P-Resolutions of Cyclic Quotients from the Toric Viewpoint

Klaus Altmann
Institut für reine Mathematik, Humboldt-Universität zu Berlin
Ziegelstraße 13A, D-10099 Berlin, Germany.
E-mail: altmann@mathematik.hu-berlin.de

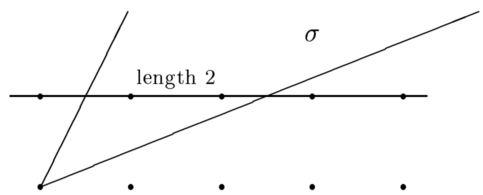
1 Introduction

(1.1) The break through in deformation theory of (two-dimensional) quotient singularities Y was Kollár/Shepherd-Barron's discovery of the one-to-one correspondence between so-called P-resolutions, on the one hand, and components of the versal base space, on the other (cf. [KS], Theorem (3.9)). It generalizes the fact that all deformations admitting a simultaneous (RDP-) resolution form one single component, the Artin component.

According to definition (3.8) in [KS], P-resolutions are partial resolutions $\pi : \tilde{Y} \to Y$ such that

- the canonical divisor $K_{\tilde{Y}|Y}$ is ample relative to π (a minimality condition) and
- $\tilde{Y}$ contains only mild singularities of a certain type (so-called T-singularities).

Despite their definition as those quotient singularities admitting a $\mathbb{Q}$-Gorenstein one-parameter smoothing ([KS], (3.7)), there are at least three further descriptions of the class of T-singularities: An explicit list of their defining group actions on $\mathbb{C}^2$ ([KS], (3.10)), an inductive procedure to construct their resolution graphs ([KS], (3.11)), and a characterization using toric language ([Al], (7.3)).
The latter one begins with the observation that affine, two-dimensional toric varieties (given by some rational, polyhedral cone $\sigma \subseteq \mathbb{R}^2$) provide exactly the two-dimensional cyclic quotient singularities. Then, T-singularities come from cones over rational intervals of integer length placed in height one (i.e. contained in the affine line $(\bullet, 1) \subseteq \mathbb{R}^2$).



If the affine interval is of length $\mu + 1$, then the corresponding T-singularity will have Milnor number μ (on the $\mathbb{Q}$-Gorenstein one-parameter smoothing).

(1.2) In [Ch] and [St] Christophersen and Stevens gave a combinatorial description of all P-resolutions for two-dimensional, cyclic quotient singularities. Using an inductive construction method (going through different cyclic quotients with step-by-step increasing multiplicity), they have shown that there is a one-to-one correspondence between P-resolutions, on the one hand, and certain integer tuples $(k_2, \ldots, k_{e-1})$ yielding zero if expanded as a (negative) continued fraction (cf. (4.2)), on the other hand.

The aim of the present paper is to provide an elementary, direct method for constructing the P-resolutions of a cyclic quotient singularity (i.e. a two-dimensional toric variety) Y_σ. Given a chain $(k_2,\dots,k_{e-1})$ representing zero, we will give a straight description of the corresponding polyhedral subdivision of σ. (In particular, the bijection between those 0-chains and P-resolutions will be proved again by a different method.)

2 Cyclic Quotient Singularities

In the following we remind the reader of basic notions concerning continued fractions and cyclic quotients as well as fix notation. References are [Od] (§1.6) or the first sections in [Ch] and [St], respectively.

(2.1) **Definition:** To integers $c_1,\dots,c_r \in \mathbb{Z}$ we will assign the continued fraction $[c_1,\dots,c_r] \in \mathbb{Q}$ if the following inductive procedure is well-defined (i.e. if no division by 0 occurs):

- $[c_r] := c_r$
- $[c_i,\dots,c_r] := c_i - 1/[c_{i+1},\dots,c_r]$.

If $c_i \geq 2$ for $i=1,\dots,r$, then $[c_1,\dots,c_r]$ is always defined and yields a rational number greater than 1. Moreover, all these numbers may be represented by those continued fractions in a unique way.

(2.2) Let $n \geq 2$ be an integer and $q \in (\mathbb{Z}/n\mathbb{Z})^*$ be represented by an integer between 0 and n. Each q provides a group action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{C}^2$ via the matrix $\begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix}$ (with ξ a primitive n-th root of unity). The quotient is denoted by $Y(n,q)$.
In toric language, $Y(n,q)$ equals the variety Y_σ assigned to the polyhedral cone $\sigma := \langle(1,0);(-q,n)\rangle$ contained in $\mathbb{R}^2$. Y_σ is defined as Spec $\mathbb{C}[\sigma^\vee \cap \mathbb{Z}^2]$ with

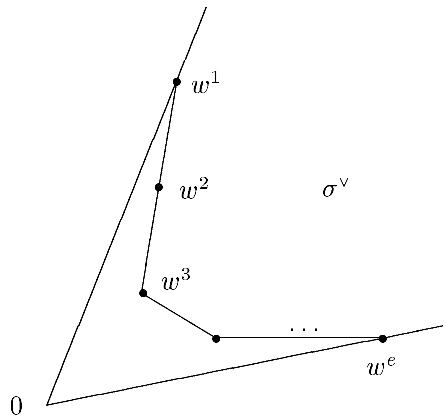
$$\sigma^\vee := \{r \in (\mathbb{R}^2)^* \mid r \geq 0 \text{ on } \sigma\} = \langle[0,1];[n,q]\rangle \subseteq (\mathbb{R}^2)^* \cong \mathbb{R}^2\,.$$

Notation: Just to distinguish between $\mathbb{R}^2$ and its dual $(\mathbb{R}^2)^* \cong \mathbb{R}^2$, we will denote these vector spaces by $N_\mathbb{R}$ and $M_\mathbb{R}$, respectively. (Hence, $\sigma \subseteq N_\mathbb{R}$ and $\sigma^\vee \subseteq M_\mathbb{R}$.) Elements of $N_\mathbb{R} \cong \mathbb{R}^2$ are written in parentheses; elements of $M_\mathbb{R} \cong \mathbb{R}^2$ are written in brackets. The natural pairing between $N_\mathbb{R}$ and $M_\mathbb{R}$ is denoted by $\langle\,,\,\rangle$, which should not be confused with the symbol indicating the generators of a cone. Finally, all these remarks apply for the lattices $N \cong \mathbb{Z}^2$ and $M \cong \mathbb{Z}^2$, too.

(2.3) Let n,q be as before. We may write $n/(n-q)$ and n/q (both are greater than 1) as continued fractions

$$n/(n-q) = [a_2,\dots,a_{e-1}] \;\text{ and }\; n/q = [b_1,\dots,b_r] \quad (a_i,b_j \geq 2).$$

The a_i's and the b_j's are mutually related by Riemenschneider's point diagram (cf. [Ri]).
Denote by $w^1,w^2,\dots,w^e$ the lattice points on the compact edges of the convex hull of $(\sigma^\vee \cap M)\setminus\{0\}$. If ordered the right way, we obtain $w^1=[0,1]$ and $w^e=[n,q]$ for the first and the last point, respectively.

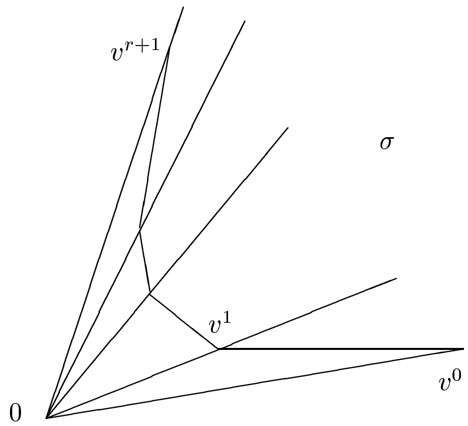


Then, $E := \{w^1, \dots, w^e\}$ is the minimal generating set (the so-called Hilbert basis) of the semigroup $\sigma^\vee \cap M$. These point are related to our first continued fraction by

$$w^{i-1} + w^{i+1} = a_i\, w^i \quad (i = 2, \dots, e-1).$$

Remark: The surjection $\mathbb{N}^E \twoheadrightarrow \sigma^\vee \cap M$ provides a minimal embedding of Y_σ. In particular, e equals its embedding dimension.

In a similar manner we can define $v^0, \dots, v^{r+1} \in \sigma \cap N$ in the original cone; now we have $v^0 = (1, 0)$, $v^{r+1} = (-q, n)$, and the relation to the continued fractions is $v^{j-1} + v^{j+1} = b_j\, v^j$ (for $j = 1, \dots, r$).



Drawing rays through the origin and each point v^j, respectively, provides a polyhedral subdivision Σ of σ. The corresponding toric variety Y_Σ is a resolution of our singularity Y_σ. The numbers $-b_j$ equal the self intersection numbers of the exceptional divisors; since $b_j \geq 2$, the resolution is *minimal*.

3 The Maximal Resolution

(3.1) Definition: ([KS], (3.12)) For a resolution $\pi : \tilde{Y} \to Y$ we may write $K_{\tilde{Y}|Y} := K_{\tilde{Y}} - \pi^* K_Y = \sum_j (\alpha_j - 1) E_j$, where the E_j denote the exceptional divisors, and $\alpha_j \in \mathbb{Q}$. Then, π will be called *maximal* if it is maximal with respect to the property $0 < \alpha_j < 1$.

The maximal resolution is uniquely determined and dominates all the P-resolutions. Hence, for our purpose, it is more important than the minimal one. It can be constructed from the minimal resolution by successively blowing up points $E_i \cap E_j$ with $\alpha_i + \alpha_j \geq 0$ (cf. Lemma (3.13) and Lemma (3.14) in [KS]).

(3.2) Proposition: *The maximal resolution of Y_σ is toric. It can be obtained by drawing rays through 0 and all interior lattice points (i.e. $\in N$) of the triangle $\Delta := \operatorname{conv}(0, v^0, v^{r+1})$, respectively.*

Proof: In order to keep track of the rational numbers α_j, we will show how they can be "seen" in an arbitrary toric resolution of Y_σ. Let $\Sigma < \sigma$ be a subdivision generated by one-dimensional rays through the points $u^0, \dots, u^{s+1} \in \sigma \cap N$. In particular, $u^0 = v^0 = (1,0)$ and $u^{s+1} = v^{r+1} = (-q, n)$; moreover, for the minimal resolution we have $s = r$ and $u^j = v^j$ ($j = 0, \dots, r+1$). Denote by $c_1, \dots, c_s$ the integers given by the relations

$$u^{j-1} + u^{j+1} = c_j\, u^j \qquad (j = 1, \dots, s).$$

In particular, $c_j = b_j$ for the minimal resolution. As usual, the numbers $-c_j$ equal the self intersection numbers of the exceptional divisors E_j in Y_Σ. Indeed, $D := \sum_i u^i E_i$ is a principal divisor (if you do not like coefficients u^i from N, evaluate them using arbitrary elements of M); hence,

$$\begin{aligned} 0 = E_j \cdot D &= E_j \cdot (u^{j-1} E_{j-1} + u^j E_j + u^{j+1} E_{j+1}) \\ &= u^{j-1} + (E_j)^2\, u^j + u^{j+1} \\ &= (c_j + (E_j)^2) \cdot u^j \qquad\qquad (j = 1, \dots, s). \end{aligned}$$

On the other hand, we can use the projection formula to obtain

$$\begin{aligned} -2 = 2\, g(E_j) - 2 &= K_{\tilde{Y}|Y} \cdot E_j + (E_j)^2 \\ &= \textstyle\sum_i (\alpha_i - 1)\, (E_i \cdot E_j) + (E_j)^2 \\ &= (\alpha_{j-1} - 1) + (\alpha_j - 1)\, (E_j)^2 + (\alpha_{j+1} - 1) + (E_j)^2\,, \end{aligned}$$

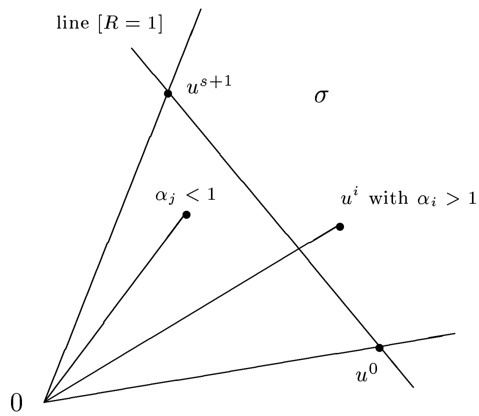
hence

$$\alpha_{j-1} + \alpha_{j+1} = c_j\, \alpha_j \qquad (j = 1, \dots, s\,;\ \alpha_0,\, \alpha_{s+1} := 1).$$

Looking at the definition of the c_j's via relations among the lattice points u^j, there has to be some $R \in M_{\mathbb{R}}$ such that

$$\alpha_j = \langle u^j,\, R \rangle \qquad\qquad (j = 0, \dots, s+1).$$

The conditions $\langle u^0,\, R\rangle = \alpha_0 = 1$ and $\langle u^{s+1},\, R\rangle = \alpha_{s+1} = 1$ determine R uniquely. Now, we can see that α_j measures exactly the quotient between the length of the line segment $\overline{0\, u^j}$ and the length of the Δ-part of the line through 0 and u^j. In particular, $\alpha_j < 1$ if and only if u^j lies below the line connecting u^0 and u^{s+1}.



This explains how to construct the maximal resolution: Starting with the minimal one, continue subdividing each small cone $\langle u^j, u^{j+1}\rangle$ into $\langle u^j, u^j + u^{j+1}\rangle \cup \langle u^j + u^{j+1}, u^{j+1}\rangle$ as long as it contains interior lattice points below the line $[R = 1]$, i.e. belonging to $\operatorname{int}\Delta$. $\square$

Corollary: *Every P-resolution is toric.*

Proof: P-resolutions are obtained by blowing down curves in the maximal resolution. $\square$

(3.3) **Example:** We take the example $Y(19, 7)$ from [KS], (3.15). Since $\sigma = \langle (1, 0), (-7, 19)\rangle$, the interior of Δ is given by the three inequalities

$$y > 0\,,\ 19x + 7y > 0\,,\ \text{and } 19x + 8y < 19\ \text{(corresponding to } R = [1, 8/19])\,.$$

The only primitive lattice points (i.e. generating rays) contained in $\operatorname{int}\Delta$ are

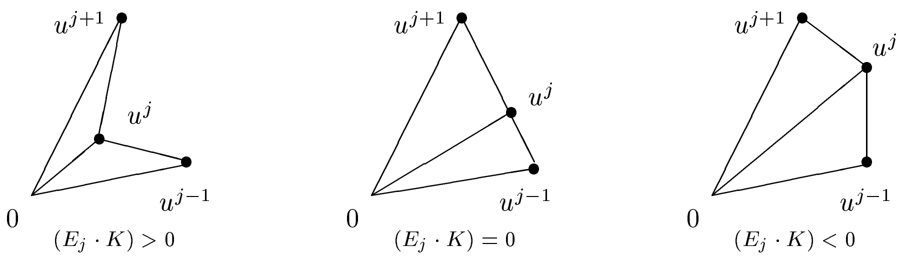
$$u^1 = (0, 1)\,,\ u^2 = (-1, 4)\,,\ u^3 = (-2, 7)\,,\ u^4 = (-1, 3)\,,\ u^5 = (-5, 14)\,,\ u^6 = (-4, 11)\,.$$

They provide the maximal resolution. The corresponding α's can be obtained by taking the scalar product with $R = [1, 8/19]$, which yields 8/19, 13/19, 18/19, 5/19, 17/19, and 12/19.
The minimal resolution uses only the rays through $u^1 = (0, 1)$, $u^4 = (-1, 3)$, and $u^6 = (-4, 11)$, respectively.

4 P-Resolutions

(4.1) In this section we will speak about *partial* toric resolutions $\pi : Y_\Sigma \to Y_\sigma$. Nevertheless, we use the same notation as we did for the maximal resolution: The fan Σ subdividing σ is generated by rays through $u^0, \dots, u^s \in \sigma \cap N$; each ray u^j corresponds to an exceptional divisor $E_j \subseteq Y_\Sigma$. However, since $u^{j-1} + u^{j+1}$ need not to be a multiple of u^j, the numbers c_j no longer make sense.

Lemma: ([Re], (4.3)) *For $K := K_{Y_\Sigma}$ or $K := K_{Y_\Sigma | Y_\sigma}$ the intersection number $(E_j \cdot K)$ is positive, zero, or negative if the line segments $\overline{u^{j-1}u^j}$ and $\overline{u^j u^{j+1}}$ form a strict concave, flat, or strict convex "roof" over the two cones, respectively.*



$(E_j \cdot K) > 0$ $\quad$ $(E_j \cdot K) = 0$ $\quad$ $(E_j \cdot K) < 0$

Proof: Using $K := K_{Y_\Sigma} = -\sum_{i=0}^{s+1} E_i$ (cf. [Od], (2.1)) we have

$$(E_j \cdot K) = -(E_j \cdot E_{j-1}) - (E_j)^2 - (E_j \cdot E_{j+1})\,.$$

On the other hand, as in the proof of Proposition (3.2), we know that

$$0 = (E_j \cdot E_{j-1})\, u^{j-1} + (E_j)^2\, u^j + (E_j \cdot E_{j+1})\, u^{j+1}\,.$$

Combining both formulas yields the final result

$$(E_j \cdot K)\, u^j = (E_j \cdot E_{j-1})\,(u^{j-1} - u^j) + (E_j \cdot E_{j+1})\,(u^{j+1} - u^j)\,. \qquad \square$$

Remark: The previous lemma together with Proposition (3.2) illustrate again the fact that all P-resolutions are dominated by the maximal resolution.

(4.2) In [Ch] Christophersen has defined the set

$$K_{e-2} := \{(k_2, \ldots, k_{e-1}) \in \mathbb{N}^{e-2} \mid [k_2, \ldots, k_{e-1}] \text{ is well defined and yields } 0\}$$

of chains representing zero. To every such chain, non-negative integers $q_1, \ldots, q_e$ are assigned. They are characterized by the following equivalent properties:

- $q_1 = 0,\ q_2 = 1$, and $q_{i-1} + q_{i+1} = k_i\, q_i \quad (i = 2, \ldots, e-1)$;
- $q_{e-1} = 1,\ q_e = 0$, and $q_{i-1} + q_{i+1} = k_i\, q_i \quad (i = 2, \ldots, e-1)$;
- $q_e = 0$ and $[k_i, \ldots, k_{e-1}] = q_{i-1}/q_i$ with $\gcd(q_{i-1}, q_i) = 1$ $(i = 2, \ldots, e-1)$.

(The two latter properties do not even use the fact that the continued fraction $[k_2, \ldots, k_{e-1}]$ yields zero.)

Remark: The elements of K_{e-2} are in one-to-one correspondence with triangulations of a (regular) $(e-1)$-gon with vertices $P_2, \ldots, P_{e-1}, P_*$. The numbers k_i tell how many triangles are attached to P_i, and the q_i have an easy meaning in this language, too.

Finally, for a given Y_σ with embedding dimension e, Christophersen defines

$$K(Y_\sigma) := \{(k_2, \ldots, k_{e-1}) \in K_{e-2} \mid k_i \le a_i\}\,.$$

Theorem: *Each P-resolution of Y_σ (i.e. the corresponding subdivision Σ of σ) is given by some $\underline{k} \in K(Y_\sigma)$ in the following way:*
(1) Σ is built from the rays that are orthogonal to $w^i/q_i - w^{i-1}/q_{i-1} \in M_{\mathbb{R}}$ (for $i = 3, \ldots, e-1$). In some sense, if the occurring divisions by zero are interpreted properly, Σ may be seen as dual to the Newton boundary generated by $w^i/q_i \in \sigma^\vee$ $(i = 1, \ldots, e)$.
(2) The affine lines $[\langle \bullet, w^i \rangle = q_i]$ form the "roofs" of the Σ-cones. In particular, the (possibly degenerate) cones $\tau^i \in \Sigma$ correspond to the elements $w^1, \ldots, w^e \in E$. The "roof" over the cone τ^i has length $\ell_i := (a_i - k_i)\, q_i$, where the length is defined via the metric induced by the lattice structure $M \subseteq M_{\mathbb{R}}$ on rational lines. In particular, τ^i is degenerate if and only if $k_i = a_i$. The Milnor number of the T-singularity Y_{τ^i} equals $(a_i - k_i - 1)$.

(4.3) **Proof:** According to the notation introduced in (4.1), the fan Σ consists of (non-degenerate) cones $\tau^j := \langle u^{j-1}, u^j \rangle$ with $j = 1, \ldots, s+1$. (Except for $u^0 = (1, 0)$ and $u^s = (-q, n)$, their generators u^j are primitive lattice points (i.e. $\in N$) contained in $\mathrm{int}\Delta \subseteq \sigma$.)

Step 1: For each τ^j there are $w \in E, d \in \mathbb{N}$ such that $\langle u^{j-1}, w \rangle = \langle u^j, w \rangle = d$.
First, it is very clear that there is a primitive lattice point $w \in M$ and a non-negative number $d \in \mathbb{R}_{\ge 0}$ admitting the desired properties. Moreover, since $u^j \in N$, d has to be an integer, and Reid's Lemma (4.1) tells us that $w \in \sigma^\vee$. It remains to show that w belongs to the Hilbert basis $E \subseteq \sigma^\vee \cap M$ as well.
Denote by ℓ the length of the line segment $\overline{u^{j-1}u^j}$ on the "roof" line $[\langle \bullet, w \rangle = d]$. Since τ^j represents a T-singularity, we know from (7.3) of [Al] (cf. (1.1) of the present paper) that $d|\ell$. In particular, $\overline{u^{j-1}u^j}$ contains the d-th multiple $d \cdot u$ of some lattice point $u \in \tau^j \cap M$ (w.l.o.g. not belonging to the boundary of σ). Hence, $\langle u, w \rangle = 1$ and $u \in \mathrm{int}\, \sigma \cap M$, and this implies $w \in E$.

Step 2: As each of the cones $\tau^1, \ldots, \tau^{s+1} \in \Sigma$ is assigned to some element $w \in E$, a renumbering will be used to make the notation more obvious: Let $\tau^i = \langle u^{i-1}, u^i \rangle$ be the cone assigned to $w^i \in E$,

and denote by d_i, ℓ_i the height and the length of its "roof" $\overline{u^{i-1}u^i}$, respectively. Some of these cones might be degenerated, i.e. $\ell_i = 0$. This it at least true for the extremal τ^1 and τ^e coinciding with the two rays spanning σ. Here we even have $d_1 = d_e = 0$; in particular $u^0 = u^1 = (1,0)$ and $u^{e-1} = u^e = (-q, n)$.
Since $d_i | \ell_i$, we may introduce integers $k_i \leq a_i$ yielding $\ell_i = (a_i - k_i)\, d_i$. For $i = 2, \ldots, e-1$ they are even uniquely determined.

Step 3: Using the following three ingredients

(i) $\langle u^{i-1},\, w^i\rangle = \langle u^i,\, w^i\rangle = d_i \quad (i = 1, \ldots, e)\,,$

(ii) $w^{i-1} + w^{i+1} = a_i\, w^i \quad (i = 2, \ldots, e-1;$ cf. (2.3)), and

(iii) $\langle u^i - u^{i-1},\, w^{i-1}\rangle = \ell_i = (a_i - k_i)\, d_i$ (since $\{w^{i-1}, w^i\}$ forms a $\mathbb{Z}$-basis of M),

we obtain

$$\begin{array}{rcl} d_{i-1} + d_{i+1} &=& (a_i\, d_i + d_{i-1}) - (a_i\, d_i - d_{i+1}) \\ &=& (a_i\, d_i + \langle u^{i-1}, w^{i-1}\rangle) - \langle u^i,\, a_i\, w^i - w^{i+1}\rangle \\ &=& a_i\, d_i + \langle u^{i-1}, w^{i-1}\rangle - \langle u^i, w^{i-1}\rangle \\ &=& a_i\, d_i + \langle u^{i-1} - u^i,\, w^{i-1}\rangle \\ &=& a_i\, d_i - (a_i - k_i)\, d_i \;=\; k_i\, d_i \quad (\text{for } i = 2, \ldots, e-1)\,. \end{array}$$

In particular, $k_i \geq 0$ (and in fact ≥ 1 for $e > 3$). Moreover, because $\{w^{i-1}, w^i\}$ forms a basis of M and $u^{i-1} \in N$ is primitive, we have $\gcd(d_{i-1}, d_i) = 1$. It follows that $d_i = q_i$, since both sequences of integers satisfy the second of the three properties mentioned in the beginning of (4.2). Finally, the third of these properties yields $[k_2, \ldots, k_{e-1}] = q_1/q_2 = d_1/d_2 = 0$, i.e. $\underline{k} \in K_{e-2}$.

The reverse direction, i.e. the fact that each $K(Y_\sigma)$-element indeed yields a P-resolution, follows from the above calculations in a similar manner. □

Remark: Subdividing each τ^i further into $(a_i - k_i)$ equal cones (with each "roof" length q_i) yields the so-called M-resolution (cf. [BC]) assigned to a P-resolution. It is defined to contain only T_0-singularities, i.e. T-singularities with Milnor number 0; in exchange, $K_{\tilde{Y}|Y}$ does not need to be relatively ample anymore. This property is replaced by "relatively nef".

Examples: (1) The continued fraction $[1, 2, 2, \ldots, 2, 1] = 0$ yields $q_1 = q_e = 0$ and $q_i = 1$ otherwise. In particular, the "roof" lines equal $[\langle \bullet,\, w^i\rangle = 1]$ (for $i = 2, \ldots, e-1$), describing the RDP-resolution of Y_σ. The assigned M-resolution equals the minimal resolution mentioned at the end of (2.3).

(2) Let us return to Example (3.3): The embedding dimension e of Y_σ is 6, the vector $(a_2, \ldots, a_{e-1})$ equals $(2, 3, 2, 3)$, and, except the trivial RDP element mentioned in (1), $K(Y_\sigma)$ contains only $(1, 3, 1, 2)$ and $(2, 2, 1, 3)$.
In both cases we already know that $q_1 = q_6 = 0$ and $q_2 = q_5 = 1$. The remaining values are given by the equation $q_3/q_4 = [k_4, k_5]$, i.e. we obtain $q_3 = 1$, $q_4 = 2$ or $q_3 = 2$, $q_4 = 3$, respectively.
Hence, in case of $(1, 3, 1, 2)$ the fan Σ is given by the additional rays through $(0, 1)$ and $(-4, 11)$. For $\underline{k} = (2, 2, 1, 3)$ we only need the one through $(-1, 4)$.

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