Analysis of the Randomized Incremental Construction of Convex Hulls in Space

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1 Clarkson's Theorem in Space

Let $P \subseteq \mathbb{R}^3$ a three-dimensional n-point set in general position (i.e., no four points from P lie on a common plane). We define the set $S_{\leq k}$ of $(\leq k)$-*sets* of P as

$$S_{\leq k} := \{Q \subseteq P \mid |Q| \leq k \text{ and } Q = P \cap h, h \text{ open halfspace}\}.$$

Clarkson's theorem bounds the number of possible $(\leq k)$-sets.

Theorem 1. *We have* $|S_{\leq k}| = O(nk^2)$.

Proof. We assume that $3 \leq k \leq n-3$ since otherwise the theorem is obvious.

We begin with a definition: let $0 \leq \ell \leq k$. A pair $(\{p,q,r\}, s) \in \binom{P}{3} \times \{+,-\}$, consisting of a set of three distinct points from P and a sign $+$ or $-$, is called ℓ-*facet* if and only if $|P \cap h^s_{pqr}| = \ell$. Here, h^+_{pqr} denotes the open halfspace above the plane that is spanned by p,q and r and h^-_{pqr} denotes the open halfspace below the plane spanned by $\{p,q,r\}$. Let $L_{\leq k}$ the set of all $(\leq k)$-facets.

We have $|S_{\leq k}| = O(|L_{\leq k}|)$. Indeed, by appropriate rotations, we can assign to each ℓ-facet a constant number of ℓ-, $(\ell+1)$- and $(\ell+2)$-sets, and we can generate every $(\leq k)$-set in this way.

Now, let $R \subseteq P$ be a random subset of P, that contains every point $p \in P$ independently with probability $1/k$. We consider the set $F(\mathrm{CH}(R))$ of facets on the convex hull of R, and we bound this expectation in two ways.

On the one hand, we have

$$\mathbf{E}[|F(\mathrm{CH}(R))|] \leq 2\mathbf{E}[|R|]) = 2n/k,$$

as the convex hull of R has at most $2|R|-4$ facets and each point of P lies in R with probability $1/k$.

Now let $X = (\{p,q,r\}, s\} \in \binom{P}{3} \times \{+,-\}$ a pair of three points from P and a sign, and let I_X be the indicator random variable for the event that X defines a facet of $\mathrm{CH}(R)$ (this means that $\{p,q,r\}$ bounds a facet of $\mathrm{CH}(P)$ and that h^s_{pqr} does *not* contain the polytope $\mathrm{CH}(R)$). Then,

$$\mathbf{E}[|F(\mathrm{CH}(R))|] = \sum_{(\{p,q,r\},s) \in \binom{P}{3} \times \{+,-\}} \mathbf{E}[I_X] \geq \sum_{X \in L_{\leq k}} \mathbf{E}[I_X],$$

by linearity of expectation. For an $(\leq k)$-facet X, the expectation $\mathbf{E}[I_X]$ is the probability that X is a facet of $\mathrm{CH}(R)$. For this, we must have that (i) $p,q,r \in R$; and (ii) $R \cap h^s_{pqr} = \emptyset$. The probability for this is at least $k^{-3}(1-1/k)^k$, as $|P \cap h^s_{pqr}| \leq k$ and the points in R where chosen independently.

It follows that

$$\mathbf{E}[|F(\mathrm{CH}(R))|] \geq \sum_{X \in L_{\leq k}} \mathbf{E}[I_X] \geq \sum_{X \in L_{\leq k}} k^{-3}(1-1/k)^k \geq |L_{\leq k}|/4k^3,$$

because $k \geq 2$. Therefore, $|L_{\leq k}| \leq 4nk^2$ and $|S_{\leq k}| \leq O(nk^2)$. $\square$

2 The Θ-Series in Space

Let P a 3-dimensional n-point set in general position. In class, we saw that the total work for updating the conflict information during the randomized incremental construction of $\mathrm{CH}(P)$ is asymptotically bounded by

$$\Theta := \sum_{(\{p,q,r\},s)\in\binom{P}{3}\times\{+,-\}} |P \cap h^s_{pqr}| \cdot [\text{The face } (\{p,q,r\},s) \text{ is created during the RIC}].$$

Here, h^s_{pqr} is defined as above, and $[Z]$ is Iverson's notation: $[Z] := 1$, if Z is true, and $[Z] := 0$ otherwise.

The randomized incremental construction of $\mathrm{CH}(P)$ first chooses a random permutation σ of P and inserts the points according to the order of σ into the hull. We will now calculate the expected conflict change. By linearity of expectation:

$$\begin{aligned}\mathbf{E}_\sigma[\Theta] &= \sum_{(\{p,q,r\},s)\in\binom{P}{3}\times\{+,-\}} |P \cap h^s_{pqr}| \cdot \Pr[\text{The facet } (\{p,q,r\},s) \text{ is created during the RIC}] \\ &= \sum_{k=1}^{n-4}\sum_{X\in L_k} k \cdot \Pr[\text{The facet } X \text{ is created during the RIC}],\end{aligned}$$

where L_k is the set of k-facets for P (see above). Since a k-facet $X = (\{p,q,r\},s)$ is created if and only if the permutation σ puts the points p, q, r before the k points in $P \cap h^s_{pqr}$, we have

$$\Pr[\text{The facet } X \text{ is created during the RIC}] = \frac{3!k!}{(k+3)!} = \frac{6}{(k+1)(k+2)(k+3)}.$$

Hence,

$$\mathbf{E}_\sigma[\Theta] = \sum_{k=1}^{n-4}\sum_{X\in L_k} \frac{6k}{(k+1)(k+2)(k+3)} \le \sum_{k=1}^{n-4} \frac{6|L_k|}{k^2}.$$

We write $|L_k| = |L_{\le k}| - |L_{\le(k-1)}|$, where $L_{\le k}$ denotes the set of ℓ-facets of P for $0 \le \ell \le k$. Using summation by parts,

$$\begin{aligned}\mathbf{E}_\sigma[\Theta] &\le \sum_{k=1}^{n-4} \frac{6}{k^2}\Big(|L_{\le k}| - |L_{\le(k-1)}|\Big) \\ &= \frac{6}{(n-3)^2}|L_{\le(n-4)}| - 6|L_{\le 0}| + \sum_{k=1}^{n-4} |L_{\le k}|\Big(\frac{6}{k^2} - \frac{6}{(k+1)^2}\Big) \le O(n) + \sum_{k=1}^{n-4}\frac{18|L_{\le k}|}{k^3},\end{aligned}$$

since $|L_{\le(n-4)}| = O(n^3)$, $|L_{\le 0}| = O(n)$ and

$$\frac{6}{k^2} - \frac{6}{(k+1)^2} = \frac{12k+6}{k^2(k+1)^2} \le \frac{18}{k^3}.$$

By Clarkson's Theorem, we have $|L_{\le k}| = O(nk^2)$, so

$$\mathbf{E}_\sigma[\Theta] = O\Big(\sum_{k=1}^{n-4}\frac{nk^2}{k^3}\Big) = O\Big(n \cdot \sum_{k=1}^{n-4}\frac{1}{k}\Big) = O(n\log n).$$

The expected conflict change, and hence the expected running time for the randomized incremental construction of convex hulls in $\mathbb{R}^3$, is $O(n \log n)$.