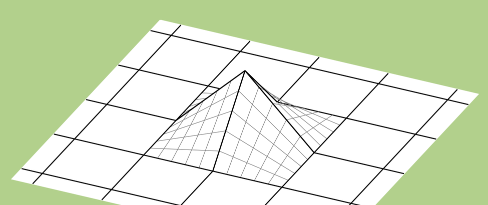

A New Projection Method for the Zero Froude Number Shallow Water Equations

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Abstract. A new semi-implicit projection method for the zero Froude number shallow water equations is presented. This method enforces the divergence constraint on the velocity field, which arises in the limit, in two steps. First, numerical fluxes of an auxiliary hyperbolic system are computed with a standard second order method. Then, these fluxes are corrected by solving two Poisson-type equations. These corrections guarantee that the new velocity field satisfies a discrete form of the divergence constraint.
The main feature of the new method is a unified discretization of the two Poisson-type equations, which rests on a *Petrov-Galerkin* finite element formulation with piecewise bilinear ansatz functions for the unknown. This ansatz naturally leads to piecewise linear ansatz functions for the momentum components. In order to show the stability of the new projection step, a *mixed* formulation is derived, which is equivalent to the Poisson-type equations of the scheme. Existence and uniqueness of the continuous problem are proven and preliminary results regarding the stability of the discrete method are presented.

Governing Equations

The shallow water equations are a hyperbolic system of conservation laws. In their nonlinear form, they are given by

$$\begin{aligned} h_t + \nabla\cdot(h\boldsymbol{v}) &= 0 \\ (h\boldsymbol{v})_t + \nabla\cdot\left(h\boldsymbol{v}\circ\boldsymbol{v} + \tfrac{1}{2\,\mathsf{Fr}^2}h^2\boldsymbol{I}\right) &= \tfrac{1}{\mathsf{Fr}^2}h\nabla h^b \end{aligned}$$

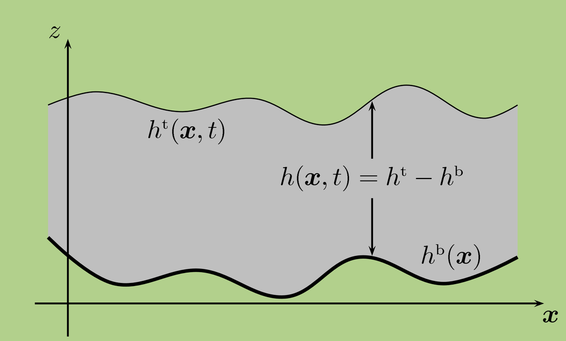
where h is the height and $\boldsymbol{v}$ the velocity. The Froude number Fr is given by the ratio between the velocity of flow and the gravity wave speed. In the zero Froude number limit spatial height variations vanish, but they do affect the velocity field at leading order. The limit equations can be obtained by an asymptotic analysis. Omitting the source terms, one has

$$\begin{aligned} h_t + \nabla\cdot(h\boldsymbol{v}) &= 0 \\ (h\boldsymbol{v})_t + \nabla\cdot(h\boldsymbol{v}\circ\boldsymbol{v}) + h\nabla h^{(2)} &= \boldsymbol{0} \end{aligned}$$

where $h = h_0(t)$ given through the boundary conditions. The spatial homogeneity of the leading order height implies an elliptic *divergence constraint* for the velocity field:

$$\int_{\partial V}(h\boldsymbol{v})\cdot\boldsymbol{n}\,d\sigma = -|V|\frac{dh_0}{dt} \qquad \text{for } V\subset\Omega$$

The zero Froude number shallow water system is no longer hyperbolic, but of *mixed elliptic-hyperbolic* type with an additional unknown $h^{(2)}$.



The Numerical Scheme

For the construction of the method, a finite volume scheme in *conservation form* is considered, i.e.

$$\mathbf{U}_V^{n+1} = \mathbf{U}_V^n - \frac{\delta t}{|V|}\sum_{I\in\mathcal{I}_{\partial V}}|I|\,\mathbf{F}_I \quad \text{with } \mathbf{F}_I := \begin{pmatrix} h(\boldsymbol{v}\cdot\boldsymbol{n}) \\ h\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{n}) + h_0\,h^{(2)}\,\boldsymbol{n}\end{pmatrix}_I$$

The numerical fluxes, which are second order accurate, are computed by

- advective fluxes from a standard explicit finite volume scheme (applied to an *auxiliary system*);
- a *(MAC)-projection*, which corrects the advection velocity divergence;
- a *second (exact) projection*, which adjusts the new time level divergence of the cell-centered velocities

In the Original Scheme [Schneider et al., 1999] both Poisson-type problems (projections) are solved for cell averages (i.e. piecewise constant data). The resulting stencils are standard *finite difference* discretizations. But the second Poisson-type problem admits local decoupling (Figure 1).

In the New (Second) Projection a *Petrov-Galerkin* finite element discretization [Süli, 1991] of the Poisson-type problem is applied with

- piecewise bilinear trial functions for the unknown $h^{(2)}$ (Figure 2), and
- piecewise constant test functions on the dual discretization.

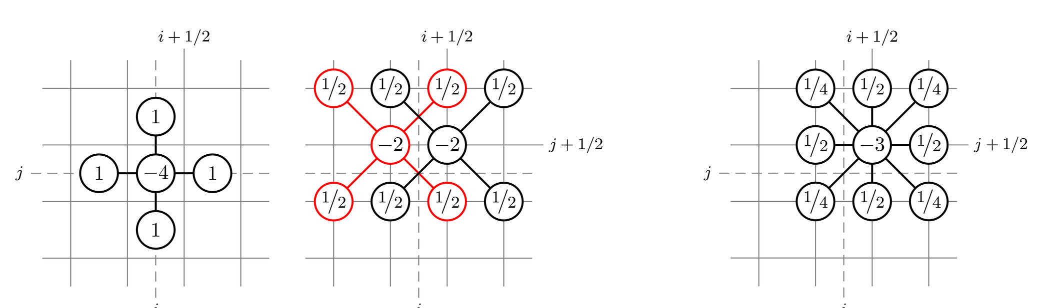


Figure 1: Stencils for the two projections of the original method (left) and for the new discretization (right).

Integration over Ω and using the divergence theorem leads to ($h_0 = const.$):

$$\delta t\,h_0\int_{\partial\bar V}\nabla h^{(2)}\cdot\boldsymbol{n}\,d\sigma = \int_{\partial\bar V}\left((h\boldsymbol{v})^{**} + (h\boldsymbol{v})^n\right)\cdot\boldsymbol{n}\,d\sigma$$

The discrete velocity space consists of piecewise linear functions, which fits with the gradient of $h^{(2)}$ (Figure 2). Using this discretization, the velocity components at the boundary of the dual cells are *piecewise linear*. Thus, the discrete divergence $\mathsf{D}(\boldsymbol{v}) := \frac{1}{|\bar V|}\int_{\partial\bar V}\boldsymbol{v}\cdot\boldsymbol{n}\,d\sigma$ can be *exactly calculated*. Furthermore, the discrete divergence, Laplacian and gradient satisfy $\mathsf{L} = \mathsf{D}(\mathsf{G})$ and the discrete Laplacian has compact stencil (Figure 1).

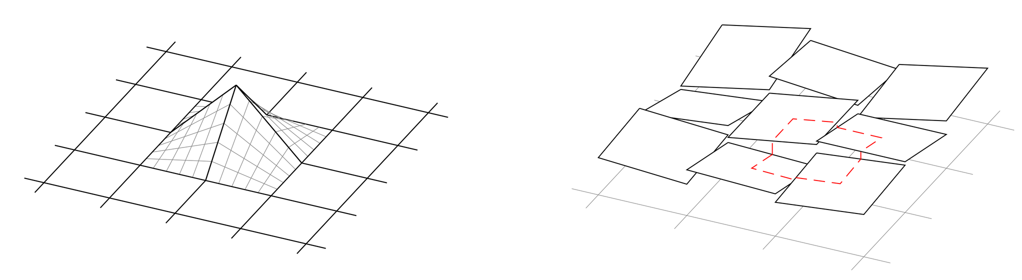


Figure 2: Piecewise bilinear trial function for the unknown $h^{(2)}$ and piecewise linear functions for the velocity components

Stability of the Second Projection

By using the momentum update and the divergence constraint:

$$\begin{aligned} (h\boldsymbol{v})^{n+1} &= (h\boldsymbol{v})^{**} - \delta t\,(h_0\nabla h^{(2)}) \\ \tfrac{1}{2}\nabla\cdot\left[(h\boldsymbol{v})^{n+1} + (h\boldsymbol{v})^n\right] &= -\frac{dh_0}{dt} \end{aligned}$$

the Poisson-type problem can be reformulated as a *generalized saddle-point problem*. The variational formulation is obtained by multiplication with test functions φ and ψ and integration over Ω. Then, the discrete problem obtained by using piecewise linear vector and piecewise constant scalar test functions is equivalent to the Poisson-type equation.

Generalized Saddle-Point Problems: Find $(u,p)\in(\mathcal{X}_2\times\mathcal{M}_1)$, such that

$$\begin{cases} a(u,v) + b_1(v,p) = \langle f,v\rangle & \forall v\in\mathcal{X}_1 \\ b_2(u,q) = \langle g,q\rangle & \forall q\in\mathcal{M}_2 \end{cases} \qquad (1)$$

The abstract theory is given by Nicolaïdes [1982] and Bernardi el al. [1988]: If $b_i(\cdot,\cdot)$ (and similarly $a(\cdot,\cdot)$) satisfies:

$$\inf_{q\in\mathcal{M}_i}\sup_{v\in\mathcal{X}_i}\frac{b_i(v,q)}{\|v\|_{\mathcal{X}_i}\|q\|_{\mathcal{M}_i}} \geq \beta_i > 0$$

Then, (1) has a *unique solution* for all f and g.

Existence & Uniqueness of the continuous problem is analyzed for the following case:

- find a solution with $(h\boldsymbol{v})^{n+1}\in H_0(\mathrm{div};\Omega)$ and $h^{(2)}\in H^1(\Omega)/\mathbb{R}$,
- the test functions are in the spaces $(L^2(\Omega))^2$ and $L^2(\Omega)$,
- the bilinear forms are given by:

$$a(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega\boldsymbol{u}\cdot\boldsymbol{v}\,d\boldsymbol{x}\,,\quad b_1(\boldsymbol{v},q) := \delta t\,h_0\int_\Omega\boldsymbol{v}\cdot\nabla q\,d\boldsymbol{x}$$
$$b_2(\boldsymbol{v},q) := \int_\Omega q\,(\nabla\cdot\boldsymbol{v})\,d\boldsymbol{x}$$

Theorem: Vater [2005] The continuous generalized saddle point problem has a unique solution $((h\boldsymbol{v})^{n+1}, h^{(2)})$.

Stability of the discrete problem:

- It could be proved that $b_1(\cdot,\cdot)$ satisfies a discrete inf-sup condition, but it is still an open question for $a(\cdot,\cdot)$ and $b_2(\cdot,\cdot)$
- Problem: the piecewise linear vector functions are not in $H(\mathrm{div};\Omega)$ in general (*nonconforming* finite elements), and common (e.g. Raviart-Thomas) elements do not match with the piecewise linear, *discontinuous* ansatz functions from the Godunov-Type method.
- The discretization by Schneider et al. [1999] can also be formulated as saddle point problem; but this problem is *unstable*.

Construction of the Fluxes

The auxiliary system

$$\begin{aligned} h_t^* + \nabla\cdot(h\boldsymbol{v})^* &= 0 \\ (h\boldsymbol{v})_t^* + \nabla\cdot\left((h\boldsymbol{v}\circ\boldsymbol{v})^* + \tfrac{1}{2}(h^*)^2\boldsymbol{I}\right) &= \boldsymbol{0} \end{aligned}$$

enjoys the following properties:

- it has the *same convective fluxes* as the zero Froude number shallow water equations;
- the system is *hyperbolic*;
- having constant height h^* and a zero velocity divergence at time t_0, solutions satisfy at $t_0+\delta t$:

$$\nabla\cdot\boldsymbol{v}^* = \mathcal{O}(\delta t)\ ,\quad (h^*\nabla h^*) = \mathcal{O}(\delta t^2)$$

The 1. Projection corrects the convective parts of the fluxes. The *divergence constraint* is imposed at a half time level $t^{n+1/2}$:

$$\frac{dh_0}{dt} = -\nabla\cdot(h\boldsymbol{v})^* + \frac{\delta t}{2}\nabla\cdot(h_0\nabla h^{(2)}) + \mathcal{O}(\delta t^3)$$

The divergence constraint is imposed on each grid cell (Figure 3). This corrects the *convective fluxes* on the boundary of each volume.

The 2. Projection adjusts the interface heights. A second application of the *divergence constraint* yields:

$$\delta t\,\nabla\cdot(h_0\nabla h^{(2)}) = \nabla\cdot(h\boldsymbol{v})^{**} + \nabla\cdot(h\boldsymbol{v})^n + 2\frac{dh_0}{dt} + \mathcal{O}(\delta t^2)$$

Here, the divergence constraint is imposed on a dual discretization (Figure 3). This corrects the momentum to obtain a correct divergence for the *new velocity field* at time t^{n+1}.

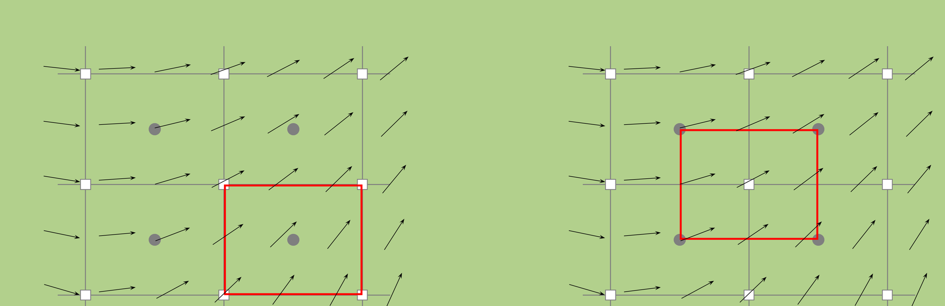


Figure 3: Application of the divergence constraint in the 1. and 2. projection.

Approximate and Exact Projection

The new discrete divergence is affected by the mean values and the *partial derivatives* u_y and v_x of the velocity field. To apply this discretization into the whole scheme, there are two possibilities:

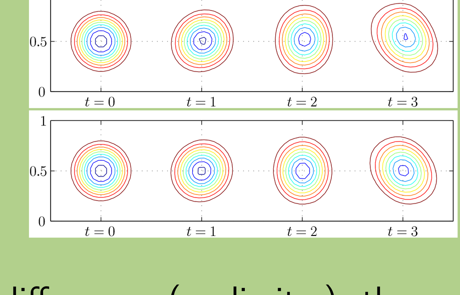
- *Approximate* projection method: Using just the mean values to correct momentum
$$(h\boldsymbol{v})_V^{n+1} = (h\boldsymbol{v})_V^{**} - \delta t\,h_0\,\overline{\mathsf{G}(h^{(2)})}$$
we obtain $\mathsf{D}(\boldsymbol{v}^{n+1}) = \mathcal{O}(\delta t\,\delta x^2)$.
- *Exact* projection method: An additional correction of the derivatives of the momentum components within one cell and their employment in the reconstruction of the next predictor step.

Numerical Results

In a Convergence Study a Taylor Vortex is advected in diagonal direction [cf. Minion, 1996; Almgren et al., 1998]. The numerical solution is computed on three different grids and compared to the exact solution at $t = 3$. It showed that:

- second order accuracy is obtained in the L^2 and the L^∞ norms;
- the absolute error obtained with the new exact projection is about *four times smaller* on fixed grids.

Advection of a Vortex: In this test a stationary vortex is advected by constant background flow on a rectangular domain with 80×20 grid cells [Gresho and Chan, 1990].



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Using central differences (no limiter), the results for the new exact projection show that there is *less deviation* from the center line of the channel and the loss in vorticity is *slightly reduced* compared to the original discretization.

References

Schneider, T., Botta, N., Geratz, K. J., and Klein, R. [1999]. Extension of Finite Volume Compressible Flow Solvers to Multi-dimensional, Variable Density Zero Mach Number Flows. *J. Comput. Phys.*, 155: pp. 248–286.

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