Book review: The mathematics of juggling, by Burkard Polster (Springer 2004)
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Let me confess at the very beginning that I did not know anything from the mathematical theory of juggling before reading Polster's book. I was not even aware of the fact that such a theory exists. My personal experience with juggling is not very impressing either. I can juggle with three balls, once or twice in my lifetime I also managed four.

For me one of the most fascinating aspect of the "mathematical theory of juggling" is the fact that it represents in a nutshell everything that a decent theory which describes certain phenomena in the real world should have. The first point is that it is necessary to consider a rather idealized situation, here this leads to the basic definition of a "juggling sequence" (the definition will be given below). And then one can start to work on this fundament. One has to discuss the first examples, one can prove theorems (the proofs of which sometimes have to be prepared by some lemmata), and from these theorems one can derive corollaries.

Having mastered the problems connected with the basic approach one can turn to more complicated situations. This gives rise to more sophisticated definitions, the theorems are harder to prove, the mathematics involved is deeper. And finally one can try to apply the theory to other areas of interest.

Juggling sequences

We start with the simplest situation. A juggler can throw a ball with his or her left or right hand, we suppose that the "actions" are equally spaced in time and that the hands alternate: left hand, right hand and so on. If we assume that at any time at most one ball can land and that the juggling lasts from the infinite past to the infinite future we arrive at the following mathematical model of juggling:

> Juggling is encoded by a function $g : \mathbb{Z} \to \mathbb{N}_0$. A positive $g(n)$ indicates that a ball is caught at time n and is thrown immediately again such that it lands after $g(n)$ time units. If $g(n) = 0$, then no action takes place at time n.
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> One assumes that g satisfies the following conditions:
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> - The function $n \mapsto n + g(n)$ is one-to-one: otherwise more than one ball would land at a certain time.
> - g is periodic: for a suitable $p > 0$ one has $g(n + p) = g(n)$ for all n.

As simple examples consider $g(n) := s$, where s is a fixed integer (this is called the *s-ball cascade*). For $s = 1$ this pattern can be juggled by everyone: simply throw the ball from the right hand to the left and vice versa. The case $s = 2$ is also simple: at even time units catch ball number one and throw it immediately again with the left hand such that it can be caught two time units later with the same hand, similarly the right hand deals with ball number two at odd time units. A more demanding juggling pattern is the three ball cascade: At every time unit a ball is caught and immediately thrown again such that it is in the air for three time units. (This seems to be the simplest juggling pattern where three balls are involved.)

If g is periodic with period p it suffices to know the values of g for p consecutive values. g can be reconstructed uniquely up to a translation. For example, "312" stands for a juggling pattern where the g-sequence looks like $\ldots 312312312 \ldots$, and the three ball cascade can be abbreviated by “3”. Finite sequences which arise in this way (like 312 and 3 in the preceding example) are called *juggling sequences*.

Theorems and corollaries

The theory starts with simple results (“If $a_0, \ldots, a_{p-1}$ is a juggling sequence then so is the sequence $a_0 + d, \ldots, a_{p-1} + d$, where d is a fixed positive integer.”), but rather soon the assertions are more interesting and the proofs are more tricky. For example, let a juggling sequence $a_0, \ldots, a_{p-1}$ be given. One can show that the number of balls needed to juggle it is precisely $(a_0 + \cdots + a_{p-1})/p$: this is intuitively clear since this number is the number of balls which are in the air “in the average”. It follows that the s-cascade “s” needs s balls (which is clear) and that one must have 2 balls to juggle “312”. This “average theorem” has a nice corollary: A finite sequence of integers will give rise to a juggling sequence only if the average is an integer. However, this condition is not sufficient. As an example, consider 321. The average is 2, but the associated function $n \mapsto n + g(n)$is not one-to-one.

Surprisingly, it can be shown that the condition is nearly sufficient: If the average of a finite sequence a of nonnegative integers is an integer, then there exists a permutation of a which is a juggling sequence. The proof is rather involved, but nevertheless – as it is to be expected – elementary.

Now suppose that someone is juggling a certain juggling sequence $a_0 \ldots a_{p-1}$ of length p. After some time it might be desirable to pass without interruption to another pattern. Fix an i and an integer d such that $0 < d \leq a_i$. If a_i is replaced by $a_{i+d} + d$ and a_{i+d} by $a_i - d$, then this will be again an admissible sequence. E.g., the ball thrown at time unit i will be caught at that moment when in the original pattern the ball thrown at time unit $i + d$ is going to land. (For example, in the case $p = 3$, $i = 0$ and $d = 1$ one will pass from 642 to 552.)

The operation to generate a new sequence in this way is called a *site swap*. As another, more elementary operation one can consider the *cyclic shift* where one replaces $a_0 \ldots a_{p-1}$ by $a_1 a_2 \cdots a_{p-1} a_0$. It is rather surprising that site swaps and cyclic shifts suffice to pass from any juggling sequence to any other provided the length p and the number of balls $(a_0 + \cdots + a_{p-1})/p$ coincide. For example, one could start with $bb \cdots b$ (which is identical with the cascade b) and arrive at $(pb)0 \cdots 0$ (this corresponds to the pattern where at every p'th time unit a ball is thrown so strong that it is in the air for pb time units).

The book also contains many results centering around *counting*: What is the number of juggling sequences with a prescribed property? As a sample theorem consider the case of *minimal juggling sequences*. (A juggling sequence $a_0 \ldots a_{p-1}$ is called minimal if it cannot be written as a repetition of smaller juggling sequences; so 441 is minimal, but 44444 is not.) It can be shown that the number of minimal juggling sequences of period p with average b is precisely

$$\frac{1}{p} \sum_{d \,|\, p} \mu\big(\frac{p}{d}\big)\big((b+1)^d - b^d\big);$$

here μ is the usual *Moebius function*, and juggling sequences which are identical up to cyclic permutations are identified.

More sophisticated definitions

Juggling sequences are the basic objects of interest, their study covers the first chapters of Polster's book. More complex notation is needed when weakening the assumptions of the first approach. How can one deal with the possibility that it is allowed to throw and to catch more than one ball at a given time unit (multiplex juggling)? What modifications are necessary if not one but several jugglers are involved (multihand juggling)?

Formally a *multiplex juggling* sequence is a finite sequence of finite nonempty ordered sets of nonnegative integers. These sets encode what kinds of throws are made on every beat. For example, $\{1,4\}\{1\}$ has to be realized as follows: On the first beat, two balls are caught, they are immediately thrown again, the first one to height one, the second to height four; on the second beat, one ball is caught and thrown to height one; the actions on beat one and two are repeated again and again.

Multihand juggling needs an even more elaborate notation, one has to pass to matrices the columns of which prescribe what has to be done at a certain time unit. For example, one can learn from the second entry of such a column what action "hand" number two is assumed to perform at the corresponding time: catch 5 balls, throw two of them such that they arrive at "hand" number one two units later and throw the remaining balls such that they are at "hand" four at the next time.

The emphasis is similar to that in the case of the above juggling sequences of a single player. One can prove average theorems, it is possible to count the number of essentially different patterns, the mutual dependence of these patterns can be visualized by graphs etc.

With the notations of multihand juggling at hand it is also possible to change the point of view. If b balls are juggled by h hands in a certain way one may interchange the roles of balls and hands: Fix the position of the balls and let the hands move! Now the balls juggle the hands using a pattern which is in a sense dual to the original one. Claude Shannon, the famous information theorist, was one of the first who has investigated this "duality theory".

The book contains much more. A survey on the history of juggling, hints for jugglers, the connections with bell ringing, to mention a few. For me it was very stimulating. I tried to juggle some of the simpler juggling sequences, also I wrote a computer program to visualize even very complicated $a_0, \ldots, a_{p-1}$ (which no human being will ever manage). The mathematics involved are very interesting, I had not expected to see so many connections with algebra, graph theory and combinatorics.

In a review of this book by Allen Knutson (Notices of the AMS, January 2004) it has been argued that the historical part is not free of errors. I do not have the background information to decide whether this criticism is justified. For me it is a fascinating book from which I learned a lot.