Stochastic dynamics and Parrondo's paradox

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Abstract. The Spanish physicist Juan Parrondo has provided two stochastic losing games such that for certain stochastic combinations one may obtain a winning game. If a large number of players is involved and if they try to play such that their gain in the next round is maximized one arrives at the problem to investigate a random walk on a certain space of measures.

The appropriate abstract setting is as follows. There is given a compact metric space (M,d), and M is written as the union of certain closed subsets $A_1,\ldots,A_r$. For every $\rho = 1,\ldots,r$ there is prescribed a contraction $\Gamma_\rho : A_\rho \to M$. A random walk $(X_m)_{m\in\mathbb{N}_0}$ on M is then defined as follows. The starting position is $X_0 = x_0$, where $x_0 \in M$ is fixed, and if the walk at the m'th step is at position $X_m \in M$, then one chooses a ρ among the ρ with $X_m \in A_\rho$ (with equal probability, say) and defines X_{m+1} as $\Gamma_\rho(X_m)$. Associated with the walk is a *gain* $\varphi(X_m)$ in every round, where $\varphi : M \to \mathbb{R}$ is a continuous function.

The aim of the present investigations is the study of the expectation G_m of $\varphi(X_m)$ as a function of m. Our main result states that the sequence (G_m) is "eventually approximately periodic" provided that all A_ρ are not only closed but also open in M: for every ε there is an $l_0 \in \mathbb{N}$ such that (G_m) is l_0-periodic up to an error of at most ε for sufficiently large m. In fact it turns out that the behaviour of our process can be described well by a finite Markov chain.

In the general case, however, the process might behave rather chaotically. We give an example where M is the unit interval. M is written as the union of two closed subsets A_1, A_2, the contractions Γ_1, Γ_2 are rather simple, but the expectations of the gains are not even Cesáro convergent.

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1. Introduction

A *random walk with reward* on $S := \{0, 1, \ldots, s-1\}$ is given by a stochastic matrix $\mathbf{P} = (p_{ij})_{i,j=0,\ldots,s-1}$ and a vector $\mathbf{w} = (w_0, \ldots, w_{s-1})^\top$. The walk starts at 0, the random steps are driven by $\mathbf{P}$, and if the walk is at $i \in S$ the player gets the reward w_i (which might be negative).

A *Parrondo game* is given by r such random walks with reward on S, i.e. by a family $(\mathbf{P}_\rho, \mathbf{w}_\rho)$, $\rho = 1, \ldots, r$. The Spanish physicist J. Parrondo has observed that it is possible that each individual $(\mathbf{P}_\rho, \mathbf{w}_\rho)$ is "fair"[1], but that the expectation of the total gain might tend to infinity if one is allowed to switch between the $(\mathbf{P}_\rho, \mathbf{w}_\rho)$ (stochastically or using a suitable pattern) in every round. For a survey on the original Parrondo paradox we refer the reader to [6], [7], [8], here we follow the more general approach introduced in [2] and [3].

As in [5] we now imagine that a large number N of players are playing such a Parrondo game. In the limit when N tends to infinity only the proportions of the players who are in state $i \in S$ are of importance. If these proportions in step m are $\nu_0, \ldots, \nu_{s-1}$ and if the players decide to play with $(\mathbf{P}_\rho, \mathbf{w}_\rho)$, then they will obtain the (normalized) reward $\langle \mathbf{w}_\rho, (\nu_0, \ldots, \nu_{s-1})^\top \rangle$, the scalar product of $\mathbf{w}$ with $(\nu_0, \ldots, \nu_{s-1})^\top$, and in the next round the proportions are the components of the vector $\mathbf{P}_\rho(\nu_0, \ldots, \nu_{s-1})^\top$.

The appropriate abstract setting is the following[2]. Consider a compact metric space M (which in the present situation is the space of probability measures on S) and contractions $\Gamma_\rho : M \to M$. In the Parrondo context these are the maps $\nu \mapsto \mathbf{P}_\rho \nu$; they are contractions if one assumes that the $\mathbf{P}_\rho$ are "sufficiently ergodic" (cf. [2] for details). In addition there are continuous maps $\varphi_\rho : M \to \mathbb{R}$, the "reward functions", in the Parrondo case one has to define $\varphi_\rho : \nu \mapsto \langle \mathbf{w}_\rho, \nu \rangle$..

Given the $M, \Gamma_1, \ldots, \Gamma_r, \varphi_1, \ldots, \varphi_r$ one considers the following walk with rewards on M: start at a fixed $x_0 \in M$ and choose $\rho_1 \in \{1, \ldots, r\}$. Obtain $\varphi_{\rho_1}(x_0)$ as a reward and move to $x_1 := \Gamma_{\rho_1}(x_0)$. Choose there ρ_2, this amounts in a gain of $\varphi_{\rho_2}(x_1)$ and the next position will be $\Gamma_{\rho_2}(x_1)$. In this way the decisions $\rho_1, \rho_2, \ldots$ induce a walk on M, and one may consider various scenarios how the ρ are chosen. [3] contains a systematic investigation of the *optimal gain*: what is the best way to choose the ρ if one knows in

[1] This means that on the long run gains and losses balance.

[2] A similarly general approach to phenomena centering around Parrondo's paradox can be found in [4]. This paper contains a systematic study of situations where "chaos+chaos=order" or "order+order=chaos" can be observed.

advance that m rounds are to be played?

Here we will treat another situation, the players choose the ρ for the next round on the basis of a *greedy strategy*. More precisely, if $x \in M$ is the position of the present round, they calculate the numbers $\varphi_\rho(x)$ for $\rho = 1, \ldots, r$ and decide to choose that ρ where $\varphi_\rho(x)$ is maximal. If there should be several ρ where the maximum is assumed they select one among them with uniform probability.

With $\varphi(x) := \max_\rho \varphi_\rho(x)$ and $A_\rho := \{x \mid \varphi(x) = \varphi_\rho(x)\}$ this is precisely the situation described in the abstract.

Let G_m be the expectation of the gain in the m'th round. (The starting position x_0 will be fixed.) Computer simulations have indicated that the sequence (G_m) has an "eventually quasiperiodic" behaviour. One of the main results of this paper is the assertion that this is in fact true if the A_ρ are not only closed but also open:

Proposition 3.1: *Under the assumption that all A_ρ are clopen the sequence (G_m) behaves as follows: for every $\varepsilon > 0$ there is an $l_0 \in \mathbb{N}$ such that*

$$|G_{m+k \cdot l_0} - G_m| \leq \varepsilon$$

for all $k \in \mathbb{N}$ and sufficiently large m.

In general – if the A_ρ are not necessarily clopen – the sequence (G_m) might be very chaotic. We prove in proposition 4.1 that nearly every zero-one pattern can be realized for suitably defined $M, \varphi, A_1, \ldots, A_r, \Gamma_1, \ldots, \Gamma_r$. A detailed case study of the simplest example with non-clopen A_ρ will also be presented (in proposition 4.2). It concerns the usual Cantor discontinuum. We will show that the (G_m)-sequence is not necessarily "eventually nearly periodic", but there remain some regularity properties: the Cesáro limit of the (G_m) exists so that one has a reasonable notion of an average gain.

The paper is organized as follows. In *section 2* we introduce some notation, also some first examples are discussed. *Section 3* contains the main result for the case of clopen A_ρ. In the proof it will be essential that our random walks can be investigated by using something like a "shadow walk" which is a random walk associated with a finite Markov chain. Finally, in *section 4*, we discuss our counterexamples.

2. Preliminaries

It will be convenient for our investigations to consider situations which are slightly more general than that described in the abstract. We replace a *uniform* selection among the admissible indices by a general probability law.

As before we start with a compact metric space M which is written as $M = \bigcup_{\rho=1}^{r} A_\rho$, where the A_ρ are (not necessarily distinct) closed subsets of M. For every ρ there is defined a contraction $\Gamma_\rho : A_\rho \to M$. Let $L < 1$ be a number such that all Γ_ρ are Lipschitz with Lipschitz constant L.

For $x \in M$ the collection of ρ with $x \in A_\rho$ will be called Δ_x. Thus the Δ_x are nonvoid subsets of $\{1, \ldots, r\}$. We now introduce *transition probabilities*. Suppose that $\Delta \subset \{1, \ldots, r\}$ is such that $\Delta = \Delta_x$ for at least one x in M. Then there is given a family $(p_{\rho,\Delta})_{\rho \in \Delta}$ of nonnegative numbers with $\sum_{\rho \in \Delta} p_{\rho,\Delta} = 1$.

If also an $x_0 \in M$ is fixed, this setting induces a discrete time Markov process $(X_m)_{m \in \mathbb{N}_0}$ on M: the process starts with $X_0 = x_0$, and if the position at "time" m is X_m, one considers $\Delta = \Delta_{X_m}$; a $\rho \in \Delta$ is chosen using the distribution defined by the $(p_{\rho,\Delta})_{\rho \in \Delta}$, and X_{m+1} is then defined as $\Gamma_\rho(X_m)$.

For the sake of easy reference we introduce the following

Definition 2.1. *The family* $\big(M, (A_\rho), (\Gamma_\rho), (p_{\rho,\Delta}), x_0\big)$ *will be called a* Markov chain induced by contractions*.*

Usually there will also be given a continuous *reward function* $\varphi : M \to \mathbb{R}$. Our results will concern the structure of the sequence $(G_m)_{m \in \mathbb{N}_0}$, where G_m is the expectation of $\varphi(X_m)$.

Examples:

1. As a first example we consider the case when $r = 1$. There is only one A and only one contraction Γ. Then the random walk is given by the sequence $x_0, \Gamma_1(x_0), \Gamma_1^2(x_0), \ldots$ It converges geometrically fast to the unique fixed point π_1 of Γ_1. Therefore the sequence (G_m) tends to $\varphi(\pi_1)$, and in particular it is "eventually quasiperiodic" (with $l_0 = 1$ for every ε).

2. Next we suppose that the A_ρ are pairwise disjoint, it follows that the A_ρ are clopen. Denote the positive number

$$\min\{d(x, y) \mid x \in A_\rho,\ y \in A_{\rho'},\ \text{where } 1 \leq \rho < \rho' \leq r\}$$

by δ_0. We observe that the $B_{\rho,\rho'} := \Gamma_\rho^{-1}(A_{\rho'})$ have mutual distance δ_0/L. Consequently, by passing from the A_ρ, Γ_ρ to the $B_{\rho,\rho'}$ and the restrictions of the Γ_ρ to the $B_{\rho,\rho'}$, we may assume without loss of generality that the range of Γ_ρ is contained in only one $A_{\rho'}$. Thus there is a function $\tau : \{1, \ldots, r\} \to \{1, \ldots, r\}$ such that Γ_ρ may be regarded as a mapping from A_ρ to $A_{\tau(\rho)}$.

For every ρ_0 the sequence $\rho_0, \tau(\rho_0), \tau^2(\rho_0), \ldots$ is eventually periodic: there are k_0, l_0 such that $\tau^{k+l_0}(\rho_0) = \tau^k(\rho_0)$ for every k with $k \geq k_0$; this follows immediately from the fact that $\{1, \ldots, r\}$ is finite. Put

$$\rho_1 := \tau^{k_0+1}(\rho_0),\ \rho_2 := \tau^{k_0+2}(\rho_0), \ldots,\ \rho_{l_0} := \tau^{k_0+l_0}(\rho_0).$$

In particular we consider ρ_0 such that $x_0 \in A_{\rho_0}$. We have $X_1 = \Gamma_{\rho_0}(x_0)$, $X_2 = \Gamma_{\tau(\rho_0)}\Gamma_{\rho_0}(x_0), \ldots$ so that (after k_0 steps) the walk (X_m) is given by applying the maps $\Gamma_{\rho_1}, \ldots, \Gamma_{\rho_{l_0}}$ again and again. Because the product of l_0 Γ's is a contraction this means that – up to an error which tends to zero fast – the (X_m) describe a cyclic walk through the fixed points of $\Gamma_{\rho_{l_0}} \circ \Gamma_{\rho_{l_0-1}} \circ \cdots \circ \Gamma_{\rho_1}$, $\Gamma_{\rho_1} \circ \Gamma_{\rho_{l_0}} \circ \Gamma_{\rho_{l_0-1}} \circ \cdots \circ \Gamma_{\rho_2}, \ldots, \Gamma_{\rho_{l_0-1}} \circ \Gamma_{\rho_{l_0-2}} \circ \cdots \circ \Gamma_{\rho_1} \circ \Gamma_{\rho_{l_0}}$.

Again it follows that the sequence (G_m) of expected gains is eventually quasiperiodic. (Here it is important that φ is uniformly continuous.)

3. Consider a finite Markov chain with state space $S = \{0, \ldots, s-1\}$ which is given by a stochastic matrix $\mathbf{P} = (p_{ij})_{i,j=0,\ldots,s-1}$; it is assumed that the starting position is $0 \in S$. We define $M := S$ and provide M with the discrete metric. M is written as the union of the singletons $\{i\}$, where each of the $\{i\}$ is repeated s times: $M = \bigcup_{\rho=1}^{r} A_\rho$, with $r = s^2$, $A_1 = \cdots = A_s = \{0\}$, $A_{s+1} = \cdots = A_{2s} = \{1\}, \ldots, A_{s^2-s+1} = \cdots = A_{s^2} = \{s-1\}$. The family of Γ_ρ is the family of all possible maps from arbitrary $\{i\}$ to arbitrary $\{j\}$: Γ_{1+is+j} maps $\{i\}$ to $\{j\}$. Finally, if Δ is a subset of $\{1, \ldots, r\}$ of the form $\{is+1, is+2, \ldots, is+s\}$, then $p_{\rho,\Delta} := p_{ij}$, if $\rho = 1 + is + j$.

Then the random walk associated with the chain is identical with the random walk induced by $\big(M, (A_\rho), (\Gamma_\rho), (p_{\rho,\Delta}), x_0 = 0\big)$.

To put it otherwise: our notion of "Markov chains induced by contractions" contains the finite Markov chains with a fixed starting state as a special case.

Now let also a vector $\mathbf{w} = (w_0, \ldots, w_{s-1})^\top$ be given (this corresponds to the function φ in our approach). Along with the walk one gets rewards, if the walk passes through i one obtains w_i. It follows easily from the elementary theory of finite Markov chains that the expected gain G_m^S in the m'th round is the inner product of $\mathbf{w}$ with the first row of $\mathbf{P}^m$, i.e.,

$$G_m^S = \langle \mathbf{w}, \mathbf{P}^m \mathbf{e}_0 \rangle, \text{ with } \mathbf{e}_0 = (1, 0, \ldots, 0)^\top.$$

To continue our analysis we recall that the state space S of a finite Markov chain can be decomposed as $T \cup S_1 \cup \cdots \cup S_l$, where T corresponds to the transient states and the S_λ are minimal invariant. Further, for each λ, all $i \in S_\lambda$ have the same period c_λ, and S_λ is the disjoint union of c_λ

subsets such that each of them is invariant with respect to P^{c_λ}, and the restriction leads to an ergodic and aperiodic chain[3]. Since for an ergodic and aperiodic chain $\mathbf{P}$ the sequence $(\mathbf{P}^n)$ converges, this has an important consequence:

> There exists an $l_0 > 0$ such that for every $\varepsilon > 0$ one can find an m_0 such that $\mathbf{P}^{m+k\cdot l_0}$ is ε-close to $\mathbf{P}^m$ whenever $m \geq m_0$ and $k = 1, 2, \ldots$ (Here "distance" refers to any matrix norm.)
>
> The period l_0 can be taken to be the smallest common multiple of the periods of the $i \in S$.

It follows that the sequence (G^S_m) is again eventually quasiperiodic[4].

4. Now we consider the case when all A_ρ coincide with M. Only the $p_{\rho,\Delta}$ with $\Delta = \{1, \ldots, r\}$ will be of importance. We denote these numbers by $p_1, \ldots, p_r$, and we assume that all of them are strictly positive.

A similar situation has been investigated in [3], there it has been shown that the set of fixed points of finite products of the Γ_ρ plays a crucial role:

> Denote, for $\rho_1, \ldots, \rho_l \in \{1, \ldots, r\}$, by $\pi_{\rho_1 \cdots \rho_l}$ the fixed point of $\Gamma_{\rho_1} \circ \cdots \circ \Gamma_{\rho_l}$. By F we mean the closure of the set
>
> $$\{\pi_{\rho_1 \cdots \rho_l} \mid l = 1, 2, \ldots,\ \rho_1, \ldots, \rho_l = 1, \ldots, r\}.$$
>
> Then the random walk (X_m) will "converge" to F: the distance of X_m to F tends to zero.
>
> Also, F is an invariant set, a walk which starts in F will never leave this set.

Thus, if only the long term behaviour is to be investigated, one may assume without loss of generality that x_0 lies in F.

Let's analyse the average gain sequence (G_m) associated with a contin-

[3]For a proof of this structure theorem we refer the reader to chapter 7 in [1].

[4]In fact the result is stronger: in this special case the period l_0 does not depend on ε.

uous gain function φ. One has

$$
\begin{aligned}
G_0 &= \varphi(x_0)\\
G_1 &= \sum_{\rho_1=1}^{r} p_{\rho_1}\varphi\big(\Gamma_{\rho_1}(x_0)\big)\\
G_2 &= \sum_{\rho_1,\rho_2=1}^{r} p_{\rho_1}p_{\rho_2}\varphi\big(\Gamma_{\rho_1}\circ\Gamma_{\rho_2}(x_0)\big)\\
\vdots &= \vdots\\
G_m &= \sum_{\rho_1,\dots\rho_m=1}^{r} p_{\rho_1}\cdots p_{\rho_m}\varphi\big(\Gamma_{\rho_1}\circ\cdots\circ\Gamma_{\rho_m}(x_0)\big).
\end{aligned}
$$

Now let $\varepsilon > 0$ be given. We first choose $\varepsilon' > 0$ such that $d(x, y) \le \varepsilon'$ implies $|\varphi(x) - \varphi(y)| \le \varepsilon$. Then a k_0 is selected with the property that $\Gamma_{\rho_1}\circ\cdots\circ\Gamma_{\rho_m}(x_0)$ is ε'-close to $\pi_{\rho_1\cdots\rho_{k_0}}$ uniformly in $\rho_1, \dots, \rho_m$ (for $m \ge k_0$); this is possible since M is bounded and the Γ_ρ are uniformly Lipschitz. Consequently

$$
\left| G_m - \sum_{\rho_1,\dots\rho_{k_0}=1}^{r} p_{\rho_1}\cdots p_{\rho_{k_0}}\varphi(\pi_{\rho_1\cdots\rho_{k_0}}) \right| \le \varepsilon
$$

for $m \ge k_0$, and this shows that (G_m) is convergent (and therefore in particular eventually quasiperiodic also in this case).

What is the limit? We provide $\{1, \dots, r\}$ with the measure P defined by $P(\{\rho\}) := p_\rho$ and consider on $\{1, \dots, r\}^{\mathbb{N}_0}$ the product measure P_∞ associated with countably many independent copies of P. As in [3] we define a map $\Phi : \{1, \dots, r\}^{\mathbb{N}_0} \to F$ by

$$
\Phi(\rho_1, \rho_2, \dots) := \lim_{l\to\infty} \pi_{\rho_1\cdots\rho_l}.
$$

Then, if μ denotes the image measure of P_∞ on M (i.e., $\mu(B) := P_\infty\big(\Phi^{-1}(B)\big)$ for Borel sets $B \subset M$), then it can deduced easily from the preceding calculations that

$$
\lim_m G_m = \int_M \varphi(x)\mu(dx).
$$

In the last example we have seen that – by passing from M to F – the essential aspects of a problem sometimes can be analysed on a much smaller set. This is true also in the present context.

To motivate the next definition consider the following situation. M is the interval $[0,2]$, and we define

$$A_1 = [0,1], \ A_2 = [1,2], \ x_0 = 0,$$

$$\Gamma_1 : A_1 \to M, \ x \mapsto 1, \ \Gamma_2 : A_2 \to M, \ x \mapsto (x+4)/3.$$

The A_1, A_2 are *not* clopen. However, if we replace A_1 by $\tilde{A}_1 := \{0,1\}$, A_2 by $\tilde{A}_2 := \{1, \Gamma_2(1), \Gamma_2^2(1), \Gamma_2^3(1), \ldots, 2\}$ and M by $\tilde{M} = \tilde{A}_1 \cup \tilde{A}_2$, then we arrive at a chain where the walk is identical with that of the original model but the $\tilde{A}_\rho$ are now clopen.

In the general case the construction is as follows:

Lemma 2.2. *Let* $\big(M, (A_\rho), (\Gamma_\rho), (p_{\rho,\Delta}), x_0\big)$*, a Markov chains induced by contractions as in definition 2.1, be given.*

1. *A finite sequence* $\rho_1, \ldots, \rho_l$ *is called* admissible*, if* $x_0 \in A_{\rho_1}$, $\Gamma_{\rho_1}(x_0) \in A_{\rho_2}$, ..., $\Gamma_{\rho_{l-1}} \circ \cdots \circ \Gamma_{\rho_1}(x_0) \in A_{\rho_l}$. *This means that in a concrete realization of the walk it might really happen that the choice of indices is* $\rho_1, \ldots, \rho_l$.
2. *Let* M' *denote the collection of* x_0 *together with all* $\Gamma_{\rho_{l-1}} \circ \cdots \circ \Gamma_{\rho_1}(x_0)$*, where* $\rho_1, \ldots, \rho_l$ *range over all admissible sequences. We further define* $\tilde{M}$ *as the closure of* M' *and* $\tilde{A}_\rho$ *as the closure of* $A_\rho \cap M'$*. The map* $\tilde{\Gamma}_\rho$ *is the restriction of* Γ_ρ *to* $\tilde{A}_\rho$.

We claim that $\big(\tilde{M}, (\tilde{A}_\rho), (\tilde{\Gamma}_\rho), (p_{\rho,\Delta}), x_0\big)$ *is a Markov chain induced by contractions which has the same stochastic behaviour as the original one.*
The new chain is called the reduced model *of the original chain* $\big(M, (A_\rho), (\Gamma_\rho), (p_{\rho,\Delta}), x_0\big)$.

Proof. It is clear that $\tilde{M}$ is the union of the $\tilde{A}_\rho$. It only remains to show that the range of the $\tilde{\Gamma}_\rho$ lies in $\tilde{M}$. But, by the definition of M', $\tilde{\Gamma}_\rho$ maps $M' \cap A_\rho$ to M'. Therefore, since Γ_ρ is continuous, the range of $\tilde{\Gamma}_\rho$ must lie in $\tilde{M}$.

Since every concrete realization of a walk stays in M', and since, for $x \in M'$, one has

$$\{\rho \mid x \in A_\rho\} = \{\rho \mid x \in \tilde{A}_\rho\},$$

the reduced model and the original one give rise to the same walks with the same probabilities. □

3. The case of clopen A_ρ

The following proposition applies if the A_ρ are particularly simple[5]. The examples 1 to 4 from the last section are special cases of this situation.

The counterexamples which will be presented in section 4 show that in the general case a similar behaviour is not to be expected.

Proposition 3.1. *Let $\big(M, (A_\rho), (\Gamma_\rho), (p_{\rho,\Delta}), x_0\big)$, a Markov chain induced by contractions, be given such that the A_ρ are clopen. Further, there is prescribed a continuous gain function $\varphi : M \to \mathbb{R}$, and as above G_m denotes the expected gain in the m'th round.*
Then, for every $\varepsilon > 0$, there are $l_0, m_0 \in \mathbb{N}$ such that

$$|G_{m+k\cdot l_0} - G_m| \leq \varepsilon$$

for $m \geq m_0$ and $k = 1, 2, \ldots$: the sequence (G_m) is eventually quasiperiodic.

The proof will be given later, it will be convenient to prove some preliminary results first. The idea is to reduce the assertion to the case of finite Markov chains (cf. example 3 of section 2).

Lemma 3.2. *Let a compact metric space (M, d) be written as $M = \bigcup_{\rho=1}^{r} A_\rho$ with clopen A_ρ. We claim that there exists a $\delta > 0$ such that $\Delta_x = \Delta_y$ for all $x, y \in M$ with $d(x, y) \leq \delta$. (As before, Δ_x denotes the set $\{\rho \mid x \in A_\rho\}$.)*

Proof. If the assertion were false one could find (x_n) and (y_n) such that $d(x_n, y_n) \to 0$, but $\Delta_{x_n} \neq \Delta_{y_n}$ for every n. Since M is compact, we may assume that there is an $\hat{x}$ such that $x_n \to \hat{x}$ and $y_n \to \hat{x}$. This already leads to a contradiction: since the A_ρ are open, one has $\Delta_{\hat{x}} \subset \Delta_{x_n}, \Delta_{y_n}$ for sufficiently large n, and since they are closed, the sets $\Delta_{x_n}, \Delta_{y_n}$ are eventually subsets of $\Delta_{\hat{x}}$. (Alternatively one could argue that the function $x \mapsto \Delta_x$ is locally constant since the A_ρ are clopen. Therefore it must be "uniformly locally constant" by the compactness of M.) □

The number δ from the preceding lemma will be fixed throughout. With an arbitrary $\eta > 0$ which satisfies $\eta \leq \delta$ (it will be specified later) we consider an η-net $z_0, \ldots, z_{s-1}$ in M: this means that for every $x \in M$ there is an i with $d(x, z_i) \leq \eta$. Such a net exists since M is compact. We suppose that $d(x_0, z_0) \leq \eta$, where x_0 is the starting position.

[5]Of course it suffices to assume that the conditions are met in the reduced model introduced in lemma 2.2.

We will define a finite Markov chain with state space $S = \{0, \ldots, s-1\}$ and starting position 0 as follows. If $i \in S$ is given, consider the set Δ_{z_i}. For $\rho \in \Delta_{z_i}$ calculate $\Gamma_\rho z_i$. We may choose a j with $d\big(\Gamma_\rho(z_i), z_j\big) \le \eta$. In fact there might be several candidates z_j, but we fix one for every pair (i, ρ) with $z_i \in A_\rho$. (To put it more formally, we fix once and for all a function

$$J : \{(i, \rho) \mid \rho \in \Delta_{z_i}\} \to S$$

such that always $d\big(\Gamma_\rho(z_i), z_{J(i,\rho)}\big) \le \eta$.)

After this preparation we can introduce the transition probabilities p_{ij}. For $i, j \in S$ we put

$$p_{i,j} := \sum_{\rho \in \Delta_{z_i},\ J(i,\rho)=j} p_{\rho, \Delta_{z_i}},$$

where as usual the empty sum is zero by definition.

> One should not be confused by the rather technical approach. The idea is simple: First one fixes once and for all a j for (i, ρ) such that $\Gamma_\rho(z_i)$ is mapped close to z_j, and then one collects probabilities; p_{ij} is the probability that for the original situation a ρ is chosen such that z_j is the distinguished approximation of $\Gamma_\rho(z_i)$.

We observe that $\mathbf{P} := (p_{ij})$ is a stochastic matrix. This follows at once from the fact that $\sum_{\rho \in \Delta_{z_i}} p_{\rho, \Delta_{z_i}} = 1$.

It now will be shown that the stochastic behaviour of the chain associated with $(S, \mathbf{P})$ approximates the behaviour of the walk (X_m) associated with $\big(M, (A_\rho), (\Gamma_\rho), (p_\rho^\Delta), x_0\big)$ provided that η is sufficiently small. More precisely: We will consider on S the gain vector $\mathbf{w} = (w_0, \ldots, w_{s-1})^\top$ defined by $w_i := \varphi(z_i)$, and we will compare the G_m with the G_m^S, where G_m^S is the gain in the m'th round associated with $(S, \mathbf{P}, \mathbf{w})$ (cf. example 3 from section 2).

Lemma 3.3. *For every $\varepsilon > 0$ one may choose $\eta > 0$ so small that*

$$\big|G_m - G_m^S\big| \le \varepsilon$$

for every m.

Proof. Recall that $L < 1$ denotes a number such that all Γ_ρ are contractions with Lipschitz constant L. Now suppose that ε is a given positive number. We choose $\varepsilon' > 0$ such that $d(x, y) \le \varepsilon'$ always yields $|\varphi(x) - \varphi(y)| \le \varepsilon$. Without loss of generality it will be assumed that $\varepsilon' \le \delta$. Thus, if $d(x, z_i) \le \varepsilon'$ for certain x, z_i, we know that x and z_i will lie in the same A_ρ. Suppose that x is the position of the walk associated with $\big(M,(A_\rho),(\Gamma_\rho),(p_{\rho,\Delta}),x_0\big)$ at some step. Then, to generate the next position, one has to consider Δ_x and to choose one of these ρ according to the probabilities p_{ρ,Δ_x}; the next position then will be $\Gamma_\rho(x)$.

But $d(x,z_i)\le\varepsilon'$ so that $d\big(\Gamma_\rho(x),\Gamma_\rho(z_i)\big)\le L\varepsilon'$. Thus, if it happens that $\eta\le(1-L)\varepsilon'$, then $d\big(\Gamma_\rho(x),z_j\big)\le\varepsilon'$ for every j such that $d\big(\Gamma_\rho(z_i),z_j\big)\le\eta$. In particular this applies for $j=J(i,\rho)$.

Therefore, under the assumption of $\eta\le(1-L)\varepsilon'\le\delta$, we have shown that the following assertion is true:
Fact 1: If $d(x,z_i)\le\varepsilon'$, then with probability $p_{i,j}$ the next position of the walk will be ε'-close to z_j.

It is this property which will be crucial for our proof. As an illustration let's compare G_m with G_m^S for $m=0$ and $m=1$.

The case $m=0$ is simple. One has $G_0=\varphi(x_0)$ and $G_0^S=\varphi(z_0)$, and from $d(x_0,z_0)\le\eta\le\varepsilon'$ it follows that $\big|G_0-G_0^S\big|\le\varepsilon$.
To calculate G_1 we observe that

$$G_1=\sum_{\rho\in\Delta_{x_0}}p_{\rho,\Delta}\varphi(x_\rho),$$

where we have defined $\Delta:=\Delta_x$ and $x_\rho:=\Gamma_\rho(x_0)$. Since $\eta\le\delta$ we have $\Delta_{x_0}=\Delta_{z_0}$. Thus we may represent Δ_{x_0} as the disjoint union of the subsets

$$\Delta_{z_0}^j:=\{\rho\in\Delta_{z_0}\mid J(0,\rho)=j\},\ j=0,\dots,s{-}1.$$

For $\rho\in\Delta_{z_0}^j$ we know by "fact 1" that x_ρ lies ε'-close to z_j so that

$$|\varphi(x_\rho)-w_j|\le\varepsilon.$$

It follows that

$$\begin{aligned}
G_1 &= \sum_j\sum_{\rho\in\Delta_{z_0}^j}p_{\rho,\Delta}\varphi(x_\rho)\\
&=_\varepsilon \sum_{j=0}^{s-1}\sum_{\rho\in\Delta_{z_0}^j}p_{\rho,\Delta}\varphi(z_j)\\
&= \sum_{j=0}^{s-1}\varphi(z_j)\sum_{\rho\in\Delta_{z_0}^j}p_{\rho,\Delta}\\
&= \sum_{j=0}^{s-1}\varphi(z_j)p_{0j}\\
&= G_1^S;
\end{aligned}$$

here "$a =_\varepsilon b$" abbreviates the fact that $|a-b| \le \varepsilon$.

To deal with a general but fixed m we introduce some further notation:

- By R_m we mean the collection of all admissible sequences $(\rho_1, \ldots, \rho_m)$; cf. lemma 2.2.
- R_m^S is defined to be the collection of all $(i_1, \ldots, i_m) \in S^m$ such that, for $k = 0, \ldots, m-1$, there is a $\rho \in \Delta_{z_k}$ such that $i_{k+1} = J(i_k, \rho)$.
- For $i, j \in S$ we denote by $\Delta_{z_i}^j$ the collection of the $\rho \in \Delta_{z_i}$ such that $J(i, \rho) = j$.
- If $\rho_1, \ldots, \rho_l$ is admissible, then $x_{\rho_1 \cdots \rho_l} := \Gamma_{\rho_l} \circ \cdots \circ \Gamma_{\rho_1}(x_0)$.

Let $(\rho_1, \ldots, \rho_m) \in R_m$ be given. ρ_1 lies in $\Delta_{x_0} = \Delta_{z_0}$ so that there is a unique i_1 with $\rho_1 \in \Delta_{z_0}^{i_1}$. Then x_{ρ_1} is ε'-close to z_{i_1} so that $\Delta_{x_{\rho_1}} = \Delta_{z_{i_1}}$. Therefore there is a unique i_2 with $\rho_2 \in \Delta_{z_{i_1}}^{i_2}$. Continuing this way we obtain a unique $(i_1, \ldots, i_m) \in S^m$ with $\rho_{l+1} \in \Delta_{z_{i_l}}^{i_{l+1}}$ for $l = 0, \ldots, s-1$[6]. Since the i's are generated by the ρ's it is clear that $(i_1, \ldots, i_m)$ lies in R_m^S, and in this way we have constructed a map $\psi : R_m \to R_m^S$ which will be used to relate G_m with G_m^S.

What will be needed in the sequel is summarized here as

Fact 2: (i) ψ is onto.

(ii) If $\psi(\rho_1, \ldots, \rho_m) = (i_1, \ldots, i_m)$ holds, then $d(x_{\rho_1 \cdots \rho_m}, z_{i_m}) \le \varepsilon'$.

(iii) For every $(i_1, \ldots, i_m) \in R_m^S$ one has

$$\sum p_{\rho_1, \Delta_{x_0}} p_{\rho_2, \Delta_{x_{\rho_1}}} \cdots p_{\rho_m, \Delta_{x_{\rho_1 \cdots \rho_{m-1}}}} = p_{0,i_1} p_{i_1,i_2} \cdots p_{i_{m-1},i_m},$$

where the sum runs through the $(\rho_1, \ldots, \rho_m) \in R_m$ with $\psi(\rho_1, \ldots, \rho_m) = (i_1, \ldots, i_m)$.

((i) follows immediately from the definition of G_m^S and (ii) can be proved by an m-fold application of fact 1. For the proof on (iii) one proceeds by induction. In the case $m = 1$ the claim reduces to the definition of the p_{0,i_1}, and for the proof of the induction step $m \to m+1$ one only has to observe that $R_m = \{(\rho_1, \ldots, \rho_m) \mid (\rho_1, \ldots, \rho_{m+1}) \in R_{m+1}\}$ and $R_m^S = \{(i_1, \ldots, i_m) \mid (i_1, \ldots, i_{m+1}) \in R_{m+1}^S.\}$)

We now are able to complete the proof of the lemma. The explicit form of G_m is

$$G_m = \sum_{(\rho_1, \ldots, \rho_m) \in R_m} p_{\rho_1, \Delta_{x_0}} p_{\rho_2, \Delta_{x_{\rho_1}}} \cdots p_{\rho_m, \Delta_{x_{\rho_1 \cdots \rho_{m-1}}}} \varphi(x_{\rho_1 \cdots \rho_m}).$$

[6] We have to put $i_0 := 0$ here.

We split this sum as

$$\sum_{(\rho_1,\ldots,\rho_m)\in R_m} = \sum_{(i_1,\ldots,i_m)\in R_m^S} \sum_{\psi(\rho_1,\ldots,\rho_m)=(i_1,\ldots,i_m)} .$$

In each of the summands associated with a particular $(i_1,\ldots,i_m)$ we replace $\varphi(x_{\rho_1\cdots\rho_m})$ by $\varphi(z_{i_m})$ which causes (by (ii)) an overall error of at most ε. In this way we arrive at

$$\begin{aligned} G_m &=_\varepsilon \sum_{(i_1,\ldots,i_m)\in R_m^S} \varphi(z_{i_m}) \sum_{\psi(\rho_1,\ldots,\rho_m)=(i_1,\ldots,i_m)} p_{\rho_1,\Delta_{x_0}} p_{\rho_2,\Delta_{x_{\rho_1}}} \cdots p_{\rho_m,\Delta_{x_{\rho_1\cdots\rho_{m-1}}}} \\ &= \sum_{(i_1,\ldots,i_m)\in R_m^S} \varphi(z_{i_m}) p_{0,i_1} p_{i_1,i_2} \cdots p_{i_{m-1},i_m} \varphi(z_{i_m}) \\ &= G_m^S. \end{aligned}$$

This completes the proof of the lemma. □

Proof of proposition 3.1: The proof can now easily be given by combining the preceding lemma with the fact that in the case of finite Markov chains with reward one always observes an eventually quasiperiodic behaviour (cf. example 3 in section 2).

4. Counterexamples

In this section we investigate in detail two examples which show how complicated the walk can behave if the A_ρ are not clopen.

In *example 1* we describe a situation where the sequence (G_m) of expectations oscillates in an arbitrarily prescribed way between zero and one. In particular it follows that one cannot guarantee that (G_m) is Cesáro convergent as it would be if some ergodicity results could be applied.

More precisely, we will prove the following

Proposition 4.1. *Let arbitrary integers $n_1, m_1, n_2, m_2 \geq 2$ be given. Then there exist M, $x_0 \in M$, closed sets A_0, A_1 with $M = A_0 \cup A_1$, contractions $\Gamma_0 : A_0 \to M$, $\Gamma_1 : A_1 \to M$ and a continuous function $\varphi : M \to R$ such that the sequence (G_m) of gains associated with $(M, x_0, A_0, A_1, \Gamma_0, \Gamma_1, \varphi)$*[7] *oscillates as follows:*

[7] For $x \in A_0 \cap A_1$ one proceeds with equal probability by using Γ_0 or Γ_1, but this will not be relevant for our example.

- *$G_0 = 0$, and $G_1 = \cdots = G_{n_1} = 1$;*
- *$G_{n_1+1} = \cdots = G_{n_1+m_1} = 0$;*
- *then $G_{n_1+m_1+1} = \cdots = G_{n_1+m_1+n_2} = 1$, etc.*

Proof. Define $M := \{0,1\}^{\mathbb{N}}$, i.e. the elements of M are sequences containing only 0's and 1's. A typical element will be written as $(a_1a_2\cdots)$, as in the case of decimal representations of rational numbers we overline elements of the sequence to indicate that these elements are repeated again and again[8].

M is provided with the following metric: if $x = (a_1a_2\cdots)$ and $y = (b_1b_2\cdots)$ are arbitrary elements of M with $x \neq y$, then $d(x,y) := 2^{-k}$, where k is the smallest index i with $a_i \neq b_i$. (Thus M is essentially the usual Cantor discontinuum; it will, however, be more convenient to work with the product representation.)

Let the n_k, m_k be given. We will construct two continuous functions $\varphi_0, \varphi_1 : M \to \mathbb{R}$ such that, with $x_0 = (\overline{01})$, the sets

$$A_0 := \{x \mid \varphi_0(x) \geq \varphi_1(x)\},\ A_1 := \{x \mid \varphi_0(x) \leq \varphi_1(x)\},$$

the contractions

$$\Gamma_0 : A_0 \to M,\ (a_1a_2\cdots) \mapsto (0a_1a_2\cdots),$$

$$\Gamma_1 : A_1 \to M,\ (a_1a_2\cdots) \mapsto (1a_1a_2\cdots)$$

and the map $\varphi : (a_1a_2\cdots) \mapsto a_1$ have the desired properties. Note that Γ_0 and Γ_1 are shifts to the right, as the first element one has to add "0" or "1", respectively. These maps satisfy $d\big(\Gamma_i x, \Gamma_i y\big) = d(x,y)/2$ so that they are in fact contractions.

We begin by describing the idea of our construction. First φ_0 and φ_1 will be defined at x_0. $\varphi_0(x_0)$ and $\varphi_1(x_0)$ will be close to zero with $\varphi_0(x_0) < \varphi_1(x_0)$. Thus the next position of the walk is $x_1 = \Gamma_1(x_0) = (1\overline{01})$, and $G_0 = \varphi(x_0) = 0$. (In this and in the following steps the walk will never be in $A_0 \cap A_1$. Therefore it will never happen that for the next position one has to make a random decision whether to apply Γ_0 or Γ_1 so that our "random" walk will be in fact deterministic.) At x_1 the function φ_1 is slightly larger than φ_0. Thus Γ_1 has to be applied again to come do $x_3 = (11\overline{01})$. One continues in a similar way until Γ_1 has been applied n_1 times. The position is then $x_{n_1} = (1\cdots 1\overline{01})$ (with n_1 1's at the beginning), and the

[8] E.g., $(001\overline{01})$ stands for $(001101010101\cdots)$.

gains are $G_1 = \cdots = G_{n_1} = 1$. At x_{n_1} the function φ_0 is slightly larger than φ_1. Therefore $x_{n_1} \in A_0$, and consequently $x_{n_1+1} = (01\cdots 1\overline{01})$ and $G_{n_1+1} = 0$. There again φ_0 dominates φ_1 so that Γ_0 will be used once more. After m_1-1 further steps we will arrive at $x_{n_1+m_1} = (0\cdots 01\cdots 1\overline{01})$ (with m_1 zeroes at the beginning). There φ_1 is the larger function which yields $x_{n_1+m_1+1} = (10\cdots 01\cdots 1\overline{01})$.

The next $n_2 - 1$ steps will also be done by applying Γ_1, then follow m_2 steps with Γ_0, n_3 steps with Γ_1 etc.

Since our walk is deterministic the number G_m always equals $\varphi(x_m)$. Thus (G_m) oscillates as desired.

It remains to show that φ_0 and φ_1 with the above properties really can be constructed. Let $x_1, x_2, \ldots$ the positions of the walk of our heuristic approach:

$$x_1 = (1\overline{01}),\ x_2 = (11\overline{01}), \ldots,\ x_{n_1} = (1\cdots 1\overline{01}),$$

$$x_{n_1+1} = (01\cdots 1\overline{01}), \ldots,\ x_{n_1+m_1} = (0\cdots 01\cdots 1\overline{01}),$$

$$x_{n_1+m_1+1} = (10\cdots 01\cdots 1\overline{01}), \ldots,\ x_{n_1+m_1+n_2} = (1\cdots 10\cdots 0\cdots 01\cdots 1\overline{01}),$$

etc. The set of elements in the first resp. second resp. ... line will be denoted by $\Delta_1, \tilde{\Delta}_1, \Delta_2, \ldots$ We we observe that the elements $x_0, x_1, x_2, \ldots$ are pairwise different. But more is true:

- $d(x_0, x_n) \geq 1/2^3 =: \varepsilon_0$ for $n \geq 1$ since by construction x_0 and x_n have at most the first two digits in common. (Here it is important that we assume that the n_k, m_k are larger than one.)
- $d(x_i, x_j) \geq 1/2^{n_1+1} =: \varepsilon_1$ for $i \in \Delta_1$ and $j \in \mathbb{N}$, $j \neq i$.
- $d(x_i, x_j) \geq 1/2^{n_1+m_1+1} =: \tilde{\varepsilon}_1$ for $i \in \tilde{\Delta}_1$ and $j \in \mathbb{N}$, $j \neq i$.
- $d(x_i, x_j) \geq 1/2^{n_1+m_1+n_2+1} =: \varepsilon_2$ for $i \in \Delta_2$ and $j \in \mathbb{N}$, $j \neq i$.
- ...

Denote, for $x \in M$ and $\delta > 0$, by $\varphi_{x,\delta} : M \to \mathbb{R}$ the "hat function"

$$\varphi_{x,\delta}(y) := \max\{0, \delta - d(x, y)\}.$$

$\varphi_{x,\delta}$ is a nonnegative Lipschitz function, the support is contained in the set $\{y \mid d(x, y) \leq \delta\}$, and $\varphi_{x,\delta}(x)$ is strictly positive.

We put $\delta_0 := \varepsilon_0/2$, and we start our construction by first setting

$$\varphi_0 := 0,\ \varphi_1 := \varphi_{x_0,\delta_0}.$$

This guarantees that the first step of the walk is as it should be: we pass from x_0 to x_1. However, the definition of φ_1 and φ_2 will have to be refined to have control also over the next steps. As a second approximation to the final definition we put

$$\varphi_0 := \varphi_{x_{n_1},\delta_1},$$
$$\varphi_1 := \varphi_{x_0,\delta_0} + \varphi_{x_1,\delta_1} + \varphi_{x_2,\delta_1} + \cdots + \varphi_{x_{n_1-1},\delta_1},$$

where $\delta_1 := \varepsilon_1/2$. With this definition the first n_1 steps of the walk are as they are supposed to be, and it should be clear how to proceed. Next one adds (with $\tilde\delta_1 := \tilde\varepsilon_1/2$) $\varphi_{x_{n_1+1},\tilde\delta_1}$, $\varphi_{x_{n_1+2},\tilde\delta_1}$, $\ldots$, $\varphi_{x_{n_1+m_1-1},\tilde\delta_1}$ to the present φ_0 and $\varphi_{x_{n_1+m_1},\tilde\delta_1}$ to φ_1. Then one works with $\delta_2 := \varepsilon_2/2$ and the $x_{n_1+m_1+1},\ldots,x_{n_1+m_1+n_2}$ etc. The final definitions are as follows:

$$\begin{aligned}\varphi_0 \ :=\ & \varphi_{x_{n_1},\delta_1} + \varphi_{x_{n_1+1},\tilde\delta_1} + \cdots + \varphi_{x_{n_1+m_1-1},\tilde\delta_1} + \\ & +\varphi_{x_{n_1+m_1+n_2},\delta_2} + \varphi_{x_{n_1+m_1+n_2+1},\tilde\delta_2} + \cdots + \varphi_{x_{n_1+m_1+n_2+m_2-1},\tilde\delta_2} + \\ & + \cdots ;\\ \varphi_1 \ :=\ & \varphi_{x_0,\delta_0} + \varphi_{x_1,\delta_1} + \varphi_{x_2,\delta_1} + \cdots + \varphi_{x_{n_1-1},\delta_1} + \\ & +\varphi_{x_{n_1+m_1},\tilde\delta_1} + \varphi_{x_{n_1+m_1+1},\delta_2} + \cdots + \varphi_{x_{n_1+m_1+n_2-1},\delta_2} + \\ & + \cdots\end{aligned}$$

These functions have the claimed properties: that they generate the walk $x_0, x_1, \ldots$ is clear from the construction, and they are continuous since the $\varphi_{x,\delta}$-functions which occur here have mutually disjoint support. □

One might suspect that the strange behaviour of the walk in the preceding example was possible only since the sets A_1, A_2 are rather complicated. But also in much simpler situations a nonperiodic behaviour of the sequence (G_m) can be observed as the following example shows.

We put $M = [\,0,1\,]$, $\Gamma_0 : x \mapsto x/3$, $\Gamma_1 : x \mapsto (x+2)/3$, $A_0 := \{x \geq \mathbf{a}_0\}$, $A_1 = \{x \leq \mathbf{a}_0\}$, $x_0 = 1/3$; here $\mathbf{a}_0 \in M$ is fixed. It will turn out that the behaviour of the walks associated with this setting very subtly depend on the choice of $\mathbf{a}_0$.

The Γ_ρ are such that the walks of the system stay in the Cantor discontinuum. It will be convenient to pass to the more appropriate representation from proposition 4.1. The present situation translates as follows:

M and Γ_0, Γ_1 are as in 4.1, M is provided with the lexicographic order "$\leq$", $\mathbf{a}_0 = (a_1, a_2, \ldots) \in M$ is fixed, $A_0 = \{x \geq \mathbf{a}_0\}$, $A_1 = \{x \leq \mathbf{a}_0\}$, and $x_0 = (0\overline{1})$. In addition we will consider the reward function φ defined as in the preceding example by $(a_1 a_2 \cdots) \mapsto a_1$.

As in proposition 4.1 we only will meet situations where the walk avoids $A_0 \cap A_1$ so that it will be deterministic.

Consider, as a first example, the case $\mathbf{a}_0 = (\overline{011})$. x_0 lies in A_0 so that the next position is $(00\overline{1})$. This element is smaller than $\mathbf{a}_0$ so that we proceed to $(100\overline{1})$. This is in A_0, and we arrive at $(0100\overline{1})$. The next steps are

$$(10100\overline{1}), (010100\overline{1}), (1010100\overline{1}), (01010100\overline{1}), (101010100\overline{1}),$$

and it is clear how this sequence continues: "0" and "1" alternate as the first new entry. Therefore the walk is essentially periodic, it oscillates between $(\overline{01})$ and $(\overline{10})$ with an error which tends to zero. As a consequence the sequence (G_m) of expected gains is eventually quasiperiodic for *any* continuous φ, in our case we even have $(G_m) = (1, 0, 1, 0, 1, 0, 1, 0, \ldots)$[9].

Strangely enough, the behaviour for this particular choice of $\mathbf{a}_0$ is not exceptional. On the contrary, all random choices which we have investigated have generated rather short periodic patterns. Therefore it was tempting to conjecture that one always would have this periodicity.

> The reader is invited to start with some $\mathbf{a}_0$ and to consider the walk as the following "game": one starts with $(0\overline{1})$, and one writes a "0" (resp. a "1") in front of the present sequence if it is larger (resp. smaller) than $\mathbf{a}_0$.
>
> One can be sure that after a short time a simple periodic 0-1-pattern is repeated again and again.

In fact there exist $\mathbf{a}_0$ which give rise to a more complicated behaviour:

Proposition 4.2. *There exist uncountably many* $\mathbf{a}_0$ *such that the* 0-1-*pattern generated by the associated walk is not periodic*

Proof. Let $\mathbf{a}_0 \in M$ be fixed. $\mathbf{a}_0 = (a_1 a_2 \cdots)$ will be constructed such that the 0-1-Pattern defined by the first entries of the x_m, which are by the definition of φ just the G_m – coincide with the $a_1, a_2, \ldots$. As a simple example consider $\mathbf{a}_0 = (\overline{10})$ which really gives rise to

$$x_1 = (10\overline{1}),\ x_2 = (010\overline{1}),\ x_3 = (1010\overline{1}),\ \ldots$$

[9]Note that the walk is deterministic so that $G_m = \varphi(x_m)$, where x_m denotes the position at "time" m.

We call $\mathbf{a}_0$ *self-generating* if $G_m = a_m$ for every m. The idea is to construct self-generating $\mathbf{a}_0$ with longer and longer periods and to define the ultimate $\mathbf{a}_0$ by a diagonal procedure.

Let $n_1, n_2, \ldots \in \mathbb{N}$ be arbirarily given. By A^{1,n_1} we denote the finite sequence $1 \cdots 10$ with n_1 ones at the beginning. It is easy to check that $\mathbf{a_0}^{(1)} = (\overline{A^{1,n_1}})$ is self-generating[10].

Next we define A^{2,n_2}: this sequence starts with n_2 copies of A^{1,n_1+1} and terminates with one copy of A^{1,n_1}. With A^{2,n_2} we define the self-generating $\mathbf{a_0}^{(2)}$ as $(\overline{A^{2,n_2}})$. Similarly we continue: A^{3,n_3} is glued together from n_3 copies of A^{2,n_2+1} and one copy of A^{2,n_2}, and $\mathbf{a}_0^{(3)} = (\overline{A^{3,n_3}})$. This sequence is also self-generating.

In general, we construct $A^{k+1,n_{k+1}}$ by starting with n_{k+1} copies of A^{k,n_k+1} and terminating with one A^{k,n_k}; $\mathbf{a}_0^{(k+1)}$ then is $(\overline{A^{k+1,n_{k+1}}})$, this is a self-generating sequence[11].

Finally, if all $\mathbf{a}_0^{(k)}$ have been constructed, we define $\mathbf{a}_0$ such that the k'th entry of $\mathbf{a}_0$is the k'th entry of $\mathbf{a}_0^{(k)}$ for every k. In this way we arrive at a sequence where the first k entries of the x_n coincide with first entries of A^{k,n_k} for every k, in particular no periodic pattern will be generated.

We note that there are uncountably many choices for $n_1, n_2, \ldots$, and therefore the proof of the proposition is complete. □

Remark: The sequence (G_m) is not periodic in these examples. However, the average reward for the first m rounds is just the proportion of 1's in the first m digits of $\mathbf{a}_0$. This converges for our particular $\mathbf{a}_0$ so that the (G_m) are always Cesáro convergent here.

The situations described in the preceding proposition are intermediate between the rather predictable behaviour of the sequence (G_m) in the case of clopen A_ρ and the chaotic (G_m) of proposition 4.1. It is an *open problem* whether "very chaotic" walks can exist in the case of "reasonable" Γ_ρ,A_ρ and φ. Of particular interest would be the answer for the case of compact convex $M, A_1, \ldots, A_r$ and continuous and affine $\Gamma_1, \ldots \Gamma_r$. These conditions are satisfied when treating the collective Parrondo games which have motivated the present investigations. (We note that for the original paradox the reduced chain has clopen A_ρ but games which lead to non-periodic walks as in proposition 4.2 can easily be found.)

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[10]Here and in the sequel we use freely the overline notation to denote periodic repetition. For example, for $n_1 = 4$ one has $\mathbf{a}_0^{(1)} = (11110111101111011110\ldots)$.

[11]The proof is elementary but lengthy, it is omitted here.

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