

Nearest-Neighbor Decompositions of Drawings*

Jonas Cleve^{†1}, Nicolas Grelier^{‡2}, Kristin Knorr^{§1}, Maarten Löffler³,
Wolfgang Mulzer^{¶1}, and Daniel Perz^{||4}

- 1 Freie Universität Berlin
`{jonascleve, knorrkri, mulzer}@inf.fu-berlin.de`
- 2 ETH Zürich, Department of Computer Science
`nicolas.grelier@inf.ethz.ch`
- 3 Utrecht University
`m.loffler@uu.nl`
- 4 TU Graz
`daperz@ist.tugraz.at`

Abstract

Let \mathcal{D} be a straight-line drawing of a graph in the plane, and let c be a positive integer. We say that \mathcal{D} has a nearest-neighbor decomposition into c parts if there are point sets P_1, \dots, P_c such that \mathcal{D} is the union of the nearest neighbor graphs on P_1, \dots, P_c . We study the complexity of deciding whether it is possible to draw \mathcal{D} as the union of c nearest-neighbor graphs.

1 Introduction

Let $P \subset \mathbb{R}^2$ be a finite planar point set, and let C be a finite set of *colors*. A *coloring* is a function $\sigma : P \rightarrow C$ that assigns a color to each point in P . For any color $c \in C$, we write $P_c = \{p \in P \mid \sigma(p) = c\}$ for the points in P that were colored with c .

The *nearest-neighbor graph for a color* $c \in C$, \mathbb{N}_c , is the embedded graph with vertex set P_c and a straight-line edge between $p, q \in P_c$ if and only if p is the nearest neighbor of q among all points in P_c , or vice versa.¹ We will consider \mathbb{N}_c both as a combinatorial graph, consisting of vertices and edges, and as a subset of the plane, consisting of the points in P_c and the line segments that represent the edges. We write $\mathbb{N} = \bigcup_{c \in C} \mathbb{N}_c$ for the union of the nearest-neighbor graphs of all colors. Again, we consider \mathbb{N} both as a graph and as a set.

We are interested in the following problem: suppose we are given a *drawing* \mathcal{D} , i.e., a graph that is embedded in the plane and whose edges are represented by (possibly crossing) straight line segments. Our general task is to construct a point set P , a set of colors C , and a color assignment σ , such that the union \mathbb{N} of the nearest-neighbor graphs for P and C equals \mathcal{D} , considered as subsets of the plane. We call \mathbb{N} an *NN-decomposition* of \mathcal{D} with vertex set P , and we call $|C|$ the *color-number* of \mathbb{N} . See Figure 1.

* This research was started at the 4th DACH Workshop on Arrangements and Drawings, February 24–28, 2020, in Malchow, Germany. We thank all participants of the workshops for valuable discussions and for creating a conducive research atmosphere.

† Supported in part by ERC StG 757609.

‡ Supported by the Swiss National Science Foundation within the collaborative DACH project *Arrangements and Drawings* as SNSF Project 200021E-171681.

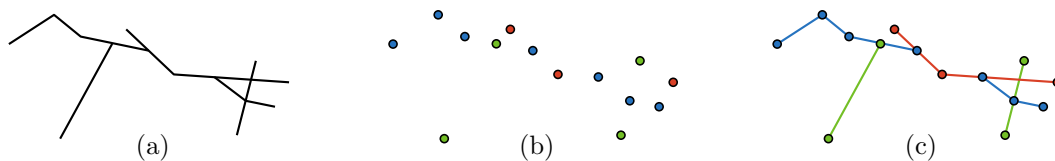
§ Supported by the German Science Foundation within the research training group ‘Facets of Complexity’ (GRK 2434).

¶ Supported in part by ERC StG 757609 and by the German Research Foundation within the collaborative DACH project *Arrangements and Drawings* as DFG Project MU 3501/3-1.

|| Partially supported by FWF within the collaborative DACH project *Arrangements and Drawings* as FWF project I 3340-N35

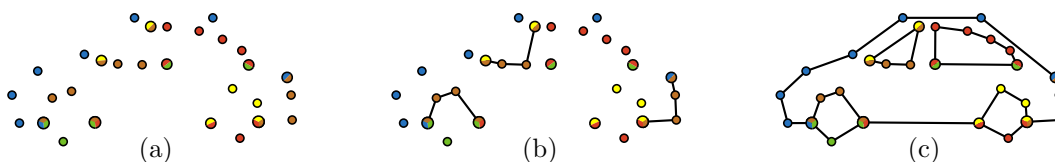
¹ We assume general position but our notion is different: Three points can be on a line but we require that the pairwise distances between the points are all different. Note that our notion of nearest-neighbor graph is undirected, but a directed version also exists.

40:2 Nearest-Neighbor Decompositions of Drawings



■ **Figure 1** (a) A drawing. (b) A set of points with 3 colors. (c) The 3 nearest-neighbor graphs.

Related work. Our problem is motivated from automated content generation for puzzle games: van Kapel introduces a version of *connect-the-dots* puzzles in which the task is to connect dots based on colors rather than numbers [7]. In this puzzle, points may have multiple colors; see Figure 2. He implemented a heuristic approach for generating such puzzles which works well for small instances, but for larger instances generates too many colors to be practical [5].



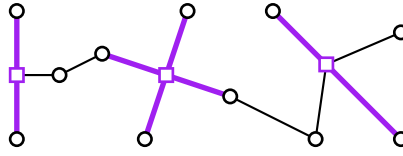
■ **Figure 2** (a) Multi-colored points with 5 colors: blue, red, green, yellow, and orange. (b) The orange nearest-neighbor graph. (c) The union of all nearest-neighbor graphs. Picture taken from [5].

The nearest-neighbor graph of a set of points in the plane is well understood [1, 6]. It is a subgraph of the relative neighborhood graph of the points [4, 6], which in turn is a subgraph of the Delaunay triangulation. The problem of recognizing whether a given abstract graph can be realized as a nearest-neighbor graph of a point set is open and conjectured to be hard. In contrast, testing whether a given embedded graph is a (single) nearest-neighbor graph is easy.

Results. We show that testing if a drawing can be decomposed into c nearest-neighbor graphs so that every vertex of the drawing is a vertex of at least one of the graphs, is polynomial for $c \leq 2$, but NP-hard for $c \geq 3$. If we allow edges of the nearest-neighbor graphs to cross without requiring a vertex at the crossing, the problem is already hard for $c = 2$.

2 Existence of NN-Decompositions on Special Points

Let \mathcal{D} be a straight-line drawing. The *segments* of \mathcal{D} are the inclusion maximal line segments in \mathcal{D} . If s is a line segment with $s \subset \mathcal{D}$ such that s is not a segment of \mathcal{D} , we say that s is *covered* by \mathcal{D} . The *special* points of \mathcal{D} are the endpoints of the line segments in \mathcal{D} and the intersection points between two segments in \mathcal{D} . We require a stronger general position assumption, namely that the pairwise distances between the special points of \mathcal{D} are all distinct. We consider the special case of our problem where the vertex set of the NN-decomposition must consist of the special points. We investigate the question under which circumstances it is possible to find such a *special-point NN-decomposition* of \mathcal{D} .



■ **Figure 3** This drawing highlights the possible violations that make a drawing non-plane.

2.1 The Plane Case

A drawing \mathcal{D} is called *plane* if its segments intersect only at their endpoints, i.e., no segment of \mathcal{D} contains a special point in its relative interior; see Figure 3 for an illustration.

► **Lemma 2.1.** *Let \mathcal{D} be a plane drawing. Suppose there is a special-point NN-decomposition \mathbb{N} of \mathcal{D} . Let σ be the underlying coloring of \mathbb{N} . Then, for any connected component \mathcal{C} of \mathcal{D} , the coloring σ assigns the same color to all special points in \mathcal{C} .*

Proof. Suppose \mathcal{D} has a connected component \mathcal{C} in which σ assigns two distinct colors. Then, \mathcal{C} contains a segment $s = uv$ whose endpoints are colored differently. However, the line segment uv must be covered by \mathbb{N} , and thus, there exists a segment t in \mathbb{N} that contains u , v , and another special point of \mathcal{D} (since the segments in \mathbb{N} are derived from nearest-neighbor relations between points of the same color). By our general position assumption, the segment t is not in \mathcal{D} , so \mathbb{N} is not an NN-decomposition of \mathcal{D} , a contradiction. ◀

Let \mathcal{C} be a connected component in a plane drawing \mathcal{D} , p a special point in \mathcal{C} . We denote by $a(p)$ the special point in $\mathcal{C} \setminus \{p\}$ that is closest to p . We denote by $b(p)$ the set of special points in \mathcal{D} that are strictly closer to p than $a(p)$. By definition, $b(p) \subset \mathcal{D} \setminus \mathcal{C}$. Let \mathcal{C}_1 and \mathcal{C}_2 be two distinct connected components. We say that \mathcal{C}_1 and \mathcal{C}_2 are *incompatible* if there is a special point $p \in \mathcal{C}_1$ such that $b(p) \cap \mathcal{C}_2 \neq \emptyset$, or vice-versa.

We call the connected component \mathcal{C} *NN-representable* if \mathcal{C} is the nearest-neighbor graph of its special points. By Lemma 2.1, if \mathcal{C} is plane, then NN-representability is equivalent to special-point NN-decomposability.

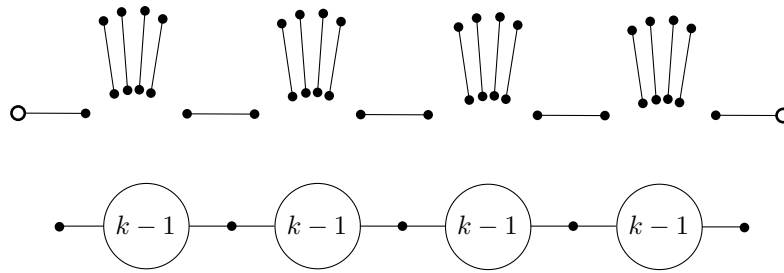
Let \mathcal{D} be a straight-line drawing. We denote by $V = \{\mathcal{C}_i\}_{1 \leq i \leq n}$ the connected components of \mathcal{D} . We define E as the set of pairs $\{\mathcal{C}_i, \mathcal{C}_j\}$ where \mathcal{C}_i and \mathcal{C}_j are incompatible. We say that the graph $G := (V, E)$ is the *incompatibility graph* of \mathcal{D} .

► **Theorem 2.2.** *Let C be a set of colors with $|C| \leq 2$. There is a polynomial time algorithm for the following task: given a plane drawing \mathcal{D} , is there a special-point NN-decomposition of \mathcal{D} with color set C ?*

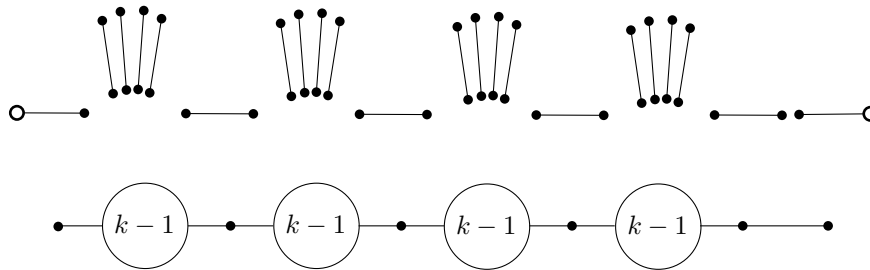
Proof. Let \mathcal{D} be a plane drawing. By Lemma 2.1 each connected component is colored with one color if there is a special-point NN-decomposition of \mathcal{D} . This means every connected component of \mathcal{D} is NN-representable if there is a special-point NN-decomposition of \mathcal{D} . This can be checked in polynomial time, as we only need to compute the nearest-neighbor graph of the special vertices. If the test is negative, the algorithm answers that there is no solution.

Otherwise, we construct the incompatibility graph G of \mathcal{D} , and we check whether G can be colored with C . This takes polynomial time since $|C| \leq 2$ (for $|C| = 2$ check whether G is bipartite, for $|C| = 1$ check that G has no edges). Note, that if G was not C -colorable, then there exists at least one incompatible pair $\{\mathcal{C}_i, \mathcal{C}_j\}$ which was assigned the same color. Since by definition $b(p) \cap \mathcal{C}_j \neq \emptyset$ for a $p \in \mathcal{C}_i$, such a coloring would create a non-existing edge between \mathcal{C}_i and \mathcal{C}_j . If possible, we give all special points in a component \mathcal{C} the color assigned to the corresponding vertex in G . Since all connected components are NN-representable, this

40:4 Nearest-Neighbor Decompositions of Drawings



■ **Figure 4** A 5-wire of length 5 and the general incompatibility graph of a k -wire of length 5.



■ **Figure 5** A 5-chain of length 5 and the general incompatibility graph of a k -chain of length 5.

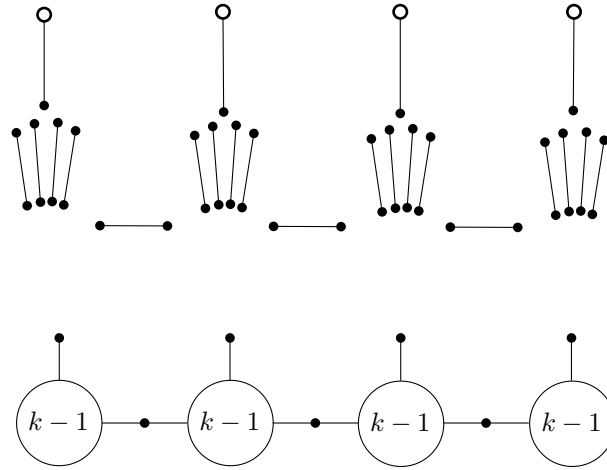
is also a special point NN-decomposition of \mathcal{D} . If G cannot be colored with C , then there is no special point NN-decomposition of \mathcal{D} . Assume to the contrary that there is a special point NN-decomposition of \mathcal{D} . Since every point of a connected component has the same color, we color the corresponding vertex in G with the same color. But then we would have a coloring of G , which contradicts that G cannot be colored with C . ◀

► **Theorem 2.3.** *Let C be a set of colors with $|C| \geq 3$. The following task is NP-complete: given a plane drawing \mathcal{D} , is there a special-point NN-decomposition of \mathcal{D} with color set C ?*

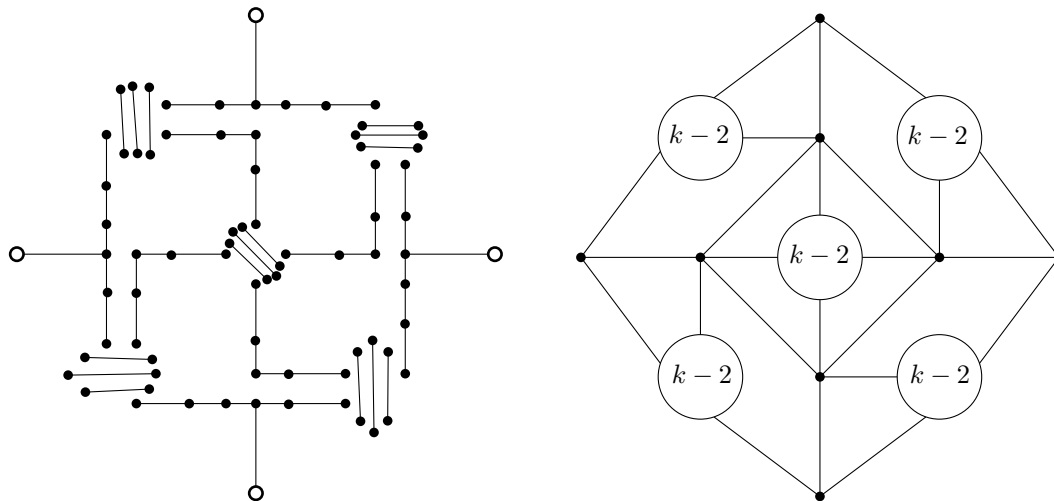
Proof. Gräf, Stumpf, and Weißenfels [3] showed how to reduce k -colorability to k -colorability of unit disk graphs. Our proof is inspired by theirs. Let $k = |C|$. We show the NP-hardness of coloring the special points of \mathcal{D} with $k \geq 3$ colors by means of a reduction from k -colorability. We make use of four types of gadgets: k -wires, k -chains, k -clones, and k -crossings. They are depicted in Figures 4 to 7, together with their incompatibility graphs. The symbol consisting of a number x in a circle denotes a clique of size x . A vertex v connected to such a symbol means that there is an edge between v and all the vertices of the clique. These incompatibility graphs are exactly the gadgets defined by Gräf, Stumpf, and Weißenfels. Note that each connected component in these gadgets is NN-representable. The gadgets shown are for $k = 5$.

In Figures 4 to 6, there are several sets of four segments that are very close and nearly vertical. For other values of k , the gadgets are analogous, but with $k - 1$ almost vertical segments instead of four. Similarly, in Figure 7, there are five sets consisting of three close segments. For other values of k , there are five sets of $k - 2$ segments. In Figures 4 and 5, k -wires and k -chains are drawn as if they were on a line, but they may also bend with a right angle. Note that in Figures 4 to 7 some vertices are specially marked with larger empty circles. These vertices will be called *extreme vertices*.

In Figure 7, there seem to be points lying on a segment between two other points. Actually, these points are shifted by a sufficiently small $\varepsilon > 0$, to ensure our general position assumption.

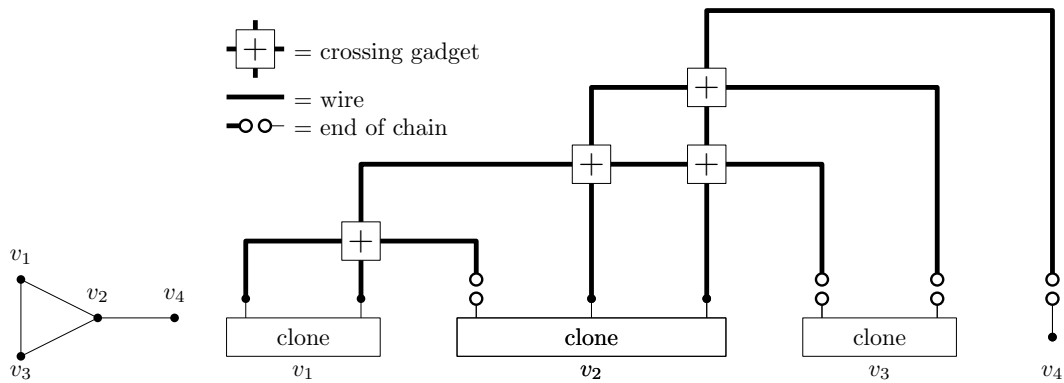


■ **Figure 6** A 5-clone of length 4 and the general incompatibility graph of a k -clone of length 4.



■ **Figure 7** The 5-crossing gadget (left); the incompatibility graph of the k -crossing gadget (right).

40:6 Nearest-Neighbor Decompositions of Drawings



■ **Figure 8** An example graph G with four vertices (left). Converting G to an NN-graph (right).

The main property of a k -wire is that in any coloring with k colors of its incompatibility graph, the extreme vertices are assigned the same color. In contrast, in a k -chain, the extreme vertices are assigned different colors. In a k -clone of length ℓ , there are ℓ extreme vertices. In any coloring with colors from C , all extreme vertices have the same color. Finally, for the k -crossing, opposite extreme vertices must have the same color; a pair of consecutive extreme vertices (e.g., top and left extreme vertices) may or may not be assigned the same color [2].

Now we follow the proof of Gräf, Stumpf, and Weißenfels. Suppose we are given a graph $G = (V, E)$. We describe a drawing \mathcal{D} whose incompatibility graph can be colored with color set C if and only if the vertices of G can be colored with C . Refer to Figure 8. For each vertex v of degree δ in G , we draw a k -clone of size δ . The clones are drawn so that they are arranged on a horizontal line and such that their upper points have the same y -coordinate. Then, for each edge $\{u, v\} \in E$, we draw it on the plane as two vertical segments, each incident to one k -clone, and one horizontal segment that connects the two upper points of the vertical segments. We do that such that for any pair of edges, their horizontal segments have distinct y -coordinates. Then we replace each crossing between a pair of edges by a k -crossing. Finally, let us consider one edge $\{u, v\} \in E$, and let us orient it arbitrarily, say toward v . We replace each part of the edge between two k -crossings by k -wires of sufficient length. If there are no crossings, we replace the edge by a k -chain. Otherwise, the part of the edge between u and the first k -crossing is replaced by a k -wire, and the part between the last k -crossing and v is replaced by a chain. As the points of distinct gadgets are sufficiently remote (except for pairs of gadgets that are connected on purpose), the incompatibility graph of this drawing is the union of the incompatibility graphs of the individual gadgets.

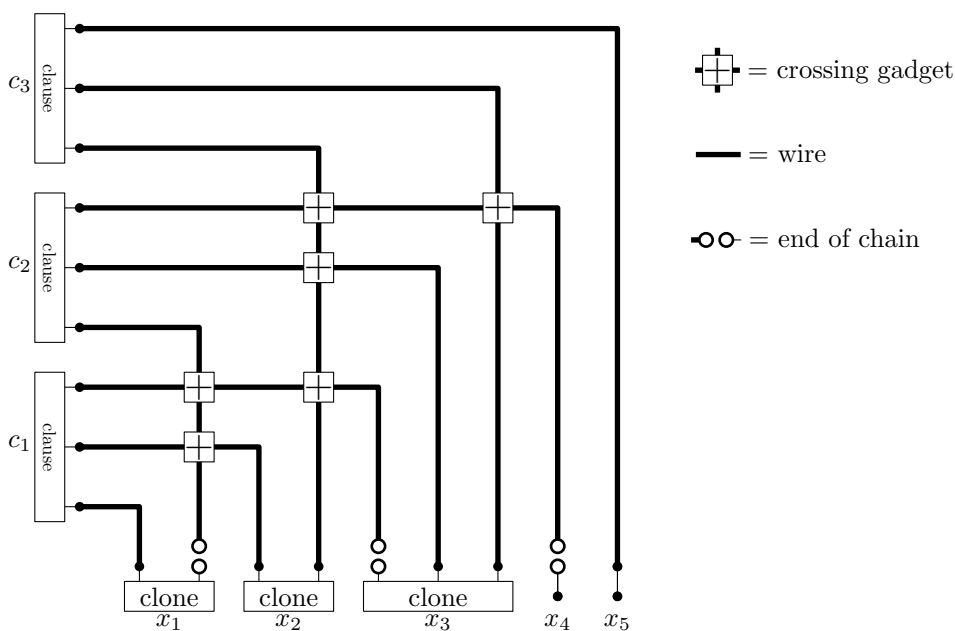
It is possible to find positions with a polynomial number of bits such that all pairwise distances are distinct but at the same time the positions are sufficiently close to the prescribed positions. This concludes the reduction. ◀

2.2 The non-plane case

We show that if drawings are not required to be plane, the problem is hard for two colors.

► **Theorem 2.4.** *Let C be a set of colors with $|C| = 2$. The following task is NP-complete: given a drawing \mathcal{D} , is there a special-point NN-decomposition of \mathcal{D} with color set C ?*

Proof. We reduce from Not-All-Equal 3SAT (NAE-3SAT). Let Φ be an NAE-3SAT formula with variable set X and clause set Y . Let G_Φ be the associated bipartite graph with vertex set $X \cup Y$, where two vertices x and y are adjacent if and only if x is a variable that appears



■ **Figure 9** Structure of the conversion of the NAE-3SAT formula with clauses $c_1 = (x_1, x_2, \neg x_3)$, $c_2 = (\neg x_1, x_3, \neg x_4)$, and $c_3 = (x_2, x_3, x_5)$ into a 2-color \mathcal{N} graph.

in clause y . We draw G_Φ as follows: clauses are represented by vertical segments on the y -axis of length 3. Variables of degree δ are represented as horizontal segments on the x -axis of length δ . Each edge $\{x, y\}$ is drawn as the union of one vertical and one horizontal segment. The vertical segment is incident to the variable gadget for x . The horizontal segment is adjacent to a clause gadget for y . See Figure 9 for an example.

We use some gadgets from the proof of Theorem 2.3. We replace each variable by a 2-clone of length δ . We replace each clause by the gadget in Figure 10 (see Figure 11 for assignments where all literals have the same color), and each crossing by the gadget in Figure 12. In Figure 12, some points have been colored. Note that this does not correspond to an assignment of truth values, but is supposed to provide visual information for the reader. The distance between a green point and a blue point is $1 - \varepsilon$, for a sufficiently small $\varepsilon > 0$. The distance between a blue point and the red point is 1. The distance between the red point and an orange point is $1 + \varepsilon$. The blue point on the left and the orange point on the right are finally shifted by a suitable $\eta > 0$ with $\eta \ll \varepsilon$, so that no two points are at the same distance from the red point. The points in the clause gadget that are on the vertical connected component on the left side are arranged so that this connected component is NN-representable. Finally, each part of an edge between two gadgets is replaced by a 2-wire of suitable length. We have thus obtained a drawing \mathcal{D} .

We claim that Φ is satisfiable if and only if there exists a special-point NN-decomposition of \mathcal{D} with two colors. First, notice a clause gadget has a special-point NN-decomposition if and only if two of the horizontal segments on the right side are assigned different colors. In the non-plane crossing gadget opposite segments are assigned the same color. All of them may be assigned the same color, as in Figure 13a, or consecutive segments might be assigned different colors, as in Figure 13b. Therefore, by associating the colors of C with truth values, \mathcal{D} has a special-point NN-decomposition if and only if Φ is satisfiable. ◀

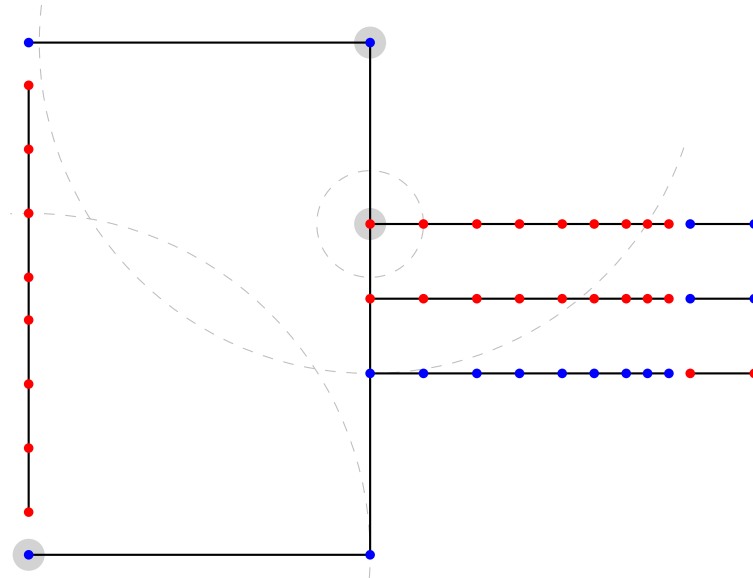
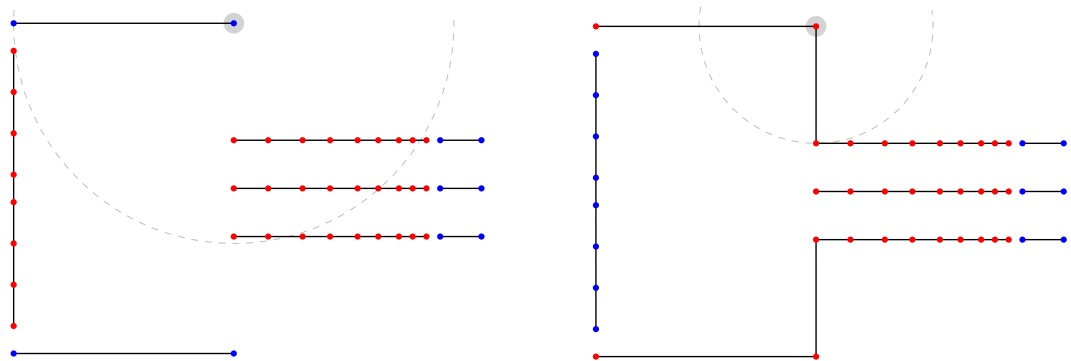


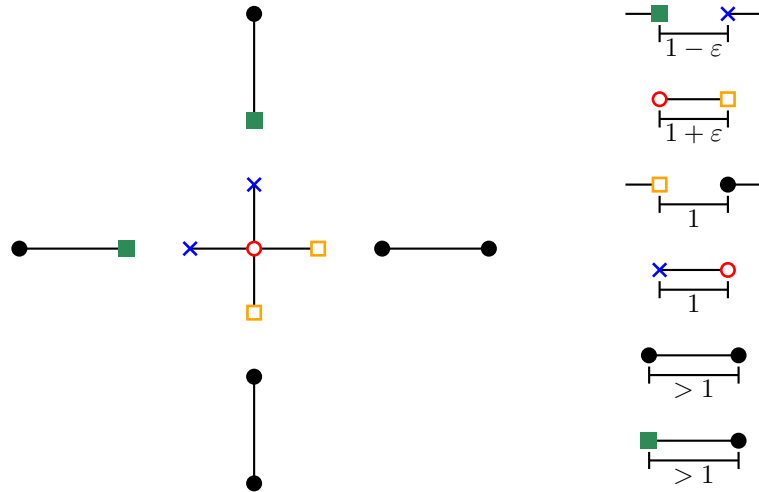
Figure 10 A clause gadget with a valid assignment. For the highlighted vertices the dashed circle indicates the distance to the nearest neighbor of the same color.



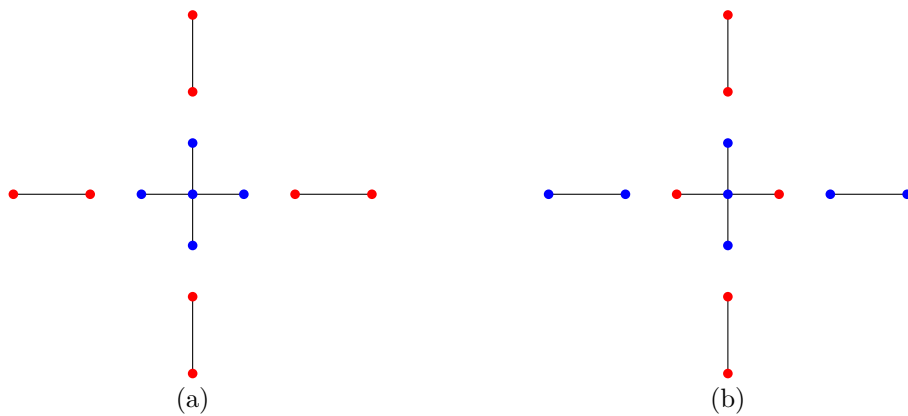
(a) The top and bottom right vertices connect to the vertices to their left.

(b) The vertical connection from the marked vertex does not span the full height.

Figure 11 Invalid assignments on the non-plane clause gadget where all three incoming wires have the same color.



■ **Figure 12** A non-plane crossing gadget.



■ **Figure 13** Two valid assignments on the non-plane crossing gadget. (a) All extreme segments are assigned the same color. (b) Opposite segments are assigned the same color.

References

- 1 D. Eppstein, M.S. Paterson, and F.F. Yao. On nearest-neighbor graphs. *Discrete Comput Geom*, 17:263–282, 1997. doi:10.1007/PL00009293.
- 2 Michael R. Garey, David S. Johnson, and Larry J. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science*, 1:237–267. doi:10/dwvqpj.
- 3 Albert Gräf, Martin Stumpf, and Gerhard Weißenfels. On coloring unit disk graphs. *Algorithmica*, 20(3):277–293, 1998.
- 4 J.W. Jaromeczyk and G.T. Toussaint. Relative neighborhood graphs and their relatives. *Proc. of the IEEE*, 80(9):1502–1517, 1992.
- 5 Maarten Löffler, Mira Kaiser, Tim van Kapel, Gerwin Klappe, Marc van Kreveld, and Frank Staals. The connect-the-dots family of puzzles: Design and automatic generation. *ACM Transactions on Graphics*, 33(4):72, 2014. doi:10.1145/2601097.2601224.
- 6 Joseph S. B. Mitchell and Wolfgang Mulzer. Proximity algorithms. In Jacob E. Goodman, Joseph O’Rourke, and Csaba D. Tóth, editors, *Handbook of Discrete and Computational Geometry*, chapter 32, pages 849–874. CRC Press, Boca Raton, 3rd edition, 2017. doi:10.1201/9781315119601.
- 7 Tim van Kapel. Connect the closest dot puzzles. Master’s thesis, Utrecht University, 2014. URL: <http://dspace.library.uu.nl/handle/1874/296600>.