Minkowski sums and homogeneous deformations of toric varieties

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Abstract

We investigate those deformations of affine toric varieties (toric singularities) that arise from embedding them into higher dimensional toric varieties as a relative complete intersection. On the one hand, many examples promise that these so-called toric deformations cover a great deal of the entire deformation theory. On the other hand, they can be described explicitly. Toric deformations are related to decompositions (into a Minkowski sum) of cross cuts of the polyhedral cone defining the toric singularity. Finally, we consider the special case of toric Gorenstein singularities. Many of them turn out to be rigid; for the remaining examples the description of their toric deformations becomes easier than in the general case.

1 Introduction

(1.1) We want to investigate germs of complex, algebraic singularities Y by describing their deformation theory. At least for isolated singularities there exists the so-called (mini-) versal deformation which induces all other ones by specialization of parameters. This flat family carries much information about the original singularity. It is a source for many numerical invariants.

(1.2) If Y is a complete intersection, then each perturbation of the defining equations defines a deformation of Y. In particular, its versal base space S_Y is smooth (and the dimension is well known). On the other hand, if $Y \subseteq \mathbb{C}^w$ is given by more equations than its codimension, then the relations among these equations may cause obstructions in creating deformations of Y. Even if Y is still Gorenstein, the versal base space might consist of several components, or it might be non-reduced.

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(1.3) **Example:** Let $Y \subseteq \mathbb{C}^5$ be the cone over the rational normal curve of degree four. Y is $\mathbb{Q}$-Gorenstein, and it is given by the equations

$$\operatorname{rank} \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \leq 1\,.$$

Destroying the symmetry by introducing three parameters $t = (t_1, t_2, t_3)$ induces a flat family $Y_t \to \mathbb{C}^3$ defined by

$$\operatorname{rank} \begin{pmatrix} y_0 & y_1 + t_1 & y_2 + t_2 & y_3 + t_3 \\ y_1 & y_2 & y_3 & y_4 \end{pmatrix} \leq 1\,.$$

On the other hand, Y could be defined by the equations

$$\operatorname{rank} \begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} \leq 1\,,$$

too. They provide a one-parameter deformation $Y_s \to \mathbb{C}$ via

$$\operatorname{rank} \begin{pmatrix} y_0 & y_1 & y_2 \\ y_1 & y_2 + s & y_3 \\ y_2 & y_3 & y_4 \end{pmatrix} \leq 1\,.$$

The versal deformation of Y equals the union of these two families. Its base space is the union of a hyperplane and a line in $\mathbb{C}^4$. (In particular, it is not possible to find any flat family over a smooth parameter space containing both deformations $Y_t \to \mathbb{C}^3$ and $Y_s \to \mathbb{C}$.)

(1.4) The previous example has been taken from the class of two-dimensional cyclic quotient singularities. They are among the easiest singularities beyond complete intersections. Many of them admit non-smooth or even non-reduced versal base spaces. Starting with Riemenschneider [Ri], many people have investigated the deformation theory of two-dimensional cyclic quotients: In [Ar] the versal family was described by explicit equations. Then, based on results of Kollár and Shephard-Barron using three-dimensional geometry, Christophersen and Stevens obtained a qualitative description at least of the reduced structure of the versal base space. Its components are smooth, they correspond one-to-one to so-called P-resolutions of the singularity, and their number can be obtained by a combinatorial approach using continued fractions (cf. [KS], [Ch 2], [St 1]).

(1.5) The class of two-dimensional cyclic quotient singularities coincides with the class of two-dimensional affine toric varieties (cf. [Od]). Moreover, as a by-product of his computations, Christophersen observed that (at least after some finite base change) the total spaces over the components of the reduced versal base space are toric varieties, too. Immediately, the following questions arise:

(1) Is it possible to generalize this result to higher dimensional toric varieties?

(2) How can the toric varieties occuring as total spaces over components (or their defining rational polyhedral cones) be obtained by combinatorial methods?

(3) What is the versal deformation of a toric variety?

(1.6) In the present paper we address the second question. Let Y be an affine toric variety. A deformation of Y is said to be toric, if the total space X together with the embedding of the special fiber Y are contained in the category of toric varieties. (Cf. Definition (2.1); see [KPR] and [Od] for basic facts about deformation theory and toric varieties, respectively.) Then, we have obtained the following results:

(i) If X is the total space of a toric deformation of Y, then $Y \hookrightarrow X$ can be defined as the zero set of a binomial regular sequence (cf. Proposition (2.3)). In particular, Y is a relative complete intersection in X. On the other hand, those regular sequences induce flat maps from X to some affine space $\mathbb{C}^m$ (with Y as special fiber). These deformations can be considered to be more or less equivalent to the original ones; at least they are induced from the same component of the versal deformation (cf. (2.4) and (2.5)).

(ii) There is an elementary geometric construction producing all possibilities to embed Y equivariantly as a hypersurface into some affine toric variety X (yielding one-parameter toric deformations of Y). Roughly speaking, this construction works as follows:

- Let $\bar{\sigma}$ denote the rational, polyhedral cone defining Y. Choose some affine hyperplane intersecting $\bar{\sigma}$ in a polyhedron Q. (We can get $\bar{\sigma}$ back by taking the cone over Q.)
- Take a decomposition $Q = R_0 + R_1$ of Q into Minkowski summands meeting certain properties, put R_0 and R_1 into parallel affine hyperplanes of some larger affine space, and define P as the convex hull of $R_0 \cup R_1$ (see Figure 1). Then, $\dim P = \dim Q + 1$, and the cone over P (call it σ) provides the toric variety X.
- We find Q back as the intersection of P with the hyperplane sitting between those containing R_0 and R_1, respectively. In particular, we can embed $\bar{\sigma}$ into σ obtaining the inclusion $Y \hookrightarrow X$.

(See §3. Actually we will discuss the more general case of so-called homogeneous toric deformations (cf. Definition (3.1)).)

(iii) The Kodaira-Spencer map of the toric deformations obtained in (ii) is computed in §5.

(1.7) In §6 we focus on the particular case of isolated, toric $\mathbb{Q}$-Gorenstein singularities of dimension at least three. Using infinitesimal calculations, we first obtain that they will be rigid,

- if they are not Gorenstein, or
- if they are at least four-dimensional.

For three-dimensional, isolated, toric Gorenstein singularities (given by some lattice polygon Q) the general theory developed so far becomes easier. Non-trivial deformations exist in one single degree only; hence, the homogeneous ones cover the whole theory. They correspond to decompositions of Q into Minkowski sums of other lattice polygons. (As will be proved in a forthcoming paper, all components of the versal deformation can be seen in that way.)

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2 Toric deformations

(2.1) Let Y be an affine toric variety over $\mathbb{C}$. We want to investigate deformations of Y with toric total space. For the notion of toric varieties see [Od].

Definition: A deformation of Y, i.e. a flat map $f : X \to S$ with an isomorphism $Y \stackrel{\sim}{\to} f^{-1}(0)$ $(0 \in S)$, is said to be *toric*, if

(i) X is an affine, toric variety,

(ii) $i : Y \stackrel{\sim}{\to} f^{-1}(0) \hookrightarrow X$ is a morphism in the category of toric varieties, i.e. it induces an algebraic group homomorphism $T_Y \hookrightarrow T_X$ between the embedded tori which makes i equivariant, and

(iii) i sends the closed T_Y-orbit in Y isomorphically onto the closed T_X-orbit in X.

Remark: The rather technical condition (iii) could be replaced by the weaker (and perhaps more natural) one that asks for mapping the closed T_Y-orbit in Y *into* the closed T_X-orbit in X. This would not essentially change the notion of a toric deformation, but the actual version of (iii) makes the theory more convenient. In most applications the closed orbits under the torus action are points, anyway. Since we are actually interested in the germs of Y, X, and S only, the condition (iii) arises quite naturally then.

Example: Both families $Y_t \to \mathbb{C}^3$ and $Y_s \to \mathbb{C}$ of (1.3) are toric deformations of the cone over the rational normal curve of degree four.

(2.2) For X and Y as before we introduce the following notation:

- Let M, N be n-dimensional, mutually dual lattices; let σ be a rational polyhedral cone in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ that does not contain any linear subspace. These objects are used to build $X = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$. ($\sigma^\vee$ denotes the dual cone of σ.)
- Analogously, we use $\bar{M}$, $\bar{N}$ (of dimension $n - m$), and $\bar{\sigma}$ to obtain $Y = \operatorname{Spec} \mathbb{C}[\bar{\sigma}^\vee \cap \bar{M}]$.

Let Y and $f : X \to S$ meet the conditions (i) and (ii) of Definition (2.1). Then, the equivariant closed embedding $i : Y \hookrightarrow X$ corresponds to an embedding

$$i : \bar{N} \hookrightarrow N$$

on the level of lattices. The dual map i^* induces a surjection

$$i^* : \sigma^\vee \cap M \longrightarrow\!\!\!\!\!\rightarrow \bar{\sigma}^\vee \cap \bar{M}$$

even between the semigroups. The kernel $L := \ker i^* \subseteq M$ is an m-dimensional sublattice, and we obtain

$$\bar{N} = N \cap L^\perp \, ; \quad \bar{\sigma} = \sigma \cap L^\perp \subseteq \bar{N}_{\mathbb{R}} \, .$$

Lemma: *The condition (iii) of Definition (2.1) is equivalent to* $\sigma^\vee \cap L_{\mathbb{R}} = \{0\}$.

Proof: The condition that i maps the closed orbit of Y into the closed orbit of X can be written as

$$i^*(\sigma^\vee \setminus \sigma^\perp) \subseteq \bar{\sigma}^\vee \setminus \bar{\sigma}^\perp .$$

Moreover, to obtain equality of both orbits, the map i^* has to induce a bijection $\sigma^\perp \cap M \xrightarrow{\sim} \bar{\sigma}^\perp \cap \bar{M}$, which is equivalent to the injectivity of i^* on $\sigma^\perp$. Now, these two conditions can be translated into

$$\sigma^\vee \cap (L + \sigma^\perp) = \sigma^\perp \;\; \text{(i.e. } \sigma^\vee \cap L \subseteq \sigma^\perp \text{)} \quad \text{and} \quad \sigma^\perp \cap L = \{0\},$$

which is equivalent to $\sigma^\vee \cap L = \{0\}$. $\square$

(2.3) Proposition: *Let $f : X \to S$ be a toric deformation of Y. Then, the germ $(S,0)$ is smooth, and the ideal $I := \ker\big(\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\bar\sigma^\vee \cap \bar M]\big)$ defining $Y \subseteq X$ can be generated by m binomials $x^{r^1} - x^{s^1}, \dots, x^{r^m} - x^{s^m} \in \mathbb{C}[\sigma^\vee \cap M]$ ($r^i, s^i \in \sigma^\vee \cap M$; $r^i - s^i \in L$). In particular, they form a binomial regular sequence, and Y is a relative complete intersection in X.*

Proof: *Step 1:* A deformation diagram

$$\begin{array}{ccc} Y & \stackrel{i}{\hookrightarrow} & X \\ \downarrow & \otimes & \downarrow f \\ \{0\} & \hookrightarrow & S \end{array}$$

induces a deformation of the corresponding torus T_Y:

$$\begin{array}{ccc} T_Y & \hookrightarrow & T_X \\ \downarrow & \otimes & \downarrow f \\ \{0\} & \hookrightarrow & S\ . \end{array}$$

T_Y and T_X are smooth. Hence, T_X splits locally into a product $T_X \cong T_Y \times S$, and the germ $(S,0)$ has to be smooth, too.

Step 2: On the level of local rings (at the general points of the closed orbits and at the special point of S, respectively) we obtain the following diagram:

$$\begin{array}{ccccc} \mathcal{O}_{X,0} & \stackrel{i^*}{\longrightarrow\!\!\!\!\!\rightarrow} & \mathcal{O}_{Y,0} & & \\ \uparrow\text{flat} & \otimes & \uparrow & & \\ \mathcal{O}_{S,0} & \longrightarrow\!\!\!\!\!\rightarrow & \mathcal{O}_{S,0}/m_{S,0} & = \mathbb{C}. & \end{array}$$

Therefore, $I \cdot \mathcal{O}_{X,0} = m_{S,0} \cdot \mathcal{O}_{X,0}$ is generated by m elements $g_1, \dots, g_m$ ($(S,0)$ is smooth), and by the Nakayama lemma we can choose these generators among the elements of the form $x^r - x^s$ ($r, s \in \sigma^\vee \cap M$; $r - s \in L$).

Step 3: Let $\tilde I := (g_1, \dots, g_m) \subseteq \mathbb{C}[\sigma^\vee \cap M]$. Then, $\tilde I \subseteq I$ are ideals in $\mathbb{C}[\sigma^\vee \cap M]$ meeting the following properties:

(i) $\tilde I$ and I are homogeneous with respect to the $\bar M$-grading;

(ii) $\tilde I = I$ in the local ring $\mathcal{O}_{X,0}$.

We want to show $\tilde I = I$. For that purpose, let $g \in I$ be an arbitrary $\bar M$-homogeneous element. By (ii) there exists an $h \in \mathbb{C}[\sigma^\vee \cap M]$ with $h \cdot g \in \tilde I$ and

$$h \notin m_0 := \bigoplus_{\substack{r \in \sigma^\vee \cap M \\ r \notin \sigma^\perp}} \mathbb{C} \cdot x^r$$

(i.e. h contains a term of $\mathbb{C}[\sigma^\perp \cap M]$); by (i) we can additionally assume h to be $\bar{M}$-homogeneous. Since $\sigma^\vee \cap L_{\mathbb{R}} = \{0\}$, this implies that h is a monomial of $\mathbb{C}[\sigma^\perp \cap M]$, i.e. h is invertible. $\square$

Remark: The m vectors $r^1 - s^1, \dots, r^m - s^m$ built from the exponents of the binomial regular sequence generating I are free generators of the sublattice $L \subseteq M$.

Proof: Let $L' := \operatorname{span}_{\mathbb{Z}}(r^1 - s^1, \dots, r^m - s^m) \subseteq L$. In particular, the ideal I is homogeneous under the M/L'-grading of $\mathbb{C}[\sigma^\vee \cap M]$.
Now, for each $l \in L$ there are $r, s \in \sigma^\vee \cap M$ such that $l = r - s$. The monomials x^r and x^s map onto equal functions in $\Gamma(Y, \mathcal{O}_Y)$, hence $x^r - x^s \in I$. Since I does not contain monomials at all, $x^r - x^s$ has to be M/L'-homogeneous itself, i.e. $r - s \in L'$. $\square$

Definition: Those binomial regular sequences defining a relative complete intersection between affine toric varieties satisfying (2.1)(ii) and (iii) (as in the previous proposition) are said to be *toric regular sequences.*

(2.4) Christophersen [Ch 1] developed the notion of relative deformations.

Definition: If $Y \hookrightarrow X$ is a relative complete intersection (i.e. given by a regular sequence in X), then a *relative deformation* (of Y in X) over S is given by a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \hookrightarrow & X \times S \\ \downarrow f & & \downarrow \mathrm{pr}_S \\ S & = & S \end{array}$$

(f is flat) and an isomorphism $Y \xrightarrow{\sim} f^{-1}(0)$ compatible with the embeddings into X. Two relative deformations are said to be equivalent, if they are so as abstract deformations (without regarding the embeddings).

Relative deformations can be obtained by arbitrary perturbations of the regular sequence. In particular, they form a smooth subfunctor $\mathrm{Def}_{Y \hookrightarrow X}$ of the deformation functor Def_Y. If Y admits a versal deformation with base space S_Y, then the versal relative deformation of $Y \hookrightarrow X$ equals the subfamily over some smooth subscheme $S_{Y \hookrightarrow X} \subseteq S_Y$.
Christophersen has worked out that $S_{Y \hookrightarrow X}$ should be a good candidate for components of the reduced base space $(S_Y)_{\mathrm{red}}$.

(2.5) Let $f : X \to S$ be a toric deformation of Y. Then, the diagram

$$\begin{array}{ccc} X & \stackrel{(\mathrm{id}, f)}{\hookrightarrow} & X \times S \\ \downarrow f & & \downarrow \mathrm{pr}_S \\ S & = & S \end{array}$$

and Proposition (2.3) show that $f : X \to S$ is a relative deformation for the embedding $Y \hookrightarrow X$ given by the toric regular sequence $g := (x^{r^1} - x^{s^1}, \ldots, x^{r^m} - x^{s^m})$. In particular, it is induced by some morphism $S \to S_{Y\hookrightarrow X}$. If, moreover, f equals the restriction of the versal family of Y to some component S of $(S_Y)_{\mathrm{red}}$, then S and $S_{Y\hookrightarrow X}$ are equal as germs.
On the other hand, the regular sequence g itself provides a special toric deformation by regarding the (flat) map $g : X \to \mathbb{C}^m$. f and g need not be equivalent in general. However, similar to the case of f, the deformation g is induced by some map $\mathbb{C}^m \to S_{Y\hookrightarrow X}$, i.e. S and $\mathbb{C}^m$ map at least into the same component of $(S_Y)_{\mathrm{red}}$. If, moreover, $S = S_{Y\hookrightarrow X}$ (for instance, if f is a component of the versal deformation), then $\mathbb{C}^m \to S_{Y\hookrightarrow X} = S$ is an isomorphism, i.e. $f = g$.

3 Constructing homogeneous, toric regular sequences

(3.1) The previous section provides motivation for looking for those pairs (Y, X) of affine toric varieties such that Y is a relative complete intersection in X, given by a toric regular sequence g.
One possibility for doing so is fixing the "big" space X and searching for those binomial regular sequences yielding a toric variety as zero set. This was the principal approach in [Al 2]. However, starting with Y and looking for toric X to map into, is a completly different story. In this section we will solve this problem, if the sequences are additionally assumed to be homogeneous:

Definition: If $g = (x^{r^1} - x^{s^1}, \ldots, x^{r^m} - x^{s^m})$ is a toric regular sequence defining $Y \hookrightarrow X$, then the common images $\bar{r}^i \in \bar{M}$ of $r^i, s^i \in M$ are called the *degrees* of g. The sequence g (and its associated toric deformation) is said to be *homogeneous of degree* $\bar{r}$, if $\bar{r} = \bar{r}^1 = \ldots = \bar{r}^m$. (Obviously, if $m = 1$, then this will always be the case.)

Lemma: *Up to $\mathbb{Z}$-linear transformations, a homogeneous, toric regular sequence g has the shape $g = (x^{r^1} - x^{r^0}, \ldots, x^{r^m} - x^{r^0})$, i.e. $r^0 = s^1 = \ldots = s^m$.*

Proof: Choose an arbitraty $r^0 \in \{r^1, \ldots, r^m, s^1, \ldots, s^m\}$. Then, since $r^i - r^0, s^i - r^0 \in L$, the $2m$ binomials $x^{r^i} - x^{r^0}, x^{s^i} - x^{r^0}$ $(i = 1, \ldots, m)$ are contained in I. Moreover, since $x^{r^i} - x^{s^i} = (x^{r^i} - x^{r^0}) - (x^{s^i} - x^{r^0})$, they generate this ideal, and we can choose m among them still doing so (cf. the proof of Proposition (2.3)).
It remains to show that this changing of generators could be done using a $\mathbb{Z}$-linear transformation only. We regard the following, more general situation: Let $x^r - x^s$ be contained in an ideal I generated by binomials $x^{r^1} - x^{s^1}, \ldots, x^{r^m} - x^{s^m}$, and assume that all exponents $r, s, r^i, s^i \in M$ map onto a single $\bar{r} \in \bar{M}$. Using the natural

$\bar{M}$-grading, all these binomials are homogeneous of degree $\bar{r}$. Hence, representing $x^r - x^s$ as a $\mathbb{C}[\sigma^\vee \cap M]$-linear combination of the generators of I can be done by using homogeneous coefficients of degree $0 \in \bar{M}$ only. Since $\sigma^\vee \cap L = \{0\}$, they have to be constants. Moreover, it is obvious that they can be taken even from $\mathbb{Z}$ then. □

Remark: If $g = (x^{r^1} - x^{r^0}, \dots, x^{r^m} - x^{r^0})$, then

$$L = \sum_{i=1}^{m} \mathbb{Z} \cdot (r^i - r^0) = \sum_{i,j=0}^{m} \mathbb{Z} \cdot (r^i - r^j) = \ker \left(\deg : \bigoplus_{i=0}^{m} \mathbb{Z} \cdot r^i \twoheadrightarrow \mathbb{Z} \right)$$

($\deg(r^i) := 1$). The elements $r^0, \dots, r^m$ are linearly independent in $M_{\mathbb{R}}$.

(3.2) Now, we approach our main issue. We will produce homogeneous toric regular sequences (yielding pairs (Y, X)) from so-called deformation data.

Definition: Let $(\mathbb{A}, \mathbb{L})$ be a pair of a real vector space $\mathbb{A}$ and a lattice $\mathbb{L} \subseteq \mathbb{A}$. A *deformation datum* of size m is a tuple $(R_0, \dots, R_m;\, C;\, p)$ admitting the following properties:

(i) $C \subseteq \mathbb{A}$ is a rational polyhedral cone with apex in $0 \in \mathbb{A}$, and $p \geq 1$ is a natural number.

(ii) $R_0, \dots, R_m \subseteq \mathbb{A}$ are rational polyhedra with C as their cone of unbounded directions, i.e. $R_i = \bar{R}_i + C$ for suitable compact polytopes $\bar{R}_i$, $i = 0, \dots, m$. (It is possible to choose these polytopes in a canonical way by taking the convex hull of the vertices of R_i.)

(3.3) For a given polyhedron $P \subseteq \mathbb{A}$ and a linear form $t \in \mathbb{A}^*$ ($t \neq 0$, t bounded below on P) we denote by $F(P, t)$ the face of P that is defined by t being minimal on it.

Definition: A deformation datum $(R_0, \dots, R_m;\, C;\, p)$ is said to be *admissible*, if it meets the following conditions:

Case 1: $p = 1$. For each $t \in C^\vee \subseteq \mathbb{A}^*$ at least m of the $m+1$ faces $F(R_i, t)$ of R_i ($i = 0, \dots, m$) contain lattice points.

Case 2: $p \geq 2$. $R_1, \dots, R_m$ are lattice polyhedra, i.e. they admit only lattice points as vertices.

The following alternative description will be useful:

Lemma: *A deformation datum $(R_0, \dots, R_m;\, C;\, p)$ is admissible, if and only if for each $t \in \mathbb{L}^* \cap C^\vee$ the values of t on at least m of the $m+1$ faces $F(R_i, t)$ of*

R_i $(i = 0, \ldots, m)$ *(or exactly on the faces* $F(R_1, t), \ldots, F(R_m, t)$*, if* $p \geq 2$*) are integers.*

Proof: Linear forms $t \in \mathbb{L}^* \cap C^\vee$ yield integers as values on lattice points. Hence, admissible deformation data always admit the property described in the lemma. To obtain the opposite implication we proceed in three steps:

Step 1: Perturbing the linear form $t \in C^\vee$ slightly (inside the cone $C^\vee$), the corresponding faces $F(R_i, t)$ of R_i will at most be replaced by smaller ones. Hence, it will be sufficient to regard only those t such that each $F(R_i, t)$ is a vertex of R_i. Moreover, if convenient, it will be possible to replace t by suitable linear forms $t' \in C^\vee$ close to t again; they provide the same vertex as t.

Step 2: Claim: Let $b^0, b^1 \in \mathbb{A} \setminus \mathbb{L}$, and let $t \in C^\vee \setminus \{0\}$. Then, there exists a linear form $t' \in (\mathbb{L}^* \cap C^\vee) \setminus \{0\}$ such that

- $t'/\|t'\|$ and $t/\|t\|$ are arbitrarily close to each other, and
- $\langle b^0, t' \rangle, \langle b^1, t' \rangle \notin \mathbb{Z}$.

Proof: First we try to meet the latter condition. Choose $t^0, t^1 \in \mathbb{L}^*$ having no integer value on b^0, b^1, respectively. If there exists a t^j among them such that both $\langle b^0, t^j \rangle, \langle b^1, t^j \rangle \notin \mathbb{Z}$, then take $t' := t^j$. Otherwise, we know that $\langle b^0, t^1 \rangle, \langle b^1, t^0 \rangle \in \mathbb{Z}$ and $\langle b^0, t^0 \rangle, \langle b^1, t^1 \rangle \notin \mathbb{Z}$. Hence, $t' := t^1 + t^2$ has the desired property.
Now, we have to improve our linear form t' to obtain the additional property $t' \in C^\vee$. If $s \in \mathbb{L}^* \cap (\text{int}\, C^\vee)$, and if $N \in \mathbb{N} \setminus \{0\}$ such that $N \cdot b^0, N \cdot b^1$ are contained in the lattice $\mathbb{L}$, then $\langle b^0, N \cdot s \rangle, \langle b^1, N \cdot s \rangle \in \mathbb{Z}$. Each element of $\mathbb{L}^*$ can be put into the cone $C^\vee$ by adding a sufficiently large multiple of s. Hence, we substitute $t' := t' + (kN) \cdot s$ (with $k \gg 0$).
Finally, we can add sufficiently large multiples of t (which are contained in the lattice and yield integers as values on b^0, b^1) to t'. This operation ensures that the directions $t'/\|t'\|$ and $t/\|t\|$ are arbitrarily close to each other.

Step 3: If a deformation datum $(R_0, \ldots, R_m;\, C;\, p)$ is not admissible, then there exists an element $t \in C^\vee$ such that two of the faces $F(R_i, t)$ $(i = 0, \ldots, m)$ equal vertices (denoted by b^0, b^1) that are not contained in the lattice (*Case* $p \geq 2$*:* one of the faces $F(R_i, t)$ $(i = 1, \ldots, m)$, denoted by $b^0 = b^1$).
Let t' be a linear form as constructed in the second step; suppose that t and t' define the same vertices of $R_0, \ldots, R_m$ (including b^0 and b^1). Then, t' violates the conditions of the lemma. $\square$

(3.4) To a given deformation datum $(R_0, \ldots, R_m;\, C;\, p)$ we associate the following objects:

(3.4.1) Define the polyhedron Q to be the Minkowski sum

$$Q := R_0 + \ldots + R_m = C + (\bar{R}_0 + \ldots + \bar{R}_m) \subseteq \mathbb{A}.$$

We embed the whole space as an affine hyperplane in a higher-dimensional space:

- $\bar{N}_{\mathbb{R}} := \mathbb{A} \times \mathbb{R}$ is a vector space containing the lattice $\bar{N} := \mathbb{L} \times \mathbb{Z}$ $(\bar{M}_{\mathbb{R}} := \bar{N}^*_{\mathbb{R}},\ \bar{M} := \bar{N}^*)$;
- $\psi_1 : \mathbb{A} \hookrightarrow \bar{N}_{\mathbb{R}}\,; \quad a \mapsto (a, p^{-1})$.

In particular, $\psi_1(Q)$ is a polyhedron in $\bar{N}_{\mathbb{R}}$. Denoting by

$$\psi : \mathbb{A} \hookrightarrow \bar{N}_{\mathbb{R}}\,; \quad a \mapsto (a, 0)$$

the associated linear embedding, we can define $Y := \operatorname{Spec} \mathbb{C}[\bar{\sigma}^\vee \cap \bar{M}]$ as the affine toric variety that is given by the cone

$$\bar{\sigma} := \overline{\mathbb{R}_{\geq 0} \cdot \psi_1(Q)} = \psi(C) \cup\ \mathbb{R}_{\geq 0} \cdot \psi_1(Q)\,.$$

(3.4.2) To define σ and X, we put the polyhedra $R_0, \ldots, R_m$ into parallel affine planes of a vector space that is large enough.

- $N_{\mathbb{R}} := \mathbb{A} \times \mathbb{R}^{m+1},\ N := \mathbb{L} \times \mathbb{Z}^{m+1};\ M_{\mathbb{R}} := N^*_{\mathbb{R}},\ M := N^*$.
  Denote by $\Phi : N_{\mathbb{R}} \longrightarrow \mathbb{R}^{m+1}$ the projection onto the second factor.
- $\phi_i : \mathbb{A} \hookrightarrow N_{\mathbb{R}}\,; \quad a \mapsto \begin{cases} (a,\, p^{-1}e^0) & \text{for } i = 0 \\ (a, e^i) & \text{for } i = 1, \ldots, m \end{cases}$
  $(e^0, \ldots, e^m$ denotes the standard basis of $\mathbb{Z}^{m+1})$.
  On the homogeneous level, the affine maps $\phi_0, \ldots, \phi_m$ correspond to the trivial embedding $\phi : \mathbb{A} \hookrightarrow N_{\mathbb{R}}$ $(a \mapsto (a, 0))$.

Now, we denote by P the convex hull

$$P := \operatorname{conv} \left(\bigcup_{i=0}^{m} \phi_i(R_i) \right) = \phi(C) + \operatorname{conv} \left(\bigcup_{i=0}^{m} \phi_i(\bar{R}_i) \right) \subseteq N_{\mathbb{R}}$$

and define $X := \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ as the affine toric variety given by the cone

$$\sigma := \overline{\mathbb{R}_{\geq 0} \cdot P} = \phi(C) \cup\ \mathbb{R}_{\geq 0} \cdot P.$$

(3.4.3) If $\operatorname{pr}_i : \mathbb{R}^{m+1} \to \mathbb{R}$ denotes the projection onto the i-th factor, we can define linear maps $r^0, \ldots, r^m : N \to \mathbb{Z}$ by

$$r^i := \begin{cases} p \cdot (\operatorname{pr}_0 \circ \Phi) & \text{for } i = 0 \\ \operatorname{pr}_i \circ \Phi & \text{for } i = 1, \ldots, m. \end{cases}$$

By construction, these maps correspond to elements $r^i \in \sigma^\vee \cap M$.

(3.4.4) $\bar{N}$ can be considered as a sublattice of N via the inclusion map

$$\bar{N} \hookrightarrow N\,, \quad (a;1) \mapsto (a;1,p,\ldots,p).$$

This embedding admits the following properties:

(i) $\bar{N} = N \cap \bigcap_{i,j} (r^i - r^j)^\perp = N \cap \bigcap_{i=1}^m (r^i - r^0)^\perp$

(ii) $\bar{\sigma} = \sigma \cap \bar{N}_{I\!R}$.

In particular, we obtain a map $Y \to X$ which sends Y into the special fiber of the morphism $X \to \mathbb{C}^m$ defined by the regular functions $x^{r^1} - x^{r^0}, \ldots, x^{r^m} - x^{r^0} \in \mathbb{C}[\sigma^\vee \cap M]$.

(3.5) **Theorem:** *Starting with an admissible deformation datum $(R_0, \ldots, R_m; C; p)$, the previous construction provides a pair (Y, X) of affine toric varieties such that $Y \subseteq X$ is given by a homogeneous toric regular sequence $x^{r^1} - x^{r^0}, \ldots, x^{r^m} - x^{r^0}$. (Looking at (3.4.4), Y equals the special fiber of $X \to \mathbb{C}^m$.) Moreover, all those pairs (Y, X) arise in that way.*

The proof is contained in §4.

Remark: Up to isomorphisms, the construction (3.4) will yield the same result, if the polyhedra R_i from the deformation datum $(R_0, \ldots, R_m; C; p)$ are shifted via vectors from $I\!L$ (for $i \geq 1$) or from $p^{-1} I\!L$ (for $i = 0$).

(3.6) Eventually, let us switch to the natural viewpoint that an affine toric variety $Y = \operatorname{Spec} \mathbb{C}[\bar{\sigma}^\vee \cap \bar{M}]$ is given, and we are asking for its (homogeneous) toric deformations.
Fixing some degree $\bar{r} \in \bar{\sigma}^\vee \cap \bar{M}$ corresponds to the choice of an affine cross cut Q of the cone $\bar{\sigma} \subseteq \bar{N}_{I\!R}$. Then, the previous theorem tells us that homogeneous toric deformations of Y arise from certain subdivisions of Q into a Minkowski sum. More precisely, we have to proceed as follows:

(i) Define the vector space $I\!A_0 := \{a \in \bar{N}_{I\!R} |\ \langle a, \bar{r} \rangle = 0\}$, which contains the lattice $I\!L_0 := I\!A \cap \bar{N}$.

(ii) Let p be the greatest common divisor of the coordinates of $\bar{r}$, i.e. $p^{-1}\bar{r}$ is a primitive element of $\bar{M}$.

(iii) Define the affine space $I\!A := \{a \in \bar{N}_{I\!R} |\ \langle a, \bar{r} \rangle = 1\}$. Fixing some point $0 \in I\!A \cap p^{-1}\bar{N}$, we obtain a sublattice via $I\!L := 0 + I\!L_0$. (In case $p = 1$, $I\!L$ equals $I\!A \cap \bar{N}$.) Moreover, we can use the point 0 to identify $(I\!A, I\!L)$ with the pair $(I\!A_0, I\!L_0)$ providing a linear structure which was assumed in (3.2).

(iv) Let $C := \bar{\sigma} \cap \mathbb{A}_0$ and $Q := \bar{\sigma} \cap \mathbb{A}$. Then, by Theorem (3.5), homogeneous regular sequences of degree $\bar{r}$ (in some larger affine toric variety X) correspond to admissible splittings of Q into a Minkowski sum $Q = R_0 + \ldots + R_m$.

4 Proof of Theorem (3.5)

(4.1) Let an admissible deformation datum $(R_0, \ldots, R_m; C; p)$ be given. We have to show that the associated data σ, $\bar{\sigma}$ and $r^0, \ldots, r^m$ (cf. (3.4)) provide indeed a toric regular sequence.

(4.1.1) $\sigma^\vee \cap M \longrightarrow\!\!\!\!\!\rightarrow \bar{\sigma}^\vee \cap \bar{M}$ *is surjective:*
Let $s \in M$ such that

$$\langle (c^0 + \ldots + c^m;\ p^{-1}, 1, \ldots, 1),\ s\rangle \geq 0 \quad \text{for } c^i \in R_i,\ i = 0, \ldots, m \quad (\text{i.e. } c^0 + \ldots + c^m \in Q)\,;$$

it means that s maps onto an element $\bar{s} \in \bar{\sigma}^\vee \cap \bar{M}$. We have to show that $\bar{s}$ can be lifted to $\sigma^\vee \cap M$.
Projecting s to the $\mathbb{A}^*$−component, we obtain an element $t = s|_{\mathbb{A}} \in \mathbb{L}^* \cap C^\vee$. Since our deformation element is admissible, we may assume that, on the faces $F(R_1, t), \ldots, F(R_m, t)$, the linear form t provides integers only. Hence, even on the embedded polyhedra $\phi_1(R_1), \ldots, \phi_m(R_m) \subseteq N_{\mathbb{R}}$, the minimal value of s is contained in $\mathbb{Z}$. Denote these values by k_i and suppose that they occur at points $(\tilde{c}^i, e^i) \in \phi_i(R_i) \subseteq \Phi^{-1}(e^i) \subseteq N_{\mathbb{R}}$ $(i = 1, \ldots, m)$.
Modifying s by $s := s - \sum_{i=1}^m k_i \cdot r^i + (\sum_{i=1}^m k_i) \cdot r^0$, we can assume that s is *non-negative* on $\phi_1(R_1), \ldots, \phi_m(R_m) \subseteq N_{\mathbb{R}}$ and, moreover, $\langle (\tilde{c}^i, e^i),\ s\rangle = 0$ for $i = 1, \ldots, m$. Now, if $c^0 \in R_0$ (embedded as $(c^0, p^{-1}e^0) \in \phi_0(R_0) \subseteq N_{\mathbb{R}}$) is given, we obtain

$$\left\langle \left(c^0, p^{-1}e^0\right),\ s\right\rangle = \left\langle \left(c^0 + \sum_{i=1}^m \tilde{c}^i,\ \ p^{-1}e^0 + \sum_{i=1}^m e^i\right),\ s \right\rangle \geq 0.$$

Hence, s is contained in $\sigma^\vee$.

(4.1.2) $\sigma^\vee \cap L_{\mathbb{R}} = \{0\}$ $(L_{\mathbb{R}} := \ker(M_{\mathbb{R}} \longrightarrow\!\!\!\!\!\rightarrow \bar{M}_{\mathbb{R}}))$:
The cone σ contains points from $\Phi^{-1}(e^0), \ldots, \Phi^{-1}(e^m)$. In particular, a non-trivial element of $L_{\mathbb{R}} = \ker\left(\deg : \bigoplus_{i=0}^m \mathbb{R} \cdot r^i \longrightarrow\!\!\!\!\!\rightarrow \mathbb{R}\right)$ cannot be contained in $\sigma^\vee$.

(4.1.3) *We show that* $I = \ker\left(\mathbb{C}[\sigma^\vee \cap M] \longrightarrow\!\!\!\!\!\rightarrow \mathbb{C}[\bar{\sigma}^\vee \cap \bar{M}]\right)$ *is generated by* $x^{r^1} - x^{r^0}, \ldots, x^{r^m} - x^{r^0}$*:*
Let $x^r - x^s \in I$ be any binomial element of I, i.e. $r, s \in \sigma^\vee \cap M$, $r - s \in L$. Hence,

$$r - s = \sum_{i=0}^m g_i \cdot r^i \quad (g_i \in \mathbb{Z},\ \sum_i g_i = 0)\,.$$

Defining

$$q := r - \sum_i g_i^+ \cdot r^i = s - \sum_i g_i^- \cdot r^i \in M \quad (g_i^+, g_i^- \in I\!N,\; g_i^+ - g_i^- = g_i,\; g_i^+ g_i^- = 0)\,,$$

this linear form will equal r or s, if it is restricted to the polyhedra $\phi_i(R_i)$ with $g_i \leq 0$ or $g_i \geq 0$, respectively. In particular, q is non-negative at P, i.e. $q \in \sigma^\vee \cap M$. Now, expressing $x^r - x^s$ as a linear combination of the generators $x^{r^i} - x^{r^0}$ is straightforward.

(4.2) It remains to prove that all homogeneous, toric regular sequences can be obtained from deformation data. Hence, in the remainder of §4 we assume that we are given such a sequence $x^{r^1} - x^{r^0}, \dots, x^{r^m} - x^{r^0}$ defining a relative complete intersection $Y \hookrightarrow X$ as in (2.1) and (2.2).

Lemma: *Let $r, s \in \sigma^\vee \cap M$ such that $r - s \in L \setminus \{0\}$. Then, there are two different indices $i, j \in \{0, \dots, m\}$ satisfying $r - r^i, s - r^j \in \sigma^\vee$.*

Proof: Since $x^r - x^s \in I$, there must be some equation

$$x^r - x^s = \sum_\mu c_\mu \, x^{t^\mu} \, (x^{r^{i(\mu)}} - x^{r^0}) \qquad (c_\mu \in \mathbb{C};\; t^\mu \in \sigma^\vee \cap M).$$

In particular, both exponents r and s have to occur somewhere on the right hand side providing the existence of the desired r^i, r^j. Moreover, if $r^i = r^j$, then we could apply that procedure to $r' := r - r^i$ and $s' := s - r^i$ again. This recursion eventually stops. □

(4.3) **Lemma:** *Denote by $\bar r \in \bar M = M/L$ the common image of the elements $r^0, \dots, r^m$ via the surjection $M \twoheadrightarrow \bar M$. We obtain:*

1. *$\bar r$ is not trivial on $\bar\sigma \subseteq \bar N_{I\!R}$.*
2. *Let $\bar r = p \cdot \bar r'$ ($p \in I\!N$, $\bar r' \in \bar M$ primitive). Then, without loss of generality, r^0 is also divisible by p and can be written as $r^0 = p \cdot \tilde r^0$. Moreover, the elements $\tilde r^0, r^1, \dots, r^m$ equal a part of a $\mathbb{Z}$-basis of the lattice M.*

Proof: *Step 1:* If $\bar r$ were trivial on $\bar\sigma$, not only $\bar r$ but also $-\bar r$ would belong to $\bar\sigma^\vee$. In particular, there would be an $r \in \sigma^\vee$ (lifting $-\bar r$) such that

$$r + r^i \in \sigma^\vee \cap L = \{0\}, \quad \text{i.e. } -r^i \in \sigma^\vee \text{ for } i = 0, \dots, m.$$

Hence, the linearly independent vectors $r^0 - r^i$ would have to be contained in $\sigma^\vee \cap L = \{0\}$, which is impossible.

Step 2: Since $\sigma^\vee \cap M \longrightarrow\!\!\!\!\rightarrow \bar{\sigma}^\vee \cap \bar{M}$ is surjective, the element $\bar{r}' \in \bar{\sigma}^\vee \cap \bar{M}$ can be lifted to some $\tilde{r} \in \overset{\vee}{\sigma} \cap M$. Then, $p \cdot \tilde{r}$ differs from $r^0, \dots, r^m$ by $I\!\!L$-elements only, and we can apply the previous lemma. There has to be an index $i \in \{0, \dots, m\}$ such that $p \cdot \tilde{r} - r^i$ is contained in $\sigma^\vee$. On the other hand, $p \cdot \tilde{r} - r^i$ is obviously contained in the lattice L, and we obtain $p \cdot \tilde{r} - r^i \in \sigma^\vee \cap L = \{0\}$. We may assume that $i = 0$.

Step 3: We will show that there are lattice elements $b^i \in N$ for $i = 0, \dots, m$ such that

$$\begin{array}{rcl} \langle b^i, r^i \rangle & = & \left\{ \begin{array}{ll} p & \text{for } i = 0 \\ 1 & \text{for } i = 1, \dots, m \end{array} \right. \qquad \text{and} \\ \langle b^i, r^j \rangle & = & 0 \quad \text{for } j \neq i. \end{array}$$

Since, for a fixed i, $(r^i - r^j)_{j \in \{0, \dots, m\} \setminus \{i\}}$ is a basis of the lattice $L \subseteq M$, it is possible to choose an element $b^i \in N$ such that $\langle b^i, r^i - r^j \rangle = 1$, i.e.

$$\langle b^i, r^i \rangle = \langle b^i, r^j \rangle + 1 \quad (\text{for } j \neq i).$$

$\bar{r}' \in \bar{M}$ is primitive, hence, there is a $b \in \bar{N} = L^\perp \cap N$ such that $\langle b, \bar{r}' \rangle = 1$. Therefore,

$$\langle b, r^j \rangle = p \quad \text{for } j = 0, \dots, m.$$

Case 1: $i \geq 1$. The equation $\langle b^i, r^i \rangle = \langle b^i, r^j \rangle + 1$ $(j \neq i)$ implies

$$\langle b^i, r^j \rangle + 1 = \langle b^i, r^i \rangle = \langle b^i, r^0 \rangle + 1 = \langle b^i, p \cdot \tilde{r}^0 \rangle + 1\,,$$

in particular, $p | \langle b^i, r^j \rangle$ for $j \neq i$. Hence, there is a $k \in Z\!\!\!Z$ such that the improved $b^i := b^i + k \cdot b$ additionally yields $\langle b^i, r^j \rangle = 0$ $(j \neq i)$.

Case 2: $i = 0$. Here, we have to use the modification $b^0 := p \cdot b^0 + k \cdot b$ (with suitable $k \in Z\!\!\!Z$) to obtain the eqations $\langle b^0, r^j \rangle = 0$ $(j \geq 1)$. $\square$

(4.4) **Lemma:** *Let $r \in M$ be such that, for each index $i = 0, \dots, m$, it can be pushed into $\sigma^\vee$ without using r^i:*

$$r + \sum_{j \neq i} \lambda^i_j \cdot r^j \in \sigma^\vee \quad (\textit{for some } \lambda^i_j \in Z\!\!\!Z).$$

Then, r itself is contained in $\sigma^\vee$.

Proof: We will proceed by induction on $\sum_j \lambda^i_j$. To do so we first have to modify the presumption of the lemma slightly:
For $r \in M$ suppose that

$$r + \sum_j \lambda^i_j \cdot r^j \in \sigma^\vee \quad \text{with } \lambda^i_j \in Z\!\!\!Z,\ \lambda^i_i \leq 0, \text{ and } \sum_j \lambda^i_j \text{ is constant in } i.$$

(The latter fact can be obtained by increasing some of the coefficients λ^i_j with $j \neq i$.) Now, on the one hand, the sum $\sum_j \lambda^i_j$ is bounded below. (Look at the vector space $\bar{M}_{I\!R}$: Subtracting $\bar{r}$ sufficiently often from a given point leads out of the cone $\bar{\sigma}^\vee$.) On the other hand, as soon as the elements $r + \sum_j \lambda^i_j \cdot r^j$ are not equal to one anothers (for different i), we can apply (4.2) to all these elements. We obtain $r + \sum_j (\lambda^i_j)' \cdot r^j \in \sigma^\vee$ with $(\lambda^i_j)' = \lambda^i_j$ or $\lambda^i_j - 1$ and $\sum_{j=0}^m (\lambda^i_j)' = \sum_{j=0}^m \lambda^i_j - 1$. Therefore, only the case that the coefficients λ^i_j do *not* depend on i is left. This implies $\lambda^i_j = \lambda^j_j \le 0$; hence, with $r + \sum_j \lambda^i_j \cdot r^j$, r has to be contained in $\sigma^\vee$, too. □

Corollary: *We can replace the fact $L \cap \sigma^\vee = \{0\}$ by an even stronger statement: If any linear combination $\sum_{j=0}^m \lambda_j \cdot r^j$ (with integer coefficients $\lambda_j \in Z\!\!\!Z$) is contained in $\sigma^\vee \cap M$, then each of the coefficients will be non-negative.*

Proof: Assume that $\lambda_i < 0$. Then, the element $r := \lambda_i \cdot r^i$ admits the following properties:

$$r + \sum_{j \neq i} \lambda_j \cdot r^j \in \sigma^\vee \quad \text{and} \quad r - \lambda_i \cdot r^i \in \sigma^\vee.$$

In particular, r fulfils the assumption of the previous lemma. We obtain $-r^i \in \sigma^\vee$, but this cannot be true (cf. Step 1 in the proof of Lemma (4.3)). □

(4.5) Now, we start with the direct construction of the deformation datum that shall induce the given toric regular sequence. Let

$$\begin{aligned} (\mathbb{A}, I\!L) &:= \left[(r^0)^\perp \cap \ldots \cap (r^m)^\perp,\ (r^0)^\perp \cap \ldots \cap (r^m)^\perp \cap N\right] \quad \text{and} \\ C &:= (r^0)^\perp \cap \ldots \cap (r^m)^\perp \cap \sigma \ \subseteq \mathbb{A}. \end{aligned}$$

Fixing some point in each of the sets

$$\begin{aligned} &\{a \in p^{-1}N \mid \langle a, r^0 \rangle = 1,\ \langle a, r^j \rangle = 0 \text{ for } j \neq 0\} \qquad \text{and} \\ &\{a \in N \mid \langle a, r^i \rangle = 1,\ \langle a, r^j \rangle = 0 \text{ for } j \neq i\} \quad (\text{for } i = 1, \ldots, m), \end{aligned}$$

we obtain $m+1$ different affine embeddings $\mathbb{A} \hookrightarrow N_{I\!R}$ which induce isomorphisms

$$\begin{aligned} \mathbb{A} &\xrightarrow{\phi_i} \{a \in N_{I\!R} \mid \langle a, r^i \rangle = 1,\ \langle a, r^j \rangle = 0 \text{ for } j \neq i\} \quad (i \ge 0) \qquad \text{and} \\ I\!L &\xrightarrow{\phi_i} \{a \in N \mid \langle a, r^i \rangle = 1,\ \langle a, r^j \rangle = 0 \text{ for } j \neq i\} \quad (\text{only for } i \ge 1 \text{ or } p = 1). \end{aligned}$$

Hence, with

$$R_i := \phi_i^{-1}\left(\sigma \cap \{a \in N_{I\!R} \mid \langle a, r^i \rangle = 1,\ \langle a, r^j \rangle = 0 \text{ for } j \neq i\}\right)$$

we have got $m+1$ polyhedra $R_0, \ldots, R_m \subseteq \mathbb{A}$ admitting C as their cone of unbounded directions (i.e. $R_i = C + \text{compact set}$).

Lemma:

(1) The polyhedra $R_0, \ldots, R_m$ are not empty.

(2) $\sigma = C \cup \mathbb{R}_{\geq 0} \cdot \operatorname{conv}\left(\bigcup_{i=0}^m \phi_i(R_i)\right)$.

Proof: (1) We define some auxiliary cones in $N_{\mathbb{R}}$:

$$\begin{aligned} C_i &:= (r^0)^\perp \cap \ldots \cap \left(\hat{r^i}\right)^\perp \cap \ldots \cap (r^m)^\perp \cap \sigma \\ &= C \cup \mathbb{R}_{\geq 0} \cdot \phi_i(R_i) \quad (\text{for } i = 0, \ldots, m). \end{aligned}$$

Then, the claim $R_i \neq \emptyset$ is equivalent to the fact that the cone C is properly contained in C_i. We will switch to the dual level showing $C_i^\vee \subset C^\vee$:
The dual cone $C^\vee \subseteq M_{\mathbb{R}}$ can be obtained as the pull back of $C^\vee$ regarded as a subset of $M_{\mathbb{R}}/\operatorname{span}_{\mathbb{R}}(r^0, \ldots, r^m)$. Moreover, the latter one equals the image of $\sigma^\vee$ via the canonical projection. Looking at $C_i^\vee$ in the same way (take $\operatorname{span}_{\mathbb{R}}(r^0, \ldots, \hat{r^i}, \ldots, r^m)$ instead of $\operatorname{span}_{\mathbb{R}}(r^0, \ldots, r^m)$), we obtain

$$C_i^\vee + \mathbb{R} \cdot r^i = C^\vee.$$

Hence, it suffices to check that $-r^i \notin C_i^\vee$:
The opposite would mean that there is an $r \in \sigma^\vee$ such that r and $-r^i$ differ by an element of $\mathcal{L}(r^0, \ldots, \hat{r^i}, \ldots, r^m)$ only, i.e.

$$r + r^i = \sum_{j \neq i} \lambda_j \cdot r^j \quad \text{in } M_{\mathbb{R}}.$$

Enlarging the coefficients λ_j, we may assume that they are integers, but then the fact $-r^i + \sum_{j\neq i} \lambda_j \cdot r^j \in \sigma^\vee$ contradicts the previous corollary.

(2) The second part of the lemma is equivalent to the fact $\sigma = \sum_{i=0}^m C_i$ (or to $\sigma^\vee = \bigcap_{i=0}^m C_i^\vee$). However, since

$$C_i^\vee = \sigma^\vee + \sum_{j \neq i} \mathbb{R} \cdot r^j \quad \text{(obtained in the first part of the proof)},$$

this was already shown in Lemma (4.4). □

(4.6) We finish the proof of Theorem (3.5) by showing that the deformation datum just constructed is admissible:

Step 1: Let $t \in \mathbb{L}^* \cap C^\vee$, and assume without loss of generality that it has no integers as values on the faces $F(R_0, t)$ and $F(R_1, t)$ (or only on $F(R_1, t)$ in case of $p \geq 2$). Let $b^0, \ldots, b^m \in N_{\mathbb{R}}$ be points of the faces $F(R_0, t), \ldots, F(R_m, t)$ in the embedded polyhedra $\phi_0(R_0), \ldots, \phi_m(R_m)$. Then, $t \in \mathbb{L}^*$ can be lifted to $T \in M$ such that

$$\langle b^0, T \rangle; \langle b^1, T \rangle \notin \mathbb{Z} \quad (\text{even in the case } p \geq 2).$$

(For $i \geq 1$ or $p = 1$, the value of t on $F(R_i, t)$ equals $\langle b^i, T\rangle$ in $\mathbb{Q}/\mathbb{Z}$. In the case $i = 0$, $p \geq 2$, the value of $\langle b^0, T\rangle$ is determined up to $p^{-1}\mathbb{Z}$ only. In particular, we can choose T in such a way that this value is not an integer.)

Step 2: Denote by $q_0, q_1 \in \mathbb{Q}$ the numbers $q_i := \langle b^i, T\rangle$. Then, there is an integer $k \geq 1$ such that $[kq_0 + kq_1] = [kq_0] + [kq_1] + 1$. ($[\ldots]$ denotes the integral part, and k can always be obtained as one of the two possibilities $k = 1$ or $k =$ (common denominator of q_0, q_1) $- 1$.) Therefore, we obtain

$$[\langle b^0, T\rangle + \langle b^1, T\rangle] = [\langle b^0, T\rangle] + [\langle b^1, T\rangle] + 1\,,$$

if t and T are replaced by kt and kT, respectively. (This operation does not change the faces $F(R_i, t)$ of R_i.)

Step 3: Now, we modify our T by adding elements of $\operatorname{span}_{\mathbb{R}}(r^0, \ldots, r^m)$; it remains a lifting of t, and the values of $\langle b^i, T\rangle$ do not change in $\mathbb{Q}/\mathbb{Z}$. It is possible to obtain the following situation:

$$\langle b^0, T\rangle \in [-1, 0) \quad \text{and} \quad \langle b^i, T\rangle \in [0, 1) \quad (\text{for } i = 1, \ldots, m),$$

i.e. T is not contained in the cone $\sigma^\vee$ and even cannot be put into $\sigma^\vee$ by adding vectors of the sublattice $L \subseteq M$ only.
On the other hand, the result of the second step yields that

$$\langle \sum_{i=0}^{m} b^i, T\rangle \geq 0.$$

This means that T is non-negative on the cone $\bar{\sigma} = C \cup \mathbb{R}_{\geq 0} \cdot (\phi_0(R_0) + \ldots \phi_m(R_m))$. Hence, we have found an element of $\bar{\sigma}^\vee \cap \bar{M}$ that cannot be lifted to $\sigma^\vee \cap M$. This is a contradiction to the fact that σ and $r^0, \ldots, r^m$ define a toric regular sequence.

5 The Kodaira-Spencer map

(5.1) Let $(R_0, \ldots, R_m;\ C;\ p)$ be an admissible deformation datum ($R_i, C \subseteq \mathbb{A}$). Then, in (3.4) we have defined cones $\bar{\sigma} \subseteq \sigma \subseteq N_{\mathbb{R}}$ and elements $r^0, \ldots, r^m \in \sigma^\vee \cap M$ such that the map

$$\operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M] \xrightarrow{\ g\ } \mathbb{C}^m \qquad \begin{array}{l}(\text{defined by the regular functions} \\ \quad x^{r^1} - x^{r^0}, \ldots, x^{r^m} - x^{r^0} \in \mathbb{C}[\sigma^\vee \cap M])\end{array}$$

yields a deformation of the special fiber $Y = \operatorname{Spec} \mathbb{C}[\bar{\sigma}^\vee \cap \bar{M}]$. Now, we will compute the Kodaira-Spencer map

$$\varrho : \mathbb{C}^m = T_0 \mathbb{C}^m \longrightarrow T^1_Y$$

associated to this deformation. (T^1_Y is the vector space of infinitesimal deformations of Y. Turning out to equal the tangent space of the versal base space S_Y, it can originally be defined as

$$T^1_Y = \left.\mathrm{Hom}\,(I/I^2, \mathbb{C}[\bar\sigma^\vee\cap\bar M])\middle/\mathrm{Hom}\,(\mathbb{C}[\bar\sigma^\vee\cap\bar M]^w, \mathbb{C}[\bar\sigma^\vee\cap\bar M])\right.;$$

here I denotes the kernel of some sujective map $\mathbb{C}[z_1,\dots,z_w] \longrightarrow\!\!\!\!\rightarrow \mathbb{C}[\bar\sigma^\vee\cap\bar M]$ providing an embedding of Y into $\mathbb{C}^w$, and I/I^2 maps to $\mathbb{C}[\bar\sigma^\vee\cap\bar M]^w$ via taking partial derivatives. For basic facts about deformation theory see [KPR].)

(5.2) For toric varieties the vector space T^1_Y is $\bar M$-graded. In [Al 1, Theorem (2.3)] we have determined its homogeneous pieces $T^1_Y(-R)$ $(R\in\bar M)$ as

$$T^1_Y(-R) = \left(L(\cup_{j=1}^N E_j^R) \Big/ \sum_{j=1}^N L(E_j^R)\right)^* \otimes_{\mathbb{R}} \mathbb{C}.$$

(Explanation of the used notation:

1) If $\bar\sigma\subseteq \bar N_{\mathbb{R}}$ is given by its fundamental generators $\bar\sigma = \langle a^1,\dots,a^N\rangle$ (i.e. $a^j\in\bar N$ are the primitive edges of $\bar\sigma$, and $\bar\sigma^\vee = \{r\in \bar M_{\mathbb{R}} \,|\, \langle a^j, r\rangle\geq 0 \text{ for } j=1,\dots,N\}$), then we define the following sets:

$$\begin{array}{rcl} E &:=& \{\bar s^1,\dots,\bar s^w\} = \text{ set of generators of the semigroup } \bar\sigma^\vee\cap\bar M;\\ E_j^R &:=& \{\bar s^v\in E \;|\; (0\leq)\langle a^j,\bar s^v\rangle < \langle a^j, R\rangle\}\quad (j=1,\dots,N)\,. \end{array}$$

2) For any subset of $\bar M_{\mathbb{R}}$ we denote by $L(\dots)$ the $\mathbb{R}$-vector space of all linear dependences among its elements. In particular, there is a canonical embedding $L(E_j^R)\hookrightarrow L(\cup_j E_j^R)$.)

Moreover, the monomials $x^{\bar s^v}$ $(\bar s^v\in E)$ generate the algebra $\mathbb{C}[\bar\sigma^\vee\cap\bar M]$, i.e. they provide a special surjection $\mathbb{C}[z_1,\dots,z_w]\longrightarrow\!\!\!\!\rightarrow \mathbb{C}[\bar\sigma^\vee\cap\bar M]$. Its kernel I is generated by binomials $z^\alpha - z^\beta$ satisfying the condition $\sum_v \alpha_v\bar s^v = \sum_v \beta_v \bar s^v$.
Now, if $\varphi: L(E)\to\mathbb{R}$ is some linear map vanishing on $\sum_j L(E_j^R)$, then via the previous formula it induces the same element of $T^1_Y(-R)$ as does the map

$$I\big/I^2 \longrightarrow \mathbb{C}[\bar\sigma^\vee\cap\bar M]\,;\qquad z^\alpha - z^\beta\mapsto \varphi(\alpha-\beta)\cdot x^{\sum_v\alpha_v\bar s^v - R}$$

via the defining equation of T^1_Y in (5.1) (see [Al 1, §3] or [Al 3, Theorem (3.4)]).

(5.3) The cone $\bar\sigma$ was defined as the cone over the polyhedron $Q = R_0+\ldots+R_m$ embedded into the affine hyperplane $\{(\bullet,\, p^{-1})\}\subseteq \mathbb{A}\times\mathbb{R} = \bar N_{\mathbb{R}}$. Hence, the elements of E can be written as

$$\begin{array}{ll} \bar s^v = [c^v,\eta^v] \quad \text{with} & c^v\in \mathbb{L}^*\cap C^\vee,\ \eta^v\in\mathbb{Z}, \qquad \text{and}\\ & \langle a, -p\,c^v\rangle \leq \eta^v \text{ for } a\in Q \quad (\text{since } \langle (a,\, p^{-1}), [c^v,\eta^v]\rangle \geq 0). \end{array}$$

Let $\{s^1, \ldots, s^w\}$ be an arbitrary lift of E to $\sigma^\vee \cap M \subseteq M = \mathbb{L}^* \times \mathbb{Z}^{m+1}$, i.e.

$$s^v = [c^v; \eta_0^v, \ldots, \eta_m^v] \quad \text{with} \quad \eta_i^v \in \mathbb{Z},\ \eta_0^v + p\,\eta_1^v + \ldots + p\,\eta_m^v = \eta^v \quad \text{and}$$
$$\langle a, -p\,c^v \rangle \le \begin{cases} \eta_0^v & \text{for } a \in R_0 \\ p\,\eta_i^v & \text{for } a \in R_i\ (i \ge 1). \end{cases}$$

Remark: Those integers η_i^v exist, because the given deformation datum is admissible. In m out of the $m+1$ possibilities $i = 0, \ldots, m$ the linear form c^v has integer value on $F(R_i, c^v)$, and we can take this value for $-\eta_i^v$.

Theorem:

(i) The Kodaira-Spencer map sends the whole space $\mathbb{C}^m$ into the homogeneous summand $T^1_Y(-\bar{r})$. (The lattice point $\bar{r} \in \bar{\sigma}^\vee \cap \bar{M}$ was defined as either the common image $[0,p]$ of $r^0, \ldots, r^m$, or by the fact that Q is contained in the affine hyperplane $[\bar{r} = 1]$ of $\bar{N}_{\mathbb{R}}$.)

(ii) Using the T^1_Y-formula of (5.2), the Kodaira-Spencer map equals

$$\varrho : \mathbb{C}^m \longrightarrow \left(L(\cup_{j=1}^N E_j^{\bar{r}}) \Big/ \sum_{j=1}^N L(E_j^{\bar{r}}) \right)^* \otimes_{\mathbb{R}} \mathbb{C}$$

induced by the bilinear map $\mathbb{R}^m \times L(E) \to \mathbb{R}$ that is described by the matrix

$$\begin{pmatrix} \eta_1^1 & \ldots & \eta_1^w \\ \vdots & & \vdots \\ \eta_m^1 & \ldots & \eta_m^w \end{pmatrix}.$$

Remark: The previous theorem justifies the notions "degree" or "homogeneous" built up in (3.1). On the other hand, it shows the limits of the concept of toric deformations; only "strictly negative" degrees (i.e. $-\bar{r}$ with $\bar{r} \in \bar{\sigma}^\vee \cap \bar{M}$) occur in the image of the Kodaira-Spencer map.

(5.4) **Proof:** *Step 1: We show that the maps $\varrho(e^i) : L(E) \to \mathbb{R}$ $(q = (q_v)_{v=1,\ldots,w} \mapsto \sum_{v=1}^w \eta_i^v\, q_v)$ are trivial on each $L(E_j^{\bar{r}})$.*
First, we notice that

$$E_j^{\bar{r}} \neq \emptyset \Leftrightarrow a^j \notin C \Leftrightarrow \langle a^j, \bar{r} \rangle > 0 \Leftrightarrow a^j \text{ corresponds to a vertex of } Q.$$

Let $a^j = (a^*, p^{-1})$ be one of the fundamental generators of $\bar{\sigma}$ that meets these properties. Then, a^* splits into a sum $a^* = a_0^* + \ldots + a_m^*$, and $a_i^* \in R_i$ are vertices defined all by the same hyperplane $t \in \mathbb{L}^* \cap C^\vee$ as $a^* \in Q$. Since our deformation datum is admissible, m of these vertices (say $a_1^*, \ldots, a_m^*$) have to be contained in the lattice $\mathbb{L}$.

For an element $q \in L(E)$ the property "$q \in L(E_j^{\bar{r}})$" means that the components q_v are allowed to be non-trivial at most for $\langle (a^*, p^{-1}), [c^v, \eta^v] \rangle < \langle (a^*, p^{-1}), \bar{r} \rangle$. Since $\bar{r} = [0, p]$, this condition is equivalent to

$$\langle a^*, -p\, c^v \rangle > \eta^v - p.$$

Restricting ourselves to those indices v, we obtain

$$\begin{array}{rcl} \eta^v - p & < & \langle a^*, -p\, c^v \rangle \;=\; \langle a_0^*, -p\, c^v \rangle + \langle a_1^*, -p\, c^v \rangle + \ldots + \langle a_m^*, -p\, c^v \rangle \\ & \leq & \eta_0^v + p\, \eta_1^v + \ldots + p\, \eta_m^v \;=\; \eta^v \,. \end{array}$$

The numbers $\langle a_i^*, -p\, c^v \rangle$ $(i \geq 1)$ are contained in $p\mathbb{Z}$; hence

$$\eta_i^v = \langle a_i^*, -c^v \rangle \quad (\text{for } i = 1, \ldots, m) \quad \text{and} \quad \eta_0^v = \eta^v + \sum_{i=1}^m \langle a_i^*, p\, c^v \rangle.$$

Therefore, if $q \in L(E_j^{\bar{r}})$, the equation $\sum_v q_v \cdot [c^v, \eta^v] = 0$ implies $\sum_v q_v\, \eta_i^v = 0$ for $i = 0, \ldots, m$.

Step 2: To compute the image of the i-th canonical unit vector $e^i \in \mathbb{C}^m$ via the Kodaira-Spencer map, we may assume without loss of generality that $i = 1$ and consider the ring

$$A := \mathbb{C}[\sigma^\vee \cap M] \Big/ \Big((x^{r^1} - x^{r^0})^2, x^{r^2} - x^{r^0}, \ldots, x^{r^m} - x^{r^0} \Big) .$$

Via $\varepsilon \mapsto (x^{r^1} - x^{r^0})$ this is a flat $\mathbb{C}[\varepsilon]/\varepsilon^2$-algebra and, moreover, $\bar{A} := A \otimes \mathbb{C}[\varepsilon]/\varepsilon$ equals the ring $\mathbb{C}[\bar{\sigma}^\vee \cap \bar{M}]$. Spec A is the infinitesimal deformation of $Y = \text{Spec}\, \bar{A}$ obtained by restricting our toric deformation to the tangent vector $e^1 \in \mathbb{C}^m = T_0\, \mathbb{C}^m$, and the corresponding class in T_Y^1 is induced from the $\bar{A}$-linear map $\psi : I/I^2 \to \bar{A}$ described by

$$\psi : P(z_1, \ldots, z_w) \mapsto \left[\frac{P(x^{s^1}, \ldots, x^{s^w})}{x^{r^1} - x^{r^0}} \text{ (computed in } A) \right] \in \bar{A}.$$

We compute ψ for $P(z) = z^\alpha - z^\beta$ (as already mentioned in (5.2), those binomials generate the ideal I). Inside the semigroup $\sigma^\vee \cap M$ it is possible to write

$$\sum_{v=1}^w (\alpha_v - \beta_v) \cdot s^v = \sum_{i=0}^m g_i \cdot r^i \quad (g_i \in \mathbb{Z},\; \sum_i g_i = 0).$$

Claim: $\psi(z^\alpha - z^\beta) = g_1 \cdot x^{(\sum \alpha_v \bar{s}^v) - \bar{r}} \;\in \bar{A}$.

Proof: Similar to (4.1.3) we know that

$$q := \sum_{v=1}^w \alpha_v \cdot s^v - \sum_{i=0}^m g_i^+ \cdot r^i = \sum_{v=1}^w \beta_v \cdot s^v - \sum_{i=0}^m g_i^- \cdot r^i$$

is contained in $\sigma^\vee \cap M$. Hence,

$$\begin{aligned} P(x^{s^1},\ldots,x^{s^w}) &= x^{\sum \alpha_v s^v} - x^{\sum \beta_v s^v} = x^q \cdot \left(x^{\sum g_i^+ r^i} - x^{\sum g_i^- r^i}\right)\\ &= x^q \cdot \left(\prod_{i=0}^m (x^{r^i})^{g_i^+} - \prod_{i=0}^m (x^{r^i})^{g_i^-}\right) \quad \text{(in the ring } \mathbb{C}[\sigma^\vee \cap M]). \end{aligned}$$

Assuming $g_1 \geq 0$, we obtain in the ring A:

$$\begin{aligned} P(x^{s^1},\ldots,x^{s^w}) &= x^q \cdot \left(x^{r^0}\right)^{(\sum g_i^+)-g_1^+} \cdot \left((x^{r^1})^{g_1^+} - (x^{r^0})^{g_1^+}\right)\\ &= \left[x^{r^1} - x^{r^0}\right] \cdot x^q \cdot \left(x^{r^0}\right)^{(\sum g_i^+)-g_1^+} \cdot \left(\sum_{\mu=1}^{g_1^+} (x^{r^1})^{g_1^+-\mu}(x^{r^0})^{\mu-1}\right). \end{aligned}$$

In particular,

$$\psi(z^\alpha - z^\beta) = x^{\bar q} \cdot (x^{\bar r})^{(\sum g_i^+)-g_1^+} \cdot \left(g_1 \cdot (x^{\bar r})^{g_1^+-1}\right) = g_1 \cdot x^{(\sum \alpha_v \bar s^v)-\bar r} \;\in \bar A.$$

Step 3: To finish the proof, we have to involve the η's. Since

$$\begin{aligned} \sum_{v=1}^w (\alpha_v - \beta_v)\, s^v &= \sum_{v=1}^w (\alpha_v - \beta_v) \cdot [c^v; \eta_0^v, \eta_1^v, \ldots, \eta_m^v]\\ &= \left(0; \sum_{v=1}^w \eta_0^v\,(\alpha_v - \beta_v), \sum_{v=1}^w \eta_1^v\,(\alpha_v - \beta_v), \ldots, \sum_{v=1}^w \eta_m^v\,(\alpha_v - \beta_v)\right)\\ &= p^{-1}\left(\sum_{v=1}^w \eta_0^v\,(\alpha_v - \beta_v)\right) \cdot r^0 + \sum_{i=1}^m \left(\sum_{v=1}^w \eta_i^v\,(\alpha_v - \beta_v)\right) \cdot r^i, \end{aligned}$$

we obtain $g_1 = \sum_v \eta_1^v\,(\alpha_v - \beta_v)$. Hence,

$$\psi(z^\alpha - z^\beta) = \sum_v \eta_1^v\,(\alpha_v - \beta_v) \cdot x^{(\sum \alpha_v \bar s^v)-\bar r} = \varrho(e^1)(\alpha - \beta) \cdot x^{(\sum \alpha_v \bar s^v)-\bar r},$$

and we are done by the last remark of (5.2). $\square$

(5.5) The set of Minkowski summands of $\mathbb{R}_{\geq 0} \cdot Q$ itself forms a polyhedral cone. Regarding summands homothetic to Q as trivial (i.e. projecting the associated points in the cone onto 0), the cone of Minkowski summands turns into a vector space denoted by $\tilde T^1(Q)$. We will compare this so-called *vector space of Minkowski summands of* Q with the homogeneous piece $T_Y^1(-\bar r)$.

Definition: Using the notation of (5.2), let F_j be the set of normalized (i.e. integral and primitive) fundamental generators of the face $\bar\sigma^\vee \cap (a^j)^\perp < \bar\sigma^\vee$. Then, we define

$$F_j^{\bar r} := F_j \cap E_j^{\bar r} = \begin{cases} F_j & \text{if } a^j \text{ corresponds to a vertex of } Q \\ \emptyset & \text{otherwise} \end{cases}$$

and, similar to the $T^1_Y(-\bar r)$-formula of (5.2),

$$\tilde T^1_Y(-\bar r) := \left(L(\cup_{j=1}^N F_j^{\bar r}) \Big/ \sum_{j=1}^N L(F_j^{\bar r}) \right)^* \otimes_{I\!R} \mathbb{C}\,.$$

Remark: The inclusions $L(F_j^{\bar r}) \subseteq L(E_j^{\bar r})$ and $L(\cup_j F_j^{\bar r}) \subseteq L(\cup_j E_j^{\bar r})$ yield a canonical linear map $\theta : T^1_Y(-\bar r) \to \tilde T^1_Y(-\bar r)$ which is, in general, neither injective nor surjective. (Compare with Lemma (6.4) in the Gorenstein case!) Composed with the Kodaira-Spencer map it equals

$$\theta \circ \varrho : \mathbb{C}^m \longrightarrow \tilde T^1_Y(-\bar r)\,; \quad e^i \mapsto [q \mapsto \sum_v q_v \cdot \max\langle R_i, -c^v\rangle]\,.$$

Proof: Only a brief notice to the latter formula is necessary. The composite map $\theta \circ \varrho$ is still given (as ϱ itself) by $(e^i,\, q) \mapsto \sum_v \eta_i^v\, q_v$. However, if $[c^v, \eta^v] \in \cup_j F_j$, the general inequality $\langle Q,\, -p\,c^v\rangle \leq \eta^v$ becomes sharp ($[c^v, \eta^v] \in F_j$ implies $\langle a^j,\, -p\,c^v\rangle = \eta^v$); hence $\eta_0^v = \max\langle R_0,\, -p\,c^v\rangle$ and $\eta_i^v = \max\langle R_i,\, -c^v\rangle$ $(i = 1, \ldots, m)$. □

If R is a Minkowski summand of some positive multiple of Q, it will be given by inequalities similar to those describing Q. We define the class $[R] \in \tilde T^1_Y(-\bar r)$ via

$$[R] :\; q = (q_v)_v \mapsto \sum_v \max\langle R,\, -c^v\rangle\,.$$

Proposition: (cf. [Sm]) *Taking the class of a Minkowski summand provides an isomorphism of vector spaces* $\tilde T^1(Q) \otimes_{I\!R} \mathbb{C} \xrightarrow{\sim} \tilde T^1_Y(-\bar r)$.

Using this identification, the modified Kodaira-Spencer map $\theta \circ \varrho : \mathbb{C}^m \to \tilde T^1(Q) \otimes \mathbb{C}$ sends the i-th unit vector e^i onto the i-th Minkowski summand R_i.

(5.6) Let $(R_0, \ldots, R_m;\, C;\, p)$ be an admissible deformation datum. Just to demonstrate that the construction described in (3.4) almost never produces trivial deformations, we will explain what it means for our deformation datum to induce a trivial Kodaira-Spencer map.

Proposition: *The Kodaira-Spencer map $\varrho : \mathbb{C}^m \to T^1_Y$ is trivial, if and only if, up to certain shifts of the polyhedra (cf. Remark (3.5)), the deformation datum fits in one of the following two classes:*

(A) With at most one single exception (which has to be R_0, if $p \geq 2$) the polyhedra R_i equal the cone C. In particular, Q equals the exceptional summand.

(B) $p = 1$*, and there exist a lattice polyhedron* R *and natural numbers* $g_i \in \mathbb{N}$ *such that* $R_i = g_i \cdot R + C$ $(i = 0, \ldots, m)$*. In particular,* $Q = (\sum_{i=0}^m g_i) \cdot R + C$*.*

Proof: It is easy to check that deformations of type (A) or (B) yield trivial Kodaira-Spencer maps. Hence, we focus on the opposite implication.
First, linear algebra tells us that, if

$$\begin{pmatrix} \eta_1^1 & \ldots & \eta_1^w \\ \vdots & & \vdots \\ \eta_m^1 & \ldots & \eta_m^w \end{pmatrix}$$

induces a trivial bilinear map $\mathbb{R}^m \times L(\cup_j E_j^{\bar{r}}) \to \mathbb{R}$, there exist elements $b_i \in \mathbb{A}$, $\beta_i \in \mathbb{R}$ such that

$$(*) \qquad \eta_i^v = \langle b_i,\ c^v \rangle + \beta_i\, \eta^v \quad \text{for } \ \bar{s}^v \in \cup_{j=1}^N E_j^{\bar{r}} \qquad (i = 0, \ldots, m).$$

The set $\cup_j E_j^{\bar{r}}$ contains all pairs $[c^v, \eta^v]$ that are necessary to define Q via the inequalities $\langle \bullet, -p\, c^v \rangle \le \eta^v$. Since the adapted inequalities $\langle \bullet, -p\, c^v \rangle \le p\, \eta_i^v$ (or $\le \eta_0^v$) characterize the Minkowski summands R_i then, we obtain

$$R_0 = -p^{-1} b_0 + \beta_0 \cdot Q + C \quad \text{ and } \quad R_i = -b_i + p\, \beta_i \cdot Q + C \quad (i \ge 1)\,.$$

(If the coefficient β_i does not vanish, then adding C in the formula for R_i is not necessary.)
Having assumed $(R_0, \ldots, R_m;\, C;\, p)$ to be admissible, we know that for every $c \in \mathbb{L}^* \cap C^\vee$ there are m special indices out of $\{0, \ldots, m\}$ (but only for $p = 1$ the choice may really depend on c) featuring certain properties described in (3.3). We distinguish two cases:

Case 1: There is a $c \in \mathbb{L}^* \cap C^\vee$ *such that* $\beta_i = 0$ *for all associated special indices* i*.*
Let $i = 1, \ldots, m$ be these special indices for c. Since $\beta_1 = \ldots = \beta_m = 0$, we obtain $R_i = -b_i + C$ for $i \ge 1$. On the other hand, admissibility implies that the b_i's have to be lattice points; we have obtained (A).

Case 2: For every $c \in \mathbb{L}^* \cap C^\vee$ *there is at least one of the associated special indices* i *such that* $\beta_i \neq 0$*.*
Assume $i = 1, \ldots, m$ to be the special indices associated to some c. We choose a vertex $a^* \in Q$ providing minimal value of the linear form c on Q. As usual, a^* splits into a sum $a^* = a_0^* + \ldots + a_m^*$, and $a_i^* \in R_i$ are vertices minimizing c on R_i.

Claim: Let $\eta \in \mathbb{Z}$ such that $\eta - p < \langle a^*, -p\, c \rangle \le \eta$. Then, $\langle a^*, -p\, c \rangle = \eta$.

Proof: The vertex a^* of Q corresponds to some of the fundamental generators $a^j = (a^*,\, p^{-1})$ of $\bar{\sigma}$. Then, the assumptions of the claim are equivalent to the facts $[c, \eta] \in \bar{\sigma}^\vee$ and $\langle a^j,\, [c, \eta] \rangle < \langle a^j, \bar{r} \rangle$, and we have to show that $\langle a^j,\, [c, \eta] \rangle = 0$.

We may additionally assume that $[c, \eta]$ is an irreducible element of the semigroup $\bar{\sigma}^\vee \cap \bar{M}$, i.e. $[c, \eta] \in E_j^{\bar{r}}$. Then, Step 1 of (5.4) shows that $\eta_i = \langle a_i^*, -c\rangle$ for $i = 1, \dots, m$. Combining this with $(*)$ and the previous formula for the R_i's (implying $a_i^* = -b_i + p\,\beta_i\, a^*$), we obtain

$$\langle b_i, c\rangle + \beta_i\, \eta = \eta_i = \langle a_i^*, -c\rangle = \langle -b_i + p\,\beta_i\, a^*,\, -c\rangle = \langle b_i, c\rangle + \beta_i\, \langle a^*,\, -p\, c\rangle\,.$$

Hence, $\beta_i\, \eta = \beta_i\, \langle a^*,\, -p\, c\rangle$ for $i = 1, \dots, m$. Since $\beta_i \neq 0$ for at least one of these indices, we are done. $(\square)$

Applying this claim to $\eta := [\langle a^*,\, -p\, c\rangle] + 1$ (the smallest integer greater than $\langle a^*,\, -p\, c\rangle$), we obtain two important facts: $p = 1$, and

$$\langle a^*, -c\rangle = \max\langle Q, -c\rangle \in \mathbb{Z}\,.$$

In particular, Q has to be a lattice polyhedron, and, by admissibility, so do the summands $R_0, \dots, R_m$. Moreover, we know that they are all homothetic to Q, and this means (B). $\square$

(5.7) *Deformation data of type A yield trivial deformations:*
The element $\bar{r} = [0,\ p] \in \bar{\sigma}^\vee \cap \bar{M}$ (cf. (5.3)) induces a regular function $x^{\bar{r}} \in \Gamma(Y, \mathcal{O}_Y)$. Now, given a deformation datum of type A, the associated toric deformation $Y \subseteq X \to \mathbb{C}^m$ can be described as follows:

- $X = Y \times \mathbb{C}^m$
- If $z_1, \dots, z_m$ denote coordinates on $\mathbb{C}^m$, then the flat projection $X \to \mathbb{C}^m$ is given by $(z_1 - x^{\bar{r}}, \dots, z_m - x^{\bar{r}})$. Hence, Y is defined by the equations $z_1 = \dots = z_m = x^{\bar{r}}$ in X.

A change of coordinates in X ($z_i' := z_i - x^{\bar{r}}$) shows that this deformation is indeed a trivial one.

(5.8) *Description of type B deformations:*
First, we should remark that non-trivial deformation data of type B (i.e. $g_i > 0$ for at least two indices i) do not exist, if Y is smooth in codimension two. (If $p = 1$ and Q is a lattice polyhedron, then this smoothness condition is equivalent to the fact that the edges do not contain interior lattice points. In particular, Q cannot equal a non-trivial multiple of some other lattice polyhedron.)
Now, if a deformation datum of type (B) is given, denote by V the affine toric Gorenstein variety induced from the polyhedron R in the same way as Y is induced from Q. Then, the relations among V, Y and X can be described in the following way:

(i) The lattice point $\bar{r} \in \bar{M}$ defines coordinate functions t and z on V and Y, respectively. On the other hand, $z : Y \to \mathbb{C}$ can be obtained from $t : V \to \mathbb{C}$ via base change $\pi : \mathbb{C} \to \mathbb{C}\,,\ z \mapsto z^g$.

(ii) The elements $r^0, \ldots, r^m \in \sigma^\vee \cap M$ define variables $z_0, \ldots, z_m$ on X. Then, $(z_0, \ldots, z_m) : X \to \mathbb{C}^{m+1}$ can be obtained from $t : V \to \mathbb{C}$ via base change $\mathbb{C}^{m+1} \to \mathbb{C}$, $(z_0, \ldots, z_m) \mapsto z_0^{g_0} \cdot \ldots \cdot z_m^{g_m}$. In particular, there is a natural map $\pi_X : X \to V$.

(iii) $Y \subseteq X$ is given by the equations $z := z_0 = \ldots = z_m$.

Example: $R = [0,1] \subseteq \mathbb{R}^1$; $m = 1$; $g_0 = g_1 = 1$, $g = 2$.
Then, we obtain

$$\begin{aligned} V &= \mathbb{C}^2(x,y) \quad \text{and } t = x\,y, \\ Y &= [z^2 = x\,y] \subseteq \mathbb{C}^3(x,y,z) \quad \text{and} \\ X &= [z_0\,z_1 = x\,y] \subseteq \mathbb{C}^4(x,y,z_0,z_1). \end{aligned}$$

The maps $\pi : Y \to V$ and $\pi_X : X \to V$ equal the projections onto the (x,y)-coordinates, respectively.

Proposition: *Let $(g_0 \cdot R, \ldots, g_m \cdot R;\ C;\ 1)$ be a deformation datum of type B. Let $Q = g \cdot R$ with $g := \sum_{i=0}^m g_i$, and let Y, V, X , z, and t be as mentioned above.*

(1) The deformation datum $(R_0 := \{0\},\ R_1 := R;\ C;\ g)$ defines a one-parameter deformation of Y (of degree $g \cdot \bar{r}$) which is equal to

$$\begin{array}{ccc} Y & \stackrel{(\pi,z)}{\hookrightarrow} & V \times \mathbb{C}(z) \\ \downarrow & \otimes & \downarrow {\scriptstyle t - z^g} \\ \{0\} & \hookrightarrow & \mathbb{C} \end{array} .$$

(2) The toric deformation induced by $(g_0 \cdot R, \ldots, g_m \cdot R;\ C;\ 1)$ can be regarded as a relative deformation of Y inside $V \times \mathbb{C}$.

Proof: All statements made in the proposition and the previous remarks are straightforward; the only thing we have to do is to describe the embedding of the original deformation $f : X \to \mathbb{C}^m$ into the trivial deformation of $V \times \mathbb{C}$:

$$\begin{array}{ccc} X & \stackrel{i}{\hookrightarrow} & (V \times \mathbb{C}) \times \mathbb{C}^m(s_1, \ldots, s_m) \\ \downarrow {\scriptstyle f = (z_i - z_0)} & & \downarrow {\scriptstyle \mathrm{pr}} \\ \mathbb{C}^m & = & \mathbb{C}^m \end{array}$$

with $i = (\pi_X,\ z_0;\ s_i := z_i - z_0)$. □

Example: We continue our previous example:
The one-parameter deformation of $Y = [z^2 = x\,y] \subseteq \mathbb{C}^3$ (described in (1) of the proposition) is given by the flat map $z^2 - x\,y : \mathbb{C}^3 \to \mathbb{C}$.
Now, we can regard $s_1 : X \to \mathbb{C}$ as a relative deformation of Y inside $\mathbb{C}^3$: The map

$i: X \hookrightarrow \mathbb{C}^4(x,y,z,s_1)$, $(x,y,z_0,z_1) \mapsto (x,y,z_0,z_1-z_0)$, makes X the closed subset of $\mathbb{C}^3 \times \mathbb{C}$ that is given by the equation

$$z^2 - x\,y + s_1\,z = z\,(s_1+z) - x\,y = 0.$$

Moreover, since $(z+s_1/2)^2 - s_1^2/4 = z\,(s_1+z)$, the deformation $s_1: X \to \mathbb{C}$ is equivalent to the relative deformation obtained from the perturbation

$$(z^2 - x\,y) - s_1^2/4 \quad \text{of} \quad (z^2 - x\,y).$$

In particular, this deformation is not a trivial one; the vanishing of the Kodaira-Spencer map comes from the occurrence of s_1^2 instead of s_1.

6 Toric $\mathbb{Q}$-Gorenstein Singularities

(6.1) A variety Y is said to be ($\mathbb{Q}$-) Gorenstein, if (the reflexive hull of some tensor power of) the dualizing sheaf ω_Y is an invertible sheaf on Y. This class of singularities can be considered the next more general one beyond complete intersections.

Fact: *Let $Y = \mathrm{Spec}\,\mathbb{C}[\bar\sigma^\vee \cap \bar M]$ be an affine toric variety given by a cone $\bar\sigma = \langle a^1, \dots, a^N\rangle$ with $a^1, \dots, a^N \in N$ primitive. Then, Y is $\mathbb{Q}$-Gorenstein, if and only if there are a primitive element $R^* \in \bar M$ and a natural number $g \in \mathbb{N}$ such that*

$$\langle a^j,\, R^*\rangle = g \quad \textit{for each } j = 1, \dots, N.$$

Moreover, Y is Gorenstein, if and only if $g = 1$.

Proof: Since toric varieties are normal, the dualizing sheaf can be obtained as the push forward of the canonical sheaf on the smooth part. Hence, in our special situation, ω_Y equals the T-invariant fractional ideal that is given by the order function mapping each fundamental generator a^j onto $1 \in \mathbb{Z}$. On the other hand, the suggested condition means that the g-th multiple of this order function equals some linear form from $\bar M$, i.e. corresponds to some invertible sheaf. (See Theorem I.9 in [Ke].) $\square$

Affine toric varieties of dimension two (the two-dimensional cyclic quotient singularities) are always $\mathbb{Q}$-Gorenstein. Their deformation theory has been well studied (cf. (1.4)) and appears to be different from that of the higher-dimensional singularities. In the present section, we will focus on the deformation theory of toric $\mathbb{Q}$-Gorenstein singularities that are smooth in codimension two; assume that $Y = \mathrm{Spec}\,\mathbb{C}[\bar\sigma^\vee \cap \bar M]$ has this property. Denoting by Q the convex hull $Q := \mathrm{conv}(a^1, \dots, a^N)$, we know that

- Q is a lattice polytope (i.e. compact lattice polyhedron) contained in the affine hyperplane $[\langle \bullet, R^* \rangle = g] \subseteq \bar{M}_{I\!R}$ (since Y is $\mathbb{Q}$-Gorenstein), and
- the one-dimensional edges of Q do not contain any interior lattice point (since Y is smooth in codimension two).

Remark: In most cases we will assume $g = 1$. Then, the previously defined polyhedron Q coincides with the older, equally named one (introduced in (3.4.1) or (3.6)) for the special degree $\bar{r} := R^*$. Since R^* will turn out to be the most interesting (and in many cases the only non-trivial) degree for the deformation theory of toric Gorenstein singularities, this slight abuse of notation should be more helpful than being a problem.

(6.2) Lemma: *Let Y be a $\mathbb{Q}$-Gorenstein variety which is smooth in codimension two. If $R \in \bar{M}$ is a degree such that $\langle a^j, R\rangle \geq 2$ for some $j \in \{1, \ldots, N\}$, then $T^1_Y(-R) = 0$.*

Proof: In general, if Y is not necessary $\mathbb{Q}$-Gorenstein but smooth in codimension two (i.e. if all two-dimensional faces $\langle a^i, a^j \rangle < \bar{\sigma}$ are spanned by a part of a $\mathbb{Z}$-basis of the lattice $\bar{N}$), there is an additional formula for T^1_Y (cf. [Al 1, (4.4)]). Let $R \in \bar{M}$; then, with

$$V_j^R := \operatorname{span}_{I\!R}(E_j^R) = \begin{cases} 0 & \text{for } \langle a^j, R\rangle \leq 0 \\ [a^j = 0] \subseteq \bar{M}_{I\!R} & \text{for } \langle a^j, R\rangle = 1 \\ \bar{M}_{I\!R} & \text{for } \langle a^j, R\rangle \geq 2\,, \end{cases}$$

it says

$$T^1_Y(-R) = \operatorname{Ker}\left[\left(V_1^R \oplus \ldots \oplus V_N^R\right) \Big/ \left(\sum_{\langle a^i, a^j\rangle < \bar{\sigma}} V_i^R \cap V_j^R \right) \twoheadrightarrow V_1^R + \ldots + V_N^R \right]^* .$$

Now, let Y and R as assumed in the lemma. Defining

$$H := \{a \in \bar{N}_{I\!R} \mid \langle a,\, g\,R - R^* \rangle = 0\}\,,$$

we obtain a hyperplane in $\bar{N}_{I\!R}$ that subdivides the set of fundamental generators of σ. The sets H^-, H, and H^+ contain the elements a^j meeting the properties $\langle a^j,\, R\rangle \leq 0$, $\langle a^j,\, R\rangle = 1$, and $\langle a^j,\, R\rangle \geq 2$, respectively. Our assumption on R implies that the latter class of generators is not empty. Moreover, we can fix a map

$$\varphi : \{j \mid \langle a^j, R\rangle = 1\} \to \{1, \ldots, N\}$$

such that for each $a^j \in H$ the element $a^{\varphi(j)}$ is contained in H^+ and adjacent to a^j (i.e. $\{a^j, a^{\varphi(j)}\} \subseteq \bar{\sigma}$ spans a two-dimensional face of $\bar{\sigma}$).
Assume that we are given an element $v = (v_1, \dots, v_N) \in V_1^R \oplus \dots \oplus V_N^R$ such that $v_1 + \dots + v_N = 0$. Adding the terms $[-v_j \cdot e^j + v_j \cdot e^{\varphi(j)}]$ (for $\langle a^j, R\rangle = 1$) does not change the equivalence class of v modulo $\sum_{\langle a^i, a^j\rangle < \sigma} V_i^R \cap V_j^R$. However, non-trivial components survive for $\langle a^j, R\rangle \geq 2$ (corresponding to $V_j^R = \bar{M}_{I\!R}$) only.
The set of those special generators a^j is connected by two-dimensional faces of $\bar{\sigma}$. Moreover, by slightly perturbing R inside $\bar{M}_{I\!R}$, we can find a unique a^* among these edges on which R is maximal. Then, each $a^j \in H^+$ is connected with a^* by an R-monotone path (consisting of two-faces of σ) inside H^+. Now, we can use the previous method of cleaning the components of v once more; the steps from a^j to $a^{\varphi(j)}$ are replaced by the steps on the path from a^j to a^*. There remains an N-tuple v which is non-trivial at most at the a^*-place. On the other hand, the components of v sum up to 0, but this yields $v = 0$. $\square$

(6.3) If $\langle a^j, R\rangle \leq 1$ for every $j \in \{1, \dots, N\}$, then equality holds on some face $\tau < \bar{\sigma}$. The cone τ is top-dimensional in the linear subspace $\tau - \tau \subseteq \bar{N}_{I\!R}$, and it defines a variety $Y_\tau = \operatorname{Spec} \mathbb{C}[(\tau^\vee \cap \bar{M})/(\tau^\perp \cap \bar{M})]$ which is even Gorenstein. The corresponding element $R_\tau^* \in \bar{M}/(\tau^\perp \cap \bar{M})$ can be obtained as the image of R as well as that of $g^{-1}R^*$ using the canonical projection $\bar{M}_{I\!R} \longrightarrow\!\!\!\!\rightarrow \bar{M}_{I\!R}/\tau^\perp$.

Lemma: *In general (even the $\mathbb{Q}$-Gorenstein assumption can be dropped), let $\tau < \bar{\sigma}$ be a face such that $\langle a^j, R\rangle \geq 1$ for $a^j \in \tau$ and $\langle a^j, R\rangle \leq 0$ otherwise. Then, $T^1_Y(-R) = T^1_{Y_\tau}(-R + \tau^\perp)$.*
In the special situation discussed previously this means $T^1_Y(-R) = T^1_{Y_\tau}(-R^*_\tau)$.

Proof: The T^1-formula of (5.2) remains true, if E is replaced by an arbitrary (not necessarily minimal) generating subset of $\bar{\sigma}^\vee \cap \bar{M}$; even a multiset (featuring its elements with multiplicities) could be allowed. Hence, for computing $T^1_{Y_\tau}(-R+\tau^\perp)$, we can use the image $\bar{E}$ of E via the projection

$$\bar{\sigma}^\vee \cap \bar{M} \longrightarrow\!\!\!\!\rightarrow \big((\bar{\sigma}^\vee + \tau^\perp) \cap \bar{M}\big) \Big/ \big(\tau^\perp \cap \bar{M}\big)\,.$$

For $a^j \in \tau$, the corresponding sets $\bar{E}_j^{R+\tau^\perp} \subseteq \bar{E}$ coincide with the images of the subsets E_j^R. For $a^j \notin \tau$, the notion $\bar{E}_j^{R+\tau^\perp}$ does not make sense, and the E_j^R are empty, anyway. It remains to show that the canonical map

$$L(\cup_{j=1}^N E_j^R) \Big/ \sum_{a^j \in \tau} L(E_j^R) \longrightarrow L(\cup_{j=1}^N \bar{E}_j^{R+\tau^\perp}) \Big/ \sum_{a^j \in \tau} L(\bar{E}_j^{R+\tau^\perp})$$

is an isomorphism.
The vector space $\tau^\perp$ is generated by $\tau^\perp \cap \left(\bigcap_{a^j \in \tau} E_j^R\right)$. Hence, by choosing a basis

among these elements, we can embed $\tau^\perp$ into $I\!R^{\tau^\perp\cap(\cap_\tau E_j)}$ to obtain a section of

$$\begin{array}{rcl} I\!R^{\tau^\perp\cap(\cap_\tau E_j)} \subseteq \bigcap_{a^j\in\tau} L(\bar{E}_j^{R+\tau^\perp}) \subseteq L(\cup_{j=1}^N \bar{E}_j^{R+\tau^\perp}) & \longrightarrow\!\!\!\!\rightarrow & \tau^\perp \\ (\dots,\lambda_r,\dots)_{r\in\cup_{j=1}^N E_j} & \mapsto & \sum_{r\in\cup_{j=1}^N E_j} \lambda_r\cdot r\in\tau^\perp\subseteq \bar{M}_{I\!R}. \end{array}$$

In particular, we obtain

$$L(\bar{E}_j^{R+\tau^\perp}) = L(E_j^R)\oplus\tau^\perp \quad (a^j\in\tau) \quad \text{and} \quad L(\cup_{j=1}^N \bar{E}_j^{R+\tau^\perp}) = L(\cup_{j=1}^N E_j^R)\oplus\tau^\perp. \qquad \square$$

(6.4) **Lemma:** *Let Y be an affine toric Gorenstein variety (i.e. $g=1$) induced from a lattice polytope Q. Then, in degree $-R^*$, the linear map $\theta: T_Y^1(-R^*)\to \tilde{T}_Y^1(-R^*)$ introduced in (5.5) is an isomorphism. In particular, the vector space $T_Y^1(-R^*)$ equals the complexified vector space $\tilde{T}^1(Q)\otimes\mathbb{C}$ of Minkowski summands of Q.*

Proof: For the special degree $R^*\in\bar{M}$ we have $E_j^{R^*} = E\cap(a^j)^\perp$. Now, let $s\in\cup_{j=1}^N E_j^{R^*}\subseteq\partial\bar{\sigma}^\vee$ be an element that is *not* a fundamental generator of $\bar{\sigma}^\vee$. Then, there is a minimal face $\alpha<\bar{\sigma}^\vee$ containing s (as a relatively interior point), and we can choose some fundamental generators $s^1,\dots,s^k\in\alpha$ providing $s=\sum_{v=1}^k \lambda_v\, s^v$ ($\lambda_v\in I\!R_{\geq 0}$). For each $E_j^{R^*}$ containing s (which is equivalent to $\alpha<(a^j)^\perp\cap\bar{\sigma}^\vee$) this defines a decomposition

$$L(E_j^{R^*}) = L(E_j^{R^*}\setminus\{s\}) \;\oplus\; I\!R\cdot[\text{relation } s=\sum_{v=1}^k \lambda_v\, s^v]\,,$$

i.e. the second summand can be reduced in the formula for $T_Y^1(-R^*)$ (cf. (5.2)). Since the map θ consists of those steps only, we are done. $\square$

(6.5) We collect the results of (6.2) - (6.4). Let Y be an affine toric $\mathbb{Q}$-Gorenstein variety which is smooth in codimension two; let $Q=\operatorname{conv}(a^1,\dots,a^N)\subseteq[\langle\bullet,R^*\rangle=g]\subseteq\bar{N}_{I\!R}$ be the corresponding lattice polytope. Using the notation of (5.5) (and $\tilde{T}^1(\emptyset):=0$), we obtain the following two equivalent descriptions of T_Y^1:

Theorem:

(1) Let $R\in M$. Then

$$T_Y^1(-R) \;=\; \begin{cases} \tilde{T}^1(\operatorname{conv}\{a^j\mid\langle a^j,R\rangle=1\})\otimes\mathbb{C} & \text{if } \langle a^j,R\rangle\leq 1\ \forall j\\ 0 & \text{otherwise}.\end{cases}$$

(2) Let $\tau<\bar{\sigma}$ be a face of $\bar{\sigma}$. Then

$$T_Y^1\big([-g^{-1}R^*+\operatorname{int}(\bar{\sigma}^\vee\cap\tau^\perp)]\cap\bar{M}\big) = \tilde{T}^1(Q\cap\tau)\,.$$

T_Y^1 vanishes in the remaining degrees.

Immediatly, we can state some applications as a corollary:

Corollary:

(1) If every two-face of Q is a triangle (for instance, if Y is smooth in codimension three), then Y is rigid, i.e. $T^1_Y = 0$.

(2) If Y is Gorenstein ($g = 1$) of dimension at least four ($\dim Q \geq 3$), then the existence of a two-face of Q that is not a triangle implies $\dim T^1_Y = \infty$.

(3) Let Y be not Gorenstein, i.e. $g \geq 2$. Then, $\dim T^1_Y < \infty$ implies $T^1_Y = 0$.

Proof: (1) Smilanski has shown that polytopes with only triangular two-faces admit at most trivial Minkowski decompositions (cf. [Sm, Corollary (5.2)]). In particular, the vector space $\tilde{T}^1$ vanishes for every face of Q.

(2) Two-dimensional polygons with at least four vertices have a non-trivial $\tilde{T}^1$. Hence, a non-triangular two-face of Q yields a proper face $\tau < \bar{\sigma}$ with $\tilde{T}^1(Q \cap \tau) \neq 0$. On the other hand, "proper" means that $[-R^* + \operatorname{int}(\bar{\sigma}^\vee \cap \tau^\perp)] \cap \bar{M}$ contains infinitely many elements, and T^1_Y is non-trivial in all those degrees.

(3) Assume that $T^1_Y \neq 0$. Then there must be a face $\tau < \bar{\sigma}$ and an element $-R \in [-g^{-1}R^* + \operatorname{int}(\bar{\sigma}^\vee \cap \tau^\perp)] \cap \bar{M}$ such that $T^1_Y(-R) = \tilde{T}(Q \cap \tau) \neq 0$.
For $\tau = \bar{\sigma}$ we would obtain $[-g^{-1}R^* + \operatorname{int}(\bar{\sigma}^\vee \cap \tau^\perp)] \cap \bar{M} = \{-g^{-1}R^*\} \cap \bar{M} = \emptyset$. Hence, $\tau < \bar{\sigma}$ must be a proper face, and the same argument as in (2) applies. □

Remark: Let Y be an isolated, toric $\mathbb{Q}$-Gorenstein singularity. The previous corollary implies that, unless Y is three-dimensional and Gorenstein, this singularity is rigid. On the other hand, if Y is a three-dimensional, isolated toric Gorenstein singularity, then

$$T^1_Y = T^1_Y(-R^*) = \tilde{T}^1(Q) \otimes \mathbb{C}\,,$$

and this vector space has dimension $N - 3$.

(6.6) In §3 we have described how to construct homogeneous toric deformations of Y in some given degree $\bar{r} \in \bar{M}$. Now, we apply this to the Gorenstein case (and degree $\bar{r} := R^*$).

Theorem: *Let Y be an affine toric Gorenstein variety induced from a lattice polytope Q. Then, toric m-parameter deformations of degree R^* correspond to Minkowski decompositions of Q into a sum $Q = R_0 + \ldots + R_m$ of $m+1$ lattice polytopes. The Kodaira-Spencer map sends the parameter space $\mathbb{C}^m$ onto the linear subspace $\operatorname{span}_{\mathbb{C}}([R_0], \ldots, [R_m]) \subseteq \tilde{T}^1(Q) \otimes \mathbb{C} = T^1_Y(-R^*) \subseteq T^1_Y$.*

Proof: As already mentioned in Remark (6.1), for $\bar{r} := R^*$ the polytope Q coincides with $\bar{\sigma} \cap \mathbb{A}$ used in (3.6). Moreover, it is obvious that $C = 0$ and $p = 1$ in this context. On the other hand, since Q is a lattice polytope, the admissibility condition

for a deformation datum $(R_0, \ldots, R_m;\, C=0;\, p=1)$ (with $Q = R_0 + \ldots + R_m$) means that $R_0, \ldots, R_m$ have to be lattice polytopes, too.
The statement concerning the Kodaira-Spencer map is a direct consequence of (5.5) and Lemma (6.4). $\square$

Remark: Let Y be a three-dimensional, isolated toric Gorenstein singularity given by some lattice polygon $Q \subseteq \mathbb{R}^2$ with primitive edges $d^j := a^{j+1} - a^j$ ($j \in \mathbb{Z}/N\mathbb{Z}$). Then, since $T^1_Y = T^1_Y(-R^*)$, the previous theorem describes all toric deformations obtained by toric regular sequences. Moreover, decompositions of Q into a Minkowski sum of $m+1$ lattice summands correspond to decompositions of $\{d^1, \ldots, d^N\}$ into a disjoint union of $m+1$ subsets each summing up to 0.

(6.7) Finally, we want to discuss a relation between the toric varieties Y and $Y^\vee$ induced by $\bar\sigma$ and its dual cone $\bar\sigma^\vee$, respectively.
Let $Q \subseteq \mathbb{A}$ be a lattice polytope with primitive edges. As usual, via embedding Q into an affine hyperplane of height one in $\bar N_{\mathbb{R}}$ (cf. (3.4.1) with $p=1$), we obtain $\bar\sigma \subseteq \bar N_{\mathbb{R}}$ as the cone over Q. Now, we have got two affine toric varieties

$$Y = \operatorname{Spec}\mathbb{C}[\bar\sigma^\vee \cap \bar M] \quad \text{and} \quad Y^\vee = \operatorname{Spec}\mathbb{C}[\bar\sigma \cap \bar N]\,;$$

Y is the toric Gorenstein singularity we know from the previous discussions, and $Y^\vee$ is the cone over the projective toric variety $\mathbb{P}(Q)$ defined by the (inner normal fan of the) polytope Q.
According to [Od, Chapter 2], there is a one-to-one correspondence between lattice summands R of positive multiples of Q on the one hand, and nef line bundles $\mathcal{L}(R)$ on $\mathbb{P}(Q)$ on the other hand. Moreover, $\mathcal{L}(R)$ is even ample, if and only if R is combinatorially equivalent to Q, i.e. if and only if both polytopes have the same inner normal fan.
For $\dim Q = 2$, this implies the following result:

Proposition:

(1) $\mathcal{L}(Q)$ is an ample line bundle on $\mathbb{P}(Q)$, and

$$T^1_Y \;=\; \left(\operatorname{Pic}(\mathbb{P}(Q)) \big/ \mathcal{L}(Q)\right) \otimes_{\mathbb{Z}} \mathbb{C} \;=\; \operatorname{Pic}\left(Y^\vee \setminus \{0\}\right) \otimes_{\mathbb{Z}} \mathbb{C}\,.$$

(2) Toric m-parameter deformations of Y correspond to decompositions of $\mathcal{L}(Q)$ into a tensor product of $m+1$ nef invertible sheaves on $\mathbb{P}(Q)$.

7 Examples

(7.1) We start with three-dimensional toric Gorenstein singularities that are defined by polygons Q containing one and only one interior lattice point ("reflex-

ive polygons"). Those polygons were classified by Batyrev and Koelman for different reasons (cf. [Ba], [Ko]); the corresponding singularities are cones over two-dimensional toric Fano varieties with Gorenstein singularities.
Our additional assumption of Y having only an isolated singularity causes that only five polygons Q survive from the original list (containing 16 items). However, including the dual polygons $Q^\vee$ (the cross cuts of the dual cones $\bar{\sigma}^\vee$), we will actually see nine of them.

(7.1.1) See Figure 2.
Y_1 is the cone over $I\!P^1 \times I\!P^1$ embedded by $\mathcal{O}(2,2)$. Q_1 is a quadrangle, hence T^1 is one-dimensional. Moreover, Q_1 is the Minkowski sum of two line segments, i.e. there exists a toric one-parameter deformation which is versal. The total space is an isolated, four-dimensional cyclic quotient singularity.

(7.1.2) See Figure 3.
Y_2 is the cone over $I\!P^2$ embedded by $\mathcal{O}(3)$. Since Q_2 is a triangle, Y_2 is rigid.

(7.1.3) See Figure 4.
Y_3 is the cone over the Del Pezzo surface of degree eight (the blowing up of $(I\!P^2, \mathcal{O}(3))$ in one point). The vector space T^1 is one-dimensional. However, there are no lattice polygons that are non-trivial Minkowski summands of Q_3. That means, Y_3 does not admit any toric deformation at all. Indeed, as Duco van Straten has computed with Macaulay, the versal base space S_3 of Y_3 equals $\operatorname{Spec} \mathbb{C}[\varepsilon]/\varepsilon^2$.

(7.1.4) See Figure 5.
Y_4 is the cone over the Del Pezzo surface of degree seven (obtained from $(I\!P^2, \mathcal{O}(3))$ by blowing up two points, or from $(I\!P^1 \times I\!P^1, \mathcal{O}(2,2))$ by blowing up one point).
T^1 is two-dimensional, but Q_4 admits one decomposition into a Minkowski sum of two lattice polygons (of a line segment and a triangle) only. This yields a one-parameter deformation of Y_4, and its total space is the cone over $I\!P(\mathcal{O}_{I\!P^2} \oplus \mathcal{O}_{I\!P^2}(1))$. The versal base space S_4 is a line with one embedded component (computed by Duco van Straten using Macaulay).

(7.1.5) See Figure 6.
Y_5 is the cone over the Del Pezzo surface of degree six (obtained by blowing up the projective variety of (7.1.4) at one more point). T^1 is three-dimensional, and Q_5 admits two different extremal Minkowski decompositions:

(i) Q_5 equals the sum of two triangles; the corresponding one-parameter family has the cone over $I\!P^1 \times I\!P^1 \times I\!P^1$ as its total space (see Figure 1).

(ii) Q_5 also equals the sum of three line segments. This corresponds to a two-parameter family with the cone over $I\!P^2 \times I\!P^2$ as its total space.

Again, Duco van Straten has computed the versal base space; it is reduced and equals the transversal union of a plane with a line. These components correspond to the toric deformations we have just seen.

(7.2) The cone over the rational normal curve of degree four (cf. Example (1.3)) is a two-dimensional cyclic quotient singularity. These singularities are toric, and this special one is given by the cone $\bar{\sigma} := \langle (1,0); (-1,4) \rangle \subseteq \mathbb{R}^2$.
The dual cone equals $\bar{\sigma}^\vee = \langle (0,1), (4,1) \rangle$, T^1 is four-dimensional, and the homogeneous pieces $T^1(-R)$ are non-trivial only for $R = [1,1], [2,1], [3,1]$.
Now, we cut $\bar{\sigma}$ with the affine hyperplanes corresponding to these values:

(i) $\mathbb{A} := [\langle \bullet, [1,1] \rangle = 1]$; $\quad \mathbb{L} := \mathbb{A} \cap \mathbb{Z}^2$ (in particular $p = 1$).
To get an isomorphism $(\mathbb{A}, \mathbb{L}) \cong (\mathbb{R}, \mathbb{Z})$ we have to choose and fix an origin (contained in $\mathbb{L}$) and a $\mathbb{Z}$-basis of $\mathbb{L}$:
$$(\mathbb{A}, \mathbb{L}) = [(0,1) + \mathbb{R} \cdot (-1,1),\ (0,1) + \mathbb{Z} \cdot (-1,1)]\,.$$
In particular, $Q := \bar{\sigma} \cap \mathbb{A} = [(1,0), (-1/3,\ 4/3)]$ corresponds to $[-1,\ 1/3] \subseteq \mathbb{R}$, and this interval admits only one admissible decomposition into a Minkowski sum:
$$[-1,\ 1/3] = [-1, 0] + [0,\ 1/3]\,.$$

(ii) Analogously, we consider $\mathbb{A} := [\langle \bullet, [3,1] \rangle = 1] = (0,1) + \mathbb{R} \cdot (-1,3) \quad (p = 1)$.
Then, $Q = [-1/3, 1]$ splits into $[-1/3, 1] = [-1/3, 0] + [0, 1]$.

(iii) Let $\mathbb{A} := [\langle \bullet, [2,1] \rangle = 1] = (0,1) + \mathbb{R} \cdot (-1,2) \quad (p = 1)$.
Then, $Q = [-1/2,\ 1/2]$ admits two different decompositions:
$$[-1/2,\ 1/2] = [-1/2, 0] + [0,\ 1/2] = \{1/2\} + [-1, 0]\,.$$

The decompositions (i), (ii) and the first one of (iii) provide the (three-dimensional) Artin component in the versal deformation of our singularity; the remaining decomposition of (iii) yields the other (one-dimensional) component. The corresponding families equal $Y_t \to \mathbb{C}^3$ and $Y_s \to \mathbb{C}$ from (1.3), respectively.

(7.3) Finally, we want to determine those two-dimensional cyclic quotient singularities that correspond to rational intervals of length one in $\mathbb{R}$. (Considering admissible Minkowski decompositions, those intervalls seem to be very interesting; they admit the funny decomposition into $[0,1]$ and an interval of length zero.)
For a given parameter $p \in \mathbb{N}$, each positive rational number can be written as a quotient $x/(pd)$ ($x, d \in \mathbb{N}$ with $\gcd(x,d) = 1$). Then, the interval $Q = [x/(pd),\ x/(pd) + 1]$ yields the cone $\bar{\sigma} = \langle (x/(pd),\ 1/p),\ (x/(pd) + 1,\ 1/p) \rangle = \langle (x,d),\ (x + pd, d) \rangle$.

(i) $\begin{vmatrix} x + pd & d \\ x & d \end{vmatrix} = xd + pd^2 - xd = pd^2.$
This is the order of the cyclic group acting on $\mathbb{C}^2$.

(ii) We will regard x as an (invertible) element of $\mathbb{Z}/d\mathbb{Z}$. Then, we obtain

$$(pdx^{-1} - 1) \cdot (x + pd, d) + (x, d) = (p^2 d^2 x^{-1},\, pd^2 x^{-1}),$$

i.e. both components are divisible by pd^2.

Hence, Q corresponds to the cyclic quotient singularity $X(pd^2, pdx^{-1} - 1)$ which equals A_{p-1} for $d = 1$ and which is called a T-singularity for $d \geq 2$. Those singularities form the fundamental bricks for building P-resolutions (cf. §3 of [KS]).
The canonical decomposition $[x/(pd),\, x/(pd) + 1] = \{x/(pd)\} + [0, 1]$ corresponds to the one-parameter deformation presenting $X(pd^2, pdx^{-1} - 1)$ as a hypersurface in a three-dimensional cyclic quotient singularity (of type $(d; x, d - x)$).

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