Deformation Theory

Klaus Altmann
Fachbereich Mathematik der Humboldt-Universität zu Berlin
Institut für Reine Mathematik, Ziegelstr. 13A, D-10099 Berlin, Germany.
E-mail: altmann@mathematik.hu-berlin.de

1 Deformations of algebraic varieties

(1.1) Assume that we are given an affine algebraic variety $Y \subseteq \mathbb{C}^N$. To obtain a qualitative description of Y or of its most interesting points – the singular points – it is useful to investigate the behavior of Y under deformations.
Is it possible to deform Y into a smooth variety or into a variety admitting weaker singularities? Which singularities can occur? "How many" different deformations of Y do exist?
If Y belongs to a certain class of singularities (quotient singularities, determinantal singularities etc.), are these classes closed under deformations? Different possibilities of deforming Y might reflect that Y belongs to different "classes" of singularities. Moreover, such classes could be defined in this way.
Assuming that it is worthwhile to investigate deformations, our first task is to give an idea how this notion can be defined in algebraic geometry. We will start with regarding some examples.

(1.2) Let $Y = Y_0 := [x\,y = 0] \subseteq \mathbb{C}^2$. The easiest way to deform Y_0 is to perturb the equation. $Y_t := [x\,y = t] \subseteq \mathbb{C}^2$ defines a family of smooth plane curves, and for $t \to 0$ this family degenerates to Y_0.

Y_0 $\qquad\qquad$ Y_t (for $t \neq 0$)

More general, this concept seems to be the right one for hypersurfaces. Let $Y = Y_0 := [f(\underline{x}) = 0] \subseteq \mathbb{C}^N$ be given by only one polynomial equation in N variables. Then, each set of polynomials $g_1(\underline{x}), \dots, g_m(\underline{x})$ induces an $m-$parameter-family of varieties adjacent to Y_0:

$$Y_t := [f(\underline{x}) + t_1\, g_1(\underline{x}) + \ldots + t_m\, g_m(\underline{x}) = 0] \subseteq \mathbb{C}^N.$$

We will use this example to introduce the usual language when dealing with deformations:

$$\begin{tikzcd} X := [f(\underline{x}) + \sum_j t_j \cdot g_j(\underline{x}) = 0] \arrow[r, "\subseteq"] \arrow[d, "\varphi"] & \mathbb{C}^N(\underline{x}) \times \mathbb{C}^m(\underline{t}) \arrow[dl, "\mathrm{pr}"] \\ S := \mathbb{C}^m(\underline{t}) & \end{tikzcd}$$

That means, X ("the total space of the deformation φ") equals the union $X = \bigcup_t Y_t$, and the varieties Y_t arise as fibers of the projection $\varphi : X \to \mathbb{C}^m$. The variety $Y = Y_0$ is called the special fiber.

(1.3) To investigate varieties Y given by more than one equation, we start with an example again: Let $Y_t := [x\,y = x\,z = y\,z = t] \subseteq \mathbb{C}^3$. Then, Y_0 equals the union of the three coordinate axes in $\mathbb{C}^3$. On the other hand, if $t \neq 0$, the equations for Y_t yield $x = y = z = \pm\sqrt{t}$, i.e. Y_t consists of two different points only. It seems to be clear that $X := \bigcup_t Y_t \to \mathbb{C}$ should not be called a deformation of Y_0 here.
What is the problem in this example? Y_0 was given by the three equations $f_1 = f_2 = f_3 = 0$ with

$$f_1 := x\,y, \quad f_2 := x\,z, \quad \text{and } f_3 := y\,z.$$

These equations are not independent, but admit linear relations such as

$$z \cdot f_1 = y \cdot f_2 = x \cdot f_3.$$

To obtain a true deformation of Y_0 (such that discrete invariants like the dimension remain constant in t), it is necessary to restrict ourselves to very special perturbations of the equations: The relations between the equations $f_i(\underline{x})$ have to lift to some relations (depending on t) between the perturbed equations $f_i(\underline{x}, \underline{t})$. In particular, this problem does not arise for complete intersections Y; if Y is given by $\operatorname{codim}_{\mathbb{C}^N}(Y)$ equations only, then these equations do not admit any non-trivial relation at all. The corresponding algebraic notion for this concept is that of flatness.

Definition: Let Y be an algebraic variety. A *deformation of* Y (with total space X and parameter space $S \ni 0$) is a flat map $\varphi : X \to S$ combined with an isomorphism $Y \xrightarrow{\sim} \varphi^{-1}(0)$.

Remark: If $S = \mathbb{C}^m$, then flatness of a map $\varphi : X \to \mathbb{C}^m$ is equivalent to the fact that the special fiber $Y \subseteq X$ is a relatively complete intersection in X. That means, the ideal of Y in X is generated from a regular sequence, i.e. from as many equations as the codimension from Y in X.
Attention: Do not mistake this property for the property of Y being a complete intersection in some $\mathbb{C}^N$!

(1.4) The deformation theory of complete intersections is relatively simple. However, affine toric varieties (which we are interested in) are mostly far away from having this property. Therefore, it is useful to have the following tool for constructing deformations of general varieties:

Let Y be an affine algebraic variety. Assume that we are given another affine variety X such that

- X contains Y and
- $Y \overset{i}{\hookrightarrow} X$ is defined by a regular sequence $f_1, \ldots, f_m \in H^0(X, \mathcal{O}_X)$.

Then, we obtain:

(i) $f = (f_1, \ldots, f_m)$ induces a flat map $f : X \to \mathbb{C}^m$ with special fiber Y, i.e. a deformation of Y with total space X. We will call this the *standard deformation* corresponding to the regular sequence f.

(ii) Perturbing the equations $f_1, \ldots, f_m$ over a parameter space S in an *arbitrary way* (we obtain $\tilde{f}_1, \ldots, \tilde{f}_m$) yields a so-called *relative deformation* of Y inside X:

$$\left[\begin{array}{ccc} Y & \hookrightarrow & \tilde{Y} \\ \downarrow & \otimes & \downarrow \\ \{0\} & \hookrightarrow & S \end{array}\right] \subseteq \left[\begin{array}{ccc} X & \overset{\text{trivial}}{\hookrightarrow} & X \times S \\ \downarrow & \otimes & \downarrow \ \mathrm{pr}_S \\ \{0\} & \hookrightarrow & S \end{array}\right]$$

The inclusion between both diagrams is given by the map $i : Y \hookrightarrow X$ and the inclusion $\tilde{Y} \subset X \times S$ defined by $\tilde{f}$, respectively.

Relative deformations of Y inside X are comparable with deformations of complete intersections in $\mathbb{C}^m$ – in particular, they are well understood. Therefore, the crucial point is to find sufficiently many varieties X containing Y as a relatively complete intersection.

(1.5) **Definition:** Let X be an affine toric variety. $f_1, \ldots, f_m \in H^0(X, \mathcal{O}_X)$ will be called a *toric regular sequence* iff

(i) $f_1, \ldots, f_m$ are binomials in $H^0(X, \mathcal{O}_X)$ and

(ii) $Y := [f_1 = \ldots = f_m = 0] \subseteq X$ is an affine toric variety of codimension m in X.

(In particular, $f_1, \ldots, f_m$ form a regular sequence in X.)

Remark: Let the affine toric varieties X and Y be given by cones σ, $\bar{\sigma}$ which are contained in real vector spaces $N_{\mathbb{R}}$ and $\bar{N}_{\mathbb{R}}$, respectively. If $f_i = x^{r_0^i} - x^{r_1^i}$ $(i = 1, \ldots, m)$, then $\bar{N}_{\mathbb{R}}$ can be regarded as the subspace of $N_{\mathbb{R}}$ that is defined by the equations $r_0^1 - r_1^1 = \ldots = r_0^m - r_1^m = 0$. Moreover, $\bar{\sigma}$ is equal to the intersection $\bar{\sigma} = \sigma \cap \bar{N}_{\mathbb{R}}$.

2 The cone over the rational normal curve of degree four

(2.1) The image of $\mathbb{P}^1_{\mathbb{C}}$ in $\mathbb{P}^4_{\mathbb{C}}$ under the Veronese map

$$(x_0 : x_1) \mapsto (x_0^4 : x_0^3 x_1 : x_0^2 x_1^2 : x_0 x_1^3 : x_1^4)$$

is called the rational normal curve of degree 4. Let $Y \subseteq \mathbb{C}^5$ be the cone over this curve; it can be described in at least three different ways:

(i) Y is given by the equations

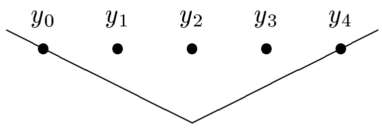
$$\operatorname{rank} \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} = 1;$$

(ii) Y is a cyclic quotient singularity obtained by dividing $\mathbb{C}^2$ by the $\mathbb{Z}/_{4\,\mathbb{Z}}$-action $1 \mapsto \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$;

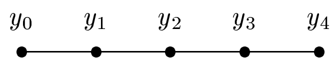
(iii) Y is the affine toric variety defined by the two-dimensional cones

$$\bar{\sigma} = \langle(-1, 2); (1, 2)\rangle \quad \text{or} \quad \bar{\sigma}^\vee = \langle[2, 1]; [-2, 1]\rangle.$$

The semigroup $\bar{\sigma}^\vee \cap \mathbb{Z}^2$ is generated by the points $[-2, 1]$, $[-1, 1]$, $[0, 1]$, $[1, 1]$, and $[2, 1]$ which correspond to the variables $y_0, \ldots, y_4$, respectively.



$\bar{\sigma}^\vee$



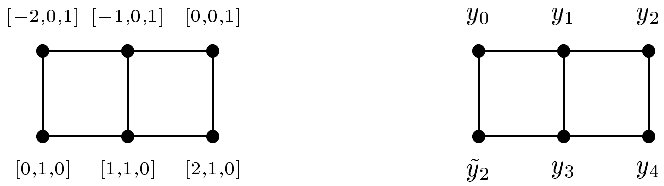
affine slice of $\bar{\sigma}^\vee$

(2.2) Let $X \subseteq \mathbb{C}^6$ be given by the equations

$$\operatorname{rank}\begin{pmatrix} y_0 & y_1 & [\tilde{y_2} := y_2 + t] & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} = 1.$$

Then, X is a three-dimensional toric variety described by the cones

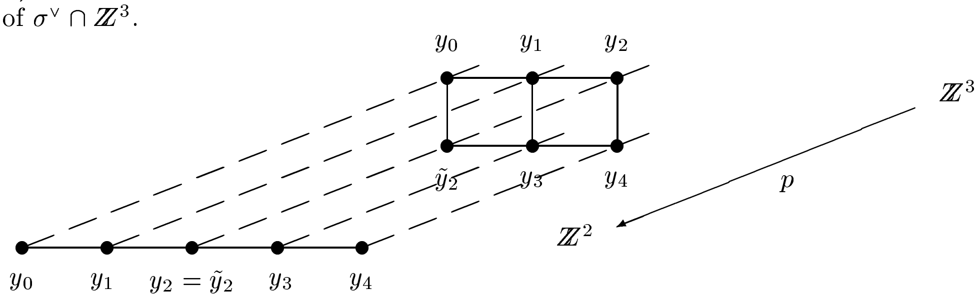
$$\sigma = \langle (-1,2,0);\, (0,0,1);\, (1,0,2);\, (0,1,0)\rangle \quad \text{and} \quad \sigma^\vee = \langle [0,0,1];\, [-2,0,1];\, [0,1,0];\, [2,1,0]\rangle.$$



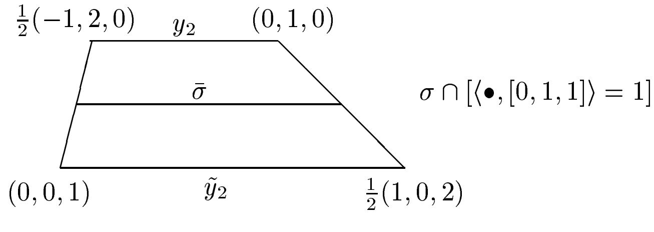
affine slice of the three-dimensional $\sigma^\vee$

The equation $t = 0$ (or equivalently $y_2 = \tilde{y_2}$) defines a closed subvariety in X which is equal to Y defined in (2.1). Since this special fiber is of codimension one in X, we have found an embedding of Y into X as a relatively complete intersection. In particular, according to the definition (1.5), $y_2 - \tilde{y_2}$ is a toric regular sequence of length one in X.

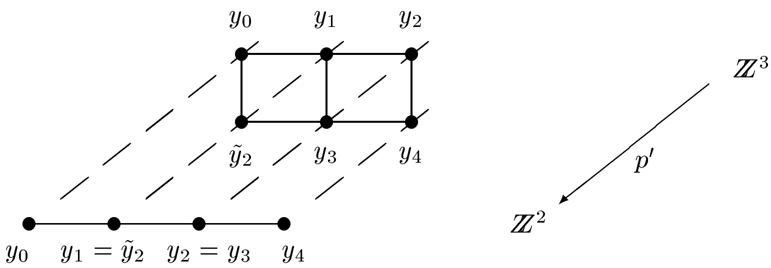
(2.3) On the level of regular functions, the closed embedding $Y \hookrightarrow X$ can be obtained by identifying the two variables y_2 and $\tilde{y_2}$ corresponding to the lattice points $[0,0,1]$ and $[0,1,0]$, respectively. This map is equivariant; the embedding of the corresponding tori $Y \cap (\mathbb{C}^*)^5 \hookrightarrow X \cap (\mathbb{C}^*)^6$ can be described by the group homomorphism $p : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2$ defined by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. The vector $[0,-1,1] = [0,0,1] - [0,1,0]$ spans the kernel of p, and $\bar{\sigma}^\vee \cap \mathbb{Z}^2$ equals the image of $\sigma^\vee \cap \mathbb{Z}^3$.



(2.4) Let us describe the situation on the *dual level*: Via the injective group homomorphism $i : \mathbb{Z}^2 \hookrightarrow \mathbb{Z}^3$ defined by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$, the cone $\bar{\sigma}$ equals $\sigma \cap \mathbb{R}^2$. Since $[0,1,1] = [0,0,1] + [0,1,0]$, the facets corresponding to $y_2 = [0,0,1]$ and $\tilde{y_2} = [0,1,0]$ become parallel in the affine slice of σ defined by $\langle \bullet,\, [0,1,1]\rangle = 1$. Moreover, in our example, $\bar{\sigma}$ equals the arithmetical mean of these two facets.



(2.5) *Forbidden projections $p : \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2$:* An essential property of the projection p described above is, roughly speaking, the following one: The dashed projecting lines drawn in (2.3) meet more than one lattice point in a single case only. (In fact, there are many such cases not drawn in our figure. However, in a certain sense, they are directly induced by the line connecting y_2 and $\tilde{y_2}$.)
Let us regard an example not admitting this property:



The projection p' defines another closed subvariety $Y' \subseteq X$; Y' is isomorphic to the cone over the rational normal curve of degree 3. The two projecting lines meeting the lattice points $y_1, \tilde{y_2}$ and y_2, y_3 correspond to two equations $y_1 = \tilde{y_2}$ and $y_2 = y_3$ of Y' in X, respectively. In particular, Y' is not a relatively complete intersection in X.
Our main theorem will describe conditions for $\bar{\sigma}, \sigma$ and p that, in the homogeneous case (cf. (3.4)(2)), are necessary and sufficient for making Y a relatively complete intersection in X.

3 Deformation elements

Let $(\boldsymbol{A}, \mathbb{L})$ be a k-dimensional real vector space with lattice.

(3.1) **Definition:** A *deformation element* of size m is a tuple $(R_0, \ldots, R_m;\ C)$ admitting the following properties:

(i) $C \subseteq \boldsymbol{A}$ is a rational polyhedral cone with apex, and

(ii) $R_0, \ldots, R_m \subseteq \boldsymbol{A}$ are rational polyhedra with cone C.

The deformation element is called *admissible* if for each $t \in C^\vee \subseteq \boldsymbol{A}^*$ at least m of the $m+1$ faces

$$F(R_i, t) := \{a \in R_i |\ \langle a,\, t\rangle = \operatorname{Min}\langle R_i,\, t\rangle\}$$

of R_i $(i = 0, \ldots, m)$ contain lattice points.

(3.2) Starting with a given deformation element $(R_0, \ldots, R_m;\ C)$, we will construct affine toric varieties X, Y such that $Y \subseteq X$ is defined by a toric regular sequence (cf. (1.5)).

(3.2.1) Define the polyhedron Q as the Minkowski sum

$$Q := R_0 + \ldots + R_m \subseteq \boldsymbol{A}.$$

On the other hand, we embed the whole space as an affine hyperplane in a higher-dimensional space:

- $\bar{N}_{\mathbb{R}} := \boldsymbol{A} \times \mathbb{R}$ is a vector space containing the lattice $\bar{N} := \mathbb{L} \times \mathbb{Z}$ ($\bar{M}_{\mathbb{R}} := \bar{N}^*_{\mathbb{R}}$, $\bar{M} := \bar{N}^*$);
- $\psi : \boldsymbol{A} \hookrightarrow \bar{N}_{\mathbb{R}};\quad a \mapsto (a, 1)$.

In particular, Q turns out to be a polyhedron in $\bar{N}_{I\!R}$ via $Q := \psi(Q)$. We define $Y := \operatorname{Spec} \mathbb{C}[\bar{\sigma}^\vee \cap \bar{M}]$ as the $(k+1)$-dimensional affine toric variety that is given by the cone

$$\bar{\sigma} := \overline{I\!R_{\geq 0} \cdot Q} = (C \times \{0\}) \cup I\!R_{\geq 0} \cdot Q$$

contained in $(\bar{N}_{I\!R}, \bar{N})$.

(3.2.2) To define σ, we put the polyhedra $R_0, \ldots, R_m$ into parallel affine planes of a vector space that is big enough:

- $N_{I\!R} := \mathbb{A} \times I\!R^{m+1}$, $N := I\!L \times \mathbb{Z}^{m+1}$; $M_{I\!R} := N_{I\!R}^*$, $M := N^*$. Denote by $\Phi : N_{I\!R} \longrightarrow I\!R^{m+1}$ the projection to the second factor.
- $\phi_i : \mathbb{A} \hookrightarrow N_{I\!R}$; $a \mapsto (a, e^i)$ for $i = 0, \ldots, m$ ($e^0, \ldots, e^m$ denotes the standard basis of $\mathbb{Z}^{m+1}$);
- $\tilde{R}_i := \phi_i(R_i) \subseteq \Phi^{-1}(e^i) \quad (i = 0, \ldots, m)$.

Now, we denote by P the convex hull $P := \operatorname{conv}\left(\bigcup_{i=0}^m \tilde{R}_i\right)$ and define $X := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ as the affine toric variety given by the cone

$$\sigma := \overline{I\!R_{\geq 0} \cdot P} = (C \times \{\underline{0}\}) \cup I\!R_{\geq 0} \cdot P.$$

In particular, $\dim X = (k+1) + m$.

(3.2.3) If $pr_i : I\!R^{m+1} \longrightarrow I\!R$ denotes the projection on the i-th factor, we can define linear maps $r^0, \ldots, r^m : N \longrightarrow \mathbb{Z}$ by $r^i := pr_i \circ \Phi$. They correspond to elements $r^i \in \sigma^\vee \cap M$.

(3.2.4) $\bar{N}$ can be considered as a sublattice of N via the inclusion map

$$\bar{N} \hookrightarrow N; \quad (a; 1) \mapsto (a; 1, \ldots, 1).$$

This embedding admits the following properties:

(i) $\bar{N} = N \cap \bigcap_{i,j} (r^i - r^j)^\perp = N \cap \bigcap_{i=1}^m (r^i - r^0)^\perp$

(ii) $\bar{\sigma} = \sigma \cap \bar{N}_{I\!R}$.

In particular, we obtain a map $Y \to X$ which sends Y into the special fiber of the morphism $X \to \mathbb{C}^m$ defined by the regular functions $x^{r^1} - x^{r^0}, \ldots, x^{r^m} - x^{r^0} \in \mathbb{C}[\overset{\vee}{\sigma} \cap M]$.

(3.3) **Theorem:** (cf. [Al 2])

(1) If $(R_0, \ldots, R_m;\ C)$ is an admissible deformation element, then $Y \to X$ is a closed embedding, and $Y \subseteq X$ is defined by the m binomials $x^{r^1} - x^{r^0}, \ldots, x^{r^m} - x^{r^0}$ which form a toric regular sequence.

(2) All toric regular sequences $f_1, \ldots, f_m$ with
   (i) $f_i = x^{r^i} - x^{r^0}$ for $i = 1, \ldots, m$ (i.e. only $m+1$ monomials are involved),
   (ii) $r^0, \ldots, r^m$ are primitive lattice points

   can be obtained from deformation elements by the previous construction.

(3.4) **Remarks:**

(1) In the second part of the previous theorem, the condition (ii) can be dropped by regarding a slightly generalized notion of "deformation elements" (an additionally parameter $p \in I\!N$ has to be involved).

(2) Let Y be given by a cone $\bar{\sigma} \subset \bar{N}_{\mathbb{R}}$; let $\bar{M}, \bar{N}$ be the corresponding pair of dual lattices. Then, the vector space T^1_Y of infinitesimal deformations of Y is $\bar{M}$-graded. Now, condition (i) is essentially equivalent to the fact that the Kodaira-Spencer-map sends our regular sequence into one homogeneous piece of T^1_Y only. Hence, toric regular sequences with property (i) will be called homogeneous.

(3) Given a deformation element, the corresponding degree equals $-[\underline{0}, 1] \in \mathbb{L}^* \times \mathbb{Z} = \bar{M}$. It is the negative of the common image of $r^0, \dots, r^m$ via the surjective map $M \longrightarrow\!\!\!\!\rightarrow \bar{M}$.
In particular, if Y is given by a cone $\bar{\sigma}$, we can find the homogeneous toric regular sequences of degree $-\bar{r} \in \bar{M}$ ($\bar{r}$ = primitive) by looking for admissible deformation elements such that $R_0 + \dots + R_m = \bar{\sigma} \cap [\langle \bullet, \bar{r} \rangle = 1]$. Deformations of non-primitive degree arise from the generalized deformation elements mentioned in (1).

(4) Toric regular sequences of length one (yielding 1-parameter-deformations) are always homogeneous.

(5) The vector space T^1_Y and the Kodaira-Spencer-map corresponding to a given deformation element can be computed explicitly, cf. [Al 1].

(6) If we are dealing with projective toric varieties Z defined by a lattice polytope P, then the cone $\tau^\vee := \mathrm{cone}(P)$ corresponds to an affine toric variety which is called the cone over Z. Starting with the cone τ, we can recover our polyhedron P as an affine slice of $\tau^\vee$. In this context, it is well known that Minkowski sums appear in the theory of toric varieties; they correspond to tensor products of invertible sheaves on Z.
Now, the surprising point of the previous theorem is that Minkowski sums occur in connection with affine slices of the cone $\bar{\sigma}$ by itself (and not of the dual one).

For more details and for the proofs of the previous theorem and remarks we refer to [Al 1] and [Al 2].

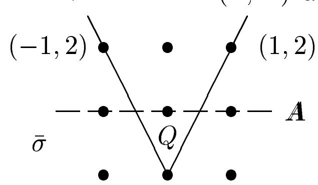
4 Back to the (-4)

(4.1) Let us return to the example of §2. Y is a two-dimensional singularity admitting a resolution graph that consists of a single rational curve only. The self intersection number (which can be defined as the degree of the normal bundle and reflects the type of embedding of this $\mathbb{P}^1$−copy) is equal to (-4). This yields the short name for Y used in the title of this section.

(4.2) Y is given by the cones $\bar{\sigma} = \langle (-1, 2); (1, 2) \rangle$ or $\bar{\sigma}^\vee = \langle [2, 1]; [-2, 1] \rangle$. The vector space T^1_Y can be computed; it is four-dimensional and splits into homogeneous pieces of degree $-[-1, 1], -[1, 1]$ (dimension 1), and $-[0, 1]$ (dimension 2). The latter one seems to be the most interesting one, and we start with looking for toric regular sequences of this special degree.

(4.2.1) Following the algorithm described in (3.4)(3), we have to define $(\mathbb{A}, \mathbb{L})$ and Q as

$$\begin{array}{rcl} \mathbb{A} & := & \{a \in \mathbb{R}^2 \,|\, \langle a, [0, 1] \rangle = 1\}\,, \\ \mathbb{L} & := & \{a \in \mathbb{Z}^2 \,|\, \langle a, [0, 1] \rangle = 1\}\,, \text{ and} \\ Q & := & \bar{\sigma} \cap \mathbb{A}. \end{array}$$



The pair $(\mathbb{A}, \mathbb{L})$ can be identified with $(\mathbb{R}, \mathbb{Z})$ $((a, 1) \mapsto a)$. Then, the line segment Q between $\frac{1}{2}(-1, 2) = (-\frac{1}{2}, 1)$ and $\frac{1}{2}(1, 2) = (\frac{1}{2}, 1)$ corresponds to the closed interval $[-\frac{1}{2}, \frac{1}{2}] \subseteq \mathbb{R}$. Now, the one-dimensional polyhedron Q ($C=0$) can be split into

$$[-\frac{1}{2},\frac{1}{2}]=[-\frac{1}{2},0]+[0,\frac{1}{2}].$$

This yields an admissible deformation element ($0\in\mathbb{Z}!$), and we can use (3.2.2) to construct the cone σ which defines X:

$$N_{\mathbb{R}}=\mathbb{R}^3;\quad \sigma=\langle\underbrace{(-\frac{1}{2};1,0),(0;1,0)}_{R_0},\underbrace{(0;0,1),(\frac{1}{2};0,1)}_{R_1}\rangle\ ,$$

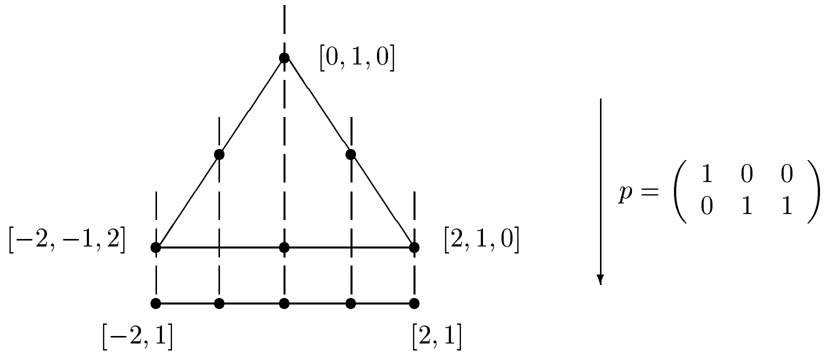
i.e. σ equals the cone of Example (2.2).

(4.2.2) The interval Q admits a second non-trivial decomposition into an admissible deformation element

$$[-\frac{1}{2},\frac{1}{2}]=\{-\frac{1}{2}\}+[0,1].$$

This shows that Y can be embedded as a hypersurface into a three-dimensional cyclic quotient singularity X'. The cone defining X' can be written as

$$\sigma'=\langle\underbrace{(-1,2,0)}_{R_0},\underbrace{(0,0,1),(1,0,1)}_{R_1}\rangle\quad\text{or } \sigma'^{\vee}=\langle[2,1,0],[0,1,0],[-2,-1,2]\rangle.$$



The relations between the lattice points of $\sigma'^{\vee}$ show that the affine variety $X'\subseteq\mathbb{C}^6$ can be defined by the equations

$$\operatorname{rank}\begin{pmatrix}y_0&y_1&y_2\\y_1&\tilde{y_2}&y_3\\y_2&y_3&y_4\end{pmatrix}=1.$$

Again, the specialization $y_2=\tilde{y_2}$ induces equations of $Y\subseteq\mathbb{C}^5$. However, these equations are different from those of (2.1)(i) ($y_0\,y_4-y_1\,y_3$ is replaced by $y_0\,y_4-y_2^2$); they just generate the same ideal.

(4.2.3) In the remaining two interesting degrees, $-[-1,1]$ and $-[1,1]$, we obtain the closed intervals

$$Q_{[-1,1]}=[-\frac{1}{3},1]\quad\text{and}\quad Q_{[1,1]}=[-1,\frac{1}{3}],$$

respectively. Both intervals admit only one non-trivial admissible Minkowski decomposition each; it is induced by the interior point $0\in\mathbb{Z}$. The corresponding toric regular sequences can be combined to get one of length two, which is no longer homogeneous. It can be written as $\{y_1-\tilde{y_1},y_3-\tilde{y_3}\}$ in the affine variety defined by

$$\operatorname{rank}\begin{pmatrix}y_0&\tilde{y_1}&y_2&\tilde{y_3}\\y_1&y_2&y_3&y_4\end{pmatrix}=1.$$

The induced equations for Y are equal to those of (2.1)(i).

5 Semiuniversal deformations

(5.1) Some deformations can be induced from other ones by substituting parameters, i.e. by making a base change. For example, the perturbation $f(\underline{x}) + (t_1^2 + t_2^3)\, g(\underline{x})$ can be obtained from $f(\underline{x}) + t\, g(\underline{x})$ by $t := t_1^2 + t_2^3$. Since neither new information, nor new fibers Y_t can arise during this process, we can restrict our searching for "all possible" deformations of a given variety to the looking for selected deformations inducing all other ones via base change.

(5.2) Let Y be an isolated singularity.

Theorem: There exists a deformation $\varphi : X \to S$ of Y such that

(i) each other deformation $\varphi' : X' \to S'$ can be induced from φ by a map $s : S' \to S$ via base change, and

(ii) on the level of tangent spaces, the map s is uniquely determined.

$\varphi : X \to S$ is uniquely determined by the conditions (i) and (ii). It is called the *semiuniversal (s.u.) deformation* of Y. The proof was given in [Gr] and [Ar] for the analytic and the algebraic case, respectively.

Remark:

(1) The second condition in the definition of a s.u. deformation is equivalent to asking S for having minimal dimension.

(2) The base space S is a powerful invariant for the given singularity Y. Information such as the dimension or the number of irreducible components of S is of high interest. The components (at least of S_{red}) correspond to the intuitive notion of "different possibilities of deforming Y" mentioned in (1.1).

(5.3) **Examples:**

(1) Let $Y = [f(\underline{x}) = 0] \subseteq \mathbb{C}^N$ be a hypersurface. The fact that deformations of Y arise from arbitrary perturbations of f means smoothness of the s.u. base space S. To construct the whole s.u. deformation, let $g_1, \dots, g_\tau$ be a basis of the τ-dimensional $\mathbb{C}$-vector space

$$\mathbb{C}[\underline{x}] \Big/ \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N}\right).$$

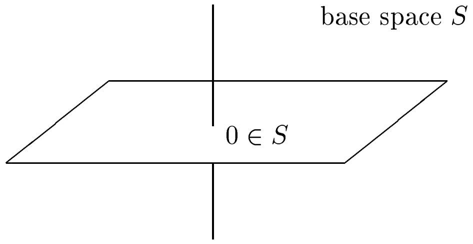
Then, we define $X := [f(\underline{x}) + \sum_j t_j \cdot g_j(\underline{x}) = 0] \subseteq \mathbb{C}^N(\underline{x}) \times \mathbb{C}^\tau(\underline{t})$ and $S := \mathbb{C}^\tau$ with the natural projection map $\varphi : X \to S$.

(2) Let Y be the cone over the rational normal curve of degree four (cf. §2 and §4). Its versal deformation was computed by Pinkham (cf. [Pi]); Y was the first example showing that the versal base space might be reducible. S consists of two smooth components of dimensions one and three; the total spaces over these components can be described by the equations

$$\operatorname{rank} \begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & \tilde{y_2} & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} = 1 \quad \text{and} \quad \operatorname{rank} \begin{pmatrix} y_0 & \tilde{y_1} & \tilde{y_2} & \tilde{y_3} \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} = 1,$$

respectively. We have already seen these deformations in (4.2.2) and (4.2.1)/ (4.2.3).

(3) Let Y be the cone over the Del Pezzo surface of degree six (arising by blowing up three points of $\mathbb{P}^2$). Then, as in the previous example, the s.u. base space S consists of two smooth components - but of dimensions one and two.



(5.4) Let us return to affine toric varieties Y.

Definition: A deformation $\varphi : X \to S$ of Y is called *toric* if

(i) Both X and the embedding $i : Y \hookrightarrow X$ belong to the category of affine toric varieties, and

(ii) i(closed T_Y-orbit of Y) = closed T_X-orbit of X.

This definition seems to be a natural one, if we work inside the category of toric varieties. (Maybe, we could replace (ii) by the weaker assumption i(closed T_Y-orbit of Y) $\subseteq$ (closed T_X-orbit of X) – but this yields some additional $(\mathbb{C})^*$-factors for X only.) On the other hand, for isolated singularities, we even conjecture that the restrictions of the semiuniversal deformation of Y to the components of the reduced base space are toric deformations. This was proved for $\dim Y = 2$ by J. Christophersen, cf. [Ch].

Theorem: Let $\varphi : X \to S$ be a toric deformation of Y. Then,

(i) S is smooth,

(ii) $Y \hookrightarrow X$ is given by a toric regular sequence, and

(iii) the deformation $\varphi : X \to S$ can be regarded as a relative deformation of Y inside X.

(For the proof cf. [Al 2].)

Since relative deformations are well understood (cf (1.4)), this attributes the study of toric deformations (and, maybe, the study of the components of the s.u. base space) to the investigation of standard toric deformations corresponding to toric regular sequences.

(5.5) Finally, we return to the last two examples of (5.3).

(2') The three-dimensional component (the so-called Artin component) of S sits in three different degrees $-[-1,1], -[0,1]$, and $-[1,1]$. Therefore, it is indeed possible to describe the corresponding total space X by writing down an explicit five-dimensional cone. However, it would be much nicer having a direct way to recognize that the three Minkowski decompositions

$$[-\frac{1}{3}, 1] = [-\frac{1}{3}, 0] + [0, 1], \quad [-\frac{1}{2}, \frac{1}{2}] = [-\frac{1}{2}, 0] + [0, \frac{1}{2}], \quad \text{and}$$

$$[-1, \frac{1}{3}] = [-1, 0] + [0, \frac{1}{3}]$$

(cf. (4.2)) fit together in a common component of S.
Both components meet the degree $-[0,1]$. Here, the two Minkowski decompositions of Q

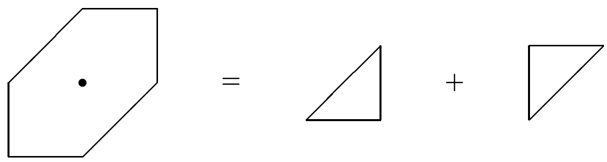
$$[-\frac{1}{2},\frac{1}{2}] = \{-\frac{1}{2}\} + [0,1] \quad \text{and} \quad [-\frac{1}{2},\frac{1}{2}] = [-\frac{1}{2},0] + [0,\frac{1}{2}]$$

are directly comparable: The fact that these decompositions do not have a common refinement reflects that both 1-parameter deformations do not fit together in a common 2-parameter deformation.

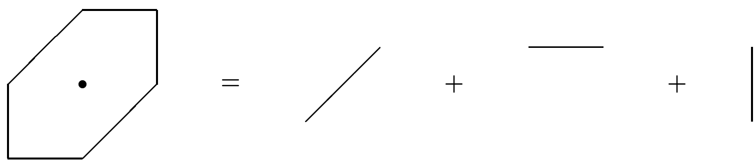
(3') This 3-dimensional example is an affine toric variety, too. It can be given by $\bar{\sigma} = \langle(0,0;1), (1,0;1), (2,1;1), (2,2;1), (1,2;1), (0,1;1)\rangle$ which equals the cone over the affine hexagon

$$Q = \operatorname{conv}((0,0),(1,0),(2,1),(2,2),(1,2),(0,1)).$$

The whole deformation theory of this variety is concentrated in the only degree $-[0,0;1]$; the corresponding affine slice $\bar{\sigma} \cap \langle \bullet, [0,0;1]\rangle = 1$ yields our hexagon again. Therefore, it is not a surprise that the splitting of S into two irreducible components corresponds to the existence of two different Minkowski decompositions of our hexagon Q:



$$Q = \operatorname{conv}((0,0),(1,0),(1,1)) + \operatorname{conv}((0,0),(0,1),(1,1)) \qquad \text{and}$$



$$Q = \operatorname{conv}((0,0),(1,0)) + \operatorname{conv}((0,0),(0,1)) + \operatorname{conv}((0,0),(1,1)).$$

In [Al 3] we have computed the semi-universal deformation of isolated toric Gorenstein singularities – a class of singularities the cone over the Del Pezzo surface belongs to. Its components arise in exactly the same way as it happened in the previous example.

References

[Al 1] Altmann, K.: Computation of the vector space T^1 for affine toric varieties. J. Pure Appl. Algebra **95** (1994), 239-259.

[Al 2] Altmann, K.: Minkowski sums and homogeneous deformations of toric varieties. Tôhoku Math. J. **47** (1995), 151-184.

[Al 3] Altmann, K.: The versal deformation of an isolated toric Gorenstein singularity. Invent. math. **128**, 443-479 (1997).

[Ar] Artin, Michael: Lectures on deformations of singularities. Bombay: Tata Institute of Fundamental Research, 1976.

[Ch] Christophersen, J.A.: Obstruction spaces for rational singularities and deformations of cyclic quotients. Thesis, University of Oslo, 1989/90.

[Gr] Grauert, H.: Über die Deformationen isolierter Singularitäten analytischer Mengen. Invent. Math. **15** (1972), 171-198.

[Pi] Pinkham, H.: Deformations of algebraic varieties with G_m-action. Asterisque **20** (1974), 1-131.