Where is pointwise multiplication in real CK-spaces locally open?

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Abstract. Let K be a compact Hausdorff space, the space CK of real-valued continuous functions on K is provided with the suprumum norm. The closed ball with center f and radius r will be denoted by $B(f,r)$.

Given $f,g \in CK$ it might be true or not that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $B(fg,\delta) \subset B(f,\varepsilon)B(g,\varepsilon)$; here fg is the pointwise product of f and g, and $B(f,\varepsilon)B(g,\varepsilon) := \{\tilde{f}\tilde{g} \mid \tilde{f} \in B(f,\varepsilon), \tilde{g} \in B(g,\varepsilon)\}$. If this is the case we say that *multiplication is locally open at* (f,g).

For the case $K = [\,0,1\,]$ a characterization of the pairs (f,g) where multiplication is locally open is known. In the present paper we extend these results to arbitrary K.

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1. Introduction

Let X,Y,Z be Banach spaces and $T : X \times Y \to Z$ a bilinear mapping. For certain rather special situations one can show that T is locally open at every pair (x,y): for all $x \in X, y \in Y$ and every $\varepsilon > 0$ one can find a positive δ such that

$$B_Z(T(x,y),\delta) \subset \{T(\tilde{x},\tilde{y}) \mid \tilde{x} \in B_X(x,\varepsilon),\ \tilde{y} \in B_Y(y,\varepsilon)\}.$$

(Balls in X,Y,Z are denoted by B_X, B_Y, B_Z, respectively.) It might even happen that, given ε, the same δ can be chosen for all x,y ; T is called *uniformly open* in this case (cf. [3]). On the other hand there are simple examples where T is *not* locally open at some (x,y), and by now it is not well understood how the interplay between the structure of the spaces under consideration and properties of the mapping T gives rise to such phenomena.

A natural candidate to be considered here is the case $X = Y = Z = A$, where A is a Banach algebra and T is the multiplication in A. Even for this seemingly simple situation surprisingly little is known. We mention here investigations of Komisarski (zero-dimensional CK-spaces, [7]) and the author ([4], multiplication on $C\,[\,0,1\,]$). For further result concerning local openness of multilinear mappings see [2], [5] and [8].

The aim of the present paper is a characterization of local openness for the case of general CK-spaces: multiplication will be locally open at f, g if and only if K "splits" suitably at those regions where both f and g are close to zero.

In order to make this precise we need some notation. We fix a nonvoid compact Hausdorff space K and two continuous functions $f, g : K \to \mathbb{R}$. The map $\phi_{f,g} : K \to \mathbb{R}^2$ will be defined by $k \mapsto \big(f(k), g(k)\big)$.

We will also deal with some special subsets of $\mathbb{R}^2$. For positive η, ε we define

$$\begin{aligned}
M_{\eta,1} &:= \{(x,y) \mid x,y \in \mathbb{R},\, xy \geq \eta\};\\
M_{\eta,2} &:= \{(x,y) \mid x,y \in \mathbb{R},\, xy \leq \eta\};\\
M_{\eta,\varepsilon,1} &:= \{(x,y) \mid x,y \in \mathbb{R},\, xy \leq \eta,\, x \geq -\varepsilon, y \leq \varepsilon\};\\
M_{\eta,\varepsilon,2} &:= \{(x,y) \mid x,y \in \mathbb{R},\, xy \leq \eta,\, x \leq \varepsilon, y \geq -\varepsilon\};\\
M_{-\eta,1} &:= \{(x,y) \mid x,y \in \mathbb{R},\, xy \leq -\eta\};\\
M_{-\eta,2} &:= \{(x,y) \mid x,y \in \mathbb{R},\, xy \geq -\eta\};\\
M_{-\eta,\varepsilon,1} &:= \{(x,y) \mid x,y \in \mathbb{R},\, xy \geq -\eta,\, x,y \leq \varepsilon\};\\
M_{-\eta,\varepsilon,2} &:= \{(x,y) \mid x,y \in \mathbb{R},\, xy \geq -\eta,\, x,y \geq -\varepsilon\}.
\end{aligned}$$

These sets are sketched in the following pictures:

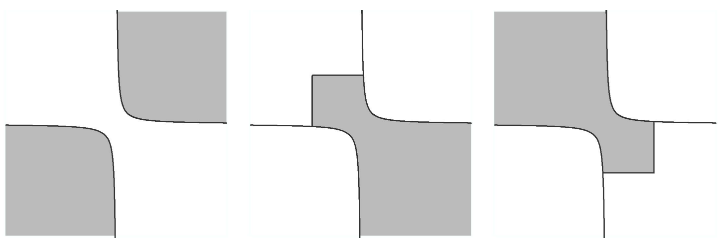

Figure 1: $M_{\eta,1}, M_{\eta,\varepsilon,1}$ and $M_{\eta,\varepsilon,2}$; $M_{\eta,2} = M_{\eta,\varepsilon,1} \cup M_{\eta,\varepsilon,2}$.

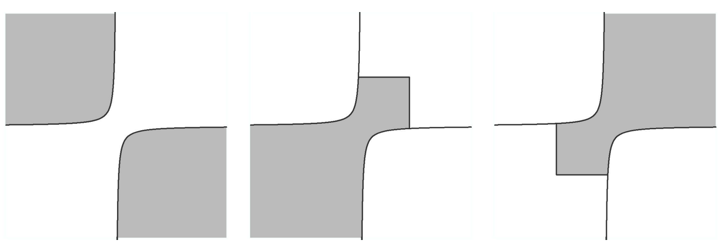

Figure 2: $M_{-\eta,1}, M_{-\eta,\varepsilon,1}$ and $M_{-\eta,\varepsilon,2}$; $M_{-\eta,2} = M_{-\eta,\varepsilon,1} \cup M_{-\eta,\varepsilon,2}$.

Since the union of the $M_{\eta,1}, M_{\eta,\varepsilon,1}, M_{\eta,\varepsilon,2}$ (resp. of the $M_{-\eta,1}, M_{-\eta,\varepsilon,1}, M_{-\eta,\varepsilon,2}$) is all of $\mathbb{R}^2$ the union of the preimages $\phi_{f,g}^{-1}(M_{\eta,1}), \phi_{f,g}^{-1}(M_{\eta,\varepsilon,1}), \phi_{f,g}^{-1}(M_{\eta,\varepsilon,2})$ (resp. the union of the $\phi_{f,g}^{-1}(M_{-\eta,1}), \phi_{f,g}^{-1}(M_{-\eta,\varepsilon,1}), \phi_{f,g}^{-1}(M_{-\eta,\varepsilon,2})$) will exhaust K.

In particular it is true that $\phi_{f,g}^{-1}(M_{\eta,2})$ (resp. $\phi_{f,g}^{-1}(M_{-\eta,2})$) is the union of $\phi_{f,g}^{-1}(M_{\eta,\varepsilon,1})$ and $\phi_{f,g}^{-1}(M_{\eta,\varepsilon,2})$ (resp. of $\phi_{f,g}^{-1}(M_{-\eta,\varepsilon,1})$ and $\phi_{f,g}^{-1}(M_{-\eta,\varepsilon,2})$). Our characterization states that multiplication is locally open at (f,g) if and only if this union can be replaced by the union of suitable disjoint closed sets:

1.1 Theorem: *Let K be a nonvoid compact Hausdorff space. For $f,g \in CK$, the space of real-valued continuous function on K, provided with the supremum norm, the following are equivalent:*

(i) Multiplication is locally open at (f,g).

(ii) The following two conditions are satisfied:

(C_1) *For every $\varepsilon_0 > 0$ there is an $\eta > 0$ with the following property: $\phi_{f,g}^{-1}(M_{\eta,2})$ can be written as the disjoint union of two closed subsets K_1, K_2 such that $\phi_{f,g}(K_1) \subset M_{\eta,\varepsilon_0,1}$ and $\phi_{f,g}(K_2) \subset M_{\eta,\varepsilon_0,2}$.*

(C_2) *For every $\varepsilon_0 > 0$ there is an $\eta > 0$ with the following property: $\phi_{f,g}^{-1}(M_{-\eta,2})$ can be written as the disjoint union of two closed subsets K_1, K_2 such that $\phi_{f,g}(K_1) \subset M_{-\eta,\varepsilon_0,1}$ and $\phi_{f,g}(K_2) \subset M_{-\eta,\varepsilon_0,2}$.*

The proof will be given in section 3, some necessary preparations are provided in section 2. In section 4, we discuss some consequences, and finally, in section 5, we collect some invitations for further research.

2. Some preparations

Positive and negative constant differences will suffice

Fix a positive δ and let α, β, γ be real numbers such that the polynomial $P : \lambda \mapsto \alpha + \beta\lambda + \gamma\lambda^2$ satisfies $P(0) = -\delta$ and $P(1) = \delta$. Then P has in fact only one free parameter: if γ is selected arbitrarily then $P(\lambda) = -\delta + (2\delta - \gamma)\lambda + \gamma\lambda^2 =: P_\gamma(\lambda)$. Figure 3 shows sketches of some P_γ where γ varies from negative values (light gray) to positive values (black):

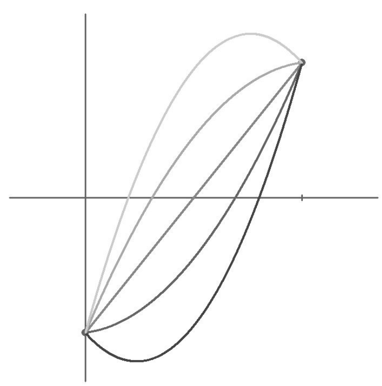

Figure 3

As a consequence of the intermediate value theorem we know that for every $a \in [-\delta, \delta]$ there will exist a $\lambda \in [0,1]$ such that $P_\gamma(\lambda) = a$. The numbers λ and a are sketched in figure 4:

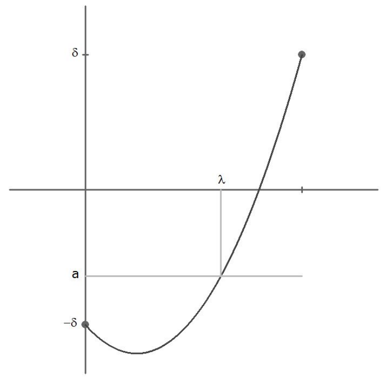

Figure 4

One sees that, for certain γ, there might be two solutions λ of $P_\gamma(\lambda) = \delta$ or $P_\gamma(\lambda) = -\delta$. We claim that nevertheless one can choose λ as a continuous function of a and γ:

2.1 Lemma: *There is a continuous function $\Lambda : [\,-\delta, \delta\,] \times \mathbb{R} \to [\,0, 1\,]$ such that for all $a \in [\,-\delta, \delta\,]$ and all $\gamma \in \mathbb{R}$ one has $P_\gamma\big(\Lambda(a, \gamma)\big) = a$.*

Proof: Our definition will be as follows. We put

$$\Lambda(a, 0) := \frac{a}{2\delta} + \frac{1}{2}$$

for all a, and for $\gamma \neq 0$ and $a \in \mathbb{R}$ we define

$$\Lambda(a, \gamma) := \frac{(\gamma - 2\delta) + \sqrt{(\gamma - 2\delta)^2 + 4\gamma(a + \delta)}}{2\gamma}.$$

It is plain that Λ satisfies $P_\gamma\big(\Lambda(a, \gamma)\big) = a$. Continuity on $[\,-\delta, \delta\,] \times (\mathbb{R} \setminus \{0\})$ is clear, and for the proof that Λ is continuous on the strip $[\,-\delta, \delta\,] \times \{0\}$ one has to use the fact that $\sqrt{1 + x} \approx 1 + x/2$, where the error term of order x^2. □

The following proposition, by which the investigations to come will be considerably simplified, is an easy consequence of this lemma:

2.2 Proposition: *For $f, g \in CK$ and positive ε, δ the following two conditions are equivalent:*

(i) $B(fg, \delta) \subset B(f, \varepsilon)B(g, \varepsilon)$.

(ii) $fg - \underline{\delta}$ and $fg + \underline{\delta}$ lie in $B(f, \varepsilon)B(g, \varepsilon)$. (For any real number α we denote by $\underline{\alpha}$ the constant function $k \mapsto \alpha$.)

Proof: We only have to show that (i) follows from (ii). So suppose that we may choose functions $d_1^-, d_2^-, d_1^+, d_2^+ \in CK$ with norm at most ε such that

$$(f + d_1^-)(g + d_2^-) = fg - \underline{\delta},\ (f + d_1^+)(g + d_2^+) = fg + \underline{\delta}.$$

A function $d \in CK$ with $-\underline{\delta} \leq d \leq \underline{\delta}$ is given, and we have to find $d_1, d_2 \in CK$ with $||d_1||, ||d_2|| \leq \varepsilon$ such that $(f + d_1)(g + d_2) = fg + d$.

This will be achieved by convex combinations of the $d_1^{\pm}, d_2^{\pm}$ that vary continuously with k. Fix $k \in K$ and consider for $\lambda \in [\,0,1\,]$ the convex combinations

$$d_1^{\lambda} := \lambda d_1^{+}(k) + (1-\lambda) d_1^{-}(k),\;\; d_2^{\lambda} := \lambda d_2^{+}(k) + (1-\lambda) d_2^{-}(k).$$

What is a suitable λ such that $\big(f(k)+d_1^{\lambda}\big)\big(g(k)+d_2^{\lambda}\big) = f(k)g(k)+d(k)$ i.e., $f(k)d_2^{\lambda} + g(k)d_1^{\lambda} + d_1^{\lambda}d_2^{\lambda} = d(k)$?

We observe that $\lambda \mapsto f(k)d_2^{\lambda} + g(k)d_1^{\lambda} + d_1^{\lambda}d_2^{\lambda}$ is a polynomial of degree at most two that assumes the value $-\delta$ (resp. δ) at 0 (resp. 1). Thus it is of the form $-\delta + (2\delta - \gamma(k))\lambda + \gamma(k)\lambda^2$; note that $\gamma(k)$ is composed from $f(k), g(k), d_1^{\pm}(k), d_2^{(}k), d(k)$ so that $\gamma(k)$ varies continuously with k.

We define $\tilde{\lambda} : K \to [\,0,1\,]$ by $\tilde{\lambda}(k) := \Lambda\big(d(k), \gamma(k)\big)$ (with Λ as in lemma 2.1). Then $d_1 := \tilde{\lambda} d_1^{-} + (1-\tilde{\lambda}) d_1^{+}$ and $d_2 := \tilde{\lambda} d_2^{-} + (1-\tilde{\lambda}) d_2^{+}$ will have the desired properties: these functions are continuous, their norm is bounded by ε, they satisfy $fd_2 + gd_1 + d_1d_2 = d$ and consequently also $(f+d_1)(g+d_2) = fg + d$. □

The strategy of our proof

Suppose that, for some given positive ε_0, δ, we want to show that $fg - \underline{\delta}$ lies in $B(f,\varepsilon_0)B(g,\varepsilon_0)$. Consider the following picture where some level curves of the function $(x,y) \mapsto H(x,y) := xy$ are depicted:

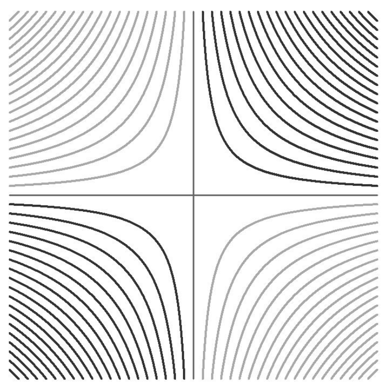

Figure 7: Some level curves of $(x,y) \mapsto xy =: H(x,y)$ (positive: black; negative: gray).

We have to find continuous $d_1, d_2 : K \to \mathbb{R}$ such that for all k the point $\big(f(k)+d_1(k), g(k)+d_2(k)\big)$ lies on a level surface where the associated value is precisely δ units smaller than the value of the level surface that contains $\phi_{f,g}(k)$. For every k there are infinitely many possibilities to choose $d_1(k), d_2(k)$, and the problem is to make these choices in such a way that they give rise to continuous functions d_1, d_2.

A natural first approach would be to define $d_1(k), d_2(k)$ such that these numbers do not depend on k but only on $\big(f(k), g(k)\big)$. (In fact this will work only for simple situations. For the general case the idea will have to be refined.)

2.3 Definition: *Let G be a subset of $\mathbb{R}^2$ and ε, δ positive numbers. A continuous function $\psi : G \to \mathbb{R}^2$ will be called an $(\varepsilon, -\delta)$* vector field *if the following two conditions are satisfied:*

(i) For all $v \in G$ one has $H\big(v+\psi(v)\big) = H(v) - \delta$.

(ii) $||\psi(v)|| \leq \varepsilon$ for all $v \in G$. (In this paper we will work with the maximum norm on $\mathbb{R}^2$.)

An (ε, δ) vector field *is defined similarly: in condition (i) the relevant equation is now $H\big(v+\psi(v)\big) = H(v) + \delta$.*

Suppose that G is such that for every $\varepsilon > 0$ there is a positive δ for which one can find an $(\varepsilon, -\delta)$ vector field ψ. We also assume that the range of $\phi_{f,g}$ is contained in G. Then it is obvious that $fg - \underline{\delta}$ lies in $B(f,\varepsilon)B(g,\varepsilon)$: one only has to define d_1, d_2 by $(d_1, d_2) := \psi \circ \phi_{f,g}$; these functions will have the desired properties.

Therefore the problem arises to find G were such ψ can be constructed. As an example consider the set $G := \{(x,y) \mid ||(x,y)|| \geq \alpha_0\}$, where α_0 is positive. It is not hard to prove that for every $\varepsilon > 0$ there exists a positive δ such that G admits (ε, δ) and $(\varepsilon, -\delta)$ vector fields so that multiplication is locally open at (f,g) whenever the range of $\phi_{f,g}$ lies in G. Due to the compactness of K this observation can be applied whenever f, g have no common zeros. (This was already observed in Komisarski's paper [7].)

Everything would be very simple if $G = \mathbb{R}^2$ were admissible. That this cannot be true follows from the fact that there are (f,g) where multiplication is *not* locally open, but a simple direct proof is also possible:

2.4 Proposition: *Let ε be positive. Then there exists for no $\delta > 0$ an (ε, δ) vector field or an $(\varepsilon, -\delta)$ vector field.*

Proof: Suppose that for some positive δ an (ε, δ) vector field ψ would exist. We have $||\psi(-2\varepsilon, -2\varepsilon)|| \leq \varepsilon$ so that $(-2\varepsilon, -2\varepsilon) + \psi(-2\varepsilon, -2\varepsilon)$ must lie in the quadrant $\{(x,y) \mid x, y \leq 0\}$. Similarly $(2\varepsilon, 2\varepsilon) + \psi(2\varepsilon, 2\varepsilon)$ will lie in $\{(x,y) \mid x, y \geq 0\}$ and consequently the curve $t \mapsto (t,t) + \psi(t,t)$ (from $[-\varepsilon, \varepsilon]$ to $\mathbb{R}^2$) will meet $\{(x,y) \mid xy = 0\}$. Therefore $t \mapsto H\big((t,t) + \psi(t,t)\big)$ vanishes at some $t \in [-2\varepsilon, 2\varepsilon]$ in contrast to the assumption that it coincides with $t \mapsto H(t,t) + \delta = t^2 + \delta$, a strictly positive function.

Similarly one shows that the existence of an $(\varepsilon, -\delta)$ vector field implies a contradiction. (This time one has to work with $t \mapsto (t,-t) + \psi(t,-t)$.) $\square$

In view of this proposition we will have to argue more subtly: our solution will be to glue together several vector fields. The main ingredient will be the following elementary topological lemma:

2.5 Lemma: *Let L, M, N be topological Hausdorff spaces and $\phi : L \to M$ a continuous maps. There are given closed subsets $L_1, \ldots, L_n$ in L such that $\bigcup_i L_i = L$ and also closed subsets $M_1, \ldots, M_n$ of M with $\bigcup_i M_i = M$. We assume that $\phi(L_i) \subset M_i$ for $i = 1, \ldots, n$.*

Further there are given continuous maps $\psi_i : M_i \to L$ such that, for all i, j, the maps ψ_i and ψ_j coincide on $M_i \cap M_j$.

We define $\Phi : L \to N$ as follows: for $l \in L_i$ (any $i \in \{1, \ldots, n\}$) we put $\Phi(l) := \psi_i\big(\phi(l)\big)$. Then Φ is well-defined and continuous.

Proof: It is clear that Φ is well-defined. The continuity follows from the fact that, for closed $A \subset N$,

$$\Phi^{-1}(A) = \bigcup_i \phi^{-1}\big(\psi_i^{-1}(A)\big),$$

and this set is closed as the union of a finite number of closed sets. □

We will use this lemma in the case $n = 3$ with $L := K,\ M := N := \mathbb{R}^2$ and $\phi := \phi_{f,g}$. The M_1, M_2, M_3 are the sets $M_{\eta,1}, M_{\eta,\varepsilon,1}$ and $M_{\eta,\varepsilon,2}$ (resp. $M_{-\eta,1}, M_{-\eta,\varepsilon,1}$ and $M_{-\eta,\varepsilon,2}$), the sets L_1, L_2, L_3 are $\phi_{f,g}^{-1}(M_{\eta,1})$ (resp. $\phi_{f,g}^{-1}(M_{-\eta,1})$) and K_1, K_2 from theorem 1; the ψ_1, ψ_2, ψ_3 will be suitable vector fields, they will be defined next.

Some suitable vector fields

A. Vector fields on $M_{\eta,1}, M_{\eta,\varepsilon,1}$ and $M_{\eta,\varepsilon,2}$

Fix $\varepsilon, \eta, \delta$ with $1 \geq \varepsilon > \eta > \delta > 0$; we will assume that $\eta \leq \varepsilon^2$. The plane $\mathbb{R}^2$ is the union of $M_{\eta,1}, M_{\eta,\varepsilon,1}$ and $M_{\eta,\varepsilon,2}$ (cf. figure 1), and we will define $(\varepsilon, -\delta)$ vector fields on these subsets.

A 1: The vector field on $M_{\eta,1}$

This vector field is sketched in the following picture:

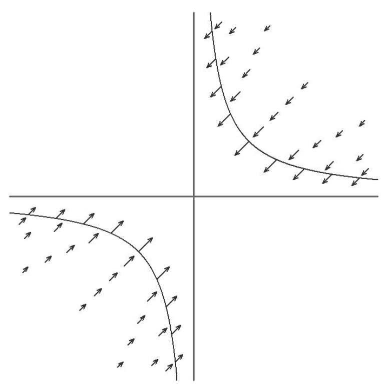

Figure 8: The vector field on $M_{\eta,1}$.

Rather than to define it by a formula we prefer here and in the sequel to describe first the rule how it is constructed: in this way the underlying ideas ar more transparent. For the case $M_{\eta,1}$ the rule is very simple:

> For an $(x, y) \in M_{\eta,1}$ such that $x, y > 0$ (resp. $x, y < 0$) one goes in the direction $(-1, -1)$ (resp. $(1, 1)$) until one reaches for the first time a point P where the H-value is precisely δ units smaller than $H(x, y)$. We define $\psi_{\eta,\varepsilon,\delta}(x, y)$ as the vector $P - (x, y)$.

2.6 Lemma: *$\psi_{\eta,\varepsilon,\delta} : M_{\eta,1} \to \mathbb{R}^2$ is an $(\varepsilon, -\delta)$ vector field.*

Proof: We will only consider the $(x, y) \in M_{\eta,1}$ with $x, y > 0$; the (x, y) with $x, y < 0$ can be treated in a similar way.

We start with the elementary observation that $xy \geq \eta$ implies $x+y \geq 2\sqrt{\eta}$, and we note that $\psi_{\eta,\varepsilon,\delta}(x,y)$ is the vector $(-t,-t)$, where t is the smallest positive solution of $(x-t)(y-t) = xy - \delta$. This t has the form

$$\frac{(x+y) - \sqrt{(x+y)^2 - 4\delta}}{2}$$

where the expression $(x+y)^2 - 4\delta$ is positive by the preceding observation.

It follows that $\psi_{\eta,\varepsilon,\delta}$ is continuous and also that the length of $\psi_{\eta,\varepsilon,\delta}(x,y)$ only depends on $x+y$.

Finally we note that $\rho : \left[2\sqrt{\delta}, \infty\right[\to \mathbb{R}$, defined by $c \mapsto (c - \sqrt{c^2 - 4\delta})/2$ has a negative derivative so that $\rho(c) \leq \rho(2\sqrt{\delta}) = \sqrt{\delta} \leq \varepsilon$ for all c, and it follows that $||\psi_{\eta,\varepsilon,\delta}(x,y)|| \leq \varepsilon$ for the (x,y) under consideration.
This proves the lemma. □

A 2: The vector field on $M_{\eta,\varepsilon,2}$

We write $M_{\eta,\varepsilon,2}$ as the union of three subsets. The vector field will be constructed on each of these separately in such in way that on the intersection of two of them the definiton is the same. Here are the subset, they will be denoted by I, II, III:

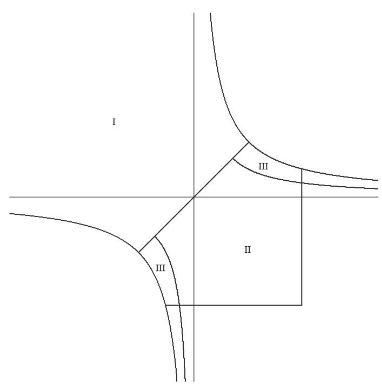

Figure 9: The partition of $M_{\eta,\varepsilon,2}$.

More precisely, we put

$$\begin{aligned}
I &:= \{(x,y) \mid (x,y) \in M_{\eta,\varepsilon,2},\ y - x \geq 0\};\\
II &:= \{(x,y) \mid x \leq \varepsilon,\ y \geq -\varepsilon,\ xy \leq \delta,\ y - x \leq 0\};\\
III &:= \{(x,y) \mid x \leq \varepsilon,\ y \geq -\varepsilon,\ \delta \leq xy \leq \eta,\ y - x \leq 0\}.
\end{aligned}$$

On I our field looks as follows:

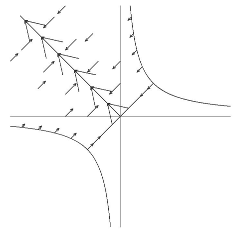

Figure 10: The field on I.

It is defined by the following rule:

1. If $x + y \geq 0$ go in the direction $(-1, -1)$.

   Case 1: Before reaching $\{(x, y) \mid x + y = 0\}$ one arrives at a point P with $H(P) = H(x, y) - \delta$. Then $\psi_I(x, y) := P - (x, y)$.

   Case 2: This is not the case, i.e., one meets $\{(x, y) \mid x + y = 0\}$ at a point Q where $H(Q) > H(x, y) - \delta$. Move from Q in the direction $(-1, 1)$ to a point P where $H(P) = H(x, y) - \delta$. Put $\psi_I(x, y) := P - (x, y)$.

2. If $x + y \leq 0$ go in the direction $(1, 1)$.

   Case 1: Before reaching $\{(x, y) \mid x + y = 0\}$ one arrives at a point P with $H(P) = H(x, y) - \delta$. Then $\psi_I(x, y) := P - (x, y)$.

   Case 2: This is not the case, i.e., one meets $\{(x, y) \mid x + y = 0\}$ at a point Q where $H(Q) > H(x, y) - \delta$. Move from Q in the direction $(-1, 1)$ to a point P where $H(P) = H(x, y) - \delta$. Put $\psi_I(x, y) := P - (x, y)$.

The sketch of the field on II is as follows:

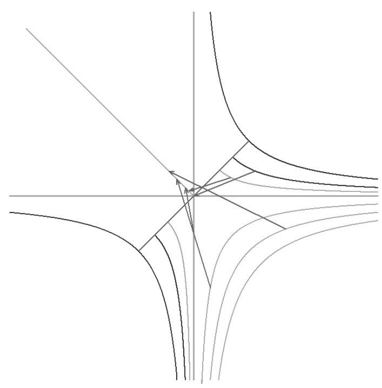

Figure 11: The field on II.

On this subset all vectors will point to some element in the diagonal

$$\{(x, y) \mid x + y = 0,\ x \leq 0,\ y \geq 0\}.$$

Thus we have only one choice:

> For $(x, y) \in II$ calculate $t := H(x, y)$: this number will lie in $\left[-\varepsilon^2, \delta\right]$. Put $s := \delta - t \in \left[0, \delta + \varepsilon^2\right]$ and $P := (-\sqrt{s}, \sqrt{s})$ (so that $H(P)$ equals $t - \delta$). We define $\psi_{II}(x, y) := P - (x, y)$ which guarantees that $H\big(\psi_{II}(x, y)\big) = H(x, y) - \delta$ as desired.

It remains to define the field on III. This has to be done in such a way that the definition coincides

- with that of $\psi_{\eta,\varepsilon,\delta}$ on $M_{\eta,1} \cap III$;
- with that of ψ_I on $I \cap III$;
- with that of ψ_{II} on $II \cap III$.

Here is a sketch:

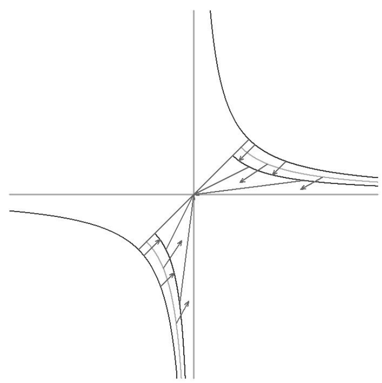

Figure 12: The field on III.

The procedure to find $\psi_{III}(x, y)$ is as follows:

> Let $(x, y) \in III$ be given and suppose first that $x, y > 0$. Then $\delta \leq H(x, y) \leq \eta$. We are looking for H-values that are smaller than $t := H(x, y)$ in the direction "south west". More precisely: for $t = \eta$ (i.e., if $x, y \in III \cap M_{1,\eta}$) we go in the direction $(-1, -1)$, for $t = \delta$ the direction points to $(0, 0)$, and for $t \in \left]\delta, \eta\right[$ the direction is interpolated continuously.
>
> Then we go in this direction to the first point P where $H(P) = H(x, y) - \delta$. $\psi_{III}(x, y)$ is the vector $P - (x, y)$. The "direction where one will find P" is chosen such that it satisfies the above conditions and depends on the same time continuously on (x, y).
>
> The $\psi_{III}(x, y)$ for the (x, y) with $x, y < 0$ are defined similarly; this time P is found in the direction "north east".

We now will put the pieces together:

2.7 Lemma: *(i) ψ_I is a $(2\varepsilon,-\delta)$ vector field on I.*

(ii) ψ_{II} is a $(2\varepsilon,-\delta)$ vector field on II.

(iii) ψ_{III} is a $(2\varepsilon,-\delta)$ vector field on III.

(iv) Define $\psi_{\eta,\varepsilon,\delta,2} : M_{\eta,\varepsilon,2} \to \mathbb{R}^2$ by $\psi_{\eta,\varepsilon,\delta,2}(x,y) := \psi_I(x,y)$ (resp. $:= \psi_{II}(x,y)$, resp. $\psi_{III}(x,y)$) if $(x,y) \in I$ (resp. $(x,y) \in II$, resp. $(x,y) \in III$). Then $\psi_{\eta,\varepsilon,\delta,2}$ is a well-defined $(2\varepsilon,\delta)$ vector field on $M_{\eta,\varepsilon,2}$. It coincides on $M_{\eta,1} \cap M_{\eta,\varepsilon,2}$ with $\psi_{\eta,\varepsilon,\delta}$.

Proof: (i) As in the proof of lemma 2.6 we start by finding the associated formulas. Let a vector $(x,y) \in I$ such that $x+y \geq 0$ be given. For $x+y \geq 2\sqrt{\delta}$ we get as above the expression $\psi_I(x,y) = (-t,-t)$, where $t = \big((x+y) - \sqrt{(x+y)^2 - 4\delta}\big)/2$.

If $x+y \in \Big[0, 2\sqrt{\delta}\Big[$ the definition is slightly more complicated. The vector Q can be written explicitly as $(-c,c)$, where $c = (y-x)/2 \geq 0$. The distance $||Q-(x,y)||$ can easily be estimated:

$$||Q-(x,y)|| = ||\big(-(x+y)/2, -(x+y)/2\big)|| = |(x+y)/2| \leq \sqrt{\delta} \leq \varepsilon.$$

It remains to move from Q to P, i.e., a positive t has to be found such that $(-\alpha-t)(\alpha+t) = -\alpha^2 - \delta'$, where $\delta' := \delta + H(Q) \in [\,0,\delta\,]$. It follows that $t = -c + \sqrt{c^2+\delta'}$, and since $c \mapsto \tilde{\rho}(c) := -c + \sqrt{c^2+\delta'}$ has a negative derivative it follows that $t \leq \tilde{\rho}(0) = \sqrt{\delta'} \leq \sqrt{\delta} \leq \varepsilon$. We conclude that

$$||\psi_I(x,y)|| = ||P-(x,y)|| \leq ||P-Q|| + ||Q-(x,y)|| \leq 2\varepsilon.$$

Since $\delta' \to 0$ for $x+y \to 2\sqrt{\delta}$ the function ψ_I is in fact continuous. For $(x,y) \in I$ with $x+y \leq 0$ we proceed similarly. On the diagonal $\{(x,y) \mid x+y=0\}$ the ψ_I-value is the same as before ($\psi_I(-x,x)$ points in the direction $(-1,1)$), and thus we have shown that ψ_I is a $(2\varepsilon,-\delta)$ vector field.

(ii) and (iii) It is proved similarly as in the case of ψ_I that both fields are continuous by giving the explicit expressions. (Of course the "change of the direction" in the definition of ψ_{III} must be prescribed by a continuous function.) For $(x,y) \in II$ one has $||(x,y)|| \leq \varepsilon$, and the norm of P is at most $\sqrt{\delta} \leq \varepsilon$; it follows that ψ_{II} is a $(2\varepsilon,-\delta)$ vector field.

For ψ_{III} both (x,y) and P are in $[\,0,\varepsilon\,] \times [\,0,\varepsilon\,]$ or in $[\,-\varepsilon,0\,] \times [\,-\varepsilon,0\,]$ so that $||\psi_{III}(x,y)|| \leq \varepsilon$.

(iv) It is an immediate consequence of the definition that the functions $\psi_I, \psi_{II}, \psi_{III}$ coincide on the intersections of their domains (so that $\psi_{\eta,\varepsilon,\delta,2}$ is well defined and continuous) and that $\psi_{\eta,\varepsilon,\delta,2} = \psi_{\eta,\varepsilon,\delta}$ on $M_{\eta,1} \cap M_{\eta,\varepsilon,2}$. □

A 3: The vector field on $M_{\eta,\varepsilon,1}$

This is much simpler: we simply "rotate" the peceding definitions by 180 degrees. More precisely, $\psi_{\eta,\varepsilon,\delta,1} := M_{\eta,\varepsilon,1} \to \mathbb{R}^2$ is defined by

$$\psi_{\eta,\varepsilon,\delta,1} := R \circ \psi_{\eta,\varepsilon,\delta,2} \circ R,$$

where R stands for the rotation $(x, y) \mapsto (-x, -y)$. This is a $(2\varepsilon, -\delta)$ vector field.

B. Vector fields on $M_{-\eta,1}, M_{-\eta,\varepsilon,1}$ and $M_{-\eta,\varepsilon,2}$

This will simply be done by a suitable rotation of the fields on $M_{\eta,1}, M_{\eta,\varepsilon,1}$ and $M_{\eta,\varepsilon,2}$ (by 90 degrees, counter-clockwise direction). We will denote these fields that are $(2\varepsilon, \delta)$ vector fields by $\psi_{-\eta,\varepsilon,\delta}$, $\psi_{-\eta,\varepsilon,\delta,1}$ and $\psi_{-\eta,\varepsilon,\delta,2}$.

3. Proof of the main theorem

We are now ready for the proof of theorem 1.

Proof of (i) ⇒ (ii)

Suppose first that multiplication is locally open at (f, g) and that $\varepsilon_0 > 0$. We have to find an $\eta > 0$ such that (ii)(C_1) and (ii)(C_2) are satisfied.

By assumption there exists a δ such that $fg - \underline{\delta}$ and $fg + \underline{\delta}$ are in $B(f, \varepsilon_0)B(g, \varepsilon_0)$. Let us first concentrate on $fg - \underline{\delta}$. We may choose $d_1, d_2 \in CK$ with $||d_1||, ||d_2|| \leq \varepsilon_0$ such that $(f + d_1)(g + d_2) = fg - \underline{\delta}$.

For the following investigations it will be useful to introduce names for the quadrants in $\mathbb{R}^2$: we will denote by Q^{++} (resp. Q^{+-} resp. Q^{-+} resp. Q^{--}) the quadrant $\{(x, y) \mid x \geq 0,\ y \geq 0\}$ (resp. $\{(x, y) \mid x \geq 0,\ y \leq 0\}$ resp. $\{(x, y) \mid x \leq 0,\ y \geq 0\}$ resp. $\{(x, y) \mid x \leq 0,\ y \leq 0\}$).

We choose any η such that $0 < \eta < \delta$. Let $(x, y) \in M_{\eta,2}$ and (x', y') be a point such that $H(x', y') = H(x, y) - \delta$. Then (x', y') must lie in $\big(Q^{+-} \cup Q^{-+}\big) \setminus \{0\}$ since in the complement of this set there are no (x', y') with this property.

Another observation will be important: if (x, y) is in $M_{\eta,2} \setminus M_{\eta,\varepsilon_0,1}$ (resp. in $M_{\eta,2} \setminus M_{\eta,\varepsilon_0,2}$) and (x', y') is such that $H(x', y') = H(x, y) - \delta$ and $||(x, y) - (x', y')|| \leq \varepsilon_0$ then (x', y') necessarily lies in Q^{-+} (resp. in Q^{+-}).

We now will show that (ii)(C_1) holds with this η. We put

$$\begin{aligned} K_1 &:= \{k \mid k \in K,\ \phi_{f,g}(k) \in M_{\eta,2},\ \big(f(k) + d_1(k), g(k) + d_2(k)\big) \in Q^{+-}\}\,; \\ K_2 &:= \{k \mid k \in K,\ \phi_{f,g}(k) \in M_{\eta,2},\ \big(f(k) + d_1(k), g(k) + d_2(k)\big) \in Q^{-+}\}\,. \end{aligned}$$

It is clear from the preceding remarks that $\phi^{-1}_{f,g}(M_{\eta,2})$ is the disjoint union of K_1 and K_2, and by the continuity of the maps under consideration these sets are closed.

Now let a $k \in K_1$ be given. If $\phi_{f,g}(k) \notin M_{\eta,\varepsilon_0,1}$ would hold, then – by the preceding observation – we would have $\big(f(k) + d_1(k), g(k) + d_2(k)\big) \in Q^{-+}$, a contradiction. This shows that $\phi_{f,g}(K_1) \subset M_{\eta,\varepsilon_0,1}$, and similarly one proves that $\phi_{f,g}(K_2) \subset M_{\eta,\varepsilon_0,2}$.

In the same way one verifies that $fg + \underline{\delta} \in B(f, \varepsilon_0)B(g, \varepsilon_0)$ implies (ii)C_2.

Proof of (ii) ⇒ (i)

Let $\varepsilon_0 > 0$ be given. By proposition 2.2 it will suffice to find a positive δ such that $fg \pm \underline{\delta}$ lie in $B(f, \varepsilon_0)B(g, \varepsilon_0)$.

By assumption there exists $\eta > 0$ such that (ii)(C_1) and (ii)(C_2) hold. A moment's reflection shows that one may replace η by a smaller number: if $0 < \eta' < \eta$, then $M_{\eta',2} \subset M_{\eta,2}$ so that $\phi_{f,g}^{-1}(M_{\eta',1})$ is the disjoint union of $K_1 \cap \phi_{f,g}^{-1}(M_{\eta',1})$ and $K_2 \cap \phi_{f,g}^{-1}(M_{\eta',1})$.

Therefore we may, wlog, assume that $\eta \leq \varepsilon_0^2/4$. We choose any δ such that $0 < \delta < \eta$.

Let us prove by using (C_1) that $fg - \underline{\delta}$ lies in $B(f,\varepsilon_0)B(g,\varepsilon_0)$. This condition implies that K can be written as $K = K' \cup K_1 \cup K_2$ with closed subsets K', K_1, K_2 such that

- $K' = \phi_{f,g}^{-1}(M_{\eta,1})$;
- K_1 and K_2 are disjoint;
- $\phi_{f,g}(K_1) \subset M_{\eta,\varepsilon_0,1}$ and $\phi_{f,g}(K_2) \subset M_{\eta,\varepsilon_0,2}$.

We have to find continuous $d_1, d_2 : K \to \mathbb{R}$ with $||d_1||, ||d_2|| \leq \varepsilon_0$ such that $(f + d_1)(g + d_2) = fg - \underline{\delta}$ holds. At this point the vector fields defined in section 2 come into play (they are defined with $\varepsilon := \varepsilon_0/2$).

Definition of d_1, d_2 on K' : Here we put $\big(d_1(k), d_2(k)\big) := \psi_{\eta,\varepsilon,\delta} \circ \phi_{f,g}(k)$.

Definition of d_1, d_2 on K_1 : For these k we define $\big(d_1(k), d_2(k)\big) := \psi_{\eta,\varepsilon,\delta,1} \circ \phi_{f,g}(k)$.

Definition of d_1, d_2 on K_2 : In this case we put $\big(d_1(k), d_2(k)\big) := \psi_{\eta,\varepsilon,\delta,2} \circ \phi_{f,g}(k)$.

Our vector fields have been constructed such that this definition gives rise to well-defined maps. From lemma 2.5 it follows that they are continuous; we emphasize that *here* one makes crucial use of the fact that $K_1 \cap K_2 = \emptyset$ [1].

Similarly it can be shown that (C_2) yields $fg + \underline{\delta} \in B(f,\varepsilon_0)B(g,\varepsilon_0)$. □

The difference between this construction and the more direct approach sketched before definition 2.3 is the following: there the vector $\big(d_1(k), d_2(k)\big)$ was a function of $\big(f(k), g(k)\big)$ whereas now there might be cases where $\big(d_1(k), d_2(k)\big)$ depends really on k: if $\big(f(k), g(k)\big) = \big(f(k'), g(k')\big)$ then $\big(d_1(k), d_2(k)\big)$ might be different from $\big(d_1(k'), d_2(k')\big)$ if, e.g., $k \in K_1$ and $k' \in K_2$.

4. Consequences

The special case $K = [\,0,1\,]$

First we will prove that for the case $K = [\,0,1\,]$ our characterization coincides with that of [4]. In this paper the relevant definition is the following. Given continuous $f, g : [\,0,1\,] \to \mathbb{R}$ we define $\gamma : [\,0,1\,] \to \mathbb{R}^2$ by $t \mapsto \big(f(t), g(t)\big)$, and we say that there is a *positive saddle point crossing* (resp. a *negative saddle point crossing*) if, for some positive $\varepsilon > 0$ one finds $t_1 < t_2$ with $||\gamma(t_1)||, ||\gamma(t_2)|| \geq \varepsilon$, $\gamma(t_1) \in Q^{++}$

[1] A k in this intersection would destroy the hope to obtain a well-defined map. One would have $\phi_{f,g}(k) \in M_{\varepsilon,\eta,1} \cap M_{\varepsilon,\eta,2}$, but $\psi_{\eta,\delta,\varepsilon,1}$ and $\psi_{\eta,\delta,\varepsilon,2}$ do not coincide there.

and $\gamma(t_2) \in Q^{--}$ or vice versa, and $\gamma(t) \in Q^{++} \cup Q^{--}$ for all $t \in [\,t_1, t_2\,]$ (resp. $\gamma(t_1) \in Q^{+-}$ and $\gamma(t_2) \in Q^{-+}$ or vice versa, and $\gamma(t) \in Q^{+-} \cup Q^{-+}$ for all $t \in [\,t_1, t_2\,]$).

4.1 Proposition: *(ii)(C_1) and (ii)(C_2) hold for all positive ε_0 if and only if there are no positive and no negative saddle point crossings.*

Proof: (Under the assumption that our arguments were and are correct there is no need for a proof. It is nevertheless included in order to illustrate the difference between both approaches.)

Suppose first that there is a positive saddle point crossing. Choose ε, t_1, t_2 as in the definition and put $\varepsilon_0 := \varepsilon/2$. We claim that (ii)$(C_1)$ cannot hold. Suppose that there were a positive η such that one could find K_1, K_2 as in this condition. $\gamma(t_1)$ must lie in K_2 and $\gamma(t_1)$ in K_1 (or vice versa). Also all $\gamma(t)$ for $t \in [\,t_1, t_2\,]$ would belong to $K_1 \cup K_2$ so that $[\,t_1, t_2\,]$ could be written as the disjoint union of two closed nonvoid subsets, a contradiction.

Similarly the existence of a negative saddle point crossing would imply that (ii)(C_2) cannot be satified.

Suppose now that, for some $\varepsilon_0 > 0$, there is no positive η such that condition (ii)(C_2) holds. We will prove that there is a positive saddle point crossing. (Similarly the negation of (ii)(C_1) would give rise to a negative saddle point crossing.)

Claim: For every $\eta > 0$ there are $t_{1,\eta}, t_{2,\eta} \in [\,0, 1\,]$ such that for all t between $t_{1,\eta}$ and $t_{2,\eta}$ one has $\gamma(t) \in M_{-2\eta,2}$, but $\gamma(t_{1,\eta}) \notin M_{-2\eta,\varepsilon_0,1}$ and $\gamma(t_{2,\eta}) \notin M_{-2\eta,\varepsilon_0,2}$. (Thus, if $\eta = 0$ were admissible, there would be a positive saddle point crossing between t_1 and t_2.)

Proof of this claim. Consider the compact set $\Delta := \{t \mid t \in [\,0, 1\,],\ \gamma(t) \in M_{-\eta,2}\}$. Δ is contained in the open set $O := \{t \mid t \in [\,0, 1\,],\ H\big(\gamma(t)\big) > -2\eta\}$, and we may choose pairwise disjoint intervals $I_i = [\,s_i, t_i\,]$ for $i = 1, \ldots, n$ with $\Delta \subset \bigcup_i I_i \subset O$.

What happens for the t in some of the I_i? We claim that there must be at least one i where $G_i := \{\gamma(t) \mid t \in I_i\}$ is *not* completely contained in $M_{-2\eta,\varepsilon_0,1}$ or in $M_{-2\eta,\varepsilon_0,2}$. If always $G_i \subset M_{-2\eta,\varepsilon_0,1}$ or $G_i \subset M_{-2\eta,\varepsilon_0,2}$ would hold we could put

$$K_1' := \bigcup\{I_i \mid G_i \subset M_{-2\eta,\varepsilon_0,1}\},\ K_2' := \bigcup\{I_i \mid G_i \subset M_{-2\eta,\varepsilon_0,2}\},$$

and $K_1 := K_1' \cap \gamma^{-1}(M_{-\eta,2})$, $K_2 := K_2' \cap \gamma^{-1}(M_{-\eta,2})$ would be a disjoint partition of $\gamma^{-1}(M_{-\eta,2})$ as in (ii)(C_2) in contradiction to our hypothesis.

Choose any I_i where G_i is neither a subset of $M_{-2\eta,\varepsilon_0,1}$ or of $M_{-2\eta,\varepsilon_0,2}$. This implies that we may can select $t_{1,\eta}, t_{2,\eta}$ with the claimed properties; $t_{1,\eta}$ will be a t with $t \in M_{-2\eta,2} \setminus M_{-2\eta,\varepsilon_0,1}$, and $t_{2,\eta}$ will be a t with $t \in M_{-2\eta,2} \setminus M_{-2\eta,\varepsilon_0,2}$.

We are now ready to prove that there is a positive saddle point crossing. We choose $t_{1,\eta}, t_{2,\eta}$ for $\eta = 1, 1/2, 1/3, \ldots$ and select an accumulation point t_1 (resp. t_2) of these $t_{1,\eta}$ (resp. of these $t_{2,\eta}$). We then know that the $\gamma(t)$ for t between t_1 and t_2 stay in $Q^{++} \cup Q^{--}$, that the norm of $\gamma(t_1), \gamma(t_2)$ is at least ε_0 and that one of these vectors lies in Q^{++} and the other in Q^{--}. □

$\boxed{\textit{The case } f = \pm g}$

It is easy to characterize those f where multiplication is locally open at (f, f):

4.1 Proposition: *For $f \in CK$ the following are equivalent:*

(i) Multiplication is locally open at (f, f).

(ii) Multiplication is locally open at $(f, -f)$.

(iii) For every $\varepsilon_0 > 0$ there are disjoint closed subsets $K_1, K_2 \subset K$ such that $K_1 \cup K_2 = K$ and $f(k_1) \geq -\varepsilon_0, f(k_2) \leq \varepsilon_0$ for $k_1 \in K_1, k_2 \in K_2$.

Proof: We will show that (i) and (iii) are equivalent, the proof of the equivalence of (ii) and (iii) is similar.

(i)⇒(iii): Supppose that (i) holds and that ε_0 is a positive number. By theorem 1.1 (ii) there exists $\eta > 0$ such that $\phi_{f,f}^{-1}(M_{-\eta,2})$ can be written as $K_1 \cup K_2$ with disjoint closed K_1, K_2 such that $\phi_{f,f}(K_1) \subset M_{-\eta.\varepsilon_0,2}$ and $\phi_{f,f}(K_2) \subset M_{-\eta.\varepsilon_0,1}$. We note that $\phi_{f,f}^{-1}(M_{-\eta,2}) = K$, i.e., K splits as $K_1 \cup K_2$, and $f(k) \geq -\varepsilon_0$ (resp. $f(k) \leq \varepsilon_0$) for $k \in K_1$ (resp. $k \in K_2$).

(iii)⇒(i): The preceding part of the proof used the fact that (iii) is a reformulation of (ii)(C_2) of theorem 1.1. But (ii)(C_1) is trivially true here: for ε_0 we put $\eta := \varepsilon^2$, and then $\phi_{f,f}^{-1}(M_{\eta,2})$ can be written as the union of $K_1 := \phi_{f,f}^{-1}(M_{\eta,2})$ and $K_2 := \emptyset$, with $\phi_{f,f}(K_1) \subset M_{\eta,\varepsilon_0,1}$ and $\phi_{f,f}(K_2) \subset M_{\eta,\varepsilon_0,2}$. □

4.2 Corollary: *Let K be a connected compact Hausdorff space. Then the following assertions are equivalent:*

(i) Multiplication is locally open at (f, f).

(ii) Multiplication is locally open at $(f, -f)$.

(iii) $f(k) \geq 0$ for all k or $f(k) \leq 0$ for all k.

Proof: We show that (i) and (iii) are equivalent, the equivalence of (ii) and (iii) is proved similarly.

(iii)⇒(i) Let us assume that, e.g., $f \geq 0$. Then we can choose in proposition 4.1(iii) $K_2 := K$ and $K_1 := \emptyset$ for arbitrary ε_0. (Here it is not essential that K is connected.)

(i)⇒(iii) Suppose that (iii) does not hold, i.e., that there are $k_1', k_2' \in K$ with $f(k_2') < 0 < f(k_1')$. Choose a positive ε_0 with $f(k_2') < -\varepsilon_0 < 0 < \varepsilon_0 < f(k_1')$. We claim that for no positive η the condition (ii)(C_2) is satisfied. If there were K_1, K_2 as in (C_2) the space K would split as $K = K_1 \cup K_2$ with disjoint closed K_1, K_2. Both sets are nonempty since $k_1' \in K_1$ and $k_2' \in K_2$, and this contradicts the fact that K is connected. □

The case $K = [-1, 1]^2$

Next we show that for $K = Q := [-1, 1]^2$ an interesting phenomenon can be observed. There – in contrast to the case of many other known bilinear mappings – the collection of "good" pairs (where multiplication is locally open) is not dense. It will be convenient to start with some preparations.

Fix a positive ε such that $\varepsilon \leq 1/3$ and define $A_1, A_2 \subset Q$ as follows:

$$A_1 := \{(x, y) \mid x \geq -\varepsilon,\ y \leq \varepsilon\};\ A_2 := \{(x, y) \mid x \leq \varepsilon,\ y \geq -\varepsilon\}\ .$$

R, S, T, U denote the points $(-1, -1), (-\varepsilon, -\varepsilon), (\varepsilon, \varepsilon), (1, 1)$, respectively.

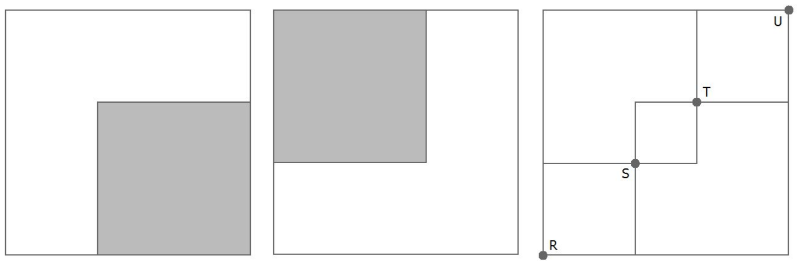

Figure 13: The sets A_1, A_2 and the points R, S, T, U.

4.3 Lemma: *Suppose that K_1, K_2 are compact subsets of Q with $K_1 \cap K_2 = \emptyset$ and $K_1 \subset (A_1)^0$, $K_2 \subset (A_2)^0$. (M^0 stands for the interior of a subset of a topological space.) Then there is a path ϕ from R to U in $Q \setminus (K_1 \cup K_2)$: the function $\phi : [\,0, 1\,] \to Q$ is continuous, it omits $K_1 \cup K_2$, and $\phi(0) = R$ and $\phi(1) = U$ hold.*

Proof: (This assertion is "obvious", it seems to be hard, however, to give a short and elegant argument.)

Since K_1 and K_2 are disjoint they have a positive distance 4τ, and since they lie in the interior of A_1 and A_2, respectively, we may assume that $K_1 + B(0, 2\tau) \subset A_1$ and $K_2 + B(0, 2\tau) \subset A_2$. (Recall that we work with the maximum norm in $\mathbb{R}^2$ so that our "balls" are in fact squares.) We will "blow up" K_1 and K_2 slightly to simplify the situation.

First step: Fix an $n \in \mathbb{N}$ so large that $1/n \leq \tau$. Let K_1' be the collection of all little squares $[\,i/n, (i+1)/n\,] \times [\,j/n, (j+1)/n\,]$ with $i, j \in \{-n, \ldots, n-1\}$ that intersect K_1. Similarly K_2' is defined.

Second step: We enlarge K_1' and K_2' further: $K_1'' := K_1' + B(0, \tau)$ and $K_2'' := K_2' + B(0, \tau)$. Then K_1'' and K_2'' are disjoint compact sets that are unions of tiny squares such that the sides are parallel to the sides of Q with $K_1 \subset K_1'' \subset A_1$ and $K_2 \subset K_2'' \subset A_2$.

We now construct a path that omits $K_1'' \cup K_2''$ (and therefore $K_1 \cup K_2$). To this end we consider the directed line L from R to U. Until S and after T it will not meet $K_1'' \cup K_2''$. If it omits this set also between S and T we are done. Otherwise we modify this path as follows:

- Suppose that L meets K_1'' at a certain component C_1 of K_1''. Then surround C_1 in the clockwise direction without touching $K_1'' \cup K_2''$ until you arrive at L again. Continue your path on L towards R until R or until you arrive at the next obstacle.

  This will be possible since $K_1'' \subset A_1$.

- Proceed in a similar way if L meets a component C_2 of K_2''. This time, however, C_2 will have to be surrounded in the counter-clockwise direction.

Since there are at most finitely many components of $K_1'' \cup K_2''$ one will arrive sooner or later at U.

(A more rigorous proof could start with a result from function theory: if a compact subset L of $\mathbb{C}$ lies in an open set $O \subset \mathbb{C}$, then there is a cycle γ in $O \setminus L$

made up from line segments such that the winding number of x with respect to γ is one for every $x \in L$; see. e.g., proposition 1.1 in chapter VIII of [6]. This should be applied here with $L = K_i$ and $O = \{x \mid d(x, K_i) < \tau\}$ for $i = 1, 2$.) □

4.4 Proposition: *Let $f_0, g_0 : Q \to \mathbb{R}$ be defined by $f_0(x,y) := x$ and $g_0(x,y) := y$. Then, whenever $f, g \in CQ$ are such that $||f - f_0||, ||g - g_0|| \le 1/9$, multiplication is* not *locally open at (f, g).*

Proof: We fix any $f, g \in CQ$ with $||f - f_0||, ||g - g_0|| \le 1/9$, and we put $\Phi := \phi_{f,g}$. The map ϕ_{f_0,g_0} is the identity on Q so that $||\Phi(q) - q|| \le 1/9$ for all q.

Now suppose that multiplication were locally open at (f, g). Choose $\eta > 0$, K_1, K_2 for $\varepsilon_0 := 1/9$ according to (C_1) of theorem 1.1: $\{q \mid \Phi(q \in M_{\eta,2})\}$ is the disjoint union of K_1 and K_2, and $\Phi(q) \in M_{\eta,\varepsilon_0,1}$ (resp. $\Phi(q) \in M_{\eta,\varepsilon_0,2}$) for $q \in K_1$ (resp. $q \in K_2$). With $\varepsilon := 3\varepsilon_0$ we consider the sets A_1, A_2 introduced before lemma 4.3.

We claim that K_1 lies in the interior of A_1. In fact, we know for $q \in K_1$ that $f(q) \ge -\varepsilon_0$ and $g(q) \le \varepsilon_0$, and from $||\Phi(q) - q|| \le \varepsilon_0$ we conclude that $f(q) \ge -2\varepsilon_0$ and $g(q) \le 2\varepsilon_0$. Similarly one proves that $K_2 \subset (A_2)^0$.

By the preceding lemma we find a continuous $\phi : [0, 1] \to Q$ with $\phi(0) = R$, $\phi(1) = U$ and $\phi(t) \notin K_1 \cup K_2$ for all t. This means that no $\Phi \circ \phi(t)$ lies in $M_{\eta,2}$: recall that $\Phi^{-1}(M_{\eta,2}) = K_1 \cup K_2$. Therefore all $\Phi \circ \phi(t)$ belong to $M_{\eta,1}$.

This leads to a contradiction. $M_{\eta,1}$ is the disjoint union of $B_1 := \{(x, y) \mid (x, y) \in M_{\eta,1}, x, y \le 0\}$ and $B_2 := \{(x, y) \mid (x, y) \in M_{\eta,1}, x, y \ge 0\}$. Consequently $[0, 1]$ would be the disjoint union of $\Phi^{-1}(B_1)$ and $\Phi^{-1}(B_2)$ where both of these closed sets are nonempty: $R \in \Phi^{-1}(B_1)$ and $R \in \Phi^{-1}(B_1)$. This is not possible by the connectedness of $[0, 1]$. □

Quantitative results

It is also sometimes simple to derive quantitative results. Suppose that multiplication is locally open at (f, g) for some $f, g \in CK$; are there, for given $\varepsilon > 0$, estimates for the $\delta > 0$ such that $B(fg, \delta) \subset B(f, \varepsilon)B(g, \varepsilon)$? As a typical result we show:

4.5 Lemma: *Let $\varepsilon, \delta_0 > 0$ be given and suppose that there is a connected subset C of K with the following properties:*

(i) $f(k)g(k) \ge -\delta_0$ for $k \in C$.

(ii) There are $k_1, k_2 \in C$ such that $f(k_1), g(k_1) > \varepsilon$ and $f(k_2), g(k_2) < -\varepsilon$.

Then, if for some positive δ one has $B(fg, \delta) \subset B(f, \varepsilon)B(g, \varepsilon)$, it follows that $\delta \le \delta_0$.

Proof: Write $fg + \underline{\delta}$ as $(f + d_1)(g + d_2)$, where $d_1, d_2 \in CK$ are such that $||d_1||, ||d_2|| \le \varepsilon$. Then $(f + d_1)(k_1) > 0$ and $(f + d_1)(k_2) < 0$ so that, since C is connected, there must exist a k_0 with $(f + d_1)(k_0) = 0$. This implies that $0 = \big(f + d_1\big)(k_0)\big(g + d_2\big)(k_0) = f(k_0)g(k_0) + \delta \ge -\delta_0 + \delta$, and it follows that $\delta \le \delta_0$. □

Complex-valued functions

It was sketched in [5] that multiplication is locally open at every pair (f,g) in $C_{\mathbb{C}}\,[\,0,1\,]$, the space of *complex* continuous functions on $[\,0,1\,]$. For the case of arbitrary K the situation is by far more complicated, we will investigate it in a separate paper. Here we only note that there are examples where – in contrast to the case $[\,0,1\,]$ – multiplication needs not to be locally open at every pair f,g:

4.6 Proposition: *Let $D = \{z \mid |z| \le 1\}$ be the complex unit disk. The functions $f,g : D \to \mathbb{C}$ are defined by $f : z \mapsto z$ and $g : z \mapsto \overline{z}$. Then the pointwise multiplication on $C_{\mathbb{C}}D$ is not locally open at (f,g).*

Proof: Put $\varepsilon := 1/3$. We claim that $B(f,\varepsilon)B(g,\varepsilon)$ does not contain any of the functions $fg + \underline{\delta}$ with $\delta > 0$ so that fg is not in the interior of $B(f,\varepsilon)B(g,\varepsilon)$.

Let contiuous $d_1, d_2 : D \to \mathbb{C}$ with $||d_1||, ||d_2|| \le \varepsilon$ and a positive δ be given. We have to show that $(f+d_1)(g+d_2)$ is different from $fg + \underline{\delta}$. The function $-d_1$ is a continuous map from D to D so that there exists by the Brouwer fixed point theorem a $z_0 \in D$ with $-d_1(z_0) = z_0$. This implies that $(f+d_1)(z_0) = 0$, and consequently $(f+d_1)(g+d_2)$ must differ from the strictly positive function $fg + \underline{\delta}$.

We note that with a similar argument one can show that multiplication is not locally open at (f,g) if we define $f : z \mapsto z^n$ and $g \mapsto \overline{z}^n$ for an arbitrary $n \in \mathbb{N}$. $\square$

5. INVITATIONS FOR FURTHER RESEARCH

We have given, for the case of real CK spaces, a characterization of those tuples where multiplication is locally open. The following two problems should be studied next:

- Find a characterization for complex CK spaces.
- In [5] a characterization of those n-tuples $(f_1,\ldots,f_n)$ with $f_1,\ldots,f_n \in C\,[\,0,1\,]$ was given where multiplication is locally open. It would be interesting to generalize this theorem to the case of arbitrary real or complex CK-spaces.

Such results would surely be important steps towards an understanding of the obstructions that are responsible for the failure of local openness of multiplication at certain tuples in arbitrary Banach algebras.

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