T(\ell)}_{p+q}\,.$$

Since $\overline{T(\ell)}$ consists of only finitely many $\mathbb{Z}^\ell_{\geq 0}$-slices, the fact that $Nb \in \overline{T(\ell)}$ for all $N \geq 1$ implies that $b \in \mathbb{Z}^\ell_{\geq 0}$. Hence, the property $a \in \overline{T(\ell)}$ yields $a + b \in \overline{T(\ell)}_{p+q} \subseteq T_{p+q}$ immediately. Moreover, by Lemma (2.5)(c), the latter two sets have to be equal, and we obtain a contradiction.

Step 3: The intersection yields I:
For every q we have to show that there is at least one ℓ such that $I_q^{(\ell)} = I_q$, i.e. such that $\overline{T(\ell)}_q = T_q$. However, if $T_q \neq \emptyset$, the latter equality is equivalent to $\overline{T(\ell)} \cap T_q \neq \emptyset$ by Lemma (2.5)(c). Hence, everything follows from $T = \bigcup_\ell T(\ell) = \bigcup_\ell \overline{T(\ell)}$.

Step 4: The ideals $I^{(\ell)}$ are cellular:
Since $\overline{T(\ell)}$ consists of only finitely many translations of $\mathbb{Z}^\ell_{\geq 0}$, there are only finitely many $\mathbb{Z}^{[n]\setminus\ell}_{\geq 0}$-elements belonging to $\overline{T(\ell)}$. Hence, for sufficiently large N, the monomials $\big(\prod_{j\notin\ell} x_j\big)^N$ must be contained in $I^{(\ell)}$.
On the other hand, we have to show that $\big(I^{(\ell)} : (\prod_{i\in\ell} x_i)^\infty\big) = I^{(\ell)}$, i.e. that for every $f \in \mathbb{C}[\underline{x}]$ with $x_i \cdot f \in I^{(\ell)}$ ($i \in \ell$) it follows that $f \in I^{(\ell)}$. Using both the $\mathbb{Z}^n/K(I)$-grading and the fact that $\dim_{\mathbb{C}} \mathbb{C}[\underline{x}]_q/I_q^{(\ell)} \leq 1$, we may assume that $f = x^a$ is a monomial. But then, with $e^i \in \mathbb{Z}^n$ denoting the i-th canonical basis vector, we already know that $e^i + a \in \overline{T(\ell)}$ if and only if $a \in \overline{T(\ell)}$. $\square$

3 Two-dimensional $\mathcal{A}$-graded monomial ideals

(3.1) If $I \subseteq \mathbb{C}[\underline{x}]$ is a *monomial* ideal, then Theorem (2.7) yields the well known formula for the primary decomposition of I into the easier looking $I^{(\ell)} = \big(x^a \mid a \notin \overline{T(\ell)}\big)$. In particular, the associated primes are

$$P^{(\ell)} = \sqrt{I^{(\ell)}} = \big(x_i \,\big|\, i \notin \ell\big) \;\text{ with } \ell \text{ such that there is a } (\bullet, \ell) \in B(T).$$

If $I \subseteq \mathbb{C}[\underline{x}]$ is additionally $\mathcal{A}$*-graded* with respect to some linear map $\mathcal{A} : \mathbb{Z}^n \to \mathbb{Z}^d$ with $(\ker \mathcal{A}) \cap \mathbb{Z}^n_{\geq 0} = 0$, cf. (1.1) for a definition, then we know that $\mathcal{A}$ induces an isomorphism $T \xrightarrow{\sim} \mathcal{A}(\mathbb{Z}^n_{\geq 0})$. In particular, every ℓ occurring in $B(T)$ fulfils $\#\ell \leq d$, and the chain property, cf. (1.2), looks as follows:
I fulfils the *chain property* for its associated primes if for any non-maximal ℓ in $B(T)$ there is another $\ell' \supseteq \ell$ in $B(T)$ with $\#\ell' = \#\ell + 1$.

Remark: An ℓ occurring in $B(T)$ via some $(\bullet, \ell)$ is maximal if and only if $(0, \ell) \in B(T)$.

We will show that the above chain property is always fulfilled for monomial, $\mathcal{A}$-graded ideals as long as $d \leq 2$, but it fails for $d \geq 3$.

(3.2) First, we need the following lemma describing the intersection behavior of two different layers $(r,\ell) = (r+\mathbb{Z}^{\ell}_{\geq 0})$, $(s,m) = (s+\mathbb{Z}^{m}_{\geq 0})$ of the base $B(T)$.

Lemma: *Let $T \subseteq \mathbb{Z}^n_{\geq 0}$ be an arbitrary subset satisfying the assumption (i) of (2.1). Then, two different $(r,\ell),(s,m) \in B(T)$ are either disjoint, or the intersection is of the form*

$$(r+\mathbb{Z}^{\ell}_{\geq 0}) \cap (s+\mathbb{Z}^{m}_{\geq 0}) = (p+\mathbb{Z}^{\ell\cap m}_{\geq 0})$$

with strict inclusions $\ell \cap m \subset \ell, m$.

Proof: Assume that the intersection is not empty. Then, outside $(\ell \cup m)$, the values of r and s coincide, and we set p as the common one. Within $(\ell \cup m)$ we define

$$p_{|(\ell\setminus m)} := s_{|(\ell\setminus m)} \quad \text{and} \quad p_{|(m\setminus\ell)} := r_{|(m\setminus\ell)}.$$

It remains to check that neither of ℓ, m is a subset of the other one. But if this was the case, say $\ell \subseteq m$, then $(r+\mathbb{Z}^{\ell}_{\geq 0}) \subseteq (s+\mathbb{Z}^{m}_{\geq 0})$ implying that $(r+\mathbb{Z}^{\ell}_{\geq 0})$ would not be a maximal subset of T, i.e., $(r,\ell) \notin B(T)$. □

(3.3) The following lemma appears in Theorem 10.10 of [St2]. Nevertheless, to make the paper more self-contained, we add a short proof here.

Lemma: *Let $I \subseteq \mathbb{C}[\underline{x}]$ be a monomial, $\mathcal{A}$-graded ideal.*

(a) *The set of all $\mathcal{A}(\mathbb{R}^{\ell}_{\geq 0})$ with $\mathbb{Z}^{\ell}_{\geq 0} \subseteq T$ forms a triangulation of the convex cone $\mathcal{A}(\mathbb{R}^{n}_{\geq 0}) \subseteq \mathbb{R}^d$. The maximal cells come from those ℓ with $(0,\ell) \in B(T)$.*

(b) *Let $(0,\ell) \in B(T)$. Then, the map $\mathcal{A}$ induces a natural bijection*

$$\{r \mid (r,\ell) \in B(T)\} \xrightarrow{\sim} \mathcal{A}(\mathbb{Z}^n)/\mathcal{A}(\mathbb{Z}^{\ell}).$$

Proof: Using Lemma (3.2) for part (a), this and the injectivity in (b) are both simple consequences from the fact that the map $\mathcal{A}$ is injective on the subset $T \subseteq \mathbb{Z}^n_{\geq 0}$.
To show the surjectivity in (b) we remark first that $\mathcal{A}(\mathbb{Q}^{\ell}) = \mathcal{A}(\mathbb{Q}^n)$. In particular, if some class $\overline{w} \in \mathcal{A}(\mathbb{Z}^n)/\mathcal{A}(\mathbb{Z}^{\ell})$ is represented by a $w = \mathcal{A}(a-b)$ with $a,b \in \mathbb{Z}^n_{\geq 0}$, then there is an $N \geq 1$ with $N \cdot \mathcal{A}(b) \in \mathcal{A}(\mathbb{Z}^{\ell})$. Hence $\overline{w}$ may be represented by $w + N\cdot\mathcal{A}(b) = \mathcal{A}\big(a+(N-1)b\big)$, i.e., by an element from $\mathcal{A}(\mathbb{Z}^n_{\geq 0})$.
Let $\overline{w}$ be now represented from $\mathcal{A}(\mathbb{Z}^n_{\geq 0})$. Since $\mathcal{A}(\mathbb{Z}^n_{\geq 0}) = \mathcal{A}(T)$, this means that

$$\forall\, a \in \mathbb{Z}^{\ell}_{\geq 0} \;\; \exists\, \big(r(a),\ell(a)\big) \in B(T): \quad w + \mathcal{A}(a) \,\in\, \mathcal{A}\big(r(a)\big) + \mathcal{A}\big(\mathbb{Z}^{\ell(a)}_{\geq 0}\big).$$

It remains to show that ℓ itself appears in the previous list of the sets $\ell(a)$. But, if not, then all elements of $w + \mathcal{A}(\mathbb{Z}^{\ell}_{\geq 0})$ would be contained in at most finitely many shifts of maximal cells different from $\mathcal{A}(\mathbb{Z}^{\ell}_{\geq 0})$. □

(3.4) **Theorem:** *Let $I \subseteq \mathbb{C}[\underline{x}]$ be a monomial, $\mathcal{A}$-graded ideal of dimension $d \leq 2$. Then I satisfies the chain property for its associated primes.*

Proof: Since there is nothing to show for one-dimensional ideals, we consider the case of $d = 2$. Let us assume that the chain condition is violated, i.e., for every $(r,\ell) \in B(T)$ we have either $\#\ell = 2$ or $\ell = \emptyset$, and, moreover, there is at least one $(r^*,\emptyset) \in B(T)$ of the second type.
Since the ℓ's with cardinality two provide a triangulation of the two-dimensional cone $\mathcal{A}(\mathbb{R}^n_{\geq 0}) \subseteq \mathbb{R}^2$, we may order them in a natural way as $\ell^1,\ldots,\ell^N$. Then Lemma (3.2) implies that adjacent sets

ℓ^{i-1}, ℓ^i share a common element, say i. Denoting the canonical basis elements of $\mathbb{Z}^n$ by e^i, this yields the following setup

$$\ell^i = \{i, i+1\} \quad \text{and} \quad \sigma_i := \mathcal{A}\big(\mathbb{R}^{\ell^i}_{\geq 0}\big) = \big\langle \mathcal{A}(e^i), \mathcal{A}(e^{i+1})\big\rangle_{\mathbb{R}_{\geq 0}} \quad \text{with} \quad i = 1, \ldots, N(< n).$$

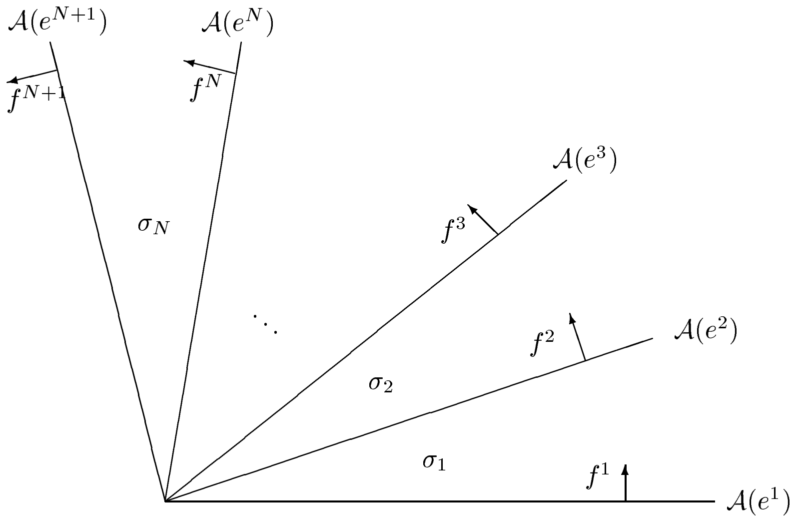

Figure 1

Using part (b) of Lemma (3.3), we may choose, for every i, a pair $(r^i, \ell^i) \in B(T)$ such that $A(r^*) \in \mathcal{A}(r^i) + \mathcal{A}(\mathbb{Z}^{\ell^i})$. On the other hand, r^* is "isolated", i.e., it does not belong to any of the sets $r^i + \mathbb{Z}^{\ell^i}_{\geq 0}$. Then the injectivity of $\mathcal{A}$ on T implies that $\mathcal{A}(r^*) \notin \mathcal{A}(r^i) + \mathcal{A}\big(\mathbb{Z}^{\ell^i}_{\geq 0}\big)$, and since $\mathcal{A}\big(\mathbb{Z}^{\ell^i}_{\geq 0}\big) = \mathcal{A}\big(\mathbb{Z}^{\ell^i}\big) \cap \mathcal{A}\big(\mathbb{R}^{\ell^i}_{\geq 0}\big)$, we obtain $\mathcal{A}(r^*) \notin \mathcal{A}(r^i) + \sigma_i$. If f^i denote linear forms on $\mathbb{Z}^2$ such that the cones σ_i are given by

$$\sigma_i = \big\{x \in \mathbb{R}^2 \;\big|\; \langle x, f^i\rangle \geq 0,\ \langle x, f^{i+1}\rangle \leq 0\big\}\,,$$

cf. Figure 1, then this statement can be translated into

$$\big\langle \mathcal{A}(r^*) - \mathcal{A}(r^i),\, f^i\big\rangle < 0 \quad \text{or} \quad \big\langle \mathcal{A}(r^*) - \mathcal{A}(r^i),\, f^{i+1}\big\rangle > 0 \quad \text{for } i = 1, \ldots, N.$$

Reorganizing these inequalities, we obtain that either

$$\big\langle \mathcal{A}(r^*),\, f^1\big\rangle < \big\langle \mathcal{A}(r^1),\, f^1\big\rangle, \quad \text{or} \quad \big\langle \mathcal{A}(r^N),\, f^{N+1}\big\rangle < \big\langle \mathcal{A}(r^*),\, f^{N+1}\big\rangle,$$

or there is an $i \in \{2, \ldots, N\}$ with

$$\big\langle \mathcal{A}(r^{i-1}),\, f^i\big\rangle < \big\langle \mathcal{A}(r^*),\, f^i\big\rangle < \big\langle \mathcal{A}(r^i),\, f^i\big\rangle$$

as depicted in Figure 2.

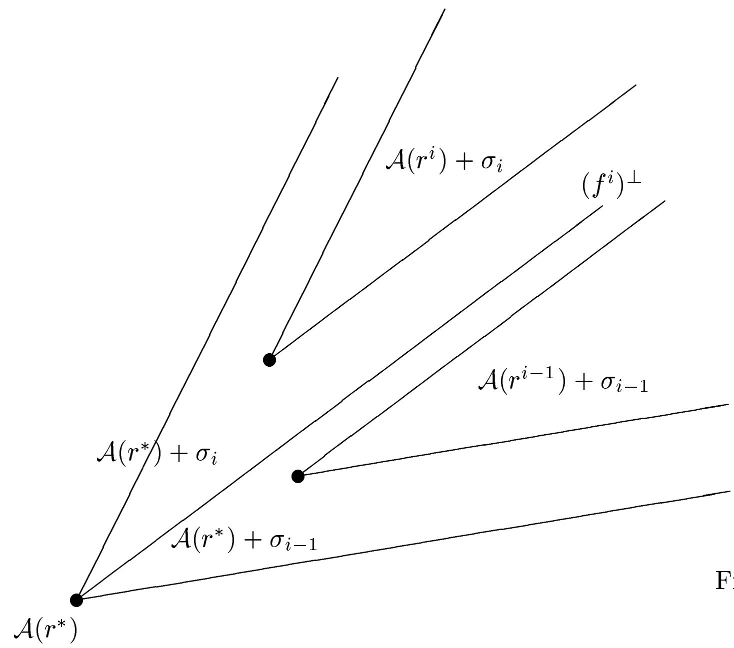

Figure 2

Assume, w.l.o.g., that the latter two inequalities apply to some i. Then we consider the series $\mathcal{A}(r^*)+\mathbb{Z}_{\geq 0}\mathcal{A}(e^i)\subseteq\mathcal{A}(r^*)+(f^i)^\perp$: All its members are contained in $\mathcal{A}(T)$. Hence, up to finitely many exceptions, they have to be contained in some $\mathcal{A}(s)+\mathcal{A}(\mathbb{Z}^\ell_{\geq 0})$ with $(s,\ell)\in B(T)$ and $\#\ell=2$. On the other hand, if $\ell\neq\ell^{i-1},\ell^i$, then any shift of the cone $\mathcal{A}\big(\mathbb{R}^\ell_{\geq 0}\big)$ intersects the ray $\mathcal{A}(r^*)+\mathbb{R}_{\geq 0}\mathcal{A}(e^i)$ in a compact set, i.e., the intersection contains at most finitely many lattice points. Thus, almost all elements of $\mathcal{A}(r^*)+\mathbb{Z}_{\geq 0}\mathcal{A}(e^i)$ are contained in sets of the form $\mathcal{A}(s)+\mathcal{A}(\mathbb{Z}^{\ell^{i-1}}_{\geq 0})$ or $\mathcal{A}(s)+\mathcal{A}(\mathbb{Z}^{\ell^i}_{\geq 0})$ with $(s,\ell^{i-1}),(s,\ell^i)\in B(T)$.
However, applying part (b) of Lemma (3.3) again, we see that there is no freedom left for the element s. It has to equal r^{i-1} or r^i, respectively. But since the sets $\mathcal{A}(r^{i-1})+\sigma_{i-1}$ and $\mathcal{A}(r^i)+\sigma_i$ do not meet the ray $\mathcal{A}(r^*)+\mathbb{R}_{\geq 0}\mathcal{A}(e^i)$ at all, we have obtained a contradiction. $\square$

4 A counter example in dimension three

(4.1) Roughly speaking, the proof of the previous theorem worked as follows: We have shown that the shifted two-dimensional cells of the triangulation create gaps that cannot be filled with isolated T-elements only. In dimension three this concept fails, since some cells might be arranged in cycles. In particular, we have

Theorem: *There exists an example of a monomial, $\mathcal{A}$-graded ideal $I\subseteq\mathbb{C}[\underline{x}]$ with $d=3$ such that the chain property for the associated primes is violated.*

Proof: Take $n=16$ with the variables e_i,f_i,g_i $(i=1,2,3)$ and k_ν $(\nu=1,\ldots,7)$. The ideal I is defined by the following 100 generators

- $f_i\,k_\nu,\ g_i\,k_\nu,\ g_i\,g_j,\ k_\nu\,k_\mu$ with $i,j\in\{1,2,3\}$ and $\mu,\nu\in\{1,\ldots,7\}$,
- $f_i^2,\ f_i\,g_{i+1},\ f_{i-1}\,f_i\,g_{i-1},$
- $f_i\,g_i\,e_{i-1},\ f_{i+1}\,g_i\,e_{i-1},\ f_i\,f_{i+1}\,e_{i-1}$, and
- $f_i\,e_{i-1}\,e_{i+1},\ g_i\,e_{i-1}\,e_{i+1}$

with $i \in \mathbb{Z}/3\mathbb{Z}$. Denoting by $\{E_1, E_2, E_3\}$ the canonical basis $\{(1,0,0),(0,1,0),(0,0,1)\}$ of $\mathbb{Z}^3$, then I is $\mathcal{A}$-graded with respect to the linear map $\mathcal{A} : \mathbb{Z}^{16} \to \mathbb{Z}^3$ given by

$$e_i \mapsto 2\,E_i, \quad f_i \mapsto E_i, \quad g_i \mapsto E_i + (1,1,1),$$

and

$$\big\{k_1, \ldots, k_7\big\} \;\mapsto\; \big\{(3,2,2),\, (2,3,2),\, (2,2,3),\, (2,3,3),\, (3,2,3),\, (3,3,2),\, (3,3,3)\big\}\,.$$

□

Remark: Since $\mathcal{A} : \mathbb{Z}^{16}_{\geq 0} \to \mathbb{Z}^3_{\geq 0}$ is surjective, the associated toric ideal defines the semigroup algebra $\mathbb{C}\big[\mathbb{Z}^3_{\geq 0}\big]$, i.e., the associated toric variety is $\mathbb{C}^3$.

(4.2) In addition to the plain presentation of the example in the previous section, we would also like to show how it really works. In particular, we rather describe the set $T = T(I)$ of standard monomials via its basis $B(T)$ – and how these sets map to $\mathbb{Z}^3$.
To improve the readability, we write $\langle \ell \rangle$ for the semigroup $\mathbb{Z}^{\ell}_{\geq 0}$. Then the ideal I has the following maximal (with respect to the partial order "$\geq$" induced by $\mathbb{Z}^{16}_{\geq 0}$) standard pairs

(i) $k_\nu + \big\langle e_1, e_2, e_3 \big\rangle$ $(\nu = 1, \ldots, 7)$,

(ii) $(f_i + f_{i+1}) + \big\langle e_i, e_{i+1} \big\rangle$, $(f_i + g_i) + \big\langle e_i, e_{i+1} \big\rangle$, $(f_{i+1} + g_i) + \big\langle e_i, e_{i+1} \big\rangle$,

(iii) $g_{i+1} + \big\langle e_i, e_{i+1} \big\rangle$, and

(iv) $(f_1 + f_2 + f_3)$

with $i \in \mathbb{Z}/3\mathbb{Z}$. Dropping the maximality condition, we have to add the standard pairs

(v) $\big\langle e_1, e_2, e_3 \big\rangle$ and

(vi) $f_i + \big\langle e_i, e_{i+1} \big\rangle$, $f_{i+1} + \big\langle e_i, e_{i+1} \big\rangle$, $g_i + \big\langle e_i, e_{i+1} \big\rangle$.

The violation of the chain property for the associated primes is caused by the existence of the standard pair $\big((f_1 + f_2 + f_3),\, \emptyset\big)$. The remaining standard pairs involve only sets $\ell \subseteq \{1, 2, \ldots, 16\}$ with $\#\ell \geq 2$.
Finally, for the $\mathcal{A}$-graded property, let us consider the $\mathcal{A}$-images:

- (i) and (v) yield all triples with entries 0 or ≥ 2.
- Assuming $[i = 1]$, the series (ii), (iii), (vi) provide $\big(2\,\mathbb{Z}_{\geq 0}, 2\,\mathbb{Z}_{\geq 0}, 0\big)$ shifted by $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, or $(3,1,1)$, $(2,1,1)$, $(1,2,1)$, $(2,2,1)$. This means that every triple having 0 or 1 as its last entry is reached, except for $\big(1, 1 + 2\,\mathbb{Z}_{\geq 0}, 1\big)$ and $\big(2\,\mathbb{Z}_{\geq 0}, 2\,\mathbb{Z}_{\geq 0}, 0\big)$ itself.
  However, the latter series already occurred in the previous point (i)/(v), and, beginning with $(1,3,1)$, the first series is included in (ii)/(iii)/(vi) with $[i = 2]$.
- The isolated (iv) yields the missing triple $(1,1,1)$.

It follows that $\mathcal{A}$ maps $T \subseteq \mathbb{Z}^{16}_{\geq 0}$ onto $\mathbb{Z}^3_{\geq 0}$. Moreover, a closer look shows that the restriction $\mathcal{A}_{|T}$ is injective, indeed.

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