CHAPTER VI

M-ideals in spaces of bounded operators

VI.1 The centralizer of $L(X, Y)$

It seems to be difficult to obtain a general description of the M-ideals of $L(X, Y)$: It is one of the aspects of Section VI.4 that there is for example no hope in trying to prove that each M-ideal in $L(X, Y)$ corresponds to an M-ideal in Y or X^* (as it will turn out for a special case in the following section). There is even no theorem known that produces in a one-to-one fashion an M-ideal in $L(X)$ for each M-ideal in X. (The foregoing section, however, gives a hint of what such a result probably could look like.)
In this section we are going to study the centralizer $Z(L(X, Y))$ and the multiplier algebra $\mathrm{Mult}(L(X, Y))$ of the space of bounded linear operators $L(X, Y)$. We first show that the centralizer $Z(Y)$ and the Cunningham algebra $\mathrm{Cun}(X)$ naturally give rise to operators in $Z(L(X, Y))$, and our main result, Theorem 1.2, essentially states the converse under the assumption that only one of these candidates is allowed to enter this game seriously, i.e. $\mathrm{Cun}(X)$ or $Z(Y)$ is trivial. This will then admit a handsome description of the M-ideals which are naturally connected with these algebras via Corollary V.3.6. The problem of characterising the product structure that arises when one treats the general case, in which both $\mathrm{Cun}(X)$ and $Z(Y)$ are supposed nontrivial, is illustrated at the end of Section VI.3, and for a little bit of more information on this the reader is referred to the Notes and Remarks section.

Let H be a closed subspace of $L(X, Y)$. We start this section with introducing some natural candidates for elements of $Z(H)$ and $\mathrm{Mult}(H)$.
In the sequel, we denote left multiplication on $L(X, Y)$ with an operator $T \in L(Y)$ by L_T and, in the same way, right multiplication with $T \in L(X)$ is denoted by R_T. Furthermore, the adjoint of an operator $T \in Z(X)$ will be denoted by $\overline{T}$. (Recall that this means that the eigenvalues of the two operators corresponding to a fixed eigenvector $p \in \mathrm{ex}\, B_{X^*}$ are conjugate to each other.) Also, recall from Theorem I.3.14 the equivalence $T \in \mathrm{Cun}(X)$

if and only if $T^* \in Z(X^*)$, and for $T \in \operatorname{Cun}(X)$ the symbol $\overline{T}$ stands for the operator whose adjoint (in the above sense) is $\overline{T^*}$.

Lemma 1.1

(a) *Let H be a closed subspace of $L(X,Y)$. If for an element T of $Z(Y)$ (or $\operatorname{Mult}(Y)$) the conditions $L_T H \subset H$ and $L_{\overline{T}} H \subset H$ ($L_T H \subset H$) are fulfilled, then L_T belongs to $Z(H)$ (or $\operatorname{Mult}(H)$, respectively).*

(b) *Similarly, if T^* is in $Z(X^*)$ (or $\operatorname{Mult}(X^*)$) and if $R_T H \subset H$ and $R_{\overline{T}} H \subset H$ ($R_T H \subset H$) then R_T belongs to $Z(H)$ (to $\operatorname{Mult}(H)$).*

(c) *In particular, the maps $S \mapsto P_\infty S$ and $S \mapsto S P_1$ are M-projections on $L(X,Y)$ and suitable subspaces H as above, if P_∞ is an M-projection on Y and P_1 is an L-projection on X.*

PROOF: The various invariance assumptions make sure that L_T resp. R_T maps H into H. Let now $T \in Z(Y)$. Since $Z(Y) = \operatorname{lin} Z_{0,1}(Y)$, we may suppose that even $T \in Z_{0,1}(Y)$. (This set of operators is defined in I.3.7.) Proposition I.3.9 yields

$$\|TUx + (Id - T)Vx\| \le 1 \qquad \forall U, V \in B_H \ \forall x \in B_X,$$

and so,

$$\|TU + (Id - T)V\| \le 1 \qquad \forall U, V \in B_H,$$

which, again by Proposition I.3.9, implies $L_T \in Z(H)$.
If $T^* \in Z_{0,1}(X^*)$ then by the above

$$\|UT + V(Id - T)\| = \|T^*U^* + (Id - T^*)V^*\| \le 1 \qquad \forall U, V \in B_H,$$

and hence $R_T \in Z(H)$.
The proof in the case that $T \in \operatorname{Mult}(Y)$ or $T^* \in \operatorname{Mult}(X^*)$ is similar when Theorem I.3.6 is used, and part (c) is a special case of the above. □

The main theorem of this section shows that, under some appropriate restrictions on H, also the converse of Lemma 1.1 holds: All elements of $Z(H)$ are of this particularly simple form.

Theorem 1.2 *Let H be a subspace of $L(X,Y)$ containing the finite rank operators.*

(a) *If $L_T H \subset H$ for each $T \in Z(Y)$ and if $\operatorname{Cun}(X)$ is trivial, then*

$$Z(H) = \{L_T \mid T \in Z(Y)\}.$$

(b) *If $R_T H \subset H$ for all $T \in \operatorname{Cun}(X)$ and if $Z(Y)$ is trivial, then*

$$Z(H) = \{R_T \mid T \in \operatorname{Cun}(X)\}.$$

If the Banach space under consideration is complex, then we have:

(a*) *If $L_T H \subset H$ for each $T \in \operatorname{Mult}(Y)$ and if $\operatorname{Mult}(X^*)$ is trivial, then*

$$\operatorname{Mult}(H) = \{L_T \mid T \in \operatorname{Mult}(Y)\}.$$

Lack of a statement (b*) in the above is due to the absence of a good substitute for $\mathrm{Cun}(X)$, when $Z(X^*)$ is replaced by $\mathrm{Mult}(X^*)$ in the statement of Theorem I.3.14(c) (see the proof of Theorem 1.2(b)).

The basic obstacle for a direct proof of Theorem 1.2 stems from the fact that, in general, only for "small" subspaces such as $H = K(X,Y)$ the extremal structure of B_{H^*} has been determined. Since we will need the description of the extreme functionals on spaces of compact operators in Section VI.3, too, we take a broader perspective and present this representation in the framework of injective tensor products of Banach spaces which we define next.

For $u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ the injective tensor norm (or ε-norm) is defined by

$$\|u\|_\varepsilon = \sup\Big\{\Big|\sum x^*(x_i)y^*(y_i)\Big| \ \Big| \ x^* \in B_{X^*},\ y^* \in B_{Y^*}\Big\},$$

and the completion of the thus normed space $X \otimes Y$ is denoted by $X\widehat{\otimes}_\varepsilon Y$. This space is called the *injective tensor product* of X and Y, and it is closely related to a space of compact operators, as follows. The linear space $X^* \otimes Y$ can clearly be identified with the space of continuous operators from X to Y of finite rank. Under this identification the ε-norm coincides with the operator norm; hence $A(X,Y)$, the space of norm limits of finite rank operators ("approximable operators"), can canonically be identified with $X^*\widehat{\otimes}_\varepsilon Y$. It is known that whenever X^* or Y has the approximation property, then $A(X,Y) = K(X,Y)$ [421, p. 32f.]. In general, however, the space of compact operators can only be represented as the so-called Schwartz ε-product $X^*\varepsilon Y$, which reduces in the Banach space setting to the space $K_{w^*}(X^{**},Y)$ of compact operators from X^{**} into Y which are weak*-weakly continuous. (The isomorphism is $T \mapsto T^{**}$.)

We also need to know the dual space of $X\widehat{\otimes}_\varepsilon Y$. By construction, $X\widehat{\otimes}_\varepsilon Y$ is isometric to a subspace of $C(B_{X^*} \times B_{Y^*})$ (where the balls are equipped with their weak* topologies). Thus, by the Hahn-Banach and Riesz representation theorems each functional $\varphi \in (X\widehat{\otimes}_\varepsilon Y)^*$ has a representation by a (in fact positive) measure μ on $S = B_{X^*} \times B_{Y^*}$ having the same norm, i.e.

$$\langle \varphi, x \otimes y\rangle = \int_S x^*(x)y^*(y)\, d\mu(x^*,y^*). \tag{1}$$

From the point of view of operator theory, (1) defines an operator $T : X \to Y^*$ by means of

$$\langle Tx, y\rangle = \int_S x^*(x)y^*(y)\, d\mu(x^*,y^*). \tag{2}$$

Operators of this form are called *integral*; they form a linear space $I(X,Y^*)$ on which

$$\|T\|_{int} = \inf\{\|\mu\| \mid (2) \text{ holds}\}$$

defines a complete norm. (Actually, the infimum is attained.) Thus, $(X\widehat{\otimes}_\varepsilon Y)^* = I(X,Y^*)$ for short. For more details we refer to [158, Chapter VIII] or [280].

Note that $K_{w^*}(X^*,Y)$, too, embeds isometrically into $C(B_{X^*} \times B_{Y^*})$ so that continuous linear functionals may be represented by measures. However, in default of density of finite rank operators $K_{w^*}(X^*,Y)^*$ cannot be thought of as a space of operators.

We now have the following representation of the extreme functionals on spaces of compact operators due to W. Ruess and C. Stegall. In both cases of Theorem 1.3 $p\otimes q$ is supposed to operate on H by $\langle p\otimes q, T\rangle := \langle p, T^*q\rangle$.

Theorem 1.3

(a) *If* $X\widehat{\otimes}_\varepsilon Y \subset H \subset K_{w^*}(X^*,Y)$, *then*

$$\operatorname{ex} B_{H^*} = \{p\otimes q \mid p \in \operatorname{ex} B_{X^*},\ q\in B_{Y^*}\}.$$

(b) *If* $X^*\widehat{\otimes}_\varepsilon Y \subset H \subset K(X,Y)$, *then*

$$\operatorname{ex} B_{H^*} = \{p\otimes q \mid p \in \operatorname{ex} B_{X^{**}},\ q\in B_{Y^*}\}.$$

For the proof of this result we need two lemmas.

Lemma 1.4 *Let* $p\in \operatorname{ex} B_{X^*}$ *and* $y\in S_Y$. *Suppose* μ *is a Radon probability on* $S = B_{X^*}\times B_{Y^*}$ *such that*

$$p(x) = \int_S x\otimes y\, d\mu$$

for all $x\in X$. *Then* $\operatorname{supp}(\mu)$, *the support of* μ, *is contained in*

$$\mathbb{S}\cdot\{p\}\times\{y^*\in B_{Y^*} \mid |\langle y^*,y\rangle| = 1\}.$$

Proof: Consider the measurable mapping $U: S\to B_{X^*}$, $U(x^*,y^*) = \langle y^*,y\rangle\cdot x^*$. Then we have for the image measure $\nu = U(\mu)$ and $x\in X$

$$\int_{B_{X^*}} x\, d\nu \;=\; \int_S x\circ U\, d\mu \;=\; \int_S x\otimes y\, d\mu \;=\; p(x).$$

Since p is extreme, there is only one probability measure on B_{X^*} representing p. (Here is a quick proof of this fact: Consider the set F of those probability measures representing p. This is a face of the set of all probability measures since p is extreme; hence its extreme points are Dirac measures. But $\delta_q\in F$ if and only if $p=q$ so that $F = \overline{\operatorname{co}}\operatorname{ex} F = \{\delta_p\}$ by the Krein-Milman theorem. For the usual proof see [7, Corollary I.2.4].) It now follows $\nu = \delta_p$. Consequently,

$$\begin{aligned} 1 &= \nu(\{p\}) \\ &= \mu(\{U^{-1}(p)\}) \\ &= \mu(\{(x^*,y^*)\in S \mid \langle y^*,y\rangle\cdot x^* = p\}) \\ &\le \mu(\mathbb{S}\cdot\{p\}\times\{y^*\in B_{Y^*} \mid |\langle y^*,y\rangle| = 1\}). \end{aligned}$$

$\square$

Lemma 1.5 *Let* $T\in I(X,Y^*)$ *with* $\|T\|_{int} = 1$. *If* $T^*(y_0^{**})\in \operatorname{ex} B_{X^*}$ *for some* $y_0^{**}\in S_{Y^{**}}$, *then* $\operatorname{ran}(T^*) = \operatorname{lin}\{T^*(y_0^{**})\}$.

PROOF: We first note that $\|i_{Y^*}T\|_{int} = 1$ [158, Theorem VIII.2.8]. Hence there is a representation of $i_{Y^*}T : X \to Y^{***}$ by means of a probability measure μ on $B_{X^*} \times B_{Y^{***}}$ such that

$$\langle Tx, y^{**}\rangle \;=\; \int x^*(x)y^{***}(y^{**})\,d\mu(x^*, y^{***}) \;=\; \int x \otimes y^{**}\,d\mu.$$

Now Lemma 1.4 applies to show that $\operatorname{supp}(\mu) \subset \mathbb{S}\cdot\{T^*(y_0^{**})\} \times B_{Y^{***}}$ which yields that the barycentre of μ has the form $(\lambda_0 T^*(y_0^{**}), y_0^{***})$ for some $|\lambda_0| = 1$, $y_0^{***} \in B_{Y^{***}}$. This shows

$$(T^*y^{**})(x) \;=\; \int x \otimes y^{**}\,d\mu \;=\; \lambda_0\,(T^*y_0^{**})(x)\cdot y_0^{***}(y^{**})$$

for all $y^{**} \in Y^{**}$, $x \in X$. Thus, the proof of the lemma is completed. □

PROOF OF THEOREM 1.3:

It is of course enough to prove (a). Let $\varphi \in \operatorname{ex} B_{H^*}$. Since H embeds isometrically into $C(B_{X^*} \times B_{Y^*}) =: C(S)$, φ has an extension to an extreme functional on $C(S)$. Thus, there exist $p \in B_{X^*}$, $q \in B_{Y^*}$ and $\lambda \in \mathbb{S}$ such that $\varphi = \lambda\cdot\delta_{(p,q)}|_H = (\lambda p)\otimes q$. Necessarily p and q must be extreme if φ is.
We come to the converse and tackle first the case $H = X\widehat{\otimes}_\varepsilon Y$. Suppose $p \in \operatorname{ex} B_{X^*}$ and $q \in B_{Y^*}$, and assume moreover that $\|p\otimes q \pm T\|_{int} \le 1$ for some integral operator T. If $y_0^{**} \in S_{Y^{**}}$ satisfies $y_0^{**}(q) = 1$, then

$$\|p \pm T^*y_0^{**}\| \;\le\; \|p\otimes q \pm T\| \;\le\; \|p\otimes q \pm T\|_{int} \;\le\; 1,$$

whence $T^*y_0^{**} = 0$, and we have for the operator $S = p\otimes q + T$ that $S^*y_0^{**} = p \in \operatorname{ex} B_{X^*}$. Lemma 1.5 entails that $\operatorname{ran} S^* = \operatorname{lin}\{p\}$, therefore

$$S \;=\; p\otimes q + T \;=\; p\otimes y_0^*$$

for some $y_0^* \in Y^*$. This shows that $T = p\otimes(y_0^* - q)$ and

$$\|q \pm (y_0^* - q)\| \;=\; \|p\otimes q \pm T\| \;\le\; \|p\otimes q \pm T\|_{int} \;\le\; 1.$$

From the extremality of q we now obtain $y_0^* = q$ and $T = 0$. This proves that $p\otimes q \in \operatorname{ex} B_{H^*}$ for $H = X\widehat{\otimes}_\varepsilon Y$.
Finally we suppose that $p \in \operatorname{ex} B_{X^*}$, $q \in B_{Y^*}$ and $X\widehat{\otimes}_\varepsilon Y =: H_0 \subset H \subset K_{w^*}(X^*, Y)$. We already know that $(p\otimes q)|_{H_0} \in \operatorname{ex} B_{H_0^*}$, hence there is an extension of this functional to an extreme functional $\varphi \in \operatorname{ex} B_{H^*}$. By the first part of the proof $\varphi = p_1 \otimes q_1$ for some p_1 and q_1. Clearly, $(p_1\otimes q_1)|_{H_0} = (p\otimes q)|_{H_0}$ implies that $p = p_1$ and $q = q_1$. Therefore $p\otimes q = \varphi$ and is hence extremal on H. □

It is worthwhile noting that, in general, it is not true that these functionals $p\otimes q$ belong to $\operatorname{ex} B_{H^*}$ for larger subspaces H. Let us interrupt the main line of reasoning to give an example of this.

Example 1.6 *For $X = c_0$ and $Y = C(K)$, K metrizable, the functional $e\otimes\delta_k$, $e \in \operatorname{ex} B_{\ell^\infty}$, is extremal if and only if k is isolated.*

Let us point out why this is so. According to [60], the space

$$\mathfrak{L}_k := L(X,Y)/J_{(k)},$$

where $J_{(k)} = \{T \in L(X,Y) \mid \limsup_{\kappa\to k} \|T^*\delta_\kappa\| = 0\}$, may be written as

$$\mathfrak{L}_k = \mathfrak{L}_k^0 \oplus_1 \mathfrak{L}_k^c \tag{$*$}$$

with

$$\mathfrak{L}_k^c = L_k^c/J_{(k)}, \qquad L_k^c = \{T \in L(X,Y) \mid \lim_{\kappa\to k} \|T^*\delta_\kappa - T^*\delta_k\| = 0\}$$

and

$$\mathfrak{L}_k^0 = L_k^0/J_{(k)}, \qquad L_k^0 = \{T \in L(X,Y) \mid \|T^*\delta_k\| = 0\}.$$

Observe that $e \otimes \delta_k \in (L_k^0)^\perp$ and so, functionals of this form belong to $(\mathfrak{L}_k^0)^*$ after canonical identification. Now, if the decomposition in $(*)$ is nontrivial, i.e. if $\mathfrak{L}_k^0 \neq \{0\}$, then $e \otimes \delta_k$ belongs to a nontrivial M-summand of the subspace $(J_{(k)})^\perp$ of $L(X,Y)^*$, which makes it impossible for $e \otimes \delta_k$ to be an extreme functional. But this is exactly what happens when k is *not* isolated:
To find an operator $T \in L_k^0 \setminus J_{(k)}$, select a sequence (k_n) converging to k as well as open and pairwise disjoint neighbourhoods U_n of k_n that do not contain k.
Furthermore, choose continuous functions φ_n of norm ≤ 1 with $\operatorname{supp}\varphi_n \subset U_n$ and $\varphi_n(k_n) = 1$ for all $n \in \mathbb{N}$. Denoting by e_n the elements of the usual Schauder basis of c_0, extend $T(e_n) := \varphi_n$ to a bounded operator (in fact, an isometry) $T : c_0 \to C(K)$. Since

$$T^*\delta_k = w^*\text{-}\lim_{\kappa\to k} T^*\delta_\kappa = 0,$$

this operator does the job. On the other hand, when k is an isolated point, then $e \otimes \delta_k$ is extreme. $(\square)$

Let us come back to the preparation of the proof of Theorem 1.2. To surmount the problem of not having a handsome representation of the elements of $\operatorname{ex} B_{H^*}$ in general, we shall need the following elementary lemma. Let us first agree on some notation.
The symbol H will always denote a space of bounded operators containing the finite rank operators, $p \otimes q$ is a functional of the type defined in Theorem 1.3, and by $\operatorname{supp}\operatorname{ex} B_{X^*}$ we denote the set of "supporting extreme functionals", i.e. those functionals in $\operatorname{ex} B_{X^*}$ attaining their norm. By the Krein-Milman theorem $\operatorname{supp}\operatorname{ex} B_{X^*}$ is weak* dense in $\operatorname{ex} B_{X^*}$. Likewise, for each $x \in X$ there is some $p \in \operatorname{supp}\operatorname{ex} B_{X^*}$ such that $p(x) = \|x\|$.

Lemma 1.7

(a) *Let $p \in \operatorname{ex} B_{E^*}$ for some subspace E of a Banach space X. Then the set N_p of norm preserving extensions of p to X is a weak* closed face of B_{X^*} and, consequently, by the Krein-Milman theorem*

$$N_p = \overline{\operatorname{co}}^{w*}(\operatorname{ex} B_{X^*} \cap N_p).$$

(b) *For all $T \in L(X,Y)$ we have*

$$\|T\| = \sup\{\langle p, T^*q\rangle \mid p \in \operatorname{supp}\operatorname{ex} B_{X^{**}}, q \in \operatorname{supp}\operatorname{ex} B_{Y^*}\}.$$

(c) *Denote by* $F_{p,q}$ *the weak** *closed face of norm preserving extensions of* $p\otimes q \in \operatorname{ex} B_{(X^*\widehat{\otimes}_\varepsilon Y)^*}$ *from* $X^*\widehat{\otimes}_\varepsilon Y$ *to* H *and define*

$$E_0 := \bigcup_{p\otimes q\in B_{(X^*\widehat{\otimes}_\varepsilon Y)^*}} \operatorname{ex} F_{p,q}.$$

Then $E_0 \subset \operatorname{ex} B_{H^*} \subset \overline{E_0}^{w*}$.

Proof: We content ourselves with a proof of (b) and (c), since (a) is obvious. To prove (b), start with an $x \in B_X$ such that $\|Tx\| > \|T\| - \varepsilon$ for some arbitrary $\varepsilon > 0$ and pick some $q \in \operatorname{supp} \operatorname{ex} B_{Y^*}$ with $\|Tx\| = q(Tx)$. Norm T^*q by $p \in \operatorname{supp}\operatorname{ex} B_{X^{**}}$ to obtain

$$p(T^*q) = \|T^*q\| \geq q(Tx) > \|T\| - \varepsilon.$$

For the proof of (c), note first that the inclusion $E_0 \subset \operatorname{ex} B_{H^*}$ is true since $F_{p,q}$ is a face of B_{H^*} and hence $\operatorname{ex} F_{p,q}$ is contained in $\operatorname{ex} B_{H^*}$. To prove the other inclusion observe that $p\otimes q$ has a natural extension to a functional on the whole of H, which we continue to denote by $p\otimes q$, and that consequently $p\otimes q \in \overline{\operatorname{co}}^{w*} E_0 =: K$. Since by (b) $\|T\| = \sup_{\psi\in K}\psi(T)$ for all $T\in H$, we have $\operatorname{ex} B_{H^*} \subset \overline{E_0}^{w*}$ by the converse of the Krein-Milman theorem. □

The following lemma essentially shows that the functionals $p\otimes q$, though not extreme in general, may nevertheless be treated as if they were, at least as far as the restrictions of operators in $\operatorname{Mult}(H)$ to $X^*\widehat{\otimes}_\varepsilon Y$ are concerned.

Lemma 1.8 *Let* $\Phi \in \operatorname{Mult}(H)$ *and identify* $X^*\widehat{\otimes}_\varepsilon Y$ *with the space of approximable operators from* X *to* Y*. Then the following assertions hold.*

(a) $\Phi\,|_{X^*\widehat{\otimes}_\varepsilon Y} = 0$ *implies that* $\Phi = 0$.

(b) *For all* $p \in B_{X^{**}}$ *and* $q \in B_{Y^*}$ *there is a number* $a_\Phi(p,q)$ *such that for each operator* $F : X \to Y$ *of finite rank*

$$\langle p, (\Phi(F))^*(q)\rangle = a_\Phi(p,q)\langle p, F^*(q)\rangle.$$

Moreover, Φ *is in* $Z_{0,1}(X)$ *precisely when* $0 \leq a_\Phi(p,q) \leq 1$ *for all* $p\otimes q$.

Proof: (a) Since $E_0 \subset \operatorname{ex} B_{H^*}$ we have by assumption for each $F \in X^*\otimes Y$ and any $\psi \in \operatorname{ex} F_{p,q}$,

$$0 = \psi(\Phi(F)) = a_\Phi(\psi)\langle p, F^*q\rangle$$

so that $a_\Phi(\psi) = 0$ for all $\psi \in E_0$. Now, the fact that $\operatorname{ex} B_{H^*} \subset \overline{E_0}^{w*}$ in combination with the weak* continuity of the function a_Φ on $\operatorname{ex} B_{H^*}$ (Lemma I.3.2) implies that $\Phi = 0$.

(b) We first show that

$$\langle p, (x^*\otimes y)^* q\rangle = 0 \qquad \forall x^* \in X^*,\ y \in Y$$

implies that

$$\langle p, (\Phi(x^*\otimes y))^* q\rangle = 0 \qquad \forall x^* \in X^*,\ y \in Y.$$

To do this, we use once more that $\operatorname{ex} F_{p,q} \subset \operatorname{ex} B_{H^*}$ and find that for all $\psi \in \operatorname{ex} F_{p,q}$

$$\psi(\Phi(x^* \otimes y)) = a_\Phi(\psi)\psi(x^* \otimes y) = a_\Phi(\psi)\langle p, (x^* \otimes y)^* q\rangle = 0.$$

Our claim then follows since

$$p \otimes q \in \overline{\operatorname{co}}^{w*} \operatorname{ex} F_{p,q}.$$

To prove the implication claimed note that we just have seen that

$$\ker\left((p\otimes q)|_{X^*\widehat{\otimes}_\varepsilon Y}\right) \subset \ker\left(\Phi^*(p\otimes q)|_{X^*\widehat{\otimes}_\varepsilon Y}\right),$$

which in turn easily implies that, just as desired, for some number $a_\Phi(p,q)$

$$a_\Phi(p,q)(p\otimes q)|_{X^*\widehat{\otimes}_\varepsilon Y} = \Phi^*(p\otimes q)|_{X^*\widehat{\otimes}_\varepsilon Y}.$$

Now let us see what happens when $\Phi \in Z_{0,1}(X)$. Fix a pair (p,q) and pick $x^* \in X^*$ and $y \in Y$ such that $\langle p, (x^*\otimes y)^* q\rangle = 1$. Approximating $p \otimes q$ in the weak* topology by convex combinations $\sum_i t_{\alpha,i}\psi_{\alpha,i}$ with $\psi_{\alpha,i} \in \operatorname{ex} F_{p,q}$ we obtain

$$a_\Phi(p,q) = \langle p, \Phi(x^*\otimes y)^* q\rangle = \lim_\alpha \sum_i t_{\alpha,i} a_\Phi(\psi_{\alpha,i}) \ \in\ [0,1].$$

For the converse, suppose that $a_\Phi(p,q) \in [0,1]$ for all p,q. By Lemma 1.7(c), the weak* closure of $\{p \otimes q \mid p \in \operatorname{ex} B_{X^{**}},\ q \in \operatorname{ex} B_{Y^*}\}$ contains $\operatorname{ex} B_{H^*}$ so that Lemma I.3.2 guarantees that

$$0 \le a_\Phi|_{\operatorname{ex} B_{H^*}} \le 1$$

which is what we have claimed. $\square$

We close this set of preparatory lemmata with a result known as the Bishop-Phelps-Bollobás theorem and one of its corollaries. Recall that

$$\Pi(X) = \{(x^*, x) \in B_{X^*} \times B_X \mid x^*(x) = 1\}.$$

Theorem 1.9 *Let X be a Banach space.*

(a) *If $x \in B_X$ and $x^* \in S_{X^*}$ with*

$$|1 - x^*(x)| \le \left(\frac{\varepsilon}{2}\right)^2,$$

there is $(x^*_\varepsilon, x_\varepsilon) \in \Pi(X)$ with

$$\|x - x_\varepsilon\| < \varepsilon \qquad \textit{and} \qquad \|x^* - x^*_\varepsilon\| < \varepsilon.$$

(b) *In particular, the functionals on X which attain their norm are norm dense.*

PROOF: [86, p. 7]. $\square$

Actually, part (a) is a special case of a more general result due to Brøndsted and Rockafellar, see [317, p. 165]. Part (b) of Theorem 1.9 is usually called the Bishop-Phelps theorem.

Lemma 1.10 *For each $(x^{**},x^*)\in\Pi(X^*)$ one can find a net (x^*_α,x_α) in $\Pi(X)$ with*

$$w^*\text{-}\lim_\alpha x_\alpha = x^{**} \qquad \textit{and} \qquad \|\,.\,\|\text{-}\lim_\alpha x^*_\alpha = x^*.$$

Proof: Choose a net (ξ_α) from B_X with $w^*\text{-}\lim_\alpha \xi_\alpha = x^{**}$ and put

$$\delta_\alpha := |x^{**}(x^*) - \xi_\alpha(x^*)|.$$

By the above, we may find a net $(x^*_\alpha,x_\alpha)\in\Pi(X)$ with

$$\|\xi_\alpha - x_\alpha\| < 2\sqrt{\delta_\alpha} \qquad \text{and} \qquad \|x^*_\alpha - x^*\| < 2\sqrt{\delta_\alpha}.$$

Since $\lim_\alpha \delta_\alpha = 0$, this is a net with the required properties. □

Proof of Theorem 1.2:

One half of the proof of Theorem 1.2 is contained in Lemma 1.1. Let us prove the missing directions.

(a), (a*) Let $\Phi\in\mathrm{Mult}(H)$ be fixed. Define $\Omega_{q,y}: X^*\to X^*$ by

$$\Omega_{q,y}(x^*) = (\Phi(x^*\otimes y))^*(q),$$

where $q\in\operatorname{ex} B_{Y^*}$, $y\in Y$ and $q(y)\neq 0$. By Lemma 1.8(b), we have for an arbitrary $p\in\operatorname{ex} B_{X^{**}}$

$$\Omega^*_{q,y}(p) = a_\Phi(p,q)q(y)p,$$

so that $\Omega_{q,y}\in\mathrm{Mult}(X^*)$. If, in addition, $\Phi\in Z_{0,1}(H)$, then by Lemma 1.8(b)

$$0\le a_\Phi(p,q) = a_{\Omega_{q,y}}(p)q(y)^{-1}\le 1,$$

whence $q(y)\Omega_{q,y}\in Z_{0,1}(X^*)$. Since always $Z(E)=\overline{\operatorname{lin}}\,Z_{0,1}(E)$ for a Banach space E, we conclude that $\Phi\in Z(H)$ implies $\Omega_{q,y}\in Z(X^*)$. In both cases, there is by assumption on these algebras (recall Theorem I.3.14(b)) a number $\alpha(q,y)$ with

$$\Omega_{q,y} = \alpha(q,y)Id.$$

Comparing equations we find

$$a_\Phi(p,q) = \alpha(q,y)q(y),$$

which in turn implies that $a_\Phi(p,q) =: a(q)$ does not depend on p. Now, take $\xi^*\in X^*$ and $\xi\in X$ with $\xi^*(\xi)=1$ and put

$$Ty := \Phi(\xi^*\otimes y)\xi.$$

By the above, $T^*q = a(q)q$ for all $q\in\operatorname{ex} B_{Y^*}$ so that $T\in\mathrm{Mult}(Y)$. Moreover, when Φ is in $Z_{0,1}(H)$ then

$$0\le a_\Phi(p,q) = a(q) = a_T(q)\le 1,$$

and so, $T\in Z_{0,1}(X)$. Therefore, T is in $Z(X)$ whenever $\Phi\in Z(H)$. By Lemma 1.1, we may conclude that $L_T\in\mathrm{Mult}(H)$ or $Z(H)$, respectively.

To finish the first part of the proof, we infer that $\Phi = L_T$. To this end, it is by Lemma 1.8(a) sufficient to show that both operators agree on $X^*\widehat{\otimes}_\varepsilon Y$. In fact, we have for all $(p,q) \in \operatorname{ex} B_{X^{**}} \times \operatorname{ex} B_{Y^*}$ and $u = \sum_{i=1}^n x_i^* \otimes y_i \in X^* \otimes Y$

$$\left\langle p, \left(\left(\Phi - L_T\right)(u)\right)^* q\right\rangle \;=\; \sum_i \big(\langle p, (\Phi(x_i^* \otimes y_i))^* q\rangle - \langle p, (x_i^* \otimes T y_i)^* q\rangle\big),$$

and the last expression becomes

$$\sum_i a_\Phi(p,q)\langle p, (x_i^* \otimes y_i)^* q\rangle - \sum_i a(q) p(x_i^*) q(y_i) = 0.$$

This is, by Lemma 1.7(b), enough to prove our claim.

(b) Let $\Phi \in Z(H)$. Similarly to the above, define $\Omega^{x^*,x} : Y \to Y$ by

$$\Omega^{x^*,x}(y) := \Phi(x^* \otimes y)x$$

We claim that $\Omega^{x^*,x} \in Z(Y)$ whenever $(x^*,x) \in \Pi(X)$. To show this, we may restrict ourselves to the case where $\Phi \in Z_{0,1}(X)$. Let $y_1, y_2 \in Y$. By Proposition I.3.9,

$$\|\Phi(x^* \otimes y_1) + (Id - \Phi)(x^* \otimes y_2)\| \le \max\{\|y_1\|, \|y_2\|\}.$$

Since $\|x\| = 1$ and $x^*(x) = 1$,

$$\begin{aligned} \|\Omega^{x^*,x}(y_1) + (Id - \Omega^{x^*,x})(y_2)\| \;&=\; \|\Phi(x^* \otimes y_1)x + (Id - \Phi)(x^* \otimes y_2)x\| \\ &\le\; \max\{\|y_1\|, \|y_2\|\}, \end{aligned}$$

and so $\Omega^{x^*,x} \in Z(Y)$, as claimed. Assuming $Z(Y)$ to be trivial, we obtain for all $(x^*,x) \in \Pi(X)$

$$\Omega^{x^*,x} = \beta(x^*,x) Id$$

for some constant $\beta(x^*,x)$. Let $q \in \operatorname{ex} B_{Y^*}$ and $y \in Y$ with $q(y) \neq 0$, put $T^{q,y}(x^*) := (\Phi(x^* \otimes y))^*(q)$ and choose for a fixed pair $(p_0, x_0^*) \in \Pi(X^*)$ with $p_0 \in \operatorname{supp} \operatorname{ex} B_{X^{**}}$ a net $((p_\alpha, x_\alpha^*))$ as indicated in Lemma 1.10. We have

$$\langle p_0, T^{q,y} x_0^*\rangle \;=\; \lim_\alpha \langle p_\alpha, T^{q,y} x_\alpha^*\rangle \;=\; \lim_\alpha \langle q, \Omega^{p_\alpha, x_\alpha^*} y\rangle \;=\; q(y) \lim_\alpha \beta(x_\alpha^*, p_\alpha)$$

so that $\beta(p_0, x_0^*) := \lim_\alpha \beta(x_\alpha^*, p_\alpha)$ exists and does not depend on the particular choice of (p_α, x_α^*). On the other hand,

$$q(y) a_\Phi(p,q) \;=\; \langle p, T^{q,y} x^*\rangle \;=\; \beta(p, x^*) q(y^*)$$

for all $p \in \operatorname{supp} \operatorname{ex} B_{X^{**}}$ and some norm attaining $x^* \in B_{X^*}$. Consequently, the functions

$$\beta(p, x^*) = a_\Phi(p,q) =: b(p)$$

are independent of their second arguments. It follows that

$$(T^{q,y})^* p = b(p) p$$

for all $p \in \operatorname{supp} \operatorname{ex} B_{X^{**}}$. Since $\operatorname{supp} \operatorname{ex} B_{X^{**}}$ is weak* dense in $\operatorname{ex} B_{X^{**}}$ we deduce from Lemma I.3.2 that $T^{q,y} \in Z(X^*)$. Applying Theorem I.3.14 we find a $T_*^{q,y} \in \operatorname{Cun}(X)$ with $(T_*^{q,y})^* = T^{q,y}$. Fix $(\eta^*,\eta) \in \Pi(Y)$ with $\eta^* \in \operatorname{ex} B_{Y^*}$ and put $T := T_*^{\eta^*,\eta}$. As was shown in Lemma 1.1, R_T belongs to $Z(H)$. Similarly to the proof of (a), we compute

$$\begin{aligned} \left\langle p, \left(\left(\Phi - R_T\right)(u)\right)^* q\right\rangle &= \sum_i \big(\langle p, (\Phi(x_i^* \otimes y_i))^* q\rangle - \langle p, (T^* x_i^* \otimes y_i)^* q\rangle\big) \\ &= \sum_i a_\Phi(p,q)\langle p, (x_i^* \otimes y_i)^* q\rangle - \sum_i b(p) p(x_i^*) q(y_i) \\ &= 0, \end{aligned}$$

which is valid for all $(p,q) \in \operatorname{supp} \operatorname{ex} B_{X^{**}} \times \operatorname{ex} B_{Y^*}$. But since these functionals form a norming subset of H^* (Lemma 1.7(b)), Φ and R_T coincide on $X^* \widehat{\otimes}_\varepsilon Y$, which in light of Lemma 1.8(a) settles our claim. □

We wish to give some applications of the result proven above. The first one takes advantage of the fact that $Z(X)$ as well as $\operatorname{Mult}(X)$ are dual spaces whenever X is (Theorem I.3.14(c)).

Corollary 1.11 *Suppose that X is a Banach space and that $Z(X^*)$ is trivial. Then $L(X,CK)$ is a dual space if and only if $C(K)$ is. The same result holds true when $\operatorname{Mult}(X^*)$ is supposed to be trivial and $C(K)$ is replaced by a function algebra $\mathfrak{A}$ throughout.*

Proof: If the space Y has a predual Y_* then $L(X,Y)$ always has a representation as a dual space, namely $L(X,Y) = (X \widehat{\otimes}_\pi Y_*)^*$ [158, Cor. VIII.2.2]. On the other hand,

$$Z(L(X,CK)) = Z(CK) = CK$$

is a dual space when $L(X,CK)$ is (Theorem I.3.14(c)). This gives the claim in the centralizer case. The proof of the other case follows the same lines, using $\operatorname{Mult}(\mathfrak{A}) \cong \mathfrak{A}$ (Example I.3.4(c)). □

It is known that $C(K)$ is isometric to a dual Banach space if and only if K is hyperstonean [385, p. 95ff.].

Corollary 1.12

(a) *If there are no nontrivial L-summands in X, then the M-projections on $L(X,Y)$ have the form $T \mapsto P_\infty T$ for some M-projection P_∞ on Y. Consequently, the M-summands of $L(X,Y)$ have the form*

$$\{T \in L(X,Y) \mid \operatorname{ran}(T) \subset J\}$$

for some M-summand J in Y.

(b) *If there are no nontrivial M-summands in Y, then the M-projections on $L(X,Y)$ have the form $T \mapsto TP_1$ for some L-projection P_1 on X. Consequently, the M-summands of $L(X,Y)$ have the form*

$$\{T \in L(X,Y) \mid \operatorname{ran}(T^*) \subset J\}$$

for some M-summand J in X^*.

PROOF: This is an immediate consequence of Theorem 1.2. For part (b) keep Theorem I.1.9 in mind. □

Note that in general, an arbitrary 1-complemented subspace of $L(X,Y)$ does not necessarily have to be of this simple form. Also, the structure of the M-ideals of $L(X,Y)$ is by far not so transparent in the setting of Corollary 1.12. We will, however, present some analogous results for the M-ideals of the operator space $A(X,Y) = X^*\widehat{\otimes}_\varepsilon Y$ in Section VI.3.
It should be noted that Corollary 1.12 still holds for subspaces H of $L(X,Y)$ which are subject to the condition that $L_{P_\infty}(H) \subset H$ resp. $R_{P_1}(H) \subset H$. For instance every closed operator ideal (such as the compact or weakly compact operators) could be considered. In order to simplify the statement of the following corollary, we do not present the most general result possible either.

Corollary 1.13 *For all Banach spaces $X \not\cong \ell^\infty_{\mathbb{R}}(2)$ and any closed two-sided ideal H of $L(X)$ with $X^*\widehat{\otimes}_\varepsilon X \subset H \subset L(X)$ either*

$$Z(H) = \{L_T \mid T \in Z(X)\}$$

or

$$Z(H) = \{R_T \mid T \in \operatorname{Cun}(X)\}$$

holds. Moreover, all M-summands J in H are either of the form

$$J = \{T \in H \mid P_\infty T = T\}$$

or

$$J = \{T \in H \mid TP_1 = T\},$$

respectively, where P_∞ is an M- and P_1 is an L-projection on X.

PROOF: The proof is a direct consequence of Theorem I.3.14(d) and Theorem 1.2. □

We finish this section with two results in the flavour of the classical Banach-Stone theorem.

Corollary 1.14 *For $i = 1,2$ let $H_i \subset L(X_i, \mathfrak{A}_i)$, where the $\mathfrak{A}_i$ are function algebras and the algebras $\operatorname{Mult}(X_i^*)$ are supposed to be trivial. If the spaces H_i contain the finite rank operators and if they are invariant under the operators L_T, $T \in \operatorname{Mult}(\mathfrak{A}_i)$, then*

$$H_1 \cong H_2 \qquad \textit{implies} \qquad \mathfrak{A}_1 \cong \mathfrak{A}_2.$$

PROOF: The proof simply follows from Theorem 1.2(a*), the fact that $W_1 \cong W_2$ implies $\mathrm{Mult}(W_1) \cong \mathrm{Mult}(W_2)$ and

$$\mathfrak{A}_1 \cong \mathrm{Mult}(\mathfrak{A}_1) \cong \mathrm{Mult}(\mathfrak{A}_2) \cong \mathfrak{A}_2,$$

where we have made use of the fact that for a function algebra $\mathfrak{A}$, $\mathrm{Mult}(\mathfrak{A})$ consists of multiplication operators with elements in $\mathfrak{A}$ (Example I.3.4(c)). □

As a special instance of the preceding corollary we obtain the following Banach-Stone theorem for operator spaces.

Corollary 1.15 *If K_1 and K_2 are compact Hausdorff spaces for which $L(CK_1)$ and $L(CK_2)$ are isometrically isomorphic, then K_1 and K_2 are homeomorphic.*

PROOF: By Corollary 1.14 we have $C(K_1) \cong C(K_2)$ so that the classical Banach-Stone theorem (see e.g. [51]) yields the result claimed. □

Corollary 1.16 *Let $H_i \subset L(L^1(\mu_i), X_i)$, $i = 1, 2$, where the μ_i are arbitrary measures, and suppose that the algebras $Z(X_i)$ are trivial. If the spaces H_i contain the finite rank operators, if they are invariant under the operators R_T, $T \in \mathrm{Cun}(L^1(\mu_i))$, and if $H_1 \cong H_2$, then one may conclude that*

$$L^1(\mu_1) \cong L^1(\mu_2).$$

PROOF: We have

$$L^1(\mu_1)^* \cong \mathrm{Cun}(L^1(\mu_1)) \cong \mathrm{Cun}(H_1) \cong \mathrm{Cun}(H_2) \cong \mathrm{Cun}(L^1(\mu_2)) \cong L^1(\mu_2)^*$$

from Theorem 1.2. Since L^1-spaces are unique preduals, i.e.

$$E^* \cong L^1(\mu)^* \quad \Rightarrow \quad E \cong L^1(\mu),$$

(see [385, p. 96f.]), we obtain the desired conclusion. □

We shall have more to say about the relation between M-structure theory and Banach-Stone type results in the Notes and Remarks section.

VI.2 M-ideals in $L(X, C(K))$ for certain X

The present section deals with an attempt to characterise the M-ideals in $L(X, Y)$ in terms of the M-ideals in Y. Unlike the previous section, no satisfactory general condition for this to be possible is known. What will be done in the following is a complete calculation of the M-ideals in $L(X, C(K))$, where X^* is either uniformly convex or else has sufficiently many one-dimensional L-summands. Note that the techniques used below apply independently of the specification of the scalar field.

We start with specifying some subspaces which are always M-ideals of the operator spaces $L(X, C(K))$.

Proposition 2.1 *Suppose D is a closed subset of the compact space K, and let X be a Banach space. Then*

$$J_{(D)} := \{T \in L(X, C(K)) \mid \limsup_{k \to k_0} \|T^*(\delta_k)\| = 0 \ \forall k_0 \in D\}$$

is an M-ideal of $L(X, C(K))$.

The proof of the above is a by now standard application of the n-ball property and will hence be left to our readers; for details we refer to [60] or [625].
Note that in the sense of Corollary V.3.6

$$J_{(D)} = L(X, C(K))_{J_D},$$

where $J_D \subset C(K) \hookrightarrow Z(L(X, C(K)))$ is the M-ideal of continuous functions vanishing on D.
Theorem 2.3 below will show that, for certain X, all M-ideals of $L(X, C(K))$ look like this. We first present an equivalent condition for this to be true. In the following lemma we call an M-ideal in a Banach space *maximal* if there is no nontrivial M-ideal which strictly contains the given one. Also, we employ the notation hT for the operator $x \mapsto h \cdot Tx$, where $T \in L(X, C(K))$, $h \in C(K)$.

Lemma 2.2 *Let X be a Banach space and K a compact Hausdorff space. Then the M-ideals of the form $J_{(D)}$ with D closed exhaust all M-ideals of $L(X, C(K))$ if and only if for all $k \in K$ the closed subspace $J_{(k)} := J_{(\{k\})}$ is a maximal M-ideal.*

PROOF: Only one implication requires a proof:
Suppose that all $J_{(k)}$, $k \in K$, are maximal M-ideals, and denote by J an arbitrary M-ideal of $L(X, C(K))$. Putting

$$D := \{k \in K | \lim_{\kappa \to k} \|T^* \delta_\kappa\| = 0 \ \ \forall T \in J\},$$

we must show that $J_{(D)} \subset J$, since the other inclusion is clear. To this end, we will first show that any $T \in J_{(D)}$ can be locally approximated by elements of J, more precisely we claim:

For all $T \in J_{(D)}$ and $k \in K$ there are a neighbourhood U of k and $S \in J$ with

$$\|(T - S)_{|U}\| := \sup_{\kappa \in U} \|(T - S)^*(\delta_\kappa)\| \le \varepsilon.$$

In fact, whenever $k \notin D$ – and only this case requires some work – the space $J + J_{(k)}$ strictly contains $J_{(k)}$ by the very definition of D and hence, $J + J_{(k)} = L(X, C(K))$, since the former space is an M-ideal in light of Proposition I.1.11. But then, for any $T \in J_{(D)}$, there is $S \in J \subset J_D$ such that $T - S \in J_{(k)}$ which yields the claim.
To obtain a global approximation of $T \in J_{(D)}$ by an element of J (which will prove the inclusion we have in mind), we fix $\varepsilon > 0$, cover K by finitely many open sets $U_1, \dots, U_n$ as above and select $S_1, \dots, S_n \in J$ accordingly with $\|(S_i - T)_{|U_i}\| < \varepsilon$, $i = 1, \dots, n$.

Of course, all this is possible by compactness and the above claim. If $h_1, \dots, h_n$ is a partition of unity subordinate to $U_1, \dots, U_n$, then

$$\left\| \sum_{i=1}^{n} h_i S_i - T \right\| < \varepsilon,$$

and we are done since $\sum_{i=1}^{n} h_i S_i \in J$ by Lemma 1.1 and Lemma I.3.5(b) (note $M_h \in Z(C(K))$).
The attentive reader will have noticed that we still owe a proof of the not obvious fact that D is closed: Pick k in the complement of D and an operator T with $\liminf_{\kappa \to k} \|T^* \delta_\kappa\| > 0$ (e.g. $T = \delta_k \otimes \mathbf{1}$). As above, $J + J_{(k)} = L(X, C(K))$ and therefore, we may write $T = T_J + T_k$ with $T_J \in J$ and $T_k \in J_{(k)}$. Clearly, $\liminf_{\kappa \to k} \|T_J^*(\delta_\kappa)\| > 0$ and so, $U \cap D = \emptyset$ for a suitable neighbourhood U of k. The proof is finished. □

The following theorem encompasses all what is known on M-ideals in $L(X, C(K))$. Note that we don't put any restriction on the scalar field.

Theorem 2.3 *Let X be Banach space and K a compact Hausdorff space. Suppose further that either*

(a) *X^* is uniformly convex,*

or else that

(b) *$\ker p$ is an M-ideal for each $p \in \operatorname{ex} B_{X^*}$.*

Then a closed subspace J of $L(X, C(K))$ is an M-ideal if and only if J is of the form $J_{(D)}$ for some closed subset D of K.

PROOF: We begin with the assumption that X^* is uniformly convex.
By Lemma 2.2 we must show that, for each $k_0 \in K$, an M-ideal J strictly containing $J_{(k_0)}$ already equals $L(X, C(K))$. Let such a J be given and, by the uniform convexity of X^*, fix for every $\varepsilon > 0$ a number $0 < \delta(\varepsilon) < \varepsilon$ such that

$$\|x^*\| \geq 1 - \delta(\varepsilon),\ \|x^* \pm y^*\| \leq 1 + \delta(\varepsilon) \quad \Longrightarrow \quad \|y^*\| \leq \varepsilon. \tag{$*$}$$

We will first show that for every $\varepsilon > 0$ there is $T_\varepsilon \in J$ with

$$1 - \varepsilon \ \leq \ \liminf_{k \to k_0} \|T_\varepsilon^* \delta_k\| \ \leq \ \limsup_{k \to k_0} \|T_\varepsilon^* \delta_k\| \ \leq \ 1 + \varepsilon.$$

To this end, pick $T \in J \backslash J_{(k_0)}$ with $\limsup_{k \to k_0} \|T^* \delta_k\| > 0$. Choosing $h \in C(K)$ properly and considering hT instead of T we may as well suppose that $\limsup_{k \to k_0} \|T^* \delta_k\| = \|T^*\| = 1$. By the 3-ball property of J, we find $T_\varepsilon \in J$ with

$$\|T_\varepsilon\| \leq 1 + \delta(\varepsilon) \leq 1 + \varepsilon$$

and $\|\pm T + (x_0^* \otimes 1 - T_\varepsilon)\| \leq 1 + \delta(\varepsilon)$, where x_0^* is any norm one functional on X. It follows by $(*)$ that $\|x_0^* - T_\varepsilon^* \delta_k\| \leq \varepsilon$ whenever $\|T^* \delta_k\| \leq 1 + \delta(\varepsilon)$. Consequently, by the lower semicontinuity of the map $k \mapsto \|x_0^* - T_\varepsilon^* \delta_k\|$ and our present assumption on T, we have that $\|x_0^* - T_\varepsilon^* \delta_{k_0}\| \le \varepsilon$ and thus $\|T_\varepsilon^* \delta_{k_0}\| \ge 1-\varepsilon$. This entails $\liminf_{k\to k_0} \|T_\varepsilon^* \delta_k\| \ge 1-\varepsilon$. To conclude the proof, fix any $A \in L(X, C(K))$, a positive number η and, by the above (applied to $\varepsilon = \delta(\eta)$), an operator $T_\eta \in J$ with

$$1 - \delta(\eta) \ \le\ \liminf_{k\to k_0} \|T_\eta^* \delta_k\| \ \le\ \limsup_{k\to k_0} \|T_\eta^* \delta_k\| \ \le\ 1 + \delta(\eta).$$

Again, by the 3-ball property of J, there is $S \in J$ with $\|S\| \le 1 + \delta(\eta)$ and $\| \pm T_\eta + (A - S)\| \le 1 + \delta(\eta)$. By $(*)$, we infer that $\|(A-S)_{|U}\| \le \eta$ in a suitable neighbourhood U of k_0. Now the rest is easy: Fix a continuous map $h : K \to [0,1]$ with $\operatorname{supp} h \subset U$ and $h(k_0) = 1$. An appeal to Lemma I.3.5(b) and Lemma 1.1 together with the fact that $J_{(k_0)} \subset J$ shows $(1-h)A + hS \in J$, and in light of

$$\|A - [(1-h)\,A + hS]\| \le \eta,$$

we see that J is dense in $L(X, C(K))$. Hence $J = L(X, C(K))$.

For the proof of the second part of the present theorem we need the following technical lemma, the proof of which we postpone until the end of this section.

Lemma 2.4 *Let Z be a Banach space, $p \in S_Z$ such that $\operatorname{lin}\{p\}$ is an L-summand with corresponding L-projection P, and fix $a_1, \dots, a_5 \ge 0$. Then we have:*

(a) $\|\theta p + z\| \le 1 + a_1$ *for every* $\theta \in \mathbb{S}$ *implies* $\|z\| \le a_1$.

(b) $\|z\| \ge 1 - a_1$ *and* $\|Pz\| \le a_2$ *imply* $\|\theta p + z\| \ge 2 - 2a_2 - a_1$ *for all* $\theta \in \mathbb{S}$.

(c) *If for all* $\theta \in \mathbb{S}$

$$\begin{aligned} \|y\| &\le 1 + a_1, & 1 - a_2 \le \|z\| &\le 1 + a_3, \\ \|\theta p + z\| &\ge 2 - a_4, & \|\theta z + (p - y)\| &\le 1 + a_5, \end{aligned}$$

then $\|p - y\| \le a_1 + 2(a_2 + a_3 + a_4 + a_5)$.

Let us now prove the second part of Theorem 2.3. By Lemma 2.2 it is enough to show:

> *If $k_0 \in K$ and J is an M-ideal in $L(X, C(K))$ properly containing $J_{(k_0)}$ then $J = L(X, C(K))$.*

The assumption $J \ne J_{(k_0)}$ yields an operator $T_1 \in J$ for which

$$\limsup_{k\to k_0} \|T_1^*(\delta_k)\| > 0.$$

However, we will need operators in J which are even worse behaved. Eventually we will prove:

> Claim: *There exist a neighbourhood U of k_0, $p \in \operatorname{ex} B_{X^*}$, a number $0 \le q < 1$ and $T_0 \in B_J$ such that*
>
> $$\|p - T_0^*(\delta_k)\| \le q \quad \forall k \in U.$$

Taking this claim for granted, we may complete the proof of the theorem as follows. Let $T \in B_{L(X,C(K))}$. By the 2-ball property (respectively, in the complex case, by Remark I.2.3(c) applied to the compact set of centres $\mathbb{S}T_0$) there is $S \in J$ such that

$$\|\theta T_0 + T - S\| \le 1 + \varepsilon \quad \forall \theta \in \mathbb{S}$$

where $\varepsilon < 1 - q$. Hence

$$\|\theta p + T^*(\delta_k) - S^*(\delta_k)\| \le 1 + \varepsilon + q$$

for $k \in K$ and all $\theta \in \mathbb{S}$ so that

$$\|T^*(\delta_k) - S^*(\delta_k)\| \le \varepsilon + q$$

for these k by Lemma 2.4(a). Finally, we consider a continuous function $h : K \to [0,1]$ which vanishes off U and takes the value 1 at k_0. Then the operator $R \mapsto M_h R$ (where M_h denotes the operator of multiplication by h) belongs to the centralizer of $L(X, C(K))$ (Lemma 1.1). Hence

$$S_0 := M_h S + M_{1-h} T$$

lies in J (by Lemma I.3.5(b) and since $M_{1-h}T \in J_{(k_0)}$), and we have

$$\begin{aligned} \|S_0 - T\| &= \|M_h(S - T)\| \\ &\le \sup_{k \in K} \|S^*(\delta_k) - T^*(\delta_k)\| \\ &\le \varepsilon + q. \end{aligned}$$

Since T was arbitrary we may now conclude that the quotient map from $L(X, C(K))$ onto $L(X, C(K))/J$ has norm $\le \varepsilon + q < 1$. By Riesz' Lemma this means $J = L(X, C(K))$.
Thus it is left to give the proof of the claim. We already know that $\limsup_{k \to k_0} \|T_1^*(\delta_k)\| > 0$ for some $T_1 \in J$. After composition with a suitable multiplication operator we may and shall assume

$$\|T_1\| = \limsup_{k \to k_0} \|T_1^*(\delta_k)\| = 1.$$

Thus, if $A = \{k \mid \|T_1^*(\delta_k)\| > 1 - \eta\}$, then $k_0 \in \overline{A}$. We shall apply this for a positive number η which will later be determined.
We now select 33 linearly independent extreme functionals $p_1, \dots, p_{33}$. By assumption on X each of them spans a one-dimensional L-summand in X^*. P_n is to denote the L-projection onto $\operatorname{lin}\{p_n\}$. Then, letting $A_n = \{k \in A \mid \|P_n(T_1^*(\delta_k))\| \le \frac{1}{33}\}$, we deduce from

$$\sum_{n=1}^{33} \|P_n(T_1^*(\delta_k))\| \le \|T_1^*(\delta_k)\|$$

the equation $A = \bigcup_{n=1}^{33} A_n$ so that $k_0 \in \overline{A_m}$ for some m. For notational convenience we assume $k_0 \in \overline{A_1}$.
As above, the n-ball property yields some $T_2 \in J$ such that

$$\|T_2\| < 1 + \eta \qquad \text{and} \qquad \|\theta T_1 + p_1 \otimes 1 - T_2\| < 1 + \eta \qquad \forall \theta \in \mathbb{S}$$

(compare Remark I.2.3(b) for the former inequality). Now we apply Lemma 2.4(c) with $z = T_1^*(\delta_k)$, $y = T_2^*(\delta_k)$ (where $k \in A_1$) and $a_1 = \eta,\ a_2 = \eta,\ a_3 = 0,\ a_4 = \frac{2}{33} + \eta$ (this is admissible by Lemma 2.4(b)), $a_5 = \eta$ to obtain

$$\|p_1 - T_2^*(\delta_k)\| \le 7\eta + \frac{4}{33} \quad \forall k \in A_1.$$

Consequently, we also have, using the weak* lower semicontinuity of the norm,

$$\|p_1 - T_2^*(\delta_{k_0})\| \le 7\eta + \frac{4}{33}. \tag{$*$}$$

From this we derive the inequality

$$\|P_2(T_2^*(\delta_k))\| \le 10\eta + \frac{4}{33} \tag{$**$}$$

on some neighbourhood U of k_0: We first note that p_1 belongs to the L-summand complementary to $\mathbb{K} \cdot p_2 = (\ker p_2)^\perp$, therefore

$$1 = \|p_1\| = \|p_1|_{\ker p_2}\|$$

(cf. I.1.12 and I.1.13). Thus, there is $x_0 \in B_X$ satisfying

$$|p_1(x_0)| > 1 - \eta \quad \text{and} \quad p_2(x_0) = 0.$$

Choose a neighbourhood U of k_0 where

$$|T_2(x_0)(k) - T_2(x_0)(k_0)| < \eta$$

so that, as a result of $(*)$,

$$|T_2(x_0)(k)| > 1 - \left(9\eta + \frac{4}{33}\right) \qquad \text{for } k \in U.$$

Since $P_2(T_2^*(\delta_k)) \in \operatorname{lin}\{p_2\}$ and $p_2(x_0) = 0$ we conclude

$$\begin{aligned} \|P_2(T_2^*(\delta_k))\| &\le \|T_2^*(\delta_k)\| - \|(Id - P_2)(T_2^*(\delta_k))(x_0)\| \\ &\le 1 + \eta - |T_2(x_0)(k)| \\ &\le 10\eta + \frac{4}{33} \end{aligned}$$

for $k \in U$, which proves $(**)$. We may also assume

$$\|T_2^*(\delta_k)\| > 1 - \left(8\eta + \frac{4}{33}\right) \quad \text{for } k \in U \tag{$***$}$$

due to the semicontinuity of $\|\cdot\|$, since $(***)$ holds for k_0 by $(*)$. We now apply the n-ball property once more to obtain an operator $T_3 \in J$ for which

$$\|\theta T_2 + p_2 \otimes 1 - T_3\| < 1 + \eta \quad \forall \theta \in \mathbb{S} \quad \text{and} \quad \|T_3\| < 1 + \eta.$$

A second application of Lemma 2.4(c) with $z = T_2^*(\delta_k)$, $y = T_3^*(\delta_k)$ $(k \in K)$ and $a_1 = \eta$, $a_2 = 8\eta + \frac{4}{33}$ (admissible by $(***)$), $a_3 = \eta$, $a_4 = 28\eta + \frac{12}{33}$ (admissible by Lemma 2.4(b) and $(**)$), $a_5 = \eta$ now shows

$$\|p_2 - T_3^*(\delta_k)\| \leq 77\eta + \frac{32}{33} \quad \text{for} \quad k \in U,$$

and the claim follows easily by a proper choice of η. □

Theorem 2.3 applies in particular to L^1-predual spaces X, e.g. $X = C(K)$ itself, as well as to function algebras (see the remarks preceding the proof of Lemma V.6.7). In fact, we did not need the full strength of the assumptions made; we merely made use of the fact that there are 33 extreme functionals for which the kernels are M-ideals. (The above proof would not have worked with 32 instead!) In other words, Theorem 2.3 remains true if X^* has an L-summand isometric to $\ell^1(33)$. (Of course, this assumption is not very natural, and the number 33 appears here only for technical reasons.)

We conclude this section with the

Proof of Lemma 2.4:
(a) and (b) are obvious consequences of the triangle inequality. For the proof of (c), we write $z = \alpha p + z_1$ and $y = \beta p + y_1$, where z_1, y_1 are in $(Id - P)Z$. We have

$$2 - a_4 \leq \|z + \theta p\| = |\theta + \alpha| + \|z_1\|$$

for every $|\theta| = 1$ so that

$$\begin{aligned} 2 - a_4 &\leq 1 - |\alpha| + \|z_1\| \\ &= 1 - |\alpha| + \|z\| - |\alpha| \\ &\leq 2 - 2|\alpha| + a_3. \end{aligned}$$

It follows that $|\alpha| \leq (1/2)(a_3 + a_4) =: a_6$, and consequently $\|z_1\| = \|z\| - |\alpha| \geq 1 - a_2 - a_6$. On the other hand we have

$$\begin{aligned} 1 + a_5 &\geq \|\theta z + (p - y)\| \\ &= |\theta\alpha + (1 - \beta)| + \|\theta z_1 - y_1\| \\ &\geq |1 - \beta| - a_6 + \|\theta z_1 - y_1\|. \end{aligned}$$

Thus $1 + a_5 + a_6 - |1 - \beta| \geq \|\theta z_1 - y_1\|$ for all $|\theta| = 1$ which yields, since $\|z_1\| \geq 1 - a_2 - a_6$, that $1 + a_5 + a_6 - |1 - \beta| \geq 1 - a_2 - a_6$, i.e. $|1 - \beta| \geq a_2 + a_5 + 2a_6$. This implies that $|\beta| \geq 1 - a_2 - a_5 - 2a_6$, and we get

$$\begin{aligned} \|p - y\| &= |1 - \beta| + \|y_1\| \\ &\leq a_2 + a_5 + 2a_6 + \|y\| - |\beta| \\ &= 2a_2 + 2a_5 + 4a_6 + a_1 \\ &\leq a_1 + 2(a_2 + a_3 + a_4 + a_5), \end{aligned}$$

which gives the claim. □

VI.3 Tensor products

The structure of M-ideals in "small" operator spaces is, due in part to Theorem 1.3, more transparent than the case of $L(X,Y)$. This will be pointed out in this section. For the sake of symmetry, we prefer in our treatment to use the notion of injective tensor products rather than that of spaces of approximable operators.
We will finally pass to the "dual" problem of finding the L-summands in the projective tensor product. Here, the central result of Section VI.1 will play a fundamental rôle.
We first show that an M-ideal in one of the factors of an injective tensor product gives rise to an M-ideal in $X\widehat{\otimes}_\varepsilon Y$ and then that, similarly to the results obtained in Section VI.1, under certain circumstances all the M-ideals arise in this way. (We have recalled the definition of the injective tensor product on p. 265.)

Proposition 3.1 *Let J be an M-ideal in Y. Then, for all Banach spaces X, $X\widehat{\otimes}_\varepsilon J$ is an M-ideal in $X\widehat{\otimes}_\varepsilon Y$.*

PROOF: Denote by $P : Y^* \to J^\perp$ the L-projection corresponding to J and recall from Lemma 1.1 that $Id_{X^{**}} \otimes P^*$ is an M-projection on $X^{**}\widehat{\otimes}_\varepsilon Y^{**}$, which gives rise to an L-projection on $(X^{**}\widehat{\otimes}_\varepsilon Y^{**})^* = I(X^{**}, Y^{***})$, the space of integral operators from X^{**} to Y^{***}. Call this projection Π. Writing $\|\cdot\|_{int}$ for the integral norm of an operator and taking advantage of the fact that $T \in I(X,Y^*)$ iff $T^{**} \in I(X^{**}, Y^{***})$ with equality of integral norms [158, Corollary VIII.2.11], we find

$$\begin{aligned} \|T\|_{int} &= \|\Pi T^{**}\|_{int} + \|T^{**} - \Pi T^{**}\|_{int} \\ &= \|P^{**}T^{**}\|_{int} + \|T^{**} - P^{**}T^{**}\|_{int} \\ &= \|PT\|_{int} + \|T - PT\|_{int}. \end{aligned}$$

(Note that the integral operators form an operator ideal in the sense of Pietsch, and hence, application of P from the left doesn't lead out of this class.) Consequently, left multiplication by P induces an L-projection on $(X\widehat{\otimes}_\varepsilon Y)^* = I(X, Y^*)$. Moreover, $T = PT$ iff $T(X) \subset P(Y^*) = J^\perp$ iff $T \in (X \otimes J)^\perp$. It follows that $X\widehat{\otimes}_\varepsilon J$ is an M-ideal in $X\widehat{\otimes}_\varepsilon Y$. □

Theorem 3.2 *Suppose X has no nontrivial M-ideal. Then every M-ideal Z in $X\widehat{\otimes}_\varepsilon Y$ has the form $Z = X\widehat{\otimes}_\varepsilon J$ with some M-ideal J in Y.*

To prepare the proof we present a result which may be of independent interest.

Proposition 3.3 *Let $p \in \operatorname{ex} B_{X^*}$ and suppose Z is an M-ideal in $X\widehat{\otimes}_\varepsilon Y$. Then the closure of $(p \otimes Id)(Z)$ is an M-ideal in Y.*

PROOF: Let E denote the L-projection from $(X\widehat{\otimes}_\varepsilon Y)^*$ onto $Z^\perp$. Given $y_0^* \in S_{Y^*}$, consider $E(p \otimes y_0^*)$. We shall prove the existence of a (uniquely determined) functional $P(y_0^*) \in Y^*$ such that

$$E(p \otimes y_0^*) = p \otimes P(y_0^*). \qquad (*)$$

To this end, represent the integral bilinear form $E(p \otimes y_0^*)$ by a positive Radon measure μ_1 on $S := B_{X^*} \times B_{Y^*}$ such that $\|E(p \otimes y_0^*)\| = \mu_1(S)$. Likewise, let $p \otimes y_0^* - E(p \otimes y_0^*)$

be represented by μ_2. Since E is an L-projection, $\mu := \mu_1 + \mu_2$ is a probability measure for which

$$\langle p \otimes y_0^*, x \otimes y \rangle = \int_S x \otimes y \; d\mu$$

for all $x \in X, y \in Y$. Assume for the moment that y_0^* attains its norm on B_Y. In this case

$$p(x) = \int_S x \otimes y_0 \; d\mu$$

for a suitable $y_0 \in S_Y$ and all $x \in X$. Then we have by Lemma 1.4

$$\operatorname{supp}(\mu_i) \subset \operatorname{supp}(\mu) \subset \mathbb{S} \cdot \{p\} \times B_{Y^*}$$

so that there are $\lambda_1 \in \mathbb{S}$, $y_1^* \in B_{Y^*}$ with

$$\begin{aligned} \langle E(p \otimes y_0^*), x \otimes y \rangle &= \int_S x \otimes y \; d\mu_1 \\ &= \langle \lambda_1 p \otimes y_1^* \; , \; x \otimes y \rangle \\ &= \langle p \otimes \lambda_1 y_1^* \; , \; x \otimes y \rangle \end{aligned}$$

for all $x \in X, y \in Y$. Thus $(*)$ is valid in case y_0^* attains its norm. In the general case, the Bishop-Phelps-Bollobás theorem 1.9 (actually the Bishop-Phelps theorem suffices for this purpose) yields a sequence of norm attaining functionals $y_n^* \in S_{Y^*}$ converging to y_0^* in norm. The validity of $(*)$ for y_n^* gives

$$E(p \otimes y_0^*) = \lim E(p \otimes y_n^*) = \lim p \otimes P(y_n^*)$$

so that $P(y_0^*) = \lim P(y_n^*)$ exists, and $(*)$ is proved in the general case, too. Now define a mapping P from Y^* into itself by $(*)$. Obviously, P is an L-projection. It is left to prove

$$P(Y^*) = (p \otimes Id)(Z)^\perp.$$

"$\subset$" is immediate from the definitions of E and P. Conversely, suppose $y^* \in Y^*$ satisfies

$$0 = \langle y^*, (p \otimes Id)(u) \rangle = \langle p \otimes y^*, u \rangle$$

for all $u \in Z$. Then $p \otimes y^* \in Z^\perp$ so that

$$p \otimes Py^* = E(p \otimes y^*) = p \otimes y^*.$$

Hence $Py^* = y^*$, and Proposition 3.3 is completely proved. □

Proof of Theorem 3.2:
As in the proof of Proposition 3.3, we denote the L-projection onto $Z^\perp$ by E. By Proposition 3.3 we may partition the set $\operatorname{ex} B_{Y^*}$ into the two subsets

$$\begin{aligned} C_1 &= \{q \in \operatorname{ex} B_{Y^*} \mid E(x^* \otimes q) = x^* \otimes q \text{ for all } x^* \in X^*\} \\ C_2 &= \{q \in \operatorname{ex} B_{Y^*} \mid E(x^* \otimes q) = 0 \text{ for all } x^* \in X^*\}, \end{aligned}$$

because X was assumed to have no nontrivial M-ideals. Next, we apply Proposition 3.3 once more to obtain a family of L-projections P_p on Y^*, indexed by the extreme functionals $p \in \operatorname{ex} B_{X^*}$, with weak* closed ranges which satisfy

$$E(p \otimes y^*) = p \otimes P_p(y^*)$$

for all $y^* \in Y^*$. Furthermore,

$$P_p(Y^*) = \{y^* \mid p \otimes y^* \in Z^\perp\} =: M_p.$$

Obviously, $C_1 \subset M_p$ for all $p \in \operatorname{ex} B_{X^*}$. On the other hand, let $q \in \operatorname{ex} B_{M_p}$. Then $q \in \operatorname{ex} B_{Y^*}$ (since M_p is an L-summand) and $p \otimes q \in Z^\perp$, that is $q \in C_1$. M_p is weak* closed and thus $M_p = \overline{C_1}^{w*}$ independently of p. In other words,

$$J := \{y \in Y \mid q(y) = 0 \text{ for all } q \in C_1\}$$

is an M-ideal in Y. To prove $X \widehat{\otimes}_\varepsilon J = Z$ it is enough to show $(X \widehat{\otimes}_\varepsilon J)^\perp = Z^\perp$. By the first part of the proof, both spaces are weak* closed L-summands, therefore it is enough to check the coincidence of the extreme points of the unit balls, which must have the form $p \otimes q$ with p and q extremal (Theorem 1.3 and Lemma I.1.5). In fact, $p \otimes q \in Z^\perp$ iff $q \in C_1$ iff $q \in J^\perp$ iff $p \otimes q \in (X \widehat{\otimes}_\varepsilon J)^\perp$. $\square$

The following Corollary is an immediate consequence of Theorem 3.2 and the canonical isometric isomorphism $C(K, X) \cong C(K) \widehat{\otimes}_\varepsilon X$ [158, p. 224f.].

Corollary 3.4 *If K denotes a compact Hausdorff space and X a Banach space without nontrivial M-ideals, then $J \subset C(K, X)$ is an M-ideal if and only if*

$$J = J_D \widehat{\otimes}_\varepsilon X = \{f \in C(K, X) \mid f|_D = 0\}$$

for some closed subset $D \subset K$.

Corollary 3.5 *Let X and Y be Banach spaces without proper M-ideals (i.e. every M-ideal is an M-summand). Then $X \widehat{\otimes}_\varepsilon Y$ fails to have proper M-ideals.*

PROOF: A Banach space without proper M-ideals is isometrically isomorphic to a c_0-sum of Banach spaces without nontrivial M-ideals; this follows as in the proof of Proposition III.2.6. If $X = c_0(X_i)$ and $Y = c_0(Y_j)$ are represented in this way, then

$$X \widehat{\otimes}_\varepsilon Y = c_0(X_i \widehat{\otimes}_\varepsilon Y_j)_{i,j},$$

and $X_i \widehat{\otimes}_\varepsilon Y_j$ has no nontrivial M-ideal by Theorem 3.2. Now an appeal to Proposition I.1.16 shows that each M-ideal in $c_0(X_i \widehat{\otimes}_\varepsilon Y_j)_{i,j}$ must be of the form $c_0(J_{i,j})$, where $J_{i,j}$ is either $\{0\}$ or the space $X_i \widehat{\otimes}_\varepsilon Y_j$. The conclusion now follows. $\square$

In the above proofs the density of the algebraic tensor product (or, in the language of operator spaces, the finite rank operators) was essential. In the case of the whole space of compact operators we have the following results; recall that $K_{w^*}(X^*, Y)$ stands for the space of compact operators which are weak*-weakly continuous and that $K_{w^*}(X^*, Y) = X \widehat{\otimes}_\varepsilon Y$ whenever X or Y has the approximation property.

Proposition 3.6 *If X and Y have no nontrivial M-ideals, then neither has $K_{w^*}(X^*, Y)$.*

PROOF: The proof of Proposition 3.3 shows that the closure of $\{T(p) \mid T \in Z\}$ is an M-ideal in Y if $p \in \operatorname{ex} B_{X^*}$ and Z is an M-ideal in $H := K_{w^*}(X^*, Y)$. As a matter of fact, the essential property to be used is that H may be embedded isometrically into $C(B_{X^*} \times B_{Y^*})$. Analogously, the closure of $\{T^*(q) \mid T \in Z\}$ is an M-ideal in X for $q \in \operatorname{ex} B_{Y^*}$. One can, therefore, partition $\operatorname{ex} B_{Y^*} = C_1 \cup C_2$ in the same way as in the proof of Theorem 3.2. Following this proof one arrives at the conclusion that

$$J = \{y \mid q(y) = 0 \text{ for all } q \in C_1\}$$

is an M-ideal in Y. Hence $J = \{0\}$ or $J = Y$. In the first case it follows that $\operatorname{ex} B_{Y^*} = C_1$ and $Z = \{0\}$. (The L-summand which is complementary to $Z^\perp$ is isometrically isomorphic to Z^*, but has no extreme points because of $C_2 = \emptyset$.) In the second case $\operatorname{ex} B_{Y^*} = C_2$ holds and thus $Z = H$. □

Corollary 3.7 *If X^* and Y have no nontrivial M-ideals, then neither has $K(X, Y)$.*

PROOF: The map $T \mapsto T^{**}$ defines an isometric isomorphism between $K(X, Y)$ and $K_{w^*}(X^{**}, Y)$. □

We now turn to the dual situation and determine the L-summands in the projective tensor product of two Banach spaces. Let us recall what that means. Let $u \in X \otimes Y$, the algebraic tensor product. The projective (or π-) norm of u is defined as

$$\|u\|_\pi := \inf \left\{ \sum_{i=1}^n \|x_i\| \, \|y_i\| \;\middle|\; u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

The completion of $X \otimes Y$ is called the *projective tensor product* and denoted by $X \widehat{\otimes}_\pi Y$. It is known that each $u \in X \widehat{\otimes}_\pi Y$ has, for every $\varepsilon > 0$, a representation as a series $u = \sum_{i=1}^\infty x_i \otimes y_i$ (converging for the π-norm) such that $\|u\|_\pi \le \sum_{i=1}^\infty \|x_i\| \, \|y_i\| + \varepsilon$. The dual space of $X \widehat{\otimes}_\pi Y$ can be identified with $L(X, Y^*)$ under the duality

$$\left\langle T, \sum_{i=1}^\infty x_i \otimes y_i \right\rangle = \sum_{i=1}^\infty (Tx_i)(y_i).$$

The reader who is not acquainted with the fundamental properties of these tensor products should consult e.g. [158, Chapter VIII], or, for full treatment, [280].
Our most general result on L-summands reads as follows.

Theorem 3.8 *Let K be a closed subspace of $X \widehat{\otimes}_\pi Y$ such that for all $\sum_{i=1}^\infty x_i \otimes y_i \in K$*

$$\sum_{i=1}^\infty x^*(x_i) y^*(y_i) = 0 \qquad \forall x^* \in X^* \; \forall y^* \in Y^*. \tag{$*$}$$

If $\operatorname{Cun}(X)$ is trivial and $(Id \otimes T)K \subset K$ for all $T \in \operatorname{Cun}(Y)$ then

$$\operatorname{Cun}((X \widehat{\otimes}_\pi Y)/K) = \{(Id \otimes T)_K \mid T \in \operatorname{Cun}(Y)\},$$

where $(Id \otimes T)_K$ denotes the natural action of $Id \otimes T$ on $(X \widehat{\otimes}_\pi Y)/K$.
A similar statement holds when $\operatorname{Cun}(Y)$ is trivial.

Concerning the assumption on K in the above theorem we would like to point out that $(*)$ expresses the requirement that $\widehat{u} = 0$ for all $u \in K$, where $\widehat{u}$ is the nuclear operator naturally assigned to the tensor u. Also, note that $u = 0$ iff $\widehat{u} = 0$ whenever X or Y has the approximation property, cf. [158, Theorem VIII.3.4].

PROOF: Consider the natural duality $(X \widehat{\otimes}_\pi Y)^* = L(X, Y^*)$ given by

$$\left\langle S\ ,\ \sum_{i=1}^{\infty} x_i \otimes y_i \right\rangle = \sum_{i=1}^{\infty} S x_i(y_i).$$

Then, by assumption, the space $K^\perp = ((X \widehat{\otimes}_\pi Y)/K)^*$ contains the finite dimensional operators between X and Y^* and $L_U K^\perp \subset K^\perp$ for all $U \in Z(Y^*) = \{T^* \mid T \in \mathrm{Cun}(Y)\}$ (see Theorem I.3.14(b), also recall the notation $L_U(T) = UT$).
Hence, the conditions imposed on $K^\perp$ are such that Theorem 1.2 may be applied to any Φ^* with $\Phi \in \mathrm{Cun}((X \widehat{\otimes}_\pi Y)/K)$. Consequently, using Theorem 1.2, we find an operator $T \in \mathrm{Cun}(Y)$ with $\Phi^* = L_{T^*}$. This gives

$$\begin{aligned}
\left\langle S\ ,\ \Phi\left(\sum_{i=1}^{\infty} x_i \otimes y_i\right)\right\rangle &= \left\langle \Phi^*(S), \sum_{i=1}^{\infty} x_i \otimes y_i \right\rangle \\
&= \left\langle T^* S\ ,\ \sum_{i=1}^{\infty} x_i \otimes y_i \right\rangle \\
&= \sum_{i=1}^{\infty} \langle S x_i, T y_i \rangle \\
&= \left\langle S\ ,\ \sum_{i=1}^{\infty} x_i \otimes T y_i \right\rangle
\end{aligned}$$

showing that Φ is of the form claimed.
Conversely, in passing to the dual $K^\perp$ of $(X \widehat{\otimes}_\pi Y)/K$, we conclude from Lemma 1.1 that $((Id \otimes T)_K)^* \in Z(K^\perp)$ and hence $(Id \otimes T)_K \in \mathrm{Cun}((X \widehat{\otimes}_\pi Y)/K)$. □

For $K = \{0\}$ the above may be rephrased as follows.

Corollary 3.9 *Let X and Y be Banach spaces. Whenever $X \widehat{\otimes}_\pi Y = L_1 \oplus_1 L_2$, there is a decomposition $X = J_1 \oplus_1 J_2$ such that $L_i = J_i \widehat{\otimes}_\pi Y$, provided that the Banach space Y cannot be decomposed as $Y = Y_1 \oplus_1 Y_2$ in a nontrivial way.*

In the above, the reader should note that J_i is a 1-complemented subspace of X so that $J_i \widehat{\otimes}_\pi Y$ is in fact a subspace of $X \widehat{\otimes}_\pi Y$.
A somewhat more involved application of Theorem 3.8 is given in the next corollary, where $N(X, Y)$ denotes the space of all nuclear operators from X to Y.

Corollary 3.10 *If $\mathrm{Cun}(X^*)$ is trivial then*

$$\mathrm{Cun}(N(X, Y)) = \{L_T \mid T \in \mathrm{Cun}(Y)\}.$$

PROOF: Write
$$N(X,Y) = (X^*\widehat{\otimes}_\pi Y)/K,$$
where K denotes the kernel of the natural surjection $u \mapsto \widehat{u}$ from $X^*\widehat{\otimes}_\pi Y$ onto $N(X,Y)$. If $u = \sum_{i=1}^\infty x_i^* \otimes y_i \in K$ then by definition
$$\widehat{u}(x) = \sum_{i=1}^\infty x_i^*(x) y_i = 0 \qquad \forall x \in X,$$
and taking adjoints, we hence obtain that
$$\sum_{i=1}^\infty x^{**}(x_i^*) y^*(y_i) = 0 \qquad \forall x^{**} \in X^{**}\, \forall y^* \in Y^*.$$
Furthermore, for all $S \in \mathrm{Cun}(Y)$ and all $\sum_{i=1}^\infty x_i^* \otimes y_i \in K$,
$$\left[(Id \otimes S)\left(\sum_{i=1}^\infty x_i^* \otimes y_i\right)\right](x) = \sum_{i=1}^\infty (x_i^* \otimes S y_i)(x) = 0,$$
and the result now follows from Theorem 3.8. ◻

Note that Corollary 3.9 implies Corollary 3.10 if Y (or X^*) has the approximation property, since under this assumption $X^*\widehat{\otimes}_\pi Y = N(X,Y)$.

Corollary 3.11 *If X_1 and X_2 have no nontrivial L-summands, then the assumption that $L^1(\mu, X_1) \cong L^1(\nu, X_2)$ implies that $L^1(\mu) \cong L^1(\nu)$.*

PROOF: We have, using Theorem 3.8 and [158, Example VIII.1.10],
$$\begin{aligned} \mathrm{Cun}(L^1(\mu)) &\cong \mathrm{Cun}(L^1(\mu)\widehat{\otimes}_\pi X_1) \;\cong\; \mathrm{Cun}(L^1(\mu, X_1)) \\ &\cong \mathrm{Cun}(L^1(\nu, X_2)) \;\cong\; \mathrm{Cun}(L^1(\nu)). \end{aligned}$$
and the claim follows as in the proof of Corollary 1.16. ◻

We finish this section with two counterexamples. The first one shows that, contrary to a rather tempting conjecture, there is at least one case in which there are nontrivial L-summands in $X\widehat{\otimes}_\varepsilon Y$:

Proposition 3.12 $\qquad \ell^2(2,\mathbb{R})\widehat{\otimes}_\varepsilon \ell^2(2,\mathbb{R}) \cong \ell^2(2,\mathbb{R}) \oplus_1 \ell^2(2,\mathbb{R}).$

PROOF: Let $X = \ell^2(2,\mathbb{R})\widehat{\otimes}_\varepsilon \ell^2(2,\mathbb{R}) = L(\ell^2(2,\mathbb{R}))$. It is well-known (see [317, p. 82]) that $\operatorname{ex} B_X = O(2,\mathbb{R})$, the group of orthogonal 2×2-matrices. Let E (respectively $\widehat{E}$) denote the linear span of all orthogonal matrices with determinant $+1$ (respectively -1). Then $X = E \oplus \widehat{E}$ and $E \cong \widehat{E} \cong \ell^2(2,\mathbb{R})$. If P denotes the projection from X onto E, then $P(\operatorname{ex} B_X) \subset \{0,1\}\cdot \operatorname{ex} B_X$ from which one can deduce that P is an L-projection; for details see [400, Theorem 4.6]. ◻

Our next example is concerned with the question of how the structure topology (see Definition I.3.11) of $X\widehat{\otimes}_\varepsilon Y$ can be derived from the respective topologies of X and Y.

Guided by Theorem 3.2 a first guess might be that the former topology should just coincide with the product of the latter ones. Note that, as an equation of sets, we always have

$$E_{X\widehat{\otimes}_\varepsilon Y} = E_X \times E_Y.$$

(Recall that E_X was defined in Section I.3 as the set of equivalence classes of $\operatorname{ex} B_{X^*}$, where the antipodal points have been glued together.) This natural conjecture, however, is false:

Proposition 3.13 *In general, the structure topology of $X\widehat{\otimes}_\varepsilon Y$ is not the product of the structure topologies of X and Y.*

PROOF: Let $X = Y = \{f \in C_0(\mathbb{R}) \mid nf(n) = f(1)\ \forall n \in \mathbb{N}\}$. By definition, this is a G-space. We now have

$$\operatorname{ex} B_{X^*} = \{\pm\delta_k \mid k \in \mathbb{R} \setminus \{2,3,4,\ldots\}\},$$

cf. p. 89. It is straightforward to check that

$$X\widehat{\otimes}_\varepsilon X = \{f \in C_0(\mathbb{R}^2) \mid mn\, f(m,n) = f(1,1)\ \forall m,n \in \mathbb{N}\}.$$

As an easy application of Proposition II.5.2 we obtain that the space E_X is homeomorphic to $\mathbb{R}/\mathbb{N}$, whereas $E_{X\widehat{\otimes}_\varepsilon X} = \mathbb{R}^2/\mathbb{N}^2$. Thus it is left to prove that $(\mathbb{R}/\mathbb{N})^2$ is not homeomorphic to $\mathbb{R}^2/\mathbb{N}^2$. To this end, denote for $m,n \in \mathbb{N}$ by $D_{m,n}$ the (open) disk with radius $(m+n)^{-1}$ centred at (m,n). Since a neighbourhood of $\mathbb{N}$ always contains a set of the form $\bigcup_{\mu\in\mathbb{N}}(a_\mu, b_\mu)$ with $\mu \in (a_\mu, b_\mu)$, we see that $\bigcup_{m,n=1}^\infty D_{m,n}$ is open in $\mathbb{R}^2/\mathbb{N}^2$, but not in $(\mathbb{R}/\mathbb{N})^2$. $\square$

As a final remark we point out that the space X considered in the above proof provides an example to show that the centralizer $Z(X\widehat{\otimes}_\varepsilon X)$ need not coincide with the norm closure of $Z(X) \otimes Z(X)$. In fact, it follows from the discussion preceding Proposition II.5.8 that

$$Z(X) = \{f \in C^b(\mathbb{R}) \mid f|_{\mathbb{N}} = \text{ const.}\} = C^b(\mathbb{R}/\mathbb{N}),$$

as well as

$$Z(X\widehat{\otimes}_\varepsilon X) = C^b(\mathbb{R}^2/\mathbb{N}^2).$$

Since a function $f \in C^b(\mathbb{R}^2)$ that vanishes on $\mathbb{R}^2 \setminus \bigcup_{m,n=1}^\infty D_{m,n}$ and attains the value 1 on $\mathbb{N}^2$ is not continuous when it is considered as a function on $(\mathbb{R}/\mathbb{N})^2$ (by the same argument as above), we obtain

$$C^b(\mathbb{R}^2/\mathbb{N}^2) \neq C^b((\mathbb{R}/\mathbb{N})^2),$$

hence the result. We remark that also c_0 yields such an example, but we will return to the above space X in the Notes and Remarks section which contains a more detailed discussion of the centralizer of an injective tensor product.

VI.4 M-ideals of compact operators

The final sections of this book are concerned with the phenomenon that for certain Banach spaces $K(X,Y)$ is an M-ideal in $L(X,Y)$; of course, here the case $X=Y$ is of special importance. One of the reasons why one is interested in cases where this happens lies in the fact that every bounded operator from X to Y must then have a best compact approximant (Proposition II.1.1). Also, the uniqueness of Hahn-Banach extensions from $K(X,Y)$ to $L(X,Y)$ in the case of an M-ideal of compact operators deserves special attention as does the fact that the functionals $T \mapsto \langle x^{**}, T^* y^* \rangle$, $x^{**} \in \operatorname{ex} B_{X^{**}}$, $y^* \in \operatorname{ex} B_{Y^*}$, are even extremal on $L(X,Y)$ (and not only on $K(X,Y)$, Theorem 1.3), which holds by Lemma I.1.5.
Another reason is that – as in other circumstances – Hilbert space provides a prominent (and in fact the first known) example of this kind; this follows from Theorem V.4.4 or Example 4.1 below. So part of what follows can be understood as a contribution to the question of how far one can go away from Hilbert space without ruining this property.
We now give the basic example of an M-ideal $K(X,Y)$.

Example 4.1 *Let $1 < p \le q < \infty$. Then $K(\ell^p, \ell^q)$ is an M-ideal in $L(\ell^p, \ell^q)$. If X is any Banach space, then $K(X, c_0)$ is an M-ideal in $L(X, c_0)$.*

PROOF: Let $1 < p \le q < \infty$. Denote the canonical coordinate projection $(a_1, a_2, \dots) \mapsto (a_1, \dots, a_n, 0, 0, \dots)$ on ℓ^p by P_n, and on ℓ^q by Q_n. We wish to verify the 3-ball property (Theorem I.2.2). So let contractive operators S_1, S_2, $S_3 \in K(\ell^p, \ell^q)$ and $T \in L(\ell^p, \ell^q)$ as well as $\varepsilon > 0$ be given. We shall prove that for large n and m

$$\|T + S_i - (Q_n T - T P_m + Q_n T P_m)\| \le 1 + \varepsilon, \qquad i = 1,2,3, \tag{$*$}$$

thus establishing that $K(\ell^p, \ell^q)$ is an M-ideal in $L(\ell^p, \ell^q)$, since $Q_n T - T P_m + Q_n T P_m$ is compact.
To prove $(*)$ note first that (P_n) and (Q_n) converge strongly to the respective identity operators, as do their adjoints. Since the convergence is uniform on relatively compact sets such as $S_i(B_{\ell^p})$ by the boundedness of these sequences, we have

$$\begin{aligned}
\lim_{n,m\to\infty} \|Q_n S_i P_m - S_i\| &\le \lim_{n,m\to\infty} \big(\|Q_n S_i - S_i\|\,\|P_m\| + \|S_i P_m - S_i\|\big) \\
&\le \lim_{n,m\to\infty} \big(\|Q_n S_i - S_i\| + \|P_m^* S_i^* - S_i^*\|\big) \\
&= 0.
\end{aligned}$$

Secondly, we have

$$\begin{aligned}
&\|T + Q_n S_i P_m - (Q_n T - T P_m + Q_n T P_m)\| \\
&\quad = \ \|(Id - Q_n) T (Id - P_m) + Q_n S_i P_m\| \ \le \ 1,
\end{aligned}$$

because

$$\begin{aligned}\big\|\big[(Id-Q_n)T(Id-P_m)+Q_nS_iP_m\big]x\big\| &= \big(\|T(Id-P_m)x\|^q+\|S_iP_mx\|^q\big)^{1/q}\\ &\leq \big(\|(Id-P_m)x\|^q+\|P_mx\|^q\big)^{1/q}\\ &\leq \big(\|(Id-P_m)x\|^p+\|P_mx\|^p\big)^{1/p}\\ &= \|x\|.\end{aligned}$$

These estimates together yield $(*)$.
The proof of the second assertion is similar, but easier. In this situation one verifies easily

$$\|T+S_i-Q_nT\|\leq 1+\varepsilon, \qquad i=1,2,3,$$

for large n and the coordinate projections Q_n on c_0. □

The above technique and the study of compact operators approximating the identity are fundamental in the theory of M-ideals of compact operators.
Note that for $1<q<p<\infty$ we have $K(\ell^p,\ell^q)=L(\ell^p,\ell^q)$ [420, Th. I.2.7] so that the above assertion extends to this case for trivial reasons. The following stability properties are easily proved with the help of the 3-ball property of Theorem I.2.2; we leave the details to our readers. A far more elaborate result than (a) will be presented in Theorem 4.19 below in the case $X=Y$.

Proposition 4.2

(a) *If $K(X,Y)$ is an M-ideal in $L(X,Y)$ and $E\subset X$ and $F\subset Y$ are 1-complemented subspaces, then $K(E,F)$ is an M-ideal in $L(E,F)$.*

(b) *The class of Banach spaces X and Y for which $K(X,Y)$ is an M-ideal in $L(X,Y)$ is closed with respect to the Banach-Mazur distance.*

A more detailed statement of (b) is this: If X and Y are Banach spaces such that whenever $\varepsilon>0$ there are spaces X_ε and Y_ε with Banach-Mazur distances $d(X,X_\varepsilon)\leq 1+\varepsilon$, $d(Y,Y_\varepsilon)\leq 1+\varepsilon$ where $K(X_\varepsilon,Y_\varepsilon)$ is an M-ideal in $L(X_\varepsilon,Y_\varepsilon)$, then $K(X,Y)$ is an M-ideal in $L(X,Y)$, too.
The next result shows that one cannot expect the compact operators to form an M-summand; the same argument shows that $K(X,Y)$ is not an L^p-summand in $L(X,Y)$ for $p>1$ unless in the trivial case. Actually, we are going to prove that $K(X,Y)$ is not an M- (or L^p-)summand in $\operatorname{lin} K(X,Y)\cup\{T\}$ for any noncompact operator T. But for $p=1$ the situation is somewhat different, see the Notes and Remarks.

Proposition 4.3 *If $K(X,Y)$ is an M-summand in $L(X,Y)$, then $K(X,Y)=L(X,Y)$.*

Proof: Assume for contradiction that $L(X,Y)=K(X,Y)\oplus_\infty R$ where $R\neq\{0\}$. Let $T\in R$ with $\|T\|=1$. Given $\varepsilon>0$ pick $x_0\in S_X$ such that $\|Tx_0\|\geq 1-\varepsilon$. Fix $x_0^*\in S_{X^*}$ with $x_0^*(x_0)=1$ and consider the compact operator $Sx=x_0^*(x)Tx_0$. Then $\|S+T\|=1$ by assumption, but on the other hand $\|(S+T)x_0\|\geq 2-\varepsilon$. This is absurd. □

We now begin our structural investigations. Before we start, let us make one general remark: All results in the sequel will be formulated in terms of $K(X,Y)$ or $K(X)$. Nevertheless, with the obvious modifications, all results, except otherwise stated, will

hold verbatim for the space $A(X,Y)$ or $A(X)$ of norm limits of finite rank operators as well. Recall that $A(X,Y) = K(X,Y)$ if X^* or Y has the approximation property [421, Th. 1.e.4 and 1.e.5].
Our first proposition gives access to the results of Chapter III.

Proposition 4.4 *If $K(X) \subset \mathfrak{L} \subset L(X)$, $Id \in \mathfrak{L}$ and $K(X)$ is an M-ideal in $\mathfrak{L}$, then X is an M-ideal in X^{**}.*

PROOF: The argument consists in quite a direct application of (both directions of) Theorem I.2.2:
Let $x^{**} \in S_{X^{**}}$, $x_1, x_2, x_3 \in B_X$ and let $\varepsilon > 0$. Pick $x^* \in B_{X^*}$ with $x^{**}(x^*) > 1 - \varepsilon$. Then, by assumption, there is $U \in K(X)$ with

$$\max_{1 \le i \le 3} \|Id + x^* \otimes x_i - U\| \le 1 + \varepsilon.$$

(Here $x^* \otimes x_i$ denotes the operator $x \mapsto x^*(x)x_i$.) But this implies

$$\max_{1 \le i \le 3} \|x^{**} + x^{**}(x^*)x_i - U^{**}x^{**}\| \le 1 + \varepsilon,$$

therefore

$$\max_{1 \le i \le 3} \|x^{**} + x_i - U^{**}x^{**}\| \le 1 + 2\varepsilon.$$

Since U is compact, we see that $U^{**}x^{**} \in X$, and hence X is an M-ideal in X^{**}. ◻

An immediate consequence of Theorems III.3.1, III.3.4, III.3.8 and III.4.6 now is:

Corollary 4.5 *Each Banach space X for which $K(X)$ is an M-ideal in $L(X)$ has property (V) as well as property (u). Furthermore, X^* has the RNP and X is weakly compactly generated. In particular, if X is separable, so is X^*.*

Recall the consequences these properties imply for X (Corollaries III.3.7 and III.4.7).
The RNP of X^* was obtained in Theorem III.3.1 as a consequence of the fact that the relative weak and weak* topologies on S_{X^*} coincide (Corollary III.2.15). In the present situation, the latter result can be strengthened.

Proposition 4.6 *If $K(X)$ is an M-ideal in $L(X)$, then the relative norm and weak* topologies on S_{X^*} coincide.*

PROOF: We formulate the proof only in the case of real scalars. Let (x^*_α) in S_{X^*} have the weak* limit $x^* \in S_{X^*}$ and fix $\varepsilon > 0$. By the above, X^* has the RNP and hence there is $e \in S_X$ as well as $t > 0$ such that the slice

$$S(e,t) = \{\xi^* \in X^* \mid 1 \ge \|\xi^*\| \ge \xi^*(e) \ge 1 - t\}$$

has diameter less than ε, see [93, Th. 4.4.1] or [496, Lemma 2.18].
Let $x \in S_X$ with $x^*(x) > 1 - t/10$. Then, for α bigger than a certain α_0, we have $x^*_\alpha(x) > 1 - t/5$. Choose $y^* \in S(e,t)$ with $y^*(e) = 1$ and fix $U \in K(X)$ with

$$\|Id \pm y^* \otimes x - U\| \le 1 + \frac{t}{5}.$$

(Note that this only requires the 2-ball property of $K(X)$ in $L(X)$.) Then, assuming $\alpha \geq \alpha_0$, we obtain

$$
\begin{aligned}
1+\frac{t}{5} &\geq \max \|[y^* \otimes x \pm (Id-U)]^*(x^*_\alpha)\| \\
&\geq \max |x^*_\alpha(x) y^*(e) \pm (x^*_\alpha - U^* x^*_\alpha)(e)| \\
&\geq 1-\frac{t}{5} + |(x^*_\alpha - U^* x^*_\alpha)(e)|
\end{aligned}
$$

and thus, $|(x^*_\alpha - U^* x^*_\alpha)(e)| \leq \frac{2t}{5}$. It follows that

$$\frac{(x^*_\alpha(x) y^* \pm (x^*_\alpha - U^* x^*_\alpha))(e)}{1+t/5} \geq \frac{1-t/5-2t/5}{1+t/5} > 1-t.$$

Using that $\operatorname{diam} S(e,t) < \varepsilon$, we get

$$
\begin{aligned}
2\|x^*_\alpha - U^* x^*_\alpha\| &\leq \|(x^*_\alpha(x) y^* + x^*_\alpha - U^* x^*_\alpha) - (x^*_\alpha(x) y^* - x^*_\alpha + U^* x^*_\alpha)\| \\
&\leq \varepsilon(1+t/5)
\end{aligned}
$$

and, consequently, $\|x^*_\alpha - U^* x^*_\alpha\| \leq \varepsilon$, which implies that $\|U^* x^* - x^*\| \leq \varepsilon$, too. Enlarging α_0, if necessary, we may assume that $\|U^* x^*_\alpha - U^* x^*\| \leq \varepsilon$, since U^* is weak* to norm continuous on B_{X^*}. Finally, for sufficiently large α, we find

$$\|x^*_\alpha - x^*\| \leq \|x^*_\alpha - U^* x^*_\alpha\| + \|U^* x^*_\alpha - U^* x^*\| + \|U^* x^* - x^*\| \leq 3\varepsilon,$$

and thus, $x^*_\alpha \to x^*$ in norm. □

Given Theorem 4.17, the previous result is contained in Proposition 4.15, too.
Before we are going to characterise the class of Banach spaces for which $K(X)$ is an M-ideal of $L(X)$, we prove two further consequences this property has. (In fact, in both cases the results hold in the case when $K(X,Y)$ is an M-ideal in $L(X,Y)$.)
The first one provides a somewhat more explicit formula for the essential norm of an operator T, i.e. the number $\|T\|_e = \inf_{K \in K(X)} \|T-K\|$. It will be useful in the following section.

Proposition 4.7 *Let X and Y be Banach spaces and put*

$$
\begin{aligned}
w(T) &:= \sup\{\limsup_\alpha \|Tx_\alpha\| \mid \|x_\alpha\| = 1,\ x_\alpha \xrightarrow{w} 0\}, \\
w^*(T) &:= \sup\{\limsup_\alpha \|T^* x^*_\alpha\| \mid \|x^*_\alpha\| = 1,\ x^*_\alpha \xrightarrow{w^*} 0\}.
\end{aligned}
$$

If $K(X,Y)$ is an M-ideal in $L(X,Y)$, then

$$\|T\|_e = \max\{w(T), w^*(T)\},$$

and at least one of the involved suprema is actually attained. If X^* and Y are separable, then sequences suffice in the definitions of w and w^*.

PROOF: Since the adjoint of any compact operator K maps bounded nets converging to zero in the weak* topology to nets which converge to zero in norm, we have $\|T-K\| \geq w^*(T)$ and similarly, $\|T-K\| \geq w(T)$. So, without any assumptions on X or Y, the inequality
$$\|T\|_e \geq \max\{w(T), w^*(T)\}$$
holds. To prove equality, fix $\psi \in \operatorname{ex} B_{K(X,Y)^\perp}$ such that $\psi(T) = \|T\|_e$. Since $K(X,Y)$ is an M-ideal in $L(X,Y)$, we must have that $\psi \in \operatorname{ex} B_{L(X,Y)^*}$ (Lemma I.1.5), and since the norm of each $T \in L(X,Y)$ is the supremum of the numbers $\langle y^* \otimes x, T\rangle = y^*(Tx)$ with $y^* \in B_{Y^*}$ and $x \in B_X$, there are, by the Hahn-Banach and the (converse to the) Krein-Milman theorem, nets (x_α) and (y^*_α), each contained in the respective unit sphere, such that the functionals $y^*_\alpha \otimes x_\alpha$ tend to ψ in the $\sigma(L(X,Y)^*, L(X,Y))$-topology. In passing to appropriate subnets, suppose that $x_\alpha \stackrel{w^*}{\longrightarrow} x^{**}$ and $y^*_\alpha \stackrel{w^*}{\longrightarrow} y^*$ so that still $x_\alpha \otimes y^*_\alpha \stackrel{w^*}{\longrightarrow} \psi$. Now, for any compact operator K mapping X into Y, we have
$$0 = \psi(K) = \lim_\alpha (K^* y^*_\alpha)(x_\alpha) = x^{**}(K^* y^*).$$
It follows that either y^* or x^{**} must be equal to zero. Suppose $x^{**} = 0$. Then $x_\alpha \to 0$ weakly and
$$\|T\|_e = \psi(T) = \lim_\alpha y^*_\alpha(Tx_\alpha) \leq \limsup_\alpha \|Tx_\alpha\| \leq w(T).$$
Should $y^* = 0$, then $y^*_\alpha \to 0$ in the weak* sense, and, as above, $\|T\|_e \leq w(T^*)$. This finishes the proof in the general case; and in the separable case we just observe that we may pass to subsequences (x_{α_n}) and $(y^*_{\alpha_n})$ above. $\square$

The next result improves in our more special situation the fact, proved by Zizler [661], that in general the operators whose adjoints attain their norms on B_{Y^*} are dense in $L(X,Y)$.

Proposition 4.8 *Suppose that $K(X,Y)$ is an M-ideal in $L(X,Y)$.*

(a) *If $T \in L(X,Y)$ has the property that T^* does not attain its norm on B_{Y^*}, then $\|T\| = \|T\|_e$.*

(b) *The set of operators whose adjoints do not attain their norm on B_{Y^*} is nowhere dense in $L(X,Y)$ with respect to the norm topology.*

PROOF: To see that (a) holds, note first that always $\psi(T) = \|T\|$ for some $\psi \in \operatorname{ex} B_{L(X,Y)^*}$. Given the hypothesis of the above proposition, we must have that $\psi \in \operatorname{ex} B_{K(X,Y)^\perp}$. Indeed, otherwise we would have $\psi \in \operatorname{ex} B_{K(X,Y)^*}$ by Lemma I.1.5 (and Remark I.1.13) and thus $\psi(T) = \langle T^{**}x^{**}, y^*\rangle$ for some $x^{**} \in \operatorname{ex} B_{X^{**}}$, $y^* \in \operatorname{ex} B_{Y^*}$ by Theorem 1.3; however, the assumption on T rules out that $\|T^*\| = \|T^* y^*\|$. But this implies
$$\|T\| = \psi(T) = \sup\{\varphi(T) \mid \varphi \in \operatorname{ex} B_{K(X,Y)^\perp}\} = \|T\|_e.$$
We now prove part (b). Since by the above the metric complement
$$K^\theta(X,Y) = \{T \in L(X,Y) \mid \|T\| = \|T\|_e\}$$

contains all operators in question, and is, in addition, norm closed, we are finished once it is shown that $K^\theta(X,Y)$ has empty interior. This follows from Proposition II.1.11 and Corollary II.1.7 since

$$\begin{aligned}\|T\| &= \sup\{|\langle Tx, y^*\rangle| \mid x \in B_X,\ y^* \in B_{Y^*}\} \\ &= \sup\{|\langle \psi, T\rangle| \mid \psi \in B_{K(X,Y)^*}\}\end{aligned}$$

for all $T \in L(X,Y)$. □

The class of Banach spaces for which X is an M-ideal in X^{**} is strictly larger than the class in which $K(X)$ is an M-ideal; this can be seen from Proposition 4.6. We now discuss another reason why this is so. We shall see in a moment that X has the metric compact approximation property if $K(X)$ is an M-ideal in $L(X)$. On the other hand, the famous Enflo-Davie subspace of c_0 fails this property [421, pp. 90ff.], but is M-embedded by Theorem III.1.6.

To prove the announced statement, we need some reformulations of the definition of the λ-compact approximation property. Recall that a Banach space X is said to have this property whenever there is a net of compact operators (K_α) on X, uniformly bounded by λ in norm, which converges strongly to the identity. Parts (ii) and (v) of the following lemma will connect us to Chapter V.

Lemma 4.9 *Let X be a Banach space. Then the following are equivalent.*

(i) *X has the λ-compact approximation property.*

(ii) *$K(X)$ contains a left λ-approximate unit.*

(iii) *There is a net (T_α) in $K(X)$ with $\|T_\alpha\| \le \lambda$ converging to the identity in the weak operator topology.*

Also the following two conditions are equivalent.

(iv) *X^* has the λ-compact approximation property with adjoint operators.*

(v) *$K(X)$ contains a right λ-approximate unit.*

In general, we have (iv) $\Rightarrow$ (iii)*, and if X is M-embedded and $\lambda = 1$, then all the above conditions are equivalent.*

Proof: (i) $\Rightarrow$ (ii): Let (T_α) be a net in $K(X)$ with $\|T_\alpha\| \le \lambda$ converging to Id_X pointwise. By the boundedness of this net, $T_\alpha \to Id_X$ uniformly on compact subsets, in particular $T_\alpha K \to K$ for a compact operator K since then $K(B_X)$ is relatively compact.

(ii) $\Rightarrow$ (iii): Evident.

(iii) $\Rightarrow$ (i): Since $L(X)$ has the same continuous linear functionals when endowed with the weak operator topology as when bearing the strong operator topology, the same convex sets must be closed with respect to both topologies, see [178, Th. VI.1.4]. Hence, after passing to convex combinations, we may suppose that $T_\alpha \to Id_X$ strongly.

(iv) $\Rightarrow$ (v): Note that

$$\|KT_\alpha - K\| = \|T_\alpha^* K^* - K^*\| \to 0 \qquad \forall K \in K(X)$$

if (T_α) is a net of compact operators such that $T_\alpha^* \to Id_{X^*}$ strongly and $\|T_\alpha\| \le \lambda$.

(v) $\Rightarrow$ (iv): For a right λ-approximative unit (T_α) and $\|x\| = 1$ we have

$$\|T_\alpha^* x^* - x^*\| = \|(x^* \otimes x)T_\alpha - x^* \otimes x\| \to 0 \qquad \forall x^* \in X^*.$$

(iv) $\Rightarrow$ (iii) is trivial, and for the remaining assertion see Proposition III.2.5. $\square$

The trick applied in the proof of the implication (iii) $\Rightarrow$ (i) will be used several times below; it will be referred to as a convex combinations argument.

Proposition 4.10 *Suppose that $\mathfrak{L}$ is a space of bounded operators such that*

$$\mathbb{K}\{Id\} + K(X) \subset \mathfrak{L} \subset L(X).$$

If $K(X)$ is an M-ideal in $\mathfrak{L}$ then there is a net (K_α) in $B_{K(X)}$ such that (K_α^*) and (K_α) converge strongly to the identity on the respective spaces. In particular, X and X^* have the metric compact approximation property. If in addition $\mathfrak{L}$ is an algebra of operators, then $K(X)$ is a two-sided inner M-ideal.

PROOF: By Remark I.1.13, $B_{K(X)}$ is $\sigma(\mathfrak{L}, K(X)^{\#})$-dense in $B_{\mathfrak{L}}$. To prove the result we first observe that the functionals

$$x^{**} \otimes x^* : T \mapsto \langle x^{**}, T^* x^* \rangle$$

belong to $K(X)^{\#}$. Hence there is a net (L_α) of compact operators such that both $L_\alpha \to Id_X$ and $L_\alpha^* \to Id_{X^*}$ for the respective weak operator topologies. As in the previous proof we obtain the desired net by taking convex combinations.
That $K(X)$ is necessarily an inner M-ideal follows from Proposition 4.4, Lemma 4.9 and Theorem V.3.2. $\square$

Let us present an application of this result.

Proposition 4.11 *Let X be a Banach space.*

(a) *If X is reflexive and $K(X)$ is an M-ideal in $L(X)$, then $K(X)^{**} \cong L(X)$ so that $K(X)$ is M-embedded.*

(b) *If $K(X)$ is M-embedded, then X is reflexive.*

PROOF: (a) It is proved in [223] that the map $V : X \widehat{\otimes}_\pi X^* \to K(X)^*$, defined by $\langle V(u), K \rangle = \sum_{i=1}^\infty x_i^*(Kx_i)$ for $u = \sum_{i=1}^\infty x_i \otimes x_i^*$ (absolutely convergent sum), is a quotient map for reflexive X. Hence, after canonical identifications,

$$K(X)^{**} = (\ker V)^\perp \subset (X \widehat{\otimes}_\pi X^*)^* = L(X, X^{**}) = L(X).$$

Thus, it remains to show that $L(X) \subset (\ker V)^\perp$. So let $T \in L(X)$ and $u \in \ker V$. Indeed we have, with (K_α) as in Proposition 4.10,

$$\begin{aligned} \langle u, T \rangle &= \sum_{i=1}^\infty x_i^*(Tx_i) \;=\; \lim_\alpha \sum_{i=1}^\infty x_i^*(K_\alpha T x_i) \\ &= \langle V(u), K_\alpha T \rangle \;=\; 0. \end{aligned}$$

(b) This follows immediately from Corollary III.3.7(e) since X^* embeds into $K(X)$. $\Box$

We now approach the most fundamental result of this section, Theorem 4.17. We are going to prepare this central result by introducing a pair of new concepts in which the geometric peculiarities of the spaces we are about to investigate are formalised.

Definition 4.12 *We say that a Banach space X has* property (M) *if whenever $u, v \in X$ with $\|u\| = \|v\|$ and (x_α) is a bounded weakly null net in X, then*

$$\limsup_\alpha \|u + x_\alpha\| = \limsup_\alpha \|v + x_\alpha\|.$$

Similarly, X is said to have property (M^*) *if whenever $u^*, v^* \in X^*$ with $\|u^*\| = \|v^*\|$ and whenever (x^*_α) is a bounded weak* null net in X^*, then*

$$\limsup_\alpha \|u^* + x^*_\alpha\| = \limsup_\alpha \|v^* + x^*_\alpha\|.$$

One can see by a gliding hump argument that the spaces ℓ^p for $1 \le p < \infty$ as well as c_0 all have property (M). More precisely, one has

$$\limsup_\alpha \|u + x_\alpha\| = \Bigl(\|u\|^p + \limsup_\alpha \|x_\alpha\|^p\Bigr)^{1/p}$$

in the ℓ^p-case and likewise for c_0. Also, easy examples involving the sequence of Rademacher functions show that $L^p[0,1]$ fails property (M) for $p \neq 2$. The relation between (M) and (M^*) will be explained in a moment.
We now state a simple lemma.

Lemma 4.13 *The following conditions are equivalent:*

(i) *X has property (M).*

(ii) *If $\|u\| = \|v\|$ and (x_α) is a bounded weakly null net such that $\lim_\alpha \|u + x_\alpha\|$ exists, then $\lim_\alpha \|v + x_\alpha\|$ exists, and*

$$\lim_\alpha \|u + x_\alpha\| = \lim_\alpha \|v + x_\alpha\|.$$

(iii) *If $\|u\| \le \|v\|$ and (x_α) is any bounded weakly null net, then*

$$\limsup_\alpha \|u + x_\alpha\| \le \limsup_\alpha \|v + x_\alpha\|.$$

(iv) *If (u_α) and (v_α) are relatively norm compact nets with $\|u_\alpha\| \le \|v_\alpha\|$ for every α and (x_α) is a bounded weakly null net, then*

$$\limsup_\alpha \|u_\alpha + x_\alpha\| \le \limsup_\alpha \|v_\alpha + x_\alpha\|.$$

Analogous results hold for property (M^*).

PROOF: The equivalence of (i) and (ii) is straightforward, and trivially (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i). To deduce (iii) from (i) choose $\lambda > 1$ so that $\|\lambda u\| = \|v\|$. Observe that $x_\alpha + u$ is a convex combination of $x_\alpha + \lambda u$ and $x_\alpha - \lambda u$ whence

$$\|u + x_\alpha\| \leq \max\{\|\lambda u + x_\alpha\|, \|\lambda u - x_\alpha\|\}.$$

It follows

$$\begin{aligned} \limsup \|u + x_\alpha\| &\leq \max\{\limsup \|v + x_\alpha\|, \limsup \|-v + x_\alpha\|\} \\ &= \limsup \|v + x_\alpha\|, \end{aligned}$$

where we employed property (M) several times. Let us conclude by deducing (iv) from (iii). Indeed, if the conclusion were false we might pass to subnets so that $\limsup_k \|u_{\alpha_k} + x_{\alpha_k}\| > \limsup_k \|v_{\alpha_k} + x_{\alpha_k}\|$ and (u_{α_k}) as well as (v_{α_k}) are convergent. This quickly leads to a contradiction to (iii) since $\|\lim u_{\alpha_k}\| \leq \|\lim v_{\alpha_k}\|$. The proofs for (M^*) are similar. $\square$

We also need:

Lemma 4.14 *Suppose X has property (M) and that $T \in L(X)$ with $\|T\| \leq 1$. If (u_α) and (v_α) are relatively norm compact nets with $\|u_\alpha\| \leq \|v_\alpha\|$ and (x_α) is a bounded weakly null net, then*

$$\limsup_\alpha \|u_\alpha + Tx_\alpha\| \leq \limsup_\alpha \|v_\alpha + x_\alpha\|.$$

PROOF: First suppose $\|T\| = 1$ and (x_α) is any bounded weakly null net. For any $\lambda < 1$ there exists w with $\|w\| = 1$ and $\|Tw\| > \lambda$. Let $w_\alpha = \|v_\alpha\| w$. Hence by Lemma 4.13

$$\begin{aligned} \limsup_\alpha \|\lambda u_\alpha + Tx_\alpha\| &\leq \limsup_\alpha \|Tw_\alpha + Tx_\alpha\| \\ &\leq \limsup_\alpha \|w_\alpha + x_\alpha\| \\ &= \limsup_\alpha \|v_\alpha + x_\alpha\|. \end{aligned}$$

Letting $\lambda \to 1$ we obtain the conclusion for this case. Now suppose $0 \leq \|T\| < 1$. Let $T = \lambda L$ where $\lambda = \|T\|$ and $\|L\| = 1$. Then

$$\begin{aligned} \limsup_\alpha \|u_\alpha + Tx_\alpha\| &\leq \max\{\limsup_\alpha \|u_\alpha + Lx_\alpha\|, \limsup_\alpha \|u_\alpha - Lx_\alpha\|\} \\ &\leq \max\{\limsup_\alpha \|v_\alpha + x_\alpha\|, \limsup_\alpha \|-v_\alpha + x_\alpha\|\} \\ &= \limsup_\alpha \|v_\alpha + x_\alpha\|. \end{aligned}$$

$\square$

The following proposition compares (M) and (M^*). Note that ℓ^1 has (M), but fails (M^*). Also, have a peek at the following Theorem 4.17, (vi) $\Longleftrightarrow$ (vii).

Proposition 4.15 *Let X be a Banach space with property (M^*). Then X has property (M) and X is an M-ideal in X^{**}. Moreover, the relative norm and weak* topologies on S_{X^*} coincide.*

PROOF: First we show that X has property (M). Suppose $\|u\| = \|v\|$ and that (x_α) is a bounded weakly null net such that $\lim_\alpha \|u + x_\alpha\| > \lim_\alpha \|v + x_\alpha\|$ (and both limits exist). Then pick $x_\alpha^* \in B_{X^*}$ so that $x_\alpha^*(u + x_\alpha) = \|u + x_\alpha\|$. By passing to a subnet we may suppose that (x_α^*) converges weak* to some x^*. Now pick v^* with $\|v^*\| = \|x^*\|$ and $v^*(v) = \|x^*\|\|v\|$. Then

$$\begin{aligned}
\lim_\alpha \|u + x_\alpha\| &= \lim_\alpha \langle u + x_\alpha, x_\alpha^* \rangle \\
&= \langle u, x^* \rangle + \lim_\alpha \langle x_\alpha, x_\alpha^* - x^* \rangle \\
&\leq \langle v, v^* \rangle + \lim_\alpha \langle x_\alpha, x_\alpha^* - x^* \rangle \\
&= \lim_\alpha \langle v + x_\alpha, v^* + x_\alpha^* - x^* \rangle \\
&\leq \limsup_\alpha \|v^* + x_\alpha^* - x^*\| \|v + x_\alpha\| \\
&\leq \lim_\alpha \|v + x_\alpha\|
\end{aligned}$$

since $\limsup_\alpha \|v^* + x_\alpha^* - x^*\| = \limsup_\alpha \|x^* + x_\alpha^* - x^*\| = 1$ by property (M^*).
Next we show that X is an M-ideal in X^{**}. Suppose $\varphi \in X^\perp \subset X^{***}$ and suppose $\psi \in X^*$ (canonically embedded in X^{***}). For $\lambda < 1$ we can pick $x^{**} \in B_{X^{**}}$ so that $\varphi(x^{**}) > \lambda\|\varphi\|$. Pick any x^* in X^* with $\|x^*\| = \|\psi\|$ and $x^{**}(x^*) > \lambda\|\psi\|$. Let (y_d^*) be a net in X^* converging weak* in X^{***} to φ and such that $\|\psi + y_d^*\| \leq \|\psi + \varphi\|$. We can suppose that $x^{**}(y_d^*) > \lambda\|\varphi\|$ for all d. Since $\varphi \in X^\perp$, (y_d^*) converges weak* in X^* to zero. Now

$$\begin{aligned}
\|\psi + \varphi\| &\geq \limsup_d \|\psi + y_d^*\| \\
&= \limsup_d \|x^* + y_d^*\| \\
&\geq \limsup_d \langle x^* + y_d^*, x^{**} \rangle \\
&\geq \lambda(\|\psi\| + \|\varphi\|),
\end{aligned}$$

and the result follows.
Let now $\|x_\alpha^*\| = \|x_0^*\| = 1$ such that $w^*\text{-}\lim x_\alpha^* = x_0^*$. By property (M^*),

$$\limsup \|x^* + (x_\alpha^* - x_0^*)\| = \limsup \|x_0^* + (x_\alpha^* - x_0^*)\| = 1 \qquad \forall x^* \in S_{X^*}.$$

If we pick a weak* strongly exposed point x^* (the existence of those objects is guaranteed by Corollary III.3.2), then the conclusion $\|x_\alpha^* - x_0^*\| \to 0$ follows immediately. $\square$

It follows in particular that a reflexive space has (M) if and only if it has (M^*) if and only if its dual has either of these properties.
We are now ready for the announced characterisation. It will be convenient to use the following notation.

Definition 4.16 *A net of compact operators (K_α) on a Banach space X will be called a* shrinking compact approximation of the identity *provided both $K_\alpha \to Id_X$ and $K_\alpha^* \to Id_{X^*}$ strongly.*

The use of the term "shrinking" in this definition is explained by the corresponding concept in the theory of Schauder bases. Proposition 4.10 says that X admits a shrinking compact approximation of the identity provided $K(X)$ is an M-ideal in $L(X)$.

Theorem 4.17 *Let X be an infinite dimensional Banach space. The following conditions are equivalent:*

(i) *$K(X)$ is an M-ideal in $L(X)$.*

(ii) *There exists a shrinking compact approximation of the identity (K_α) such that*

$$\limsup_\alpha \|SK_\alpha + T(Id - K_\alpha)\| \le \max\{\|S\|, \|T\|\} \qquad \forall S, T \in L(X).$$

(iii) *There exists a shrinking compact approximation of the identity (K_α) such that*

$$\limsup_\alpha \|K_\alpha S + (Id - K_\alpha)T\| \le \max\{\|S\|, \|T\|\} \qquad \forall S, T \in L(X).$$

(iv) *$K(X)$ is an M-ideal in $\mathbb{K}\{Id\} \oplus K(X)$.*

(v) *There exists a shrinking compact approximation of the identity (K_α) with $\|K_\alpha\| \le 1$ such that*

$$\limsup_\alpha \|S + Id - K_\alpha\| \le 1 \qquad \forall S \in B_{K(X)}.$$

(vi) *X has property (M), and there is a shrinking compact approximation of the identity (K_α) such that*

$$\lim_\alpha \|Id - 2K_\alpha\| = 1.$$

(vii) *X has property (M^*), and there is a shrinking compact approximation of the identity (K_α) such that*

$$\lim_\alpha \|Id - 2K_\alpha\| = 1.$$

PROOF: The implications (i) $\iff$ (ii) $\iff$ (iii) $\Rightarrow$ (iv) $\iff$ (v) are clear by Theorem V.3.2, Lemma 4.9 and Proposition 4.10.

(v) $\Rightarrow$ (vi): By equivalence of (iv) and (v), we can apply Theorem V.3.2 with $\mathfrak{A} = \mathbb{K}\{Id\} \oplus K(X)$, $\mathfrak{J} = K(X)$ and obtain a shrinking compact approximation of the identity (K_α) such that

$$\limsup_\alpha \|SK_\alpha + T(Id - K_\alpha)\| \le \max\{\|S\|, \|T\|\} \qquad \forall S, T \in \mathfrak{A}.$$

Now pick $S = -Id$, $T = Id$ to see that $\limsup \|Id - 2K_\alpha\| \le 1$. Another way to derive that conclusion is to apply Theorem V.5.4 and to use a convex combinations argument. It is left to prove that X has property (M). Suppose (x_i) is a bounded weakly null net and that $\|u\| \le \|v\|$. Then there is a rank-one operator S with $\|S\| \le 1$ and $Sv = u$. Fix an index β. Then, since $\|Kx_i\| \to 0$ if K is compact,

$$\begin{aligned}
\limsup_i \|u + x_i\| &= \limsup_i \|S(v + x_i) + (Id - K_\beta)x_i\| \\
&\le \|S + Id - K_\beta\| \limsup_i \|v + x_i\| + \|(Id - K_\beta)v\|,
\end{aligned}$$

and, taking limits in β, it turns out that X must have property (M), by Lemma 4.13.

(vi) $\Rightarrow$ (ii): We first prove that for each β

$$\limsup_\alpha \|K_\beta + Id - K_\alpha\| \le \|Id - 2K_\beta\|.$$

In fact, $(Id - 2K_\alpha)(Id - 2K_\beta) = Id - 2K_\alpha - 2K_\beta + 4K_\alpha K_\beta$ and hence

$$\begin{aligned} \|Id - K_\alpha - K_\beta + 2K_\alpha K_\beta\| &= \frac{1}{2}\big(Id + (Id - 2K_\alpha)(Id - 2K_\beta)\big) \\ &\le \|Id - 2K_\alpha\|\|Id - 2K_\beta\|, \end{aligned}$$

note that $\|Id{-}2K_\alpha\| \ge 1$. Thus the assertion follows by taking limits, since $\lim_\alpha \|K_\alpha K_\beta - K_\beta\| = 0$. Let us now prove (ii). We first note that, as the proof of Theorem V.3.2 reveals, it is sufficient to show that

$$\limsup_\alpha \|S + T(Id - K_\alpha)\| \le 1$$

whenever $S \in B_{K(X)}$ and $T \in B_{L(X)}$. Accordingly, suppose $S \in B_{K(X)}$ and $T \in L_{K(X)}$ are given. Fix β. We may pick $x_\alpha \in B_X$ so that

$$\limsup_\alpha \|SK_\beta + T(Id - K_\alpha)\| = \limsup_\alpha \|SK_\beta x_\alpha + T(Id - K_\alpha)x_\alpha\|.$$

The net $(K_\beta x_\alpha)_\alpha$ is contained in a relatively compact set. Hence, noting that, as a result of $\lim_\alpha K_\alpha^* x^* = x^*$ for all $x^* \in X^*$, both $((Id - K_\alpha)x_\alpha)$ and $(T(Id - K_\alpha)x_\alpha)$ are weakly null, we obtain from Lemmas 4.13 and 4.14

$$\begin{aligned} \limsup_\alpha \|SK_\beta x_\alpha + T(Id - K_\alpha)x_\alpha\| &\le \limsup_\alpha \|K_\beta x_\alpha + T(Id - K_\alpha)x_\alpha\| \\ &\le \limsup_\alpha \|(K_\beta + Id - K_\alpha)x_\alpha\| \\ &\le \|Id - 2K_\beta\|. \end{aligned}$$

Thus

$$\limsup_\alpha \|S + T(Id - K_\alpha)\| \le \|S - SK_\beta\| + \|Id - 2K_\beta\|,$$

and choosing β large enough yields the assertion.

(vii) $\Rightarrow$ (vi): This follows immediately from Proposition 4.15.

(v) $\Rightarrow$ (vii): The proof proceeds similar to that of the implication (v) $\Rightarrow$ (vi). This time we suppose that (x_i^*) is a bounded weak* null net and that $\|u^*\| < \|v^*\|$. Then there is a weak* continuous rank-one contraction S^* mapping v^* onto u^*. Now one can follow the above pattern, observing $\lim_i \|K_\beta^* x_i^*\| = 0$. □

We note that in the separable case one can get sequences of compact operators rather than nets, and it is also enough to work with the sequential versions of (M) and (M^*); this results from the above proof. Actually, one can easily check that a space with a separable dual has (M) if and only if it has the sequential version of (M); the corresponding

equivalence holds for (M^*) in the case of separable spaces. (The point is that B_X is weakly metrizable if X^* is separable, and B_{X^*} is weak* metrizable if X is separable.)
We recall that ℓ^p $(1 < p < \infty)$ and c_0 have (M). Furthermore, the coordinate projections P_n of these spaces satisfy condition (vi) of the above characterisation theorem; in fact, the norm condition $\|Id - 2P_n\| = 1$ expresses nothing but the 1-unconditionality of the canonical Schauder basis, see below for the definition. (The case $p = 1$ has to be excluded since the ℓ^1-basis is not shrinking.) Thus we regain Example 4.1. However, we point out that the coordinate projections on ℓ^p, $p < \infty$, do not work in parts (ii) and (iii). In fact, for $S = Id$ and T the shift to the right one obtains

$$\begin{aligned}\limsup_n \|P_n S + (Id - P_n)T\| &\geq \limsup_n \|P_n e_n + (Id - P_n)e_{n+1}\| \\ &\geq \limsup_n \|e_n + e_{n+1}\| \;=\; 2^{1/p}.\end{aligned}$$

A similar counterexample, with T the shift to the left, works for (ii). We further mention ℓ^1 as a Banach space having (M) and the metric approximation property for which the compact operators do not form an M-ideal.
Here is a first application of Theorem 4.17.

Corollary 4.18 *If $K(X)$ is an M-ideal in $L(X)$ and $K(Y)$ is an M-ideal in $L(Y)$, then $K(X,Y)$ is an M-ideal in $L(X,Y)$.*

PROOF: Let (K_α) be a shrinking compact approximation of the identity with $\lim_\alpha \|Id - 2K_\alpha\| = 1$. Further let $S \in K(X,Y)$ and $T \in L(X,Y)$ with $\|S\| \leq 1$, $\|T\| \leq 1$. We shall show that

$$\limsup \|S + T(Id - K_\alpha)\| \leq 1 \qquad (*)$$

which proves our claim in view of the 3-ball property, Theorem I.2.2.
The proof of $(*)$ is an adaptation of the argument for the implication (vi) $\Rightarrow$ (ii) of Theorem 4.17. Fix β and pick $x_\alpha \in B_X$ such that

$$\limsup_\alpha \|SK_\beta x_\alpha + T(Id - K_\alpha)x_\alpha\| = \limsup_\alpha \|SK_\beta + T(Id - K_\alpha)\|.$$

Observe, as in the previous proof, that $(Id - K_\alpha)x_\alpha \to 0$ weakly. Now, arguments as in Lemmas 4.13 and 4.14 yield that

$$\limsup_\alpha \|SK_\beta x_\alpha + T(Id - K_\alpha)x_\alpha\| \leq \limsup_\alpha \|K_\beta x_\alpha + (Id - K_\alpha)x_\alpha\|,$$

since X and Y have (M) by Theorem 4.17. The desired inequality $(*)$ can now be derived as above. ☐

Next we provide a stability result for Banach spaces for which $K(X)$ is an M-ideal in $L(X)$ which is much more subtle than that in Proposition 4.2(a).

Theorem 4.19 *Let X be a Banach space such that $K(X)$ is an M-ideal in $L(X)$. Let E be a subspace of a quotient space of X. Then $K(E)$ is an M-ideal in $L(E)$ if and only if E has the metric compact approximation property.*

PROOF: For the "only if" part see Proposition 4.10. The proof of the "if" part relies on the equivalences of Theorem 4.17. First we note that E is an M-ideal in its bidual since X is (Proposition 4.4, Theorem III.1.6). It follows that if E has the metric compact approximation property it has a shrinking compact approximation of the identity (K_α); use Proposition III.2.5 and a convex combinations argument. We also note that, for $E \subset X/V$, E has property (M) since X/V trivially inherits (M^*) from X and hence has (M) by Proposition 4.15.

It remains to check that E has a shrinking compact approximation of the identity (H_α) such that $\lim_\alpha \|Id - 2H_\alpha\| = 1$. Suppose (L_α) is a shrinking compact approximation of the identity for X such that $\lim_\alpha \|Id - 2L_\alpha\| = 1$. (It is legitimate to suppose that (K_α) and (L_α) are indexed by the same set, since one can always switch to a product index set with the product ordering.) Let $J_E : E \to X/V$ be the inclusion and $Q : X \to X/V$ the quotient map. Let $Y = Q^{-1}(E)$ and denote by $J_Y : Y \to X$ the inclusion and by $Q_Y : Y \to E$ the restriction of Q so that $QJ_Y = J_E Q_Y$. We claim that the net $(QL_\alpha J_Y - J_E K_\alpha Q_Y)$ converges to 0 in $\sigma(K(Y, X/V), K(Y, X/V)^*)$. In fact, the linear span of the functionals $T \mapsto \langle y^{**}, T^* x^* \rangle$, $y^{**} \in Y^{**}$ and $x^* \in (X/V)^*$, is dense in $K(Y, X/V)^*$; this is because $(X/V)^* \subset X^*$ has the RNP (Corollary 4.5), see [223] and [364]. Clearly, $K_\alpha^* J_E^* x^* \to J_E^* x^*$ and $L_\alpha^* Q^* x^* \to Q^* x^*$. Thus

$$\langle y^{**}, (J_Y^* L_\alpha^* Q^* - Q_Y^* K_\alpha^* J_E^*) x^* \rangle \to \langle y^{**}, (J_Y^* Q^* - Q_Y^* J_E^*) x^* \rangle = 0,$$

and since $(QL_\alpha J_Y - J_E K_\alpha Q_Y)$ is bounded, we see that this net converges weakly to 0. Hence there exist $H_\alpha \in \operatorname{co}\{K_\beta \mid \beta \geq \alpha\}$ and $M_\alpha \in \operatorname{co}\{L_\beta \mid \beta \geq \alpha\}$ such that $\lim_\alpha \|QM_\alpha J_Y - J_E H_\alpha Q_Y\| = 0$. Thus $\lim_\alpha \|Q(Id - 2M_\alpha)J_Y - J_E(Id - 2H_\alpha)Q_Y\| = 0$ and so $\lim_\alpha \|Id - 2H_\alpha\| = 1$. □

We remark that the previous theorem could be formulated for quotients of subspaces rather than subspaces of quotients as well, since the two concepts coincide. In fact, one direction was observed in the above proof, and on the other hand, for $V \subset W \subset X$, the quotient W/V is clearly a subspace of the quotient X/V. Hence further iteration of forming subspaces and quotient spaces does not lead to any new results.

Combining the above observations on ℓ^p with Theorem 4.19 and the fact that for separable reflexive spaces the compact approximation property already implies the metric compact approximation property (the proof of the corresponding result on the metric approximation property in [421, p. 40] shows this, too) one obtains the following corollary.

Corollary 4.20 *A subspace X of a quotient of ℓ^p $(1 < p < \infty)$ has the compact approximation property if and only if $K(X)$ is an M-ideal of $L(X)$. If X is a subspace of a quotient of c_0, then the metric compact approximation property of X is equivalent to $K(X)$ being an M-ideal of $L(X)$.*

For further examples, the reader is referred to the following sections.

For the following, we have to recall some definitions: A Banach space is said to have a *(Schauder) basis* if there is a sequence (e_i) such that each $x \in X$ has a unique representation of the form

$$x = \sum_{i=1}^{\infty} \alpha_i e_i$$

for some sequence of scalars (α_i). The basis (e_i) is called λ-*unconditional* if

$$\sup_{|\varepsilon_i|=1} \left\| \sum_{i=1}^{n} \varepsilon_i \alpha_i e_i \right\| \le \lambda \left\| \sum_{i=1}^{n} \alpha_i e_i \right\| \qquad \forall n \in \mathbb{N}.$$

(Of course, the expression "unconditional" in this definition has its origin in the unconditional convergence of the series $\sum_{i=1}^{\infty} \alpha_i e_i$, cf. [421, Section 1.c].) Replacing the one-dimensional "fibers" in these definitions by finite dimensional spaces, one arrives at the concept of a so-called *finite dimensional (Schauder) decomposition* of a Banach space, i.e., a sequence (X_k) of finite dimensional subspaces of X such that each $x \in X$ has a unique representation of the form

$$x = \sum_{i=1}^{\infty} x_i,$$

where $x_i \in X_i$. In the same vein, a finite dimensional decomposition is called λ-unconditional if

$$\sup_{|\varepsilon_i|=1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \le \lambda \left\| \sum_{i=1}^{n} x_i \right\| \qquad \forall n \in \mathbb{N}.$$

Here comes the reason for recalling these definitions; this is one of the few instances where we must restrict our attention to the case of approximable operators.

Theorem 4.21 *If X is separable and $A(X)$ is an M-ideal in $L(X)$, then, for every $\varepsilon > 0$, the space X is isometric to a $(1+\varepsilon)$-complemented subspace of a space with a $(1+\varepsilon)$-unconditional finite dimensional Schauder decomposition.*

We found it convenient to formulate a lemma, which contains the core of the proof of this theorem. In the Notes and Remarks section we shall comment on this lemma from the point of view of the so-called u-ideals.

Lemma 4.22 *If X is separable and $A(X)$ is an M-ideal in $L(X)$, then, for every $\varepsilon > 0$, the space X admits a $(1+\varepsilon)$-unconditional expansion of the identity; that is, there are finite rank operators $F_1, F_2, \dots$ such that*

$$\left\| \sum_{i=1}^{n} \varepsilon_i F_i \right\| \le 1 + \varepsilon \qquad \forall n \in \mathbb{N},\ |\varepsilon_i| = 1, \tag{1}$$

$$\sum_{i=1}^{\infty} F_i(x) = x \qquad \forall x \in X. \tag{2}$$

Note that the series (2) converges unconditionally by (1).

PROOF: Since X is separable, so is X^* (Corollary 4.5). Consequently $A(X)$ is a separable M-ideal. In this case Theorem I.2.10 implies the existence of finite rank operators S_n

such that

$$\begin{aligned}\left\|\sum_{i=1}^{n}\varepsilon_i S_i\right\| &\leq 1+\varepsilon && \forall n\in\mathbb{N},\ |\varepsilon_i|=1,\\ \varphi(Id) &= \sum_{i=1}^{\infty}\varphi(S_i) && \forall\varphi\in A(X)^*,\end{aligned}$$

in particular $x^*(x)=\sum_{i=1}^{\infty}x^*(S_i x)$ for all $x\in X$, $x^*\in X^*$, hence $\lim_{n\to\infty}\sum_{i=1}^{n}S_i=Id$ in the weak operator topology. One can now take convex combinations to obtain a strongly convergent sequence of finite rank operators $T_n=\sum_{k=p_n+1}^{p_{n+1}}\lambda_k^{(n)}\sum_{i=1}^{k}S_i$, so that we get, letting $F_1=T_1$ and $F_{n+1}=T_{n+1}-T_n$, $x=\sum_{i=1}^{\infty}F_i x$ for all $x\in X$. To see that (1) holds one just has to repeat the calculation performed in the proof of Lemma I.2.9. □

Proof of Theorem 4.21: Let $\varepsilon>0$ and $1<(1+\delta)^2\leq 1+\varepsilon$. Pick F_n according to Lemma 4.22, with δ in place of ε. Consider the space

$$U=\left\{(x_n)\mid x_n\in\operatorname{ran}(F_n),\ \sum x_n \text{ converges unconditionally}\right\},$$

endowed with the norm

$$|(x_n)|=\sup_n\sup_{|\varepsilon_i|=1}\left\|\sum_{i=1}^{n}\varepsilon_i x_i\right\|.$$

It is clear that U has a 1-unconditional finite dimensional decomposition and that the operator $I\colon X\to U$, $I(x)=(F_n x)$ satisfies

$$\|x\|\leq|I(x)|\leq(1+\delta)\|x\|$$

by (1) of Lemma 4.22. Moreover, the operator $Q:U\to U$, $(x_n)\mapsto\big(F_n(\sum_i x_i)\big)$ is a projection onto $\operatorname{ran} I$ with $|Q|\leq 1+\delta$.
If we renorm U such that the new unit ball becomes the closed convex hull of the old one and $I(B_X)$ and call the new norm $\|\,.\,\|$, then I becomes an isometry, the canonical finite dimensional decomposition of U becomes $(1+\delta)$-unconditional and we get $\|Q\|\leq(1+\delta)^2$. Thus Theorem 4.21 is proven. □

Corollary 4.23 *A Banach space X which satisfies the assumptions of Theorem 4.21 is isometric to a subspace of a space with a $(1+\varepsilon)$-unconditional basis.*

Proof: This follows by an adaptation of the proof of [421, Th. 1.g.5] □

It is worthwhile remarking that one can even embed X into a space with a shrinking $(1+\varepsilon)$-unconditional basis [396].
We conclude this section with a result that shows that among spaces with a 1-symmetric basis there are only very few examples for which $K(X)$ is an M-ideal of $L(X)$. A basis (e_i) of X is called 1-symmetric if

$$\left\|\sum_{i=1}^{\infty}\varepsilon_i\xi_{\pi(i)}e_{\pi(i)}\right\|=\left\|\sum_{i=1}^{\infty}\xi_i e_i\right\|$$

for all $x = \sum_{i=1}^{\infty} \xi_i e_i \in X$, where the ε_i range over all scalars of modulus one and π over all permutations of $\mathbb{N}$; see [421, Ch. 3] for more details. Clearly the canonical bases in ℓ^p and c_0 are 1-symmetric. We also note that a 1-symmetric basis (e_n) of X is 1-unconditional. If in addition X^* is separable, then $e_n \to 0$ weakly. This well-known result follows for instance from [421, Th. 1.c.9] and will be used below.

Proposition 4.24 *Suppose X is a Banach space with a 1-symmetric basis. Then $K(X)$ is an M-ideal in $L(X)$ if and only if X is isometric to c_0 or ℓ^p for some $1 < p < \infty$.*

PROOF: In light of Example 4.1 we must show that if $K(X)$ is an M-ideal in $L(X)$, then X is isometric to c_0 or ℓ^p for some $1 < p < \infty$. To this end we will use the following result, due to Bohnenblust:

> A Banach space X with a normalised basis (e_n) is isometric to either c_0 or ℓ^p for $1 \le p < \infty$, if (and only if) for any normalised elements $x_1, \dots, x_n$, with $x_i = \sum_{j=n_i}^{m_i} a_j e_j$ and $n_1 \le m_1 < n_2 \le m_2 < n_3 \le \dots$,
>
> $$\left\| \sum_{i=1}^{n} \alpha_i e_i \right\| = \left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \tag{$*$}$$
>
> for all $n \in \mathbb{N}$ and all scalars $\alpha_1, \dots, \alpha_n$.

For a proof see [421, Th. 2.a.9] and put $M = 1$ in that reference. Actually, this result goes back to Kolmogorov and Nagumo, see [447, p. 154].

Now, if $K(X)$ is an M-ideal in $L(X)$, X has (M) by Theorem 4.17, and X^* is separable by Corollary 4.5. Together with the 1-symmetry of the basis, this implies:

> If x and x' (resp. y and y') are finitely and disjointly supported elements in X with $\|x\| = \|y\|$ and $\|x'\| = \|y'\|$, then
>
> $$\|x + x'\| = \|y + y'\|. \tag{$**$}$$

Here the support of $x = \sum_{i=1}^{\infty} \xi_i e_i$ is understood as the set $\{i \mid \xi_i \neq 0\}$. To prove $(**)$ we introduce the shift operator $\sigma : \sum_{i=1}^{\infty} \xi_i e_i \mapsto \sum_{i=1}^{\infty} \xi_i e_{i+1}$. By 1-symmetry of the basis, this is an isometry, and moreover we have $\sigma^m(x) \to 0$ weakly for each finitely supported x, since $e_n \to 0$ weakly, as was noted above. Again by 1-symmetry we obtain that $\|x + \sigma^r(x')\|$ does not depend on r, provided r is sufficiently large, so that we get from property (M) for sufficiently large r and s

$$\begin{aligned} \|x + x'\| &= \|x + \sigma^r(x')\| \\ &= \|y + \sigma^r(x')\| \\ &= \|\sigma^s(y) + \sigma^r(x')\| \\ &= \|\sigma^s(y) + \sigma^r(y')\| \\ &= \|y + y'\|. \end{aligned}$$

Clearly, $(**)$ inductively implies $(*)$. Since $K(\ell^1)$ is not an M-ideal in $L(\ell^1)$ by Corollary 4.5, the proof of the proposition is completed. $\square$

VI.5 Banach spaces for which $K(X)$ forms an M-ideal: The (M_p)-spaces

In this section we develop the theory of a certain class of spaces X for which $K(X)$ is an M-ideal in $L(X)$. These spaces can be characterised in terms of an ℓ^p-type version of the approximation property (Theorem 5.3), and their position within the class of all Banach spaces for which $K(X)$ is an M-ideal in $L(X)$ will eventually be elucidated in Theorem 6.6. We also present some applications to the study of smooth points in operator spaces.
We remind the reader that, as in the previous section, similar results can be formulated for norm limits of finite rank operators (instead of compact operators).

Definition 5.1 *Let $1 \le p \le \infty$. We say that a Banach space X has property (M_p) or is an (M_p)-space if $K(X \oplus_p X)$ is an M-ideal of $L(X \oplus_p X)$.*

Let us point out immediately that (M_1)-spaces are finite dimensional (and thus not really of interest in the present context). In fact, otherwise there is a net (x^*_α) in the dual unit sphere weak* converging to 0. Thus, for arbitrary $x^* \in S_{X^*}$, we have $(x^*, x^*_\alpha) \to (x^*, 0) \in S_{(X\oplus_1 X)^*}$ weak*, but not in norm. By Proposition 4.6 $K(X \oplus_1 X)$ is not an M-ideal in $L(X \oplus_1 X)$.
Clearly, the basic example of an (M_p)-space is ℓ^p (for $1 < p < \infty$) by Example 4.1 and since $\ell^p \oplus_p \ell^p \cong \ell^p$. Likewise, c_0 has the (M_∞)-property.
For convenience we collect some observations on these spaces.

Proposition 5.2

(a) *If $1 < p \le \infty$ and X is an (M_p)-space, then $K(X)$ is an M-ideal in $L(X)$.*
(b) *If $1 < p < \infty$ and X is an (M_p)-space, then X is reflexive.*
(c) *If $1 < p < \infty$ and $1/p + 1/p^* = 1$, then X is an (M_p)-space if and only if X^* is an (M_{p^*})-space.*

PROOF: (a) follows from Proposition 4.2.
(b) Otherwise, $X^{**} \oplus_p X^{**}$ would contain a nontrivial L^p-summand and the nontrivial M-ideal $X \oplus_p X$ (Proposition 4.4) which is impossible as shown in [47].
(c) In this case $L(X \oplus_p X)$ and $L(X^* \oplus_{p^*} X^*)$ are isometric with the isomorphism mapping compact operators onto compact operators, since X is reflexive by (b). □

Theorem 5.3 *Let X be a Banach space and suppose $1 < p \le \infty$.*

(a) *Whenever $1 < p < \infty$, X has (M_p) if and only if there is a net (K_α) in the unit ball of $K(X)$ such that*

$$K_\alpha x \to x \quad \forall x \in X \qquad \text{and} \qquad K^*_\alpha x^* \to x^* \quad \forall x^* \in X^*, \tag{1}$$

and for all $\varepsilon > 0$ there is α_0 such that for $\alpha > \alpha_0$ we have for all $x, y \in X$

$$\|K_\alpha x + (Id - K_\alpha)y\|^p \le (1+\varepsilon)^p(\|x\|^p + \|y\|^p) \tag{2a}$$

as well as

$$\|K_\alpha x\|^p + \|x - K_\alpha x\|^p \le (1+\varepsilon)^p\|x\|^p. \tag{2b}$$

(b) *X has (M_∞) if and only if there is a net (K_α) in the unit ball of $K(X)$ satisfying (1) and*

$$\|K_\alpha x + (Id - K_\alpha)y\| \le (1+\varepsilon)\max\{\|x\|, \|y\|\} \qquad \forall x, y \in X \tag{3}$$

provided α is large enough.

Proof: (a) Suppose $K(X \oplus_p X)$ is an M-ideal in $L(X \oplus_p X)$. By Theorem 4.17 there is a net of operators (S_α) in the unit ball of $K(X \oplus_p X)$ converging to the identity operator on the square with respect to the strong operator topology as do their adjoints such that

$$\limsup \|S_\alpha A + (Id - S_\alpha)B\| \le \max\{\|A\|, \|B\|\} \qquad \forall A, B \in L(X \oplus_p X). \tag{4}$$

The S_α may be represented as 2×2 matrices of operators on X, say $S_\alpha = (S_\alpha^{kl})_{k,l=1,2}$. Now (S_α^{12}), (S_α^{21}) and $(S_\alpha^{11} - S_\alpha^{22})$ tend strongly to 0, and the technique employed in the proof of Theorem 4.19 shows that these nets tend to 0 for the weak topology of the Banach space $K(X \oplus_p X)$. Hence a convex combinations argument permits us to assume that S_α is of the particular form

$$S_\alpha = \begin{pmatrix} K_\alpha & 0 \\ 0 & K_\alpha \end{pmatrix}.$$

The (K_α) fulfill (1), and checking (4) on the operators

$$A = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & Id \\ 0 & 0 \end{pmatrix}$$

and, respectively,

$$A = \begin{pmatrix} Id & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ Id & 0 \end{pmatrix}$$

will immediately yield (2a) and (2b).

For the converse suppose a net (K_α) given as stated. To prove the M-ideal property we wish to establish the restricted 3-ball property (Theorem I.2.2). That is, given $\varepsilon > 0$, compact contractions K_1, K_2, K_3 and a bounded contraction T on $X \oplus_p X$, we must find a compact operator U such that $\|K_r + T - U\| \le 1 + \varepsilon$. To this end consider the compact operator

$$U_\alpha = \big(T^{kl} - (Id - K_\alpha)T^{kl}(Id - K_\alpha)\big)_{k,l=1,2}\ .$$

(Again we represent operators on $X \oplus_p X$ by 2×2-matrices with entries in $L(X)$; note that $\|(T^{kl})\| \le 1$ translates into

$$\|T^{11}x + T^{12}y\|^p + \|T^{21}x + T^{22}y\|^p \le \|x\|^p + \|y\|^p$$

for all $x, y \in X$.) We wish to show that $U = U_\alpha$ will do for large enough α. First of all (1) implies

$$\|K_\alpha K_r^{kl} K_\alpha - K_r^{kl}\| \le \varepsilon$$

for $r = 1,2,3$ and $k,l = 1,2$, provided α is sufficiently large. Thus it is acceptable to pretend that actually $K_\alpha K_r^{kl} K_\alpha = K_r^{kl}$ in the following computation which shows the validity of the 3-ball condition:

$$\begin{aligned}
&\|(K_r + T - U_\alpha)(x,y)\|^p \\
&\quad= \Big\|\big[K_\alpha K_r^{11} K_\alpha + (Id - K_\alpha)T^{11}(Id - K_\alpha)\big]x \\
&\qquad\qquad + \big[K_\alpha K_r^{12} K_\alpha + (Id - K_\alpha)T^{12}(Id - K_\alpha)\big]y\Big\|^p \\
&\qquad + \Big\|\big[K_\alpha K_r^{21} K_\alpha + (Id - K_\alpha)T^{21}(Id - K_\alpha)\big]x \\
&\qquad\qquad + \big[K_\alpha K_r^{22} K_\alpha + (Id - K_\alpha)T^{22}(Id - K_\alpha)\big]y\Big\|^p \\
&\quad\le (1+\varepsilon)^p \big[\|K_r^{11} K_\alpha x + K_r^{12} K_\alpha y\|^p \\
&\qquad\qquad + \|T^{11}(Id - K_\alpha)x + T^{12}(Id - K_\alpha)y\|^p \\
&\qquad\qquad + \|K_r^{21} K_\alpha x + K_r^{22} K_\alpha y\|^p \\
&\qquad\qquad + \|T^{21}(Id - K_\alpha)x + T^{22}(Id - K_\alpha)y\|^p\big] \\
&\quad\le (1+\varepsilon)^p \big[\|K_\alpha x\|^p + \|K_\alpha y\|^p + \|(Id - K_\alpha)x\|^p + \|(Id - K_\alpha)y\|^p\big] \\
&\quad\le (1+\varepsilon)^{2p}(\|x\|^p + \|y\|^p)
\end{aligned}$$

provided α is large enough.

(b) The necessity of the conditions (1) and (3) can be proved in much the same way as in part (a). Although this is basically true for the sufficiency argument, too, we would like to be slightly more precise. Following the above notation we may now suppose that $K_\alpha K_r^{kl} = K_r^{kl}$ for large α, and we then have for $U = \big(K_\alpha T^{kl}\big)_{k,l=1,2}$

$$\begin{aligned}
&\|(K_r + T - U)(x,y)\| \\
&\quad= \max\Big\{\|[K_\alpha K_r^{11} + (Id - K_\alpha)T^{11}]x + [K_\alpha K_r^{12} + (Id - K_\alpha)T^{12}]y\|, \\
&\qquad\qquad \|[K_\alpha K_r^{21} + (Id - K_\alpha)T^{21}]x + [K_\alpha K_r^{22} + (Id - K_\alpha)T^{22}]y\|\Big\} \\
&\quad\le (1+\varepsilon)\max\Big\{\|K_r^{11}x + K_r^{12}y\| \vee \|T^{11}x + T^{12}y\|, \\
&\qquad\qquad \|K_r^{21}x + K_r^{22}y\| \vee \|T^{21}x + T^{22}y\|\Big\} \\
&\quad\le 1+\varepsilon.
\end{aligned}$$

□

By a straightforward verification, one checks that the coordinate projections on $X = \ell^p$ (or more generally, the net of projections belonging canonically to an ℓ^p-sum of finite dimensional spaces) satisfy the above conditions. Therefore the following corollary is valid.

Corollary 5.4 *For every family (X_α) of finite dimensional Banach spaces, $K(\ell^p(X_\alpha))$ and $K(c_0(X_\alpha))$ are M-ideals in the respective spaces of bounded operators.*

The proof of part (b) of Theorem 5.3 yields a bit more than stated above. Namely, if T is an operator into $(X \oplus_\infty X)^{**}$ rather than into $X \oplus_\infty X$ and if we define $U = \big(K_\alpha^{**}T^{kl}\big)$, then U is an $X \oplus_\infty X$-valued compact operator, and the above proof shows that $K(X \oplus_\infty X)$ is an M-ideal in $L(X \oplus_\infty X, (X \oplus_\infty X)^{**})$. The following proposition will assert this, but its formulation is more symmetric. We denote

$$K^{\mathrm{ad}}(X^*, Y^*) = \{T^* \mid T \in K(Y, X)\}$$

and observe that $L(Y, X^{**}) \cong L(X^*, Y^*)$.

Proposition 5.5 *A Banach space X is an (M_∞)-space if and only if either of the following holds:*

(i) *$K^{\mathrm{ad}}(X^* \oplus_1 X^*)$ is an M-ideal in $L(X^* \oplus_1 X^*)$.*

(ii) *There is a net (S_α) in the unit ball of $K^{\mathrm{ad}}(X^*)$ converging to the identity in the strong operator topology such that*

$$\limsup_\alpha \sup_{\|x^*\| \le 1} \big(\|S_\alpha x^*\| + \|(Id - S_\alpha)x^*\|\big) \le 1. \tag{5}$$

Proof: This was in essence observed in the preceding remarks. It is left to point out that (5) and (3) are easily seen to be equivalent by a simple dualization argument, that, by a convex combinations argument, if $L_\alpha^* = S_\alpha$ and $S_\alpha \to Id_{X^*}$ strongly, then some convex combinations of the L_α converge strongly to Id_X and that both (3) and (5) remain in effect after having taken convex combinations. □

Condition (i) can be thought of as a dual version of the (M_∞)-property.
We now investigate the (M_∞)-spaces more closely. Recall that (iii) below means that $K(Y, X)$ is an M-ideal in $L(Y, X^{**})$ for all Y.

Proposition 5.6 *For a Banach space X the following assertions are equivalent:*

(i) *X is an (M_∞)-space.*

(ii) *For all Banach spaces Y, $K(Y, X)$ is an M-ideal in $L(Y, X)$.*

(iii) *For all Banach spaces Y, $K^{\mathrm{ad}}(X^*, Y^*)$ is an M-ideal in $L(X^*, Y^*)$.*

Proof: (i) ⇒ (iii): Pick a net $(S_\alpha) = (K_\alpha^*)$ as in Proposition 5.5. We may assume that (K_α) satisfies (1) of Theorem 5.3, due to the by now standard convex combinations argument. Then, if $T \in B_{L(X^*, Y^*)}$, $K_1, K_2, K_3 \in B_{K(Y,X)}$ and $\varepsilon > 0$, we have for sufficiently large α

$$\|K_\alpha K_r - K_r\| \le \frac{\varepsilon}{2}, \qquad r = 1, 2, 3$$

and

$$\sup_{\|x_1\|, \|x_2\| \le 1} \|K_\alpha x_1 + (Id - K_\alpha)x_2\| \le 1 + \frac{\varepsilon}{2}.$$

Putting $U = K_\alpha^{**}T^*|_Y \in K(Y, X)$ (for K_α^{**} is X-valued, since K_α is compact) we obtain for large α

$$\|K_r^* + T - U^*\| \quad = \quad \|K_r^* + T(Id - K_\alpha^*)\|$$

$$
\begin{aligned}
&\leq \|K_r^* K_\alpha^* + T(Id - K_\alpha^*)\| + \frac{\varepsilon}{2} \\
&\leq \sup_{\|x^*\| \leq 1} \big(\|K_\alpha^* x^*\| + \|(Id - K_\alpha^*)x^*\| \big) + \frac{\varepsilon}{2} \\
&\leq 1 + \varepsilon.
\end{aligned}
$$

(iii) $\Rightarrow$ (ii): This is clear since $K(Y,X) \cong K^{\mathrm{ad}}(X^*,Y^*)$ and $L(Y,X) \overset{\cong}{\hookrightarrow} L(X^*,Y^*)$ canonically.

(ii) $\Rightarrow$ (i): By assumption, $K(X \oplus_\infty X, X)$ is an M-ideal in $L(X \oplus_\infty X, X)$, and now, the statement that X possesses the (M_∞)-property follows from two general facts: The first is that we have (Lemma 1.1(c))

$$
\begin{aligned}
L(X \oplus_\infty X) &= L(X \oplus_\infty X, X) \oplus_\infty L(X \oplus_\infty X, X), \\
K(X \oplus_\infty X) &= K(X \oplus_\infty X, X) \oplus_\infty K(X \oplus_\infty X, X),
\end{aligned}
$$

and the second consists in the observation that $J \oplus_\infty J$ remains an M-ideal in $E \oplus_\infty E$ whenever J is an M-ideal in the Banach space E. $\square$

There seem to be only few (M_∞)-spaces; in fact, the closed subspaces and quotients of $c_0(I)$ with the metric compact approximation property (see Corollary 5.12) are basically the only examples we know. In this regard the following result is of interest.

Theorem 5.7 *Let X be a Banach space. Then the following conditions are equivalent:*

(i) *For every $\varepsilon > 0$ the space X is $(1+\varepsilon)$-close with respect to the Banach-Mazur distance to a subspace $Y = Y_\varepsilon$ of c_0 enjoying the metric compact approximation property.*

(ii) *The space X is, for every $\varepsilon > 0$, $(1+\varepsilon)$-close with respect to the Banach-Mazur distance to a space $Y = Y_\varepsilon$ admitting a sequence (K_n) of compact operators converging strongly to the identity on Y with the property that there is a sequence (δ_n) of nonnegative numbers tending to zero with*

$$
\|K_n y_1 + (Id - K_n) y_2\| \leq 1 + \delta_n \|y_1 - y_2\| \qquad \forall y_1, y_2 \in B_Y.
$$

Note that the approximation condition in (ii) is at least formally stronger than (3) of Theorem 5.3, but we don't have an example to show that the two conditions really differ. For the proof we need two auxiliary results. The first one can be regarded as an isomorphic version of Proposition I.3.9.

Proposition 5.8 *Let T be an operator on a Banach space X and let $\delta > 0$. Then the following statements are equivalent:*

(i) *For all $p \in \operatorname{ex} B_{X^*}$ there is a number α in the unit interval with*

$$
\|T^* p - \alpha p\| \leq \delta.
$$

(ii) *For all pairs of elements x_1, x_2 in the unit ball of X the following estimate holds:*

$$
\|T x_1 + (Id - T) x_2\| \leq 1 + \delta \|x_1 - x_2\|.
$$

(iii) *For each $x^* \in X^*$ there is an element $y^* \in X^*$ such that $\|y^*\| \le \delta\|x^*\|$ and*

$$\|T^*x^* - y^*\| + \|(Id - T^*)x^* + y^*\| = \|x^*\|.$$

PROOF: We first show that (i) implies (ii). To do this, let $x_1,\ x_2 \in B_X$ and pick $p \in \operatorname{ex} B_{X^*}$ with

$$\|Tx_1 + (Id - T)x_2\| = p(Tx_1 + (Id - T)x_2).$$

By (i), there is $\alpha \in [0,1]$ such that

$$\begin{aligned} p(Tx_1 + (Id - T)x_2) &\le p(x_2) + \alpha p(x_1 - x_2) + \delta\|x_1 - x_2\| \\ &= \alpha p(x_1) + (1-\alpha)p(x_2) + \delta\|x_1 - x_2\|, \end{aligned}$$

whence

$$\|Tx_1 + (Id - T)x_2\| \le 1 + \delta\|x_1 - x_2\|$$

as claimed.

Let us now see how one can conclude (iii) from (ii). Endow $X \oplus X$ and $X \oplus X \oplus X$ with norms

$$\|(x_1, x_2)\| = \max\{\|x_1\|, \|x_2\|\} + \delta\|x_1 - x_2\|$$

and, respectively,

$$\|(x_1, x_2, x_3)\| = \max\{\|x_1\|, \|x_2\|\} + \delta\|x_3\|.$$

Using the embedding $(x_1, x_2) \mapsto (x_1, x_2, x_1 - x_2)$ it is easy to check that the dual norm on $(X \oplus X)^* = X^* \oplus X^*$ is given by

$$\|(x_1^*, x_2^*)\| = \inf_{y^* \in Y^*} \max\left\{\|x_1^* - y^*\| + \|x_2^* + y^*\|, \delta^{-1}\|y^*\|\right\}.$$

As a benefit of the weak* lower semicontinuity of the norm and the weak* compactness of the dual unit ball the infimum in the last equation is actually attained. Hence, a pair (x_1^*, x_2^*) belongs to the unit ball of $X^* \oplus X^*$ if and only if

$$\min_{\|y^*\| \le \delta} \|x_1^* - y^*\| + \|x_2^* + y^*\| \le 1.$$

The reason for norming $X \oplus X$ in this way is the fact that the operator

$$\widehat{T} : X \oplus X \to X, \quad (x_1, x_2) \mapsto Tx_1 + (Id - T)x_2$$

now has norm one and that, consequently, in passing to adjoints,

$$\begin{aligned} \|x^*\| &\le \inf_{\|y^*\| \le \delta} \big(\|T^*x^* - y^*\| + \|(Id - T^*)x^* + y^*\|\big) \\ &= \min_{\|y^*\| \le \delta} \big(\|T^*x^* - y^*\| + \|(Id - T^*)x^* + y^*\|\big) \\ &= \|\widehat{T}^*(x^*)\| \\ &\le \|x^*\| \end{aligned}$$

for any $x^* \in B_{X^*}$. (This also shows that $\|\widehat{T}\| = 1$.) Since the case of an arbitrary $x^* \in X^*$ follows by normalization, we have shown (iii).

To finish the proof we must show that (i) is a consequence of (iii). So let us suppose that (iii) holds and pick $y^* \in X^*$ such that for a fixed functional $p \in \operatorname{ex} B_{X^*}$

$$\|T^*p - y^*\| + \|(Id - T^*)p + y^*\| = 1$$

and $\|y^*\| \le \delta$. We certainly may suppose that the numbers $\|T^*p - y^*\|$ and $\|(Id-T^*)p + y^*\|$ are different from zero. (Otherwise nothing has to be shown.) Writing

$$p = \frac{T^*p - y^*}{\|T^*p - y^*\|}\|T^*p - y^*\| + \frac{(Id - T^*)p + y^*}{\|(Id - T^*)p + y^*\|}\|(Id - T^*)p + y^*\|,$$

we find

$$T^*p - y^* = \|T^*p - y^*\|p,$$

and letting $\alpha := \|T^*p - y^*\|$ it turns out that

$$\|T^*p - \alpha p\| = \|y^*\| \le \delta,$$

which brings the proof to an end. □

Lemma 5.9 *Suppose that K is a compact operator defined on X, which satisfies one of the equivalent conditions of the above proposition for the constant $\delta > 0$. Then, for any $\alpha > \delta$ and $\varepsilon > 0$, there is a finite $(\varepsilon + \delta(\alpha - \delta)^{-1})$-net for the set*

$$\{p \in \operatorname{ex} B_{X^*} \mid \|K^*p\| \ge \alpha\}.$$

Proof: Write C_δ for the set of all points at a distance less than δ from the set C and denote by $[a, b]$ the convex hull of two real numbers a and b. An appeal to Proposition 5.8 then yields $a(p) \in [0, 1]$ which are "almost eigenvalues" of K corresponding to the "almost eigenvector" $p \in \operatorname{ex} B_{X^*}$. It follows

$$\begin{aligned} \{p \in \operatorname{ex} B_{X^*} \mid \|K^*p\| \ge \alpha\} &\subset \{p \in \operatorname{ex} B_{X^*} \mid a(p) \ge \alpha - \delta\} \\ &\subset [1, (\alpha - \delta)^{-1}]\,\{a(p)p \mid p \in \operatorname{ex} B_{X^*}, a(p) \ge \alpha - \delta\} \\ &\subset [1, (\alpha - \delta)^{-1}]\,\{K^*p \mid p \in \operatorname{ex} B_{X^*}, a(p) \ge \alpha - \delta\}_\delta \\ &\subset \Big([1, (\alpha - \delta)^{-1}]K^*(B_{X^*})\Big)_{\frac{\delta}{\alpha-\delta}}, \end{aligned}$$

and since the set $K^*(B_{X^*})$ is norm compact by assumption, we are done. □

Let us now come to the

Proof of Theorem 5.7:
To demonstrate that condition (i) implies (ii) it certainly suffices to show that the metric compact approximation property of subspaces of c_0 has the announced asymptotic behaviour. So let X be a subspace of c_0 with the metric compact approximation property and suppose that (S_n) is a sequence of norm-one compact operators converging strongly to the identity. If (P_n) is the sequence of coordinate projections on c_0 then, for every n,

$$\sup_{\|x_1\|, \|x_2\| \le 1} \|P_n x_1 + (Id - P_n)x_2\| \le 1$$

and $JS_n - P_nJ \to 0$ strongly, where J denotes the embedding of X into c_0. Now proceed as in Theorem 4.19 to see that there are sequences (S_n^c) and (P_n^c) of convex combinations of (S_n) and (P_n), respectively, such that $JS_n^c - P_n^cJ \to 0$ in norm. For the obvious choice of the sequence (δ_n) we find for $x_1,\, x_2 \in B_X$

$$\begin{aligned} \|S_n^c x_1 + (Id - S_n^c)x_2\| &\leq \|P_n^c J(x_1) + (Id - P_n^c)J(x_2)\| + \|JS_n^c - P_n^c J\| \|x_1 - x_2\| \\ &\leq 1 + \delta_n \|x_1 - x_2\|. \end{aligned}$$

This proves the desired implication.
For the implication (ii) $\Rightarrow$ (i), it is sufficient to prove that any Banach space X admitting a sequence (K_n) as stated in the theorem is, for any $\varepsilon > 0$, $(1+\varepsilon)$-close to a subspace of c_0. Invoking Proposition 5.8, we see that there is a sequence (δ_n) of nonnegative numbers tending to zero such that for every $p \in \operatorname{ex} B_{X^*}$ there exists $a_n(p) \in [0,1]$ with

$$\|K_n^* p - a_n(p)p\| \leq \delta_n.$$

Fix a natural number $N \in \mathbb{N}$. Applying Lemma 5.9 (with $\varepsilon = (2N)^{-1}$, $\delta = \delta_n$ and $\alpha = 2(N+1)\delta_n$), we successively define sets $C_n \subset \operatorname{ex} B_{X^*}$ together with functionals $p_n \in \operatorname{ex} B_{X^*}$ as follows:
Let $C_1 := \{p \in \operatorname{ex} B_{X^*} \mid \|K_1^* p\| \geq (2N+1)\delta_1\}$ and $p_1, \ldots, p_{k_1}$ be an N^{-1}-net of C_1. Having performed the n^{th} step, put

$$C_{n+1} := \{p \in \operatorname{ex} B_{X^*} \mid \|K_{n+1}^* p\| \geq (2N+1)\delta_{n+1}\} \setminus \bigcup_{\nu=1}^{n} C_\nu$$

and choose $p_{k_n+1}, \ldots, p_{k_{n+1}}$ such that $\{p_{k_n+1}, \ldots, p_{k_{n+1}}\}$ is an N^{-1}-net for C_{n+1}. Note that whenever $k > k_n$ it follows that

$$\|K_n^* p_k\| < (2N+1)\delta_n. \tag{1}$$

Define $I : X \to \ell^\infty$ by $(Ix)(n) = p_n(x)$ and note that $I(X) \subset c_0$: In fact, whenever $x \in B_X$ and $\eta > 0$ is fixed, we pick $n \in \mathbb{N}$ such that $(4N+2)\delta_n \leq \eta$ and $\|K_n x - x\| \leq \frac{\eta}{2}$. By (1) we then may conclude that for all $k \geq k_n + 1$

$$|I(x)(k)| \leq |K_n^* p_k(x)| + |p_k(x) - K_n^* p_k(x)| \leq (2N+1)\delta_n + \frac{\eta}{2} \leq \eta.$$

Let us estimate some norms. Again, fix $\eta > 0$ and $x \in X$ with norm one, and choose $n \in \mathbb{N}$ such that $\|x\| \leq \|K_n x\| + \frac{\eta}{2}$ and $\delta_n \leq \frac{\eta}{2}$ simultaneously. We have

$$1 = \|x\| \leq \|K_n x\| + \frac{\eta}{2} = p(K_n x) + \frac{\eta}{2}$$

for some $p \in \operatorname{ex} B_{X^*}$, which must be N^{-1}-close to some p_k from the sequence constructed above, because $\lim_n \|K_n^* p\| = 1$ and therefore $p \in C_n$ for some $n \in \mathbb{N}$. Consequently, the right hand side in the above inequality can be estimated from above by

$$a_n(p_k)p_k(x) + N^{-1} + \delta_n + \frac{\eta}{2} \leq \|I(x)\| + N^{-1} + \eta.$$

It follows that $\|I(x)\| \geq (1-\eta-N^{-1})\|x\|$ for all $x \in X$ and hence

$$\|I(x)\| \leq \|x\| \leq \frac{N}{N-1}\|I(x)\|. \tag{2}$$

Let us finally observe that $I(X)$ has the metric compact approximation property. In fact, letting $\widetilde{K}_n : (p_k(x)) \mapsto (p_k(K_n x))$, we have $\lim_n \widetilde{K}_n \xi = \xi$ for all $\xi \in I(X)$ and $\limsup_n \|\widetilde{K}_n\| \leq 1$. For, whenever $\sup_{k\in\mathbb{N}} |p_k(x)| \leq 1$ we have by (2) the inequality $\|x\| \leq N(N-1)^{-1}$ and therefore

$$\begin{aligned} \sup_{k\in\mathbb{N}} |p_k(K_n x)| &\leq |(K_n^* p_k - a_n(p_k)p_k)(x)| + |a_n(p_k)p_k(x)| \\ &\leq \delta_n \frac{N}{N-1} + 1. \end{aligned}$$

Further, $\widetilde{K}_n$ is compact since K_n is. The theorem now follows from (2) and the fact that $N \in \mathbb{N}$ was arbitrary. □

We now show that the (M_p)-spaces contain lots of complemented copies of ℓ^p; the existence of ℓ^p-subspaces will be proved under less stringent assumptions in Theorem 6.4.

Theorem 5.10 *Suppose X is an infinite dimensional Banach space which has property (M_p) for some $1 < p \leq \infty$.*

(a) *Every normalized sequence (x_n) in X tending to 0 weakly has, for every $\lambda > 1$, a subsequence (x_{n_k}) which is λ-equivalent to the unit vector basis in ℓ^p (c_0 if $p = \infty$) such that $\overline{\mathrm{lin}}\,\{x_{n_k} \mid k \in \mathbb{N}\}$ is complemented in X.*

(b) *If $p = \infty$, then every normalized sequence (x_n^*) in X^* tending to 0 in the weak* sense has, for every $\lambda > 1$, a subsequence $(x_{n_k}^*)$ which is λ-equivalent to the unit vector basis in ℓ^1 and such that $\overline{\mathrm{lin}}\,\{x_{n_k}^* \mid k \in \mathbb{N}\}$ is complemented in X^*.*

PROOF: (a) We first treat the case $p < \infty$. Let (ε_n) be a sequence of positive numbers such that $\prod_{n=1}^\infty (1+3\varepsilon_n) \leq \lambda$ and $\prod_{n=1}^\infty (1-3\varepsilon_n) \geq \lambda^{-1}$. Let $(K_\alpha) \subset K(X)$ be a net as in Theorem 5.3. To begin with, put $y_1 = x_1$. Then

$$\|K_{\alpha_1} y_1 - y_1\| \leq \varepsilon_1$$

for some α_1 so large that (2a) and (2b) are fulfilled with $\varepsilon = \varepsilon_1$. Since K_{α_1} is compact, $\lim_{n\to\infty} K_{\alpha_1} x_n = 0$, and we hence may find $n_2 > 1$, such that for $y_2 = x_{n_2}$

$$\|K_{\alpha_1} y_2\| \leq \varepsilon_1.$$

Next, we choose $\alpha_2 > \alpha_1$ such that for all y in the span of y_1 and y_2

$$\|K_{\alpha_2} y - y\| \leq \varepsilon_2 \|y\|$$

and that, additionally, (2a) and (2b) hold for $\varepsilon = \varepsilon_2$. Then choose $n_3 > n_2$ so that for $y_3 = x_{n_3}$

$$\|K_{\alpha_2} y_3\| \leq \varepsilon_2.$$

Thus, an induction procedure yields an increasing sequence of indices (α_k) and a subsequence (y_k) of the x_n such that K_{α_k} fulfills (2a) and (2b) for $\varepsilon = \varepsilon_k$ as well as

$$\|K_{\alpha_k} y - y\| \le \varepsilon_k \|y\| \qquad \forall y \in \operatorname{lin}\{y_1, \dots, y_k\} \tag{$*$}$$

and

$$\|K_{\alpha_k} y_{k+1}\| \le \varepsilon_k. \tag{$**$}$$

We claim that the y_k are λ-equivalent to the unit vector basis of ℓ^p. To prove an upper ℓ^p-estimate, we now consider the numerical sequence defined by

$$\begin{aligned} C_1 &= 1, \\ C_n &= \prod_{k=1}^{n-1} (1 + 3\varepsilon_k). \end{aligned}$$

Note that $\sup_n C_n \le \lambda$. An induction argument then yields

$$\left\| \sum_{k=1}^n a_k y_k \right\| \le C_n \left(\sum_{k=1}^n |a_k|^p \right)^{1/p} :$$

Indeed, we find by (2a), $(*)$ and $(**)$

$$\begin{aligned} \left\| \sum_{k=1}^{n+1} a_k y_k \right\| &\le \left\| K_{\alpha_n} \left(\sum_{k=1}^n a_k y_k \right) + (Id - K_{\alpha_n})\,(a_{n+1} y_{n+1}) \right\| \\ &\quad + \varepsilon_n \left\| \sum_{k=1}^n a_k y_k \right\| + \varepsilon_n |a_{n+1}| \\ &\le (1 + \varepsilon_n) \left(\left\| \sum_{k=1}^n a_k y_k \right\|^p + |a_{n+1}|^p \right)^{1/p} \\ &\quad + \varepsilon_n \left\| \sum_{k=1}^n a_k y_k \right\| + \varepsilon_n |a_{n+1}|, \end{aligned}$$

and by induction hypothesis and since $C_n \ge 1$ this last expression may be be estimated by

$$\le (1 + \varepsilon_n)\, C_n \left(\sum_{k=1}^{n+1} |a_k|^p \right)^{1/p} + 2\varepsilon_n C_n \left(\sum_{k=1}^{n+1} |a_k|^p \right)^{1/p} = C_{n+1} \left(\sum_{k=1}^{n+1} |a_k|^p \right)^{1/p} .$$

We end up with the upper ℓ^p-estimate

$$\left\| \sum_{k=1}^n a_k y_k \right\| \le \lambda \cdot \left(\sum_{k=1}^n |a_k|^p \right)^{1/p}$$

which holds for all $n \in \mathbb{N}$. For the proof of the lower ℓ^p-estimate we put

$$\begin{aligned} c_1 &= 1, \\ c_n &= \prod_{k=1}^{n-1} (1 - 3\varepsilon_k) \end{aligned}$$

so that $\inf_n c_n \geq \lambda^{-1}$. As before, we obtain inductively

$$\left\| \sum_{k=1}^{n} a_k y_k \right\| \geq c_n \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} :$$

$$\begin{aligned}
&(1+\varepsilon_n) \left\| \sum_{k=1}^{n+1} a_k y_k \right\| \\
\underset{(2\mathrm{b})}{\geq} \quad & \left\{ \left| \left\| K_{\alpha_n} \left(\sum_{k=1}^{n} a_k y_k \right) \right\| - \| K_{\alpha_n} (a_{n+1} y_{n+1}) \| \right|^p \right. \\
& \quad + \left. \left| \| (Id - K_{\alpha_n})(a_{n+1} y_{n+1}) \| - \left\| (Id - K_{\alpha_n}) \left(\sum_{k=1}^{n} a_k y_k \right) \right\| \right|^p \right\}^{1/p} \\
\geq \quad & \left\{ \left\| K_{\alpha_n} \left(\sum_{k=1}^{n} a_k y_k \right) \right\|^p + \| (Id - K_{\alpha_n})(a_{n+1} y_{n+1}) \|^p \right\}^{1/p} \\
& \quad - \left\{ \| K_{\alpha_n} (a_{n+1} y_{n+1}) \|^p + \left\| (Id - K_{\alpha_n}) \left(\sum_{k=1}^{n} a_k y_k \right) \right\|^p \right\}^{1/p} \\
& \quad \text{by the triangle inequality for the } \ell^p\text{-norm in } \mathbb{R}^2 \\
\underset{(*)(**)}{\geq} \quad & (1 - 2\varepsilon_n) \left(\left\| \sum_{k=1}^{n} a_k y_k \right\|^p + |a_{n+1}|^p \right)^{1/p} \\
\geq \quad & c_n (1 - 2\varepsilon_n) \left(\sum_{k=1}^{n+1} |a_k|^p \right)^{1/p} \\
& \quad \text{by induction hypothesis and since } c_n \leq 1 \\
\geq \quad & (1+\varepsilon_n) c_{n+1} \left(\sum_{k=1}^{n+1} |a_k|^p \right)^{1/p}.
\end{aligned}$$

This gives

$$\left\|\sum_{k=1}^{n} a_k y_k\right\| \geq \lambda^{-1} \cdot \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p}.$$

Thus our claim is proved.

We now complete the proof for $1 < p < \infty$. Let $(x_n^*) \subset X^*$ be a sequence biorthogonal to (y_n) with $\sup \|x_n^*\| < \infty$. By passing to a subsequence of (x_n^*) if necessary, we may suppose that $x_n^* \to x^*$ weakly, recall that X is reflexive (Proposition 5.2). Since $(x_n^* - x^*)$ remains biorthogonal to (y_n), we may even assume that $x_n^* \to 0$ weakly. Now X^* has the (M_{p^*})-property (Proposition 5.2 again), and in passing to a further subsequence we may, by the above, think of (x_n^*) as being equivalent to the unit vector basis of ℓ^{p^*}. But then $\overline{\mathrm{lin}}\,\{y_1, y_2, \ldots\}$ is complemented in X according to [447, Theorem 3.7].

The announced result for $p = \infty$ follows by first producing a λ-isometric copy of c_0 as in the first part, which must be complemented by Corollary III.4.7; recall Proposition 4.4.

(b) Again by Corollary III.4.7, we may suppose that X and, consequently, X^* are separable. As in the proof of part (a) we first extract a subsequence (y_n^*) of the x_n^* which is λ-equivalent to the unit vector basis of ℓ^1. More precisely, our construction yields a sequence of compact operators on X such that $K_n^* \to Id_{X^*}$ strongly and, for the ε_n defined in part (a),

$$\begin{aligned} \|K_n^* y_m^* - y_m^*\| &\leq \varepsilon_n \quad \forall n \geq m, \\ \|K_n^* y_m^*\| &\leq \varepsilon_n \quad \forall m > n. \end{aligned} \tag{$*$}$$

We wish to produce a subsequence of the y_n^* and a corresponding biorthogonal sequence in X^{**} equivalent to the unit vector basis of c_0. Once this is achieved, an appeal to [447, Proposition 3.5] yields the assertion of the present theorem.

We start with any bounded sequence (x_n^{**}) in X^{**} biorthogonal to (y_n^*). Now X^* is separable, hence there is a weak* convergent subsequence, say $x_{n_k}^{**} \to x^{**}$. Since $(x_{n_k}^{**} - x^{**})$ remains biorthogonal to $(y_{n_k}^*)$, which remains λ-equivalent to the unit vector basis of ℓ^1, there is no loss of generality in assuming that $x_n^{**} \to 0$ (weak*) from the outset.

Next we observe that $w^*\text{-}\lim_{j\to\infty}(K_j^{**} x_n^{**} - x_n^{**}) = 0$ for each n, and that, as a result of $(*)$,

$$|\langle K_j^{**} x_n^{**} - x_n^{**}, y_m^* \rangle| \leq \varepsilon_j \|x_n^{**}\| \qquad \forall j > n$$

uniformly in m. It follows that there is a sequence (j_n) of natural numbers such that

$$K_{j_n}^{**} x_n^{**} - x_n^{**} \xrightarrow{w^*} 0 \tag{1}$$

and

$$\left\| (K_{j_n}^{**} x_n^{**} - x_n^{**})|_{\mathrm{lin}\,\{y_1^*, y_2^*, \ldots\}} \right\| \leq 2^{-n}. \tag{2}$$

Let $\xi_n = K_{j_n}^{**} x_n^{**}$ and note that $\xi_n \in X$. Moreover, $\xi_n \to 0$ weakly by (1), and $\inf \|\xi_n\| > 0$ by (2). By part (a), we may suppose that (ξ_n) is λ-equivalent to the unit vector basis of c_0. Finally we apply the Hahn-Banach theorem to obtain a sequence (ξ_n^{**}) such that $\|\xi_n - \xi_n^{**}\| \leq 2^{-n}$ and $\langle \xi_n - x_n^{**}, y_m^* \rangle = \langle \xi_n - \xi_n^{**}, y_m^* \rangle$ for all m. Consequently, the sequence (ξ_n^{**}) is biorthogonal to (y_n^*), and a straightforward calculation shows that it is equivalent to the unit vector basis of c_0.

This completes the proof of Theorem 5.10. □

Let us draw some consequences of the results presented so far.

Corollary 5.11 *If X has (M_p) and $Y \subset X$ is infinite dimensional, then, for every $\varepsilon > 0$, Y contains a $(1+\varepsilon)$-copy of ℓ^p which is complemented in X if $1 < p < \infty$ and of c_0 if $p = \infty$.*

PROOF: If $p < \infty$, then Y is reflexive (Proposition 5.2) and hence, by the Eberlein-Smulyan theorem, contains a normalized weakly null sequence. The assertion now follows from Theorem 5.10. If $p = \infty$ and Y were reflexive, then the same argument would produce a c_0-copy inside Y, which is absurd. Hence Y is a nonreflexive M-embedded space, and the result follows from Corollary III.4.7(e). (For $p = \infty$ one might also invoke Rosenthal's ℓ^1-theorem to show that there exists a normalized weakly null sequence and apply Theorem 5.10 directly.) $\square$

Corollary 5.12

(a) *If X has (M_p) and E is a subspace or quotient with the metric compact approximation property, then E has (M_p).*

(b) *If X and Y have (M_p), then so has $X \oplus_p Y$.*

PROOF: (a) This follows from Theorem 4.19 and the definition of the (M_p)-property.
(b) If (K_α) (resp. (L_β)) are chosen according to Theorem 5.3 for X resp. Y, then

$$\begin{pmatrix} K_\alpha & 0 \\ 0 & L_\beta \end{pmatrix}$$

works for $X \oplus_p Y$. $\square$

Corollary 5.13 *An infinite dimensional Banach space has at most one of the properties (M_p).*

PROOF: This is implied by the fact that the ℓ^p-spaces are totally incomparable [420, Theorem I.2.7] and Corollary 5.11. $\square$

Corollary 5.14 *Let X and Y be infinite dimensional Banach spaces and suppose that X belongs to the (M_{p_1})- and Y to the (M_{p_2})-class, where $1 < p_1, p_2 \le \infty$.*

(a) *If $p_1 \le p_2$, then $K(X,Y)$ is an M-ideal in $L(X,Y)$.*

(b) *If $p_1 > p_2$, then $K(X,Y) = L(X,Y)$.*

PROOF: (a) This is a special case of Corollary 4.18. Alternatively, this follows by performing a calculation similar to the one in the proof of Example 4.1; also recall Proposition 5.6 in the case $p_2 = \infty$.
(b) We first deal with the case $p_1 < \infty$. Assume that $T \in L(X,Y)$ is not compact. Then, by the reflexivity of X (Proposition 5.2), there is a sequence (x_n) in X with $x_n \to 0$ weakly, yet $\|Tx_n\| \ge \alpha > 0$ for all $n \in \mathbb{N}$. Applying Theorem 5.10 we are in a position to suppose that both sequences (x_n) and (Tx_n) are equivalent to the respective unit vector bases of ℓ^{p_1} and ℓ^{p_2}, and we derive that the identity operator maps ℓ^{p_1} continuously into ℓ^{p_2}, which is clearly absurd.
The argument for $p_1 = \infty$ is similar, when we pass to the adjoint operator. $\square$

In the following we will show how our knowledge on the classes (M_p) can be used in order to characterise smooth points of some spaces of bounded operators. Note that the following applies to ℓ^p-spaces and, most particularly, to Hilbert spaces.
Recall that a smooth point x of the unit sphere of a Banach space X is defined by the requirement that $x^*(x) = 1$ for a uniquely determined $x^* \in B_{X^*}$. In this case the norm of X is Gâteaux differentiable at x with derivative x^*.

Theorem 5.15 *Suppose that X and Y are infinite dimensional and separable Banach spaces belonging to the classes (M_p) and (M_q) for some $1 < p, q \le \infty$. Then T is a smooth point of the unit sphere of $L(X,Y)$ if and only if*

(1) *the essential norm of T, i.e. the number $\|T\|_e = \inf_{K\in K(X)} \|T - K\|$, is strictly less than one,*
(2) *there is exactly one y_0^* in the unit ball of Y^* (up to multiplication with scalars of modulus 1) for which $\|T^*(y_0^*)\| = 1$,*
(3) *the point $T^*y_0^*$ is smooth.*

Before we come to the proof of this theorem, let us give an example. Put $X = Y = c_0$ and represent a norm one operator $T \in L(c_0)$ by a matrix (a_{ij}). Since T^* attains its norm at an extreme point of B_{ℓ^1}, there is, for a smooth point T, exactly one row $(a_{i_0 j})_j$ with $\sum_{j=1}^\infty |a_{i_0 j}| = 1$. Furthermore, condition (1) of the above statement is equivalent to the fact that there is $\varepsilon > 0$ with $\sum_{j=1}^\infty |a_{ij}| < 1 - \varepsilon$ for all $i \ne i_0$. Finally, a point $(\lambda_n) \in \ell^1$ is smooth iff $\lambda_n \ne 0$ for all n. So we have found:

Corollary 5.16 *A norm one operator $T = (a_{ij})$ on c_0 is a smooth point of the unit sphere if and only if there is $i_0 \in \mathbb{N}$ such that $a_{i_0 j} \ne 0$ for all j, $\sum_{j=1}^\infty |a_{i_0 j}| = 1$, and, for some $\varepsilon > 0$, $\sum_{j=1}^\infty |a_{ij}| < 1 - \varepsilon$ for all $i \ne i_0$.*

We are going to prepare the proof of Theorem 5.15 by two lemmas. In the following, we write $x + J$ for the equivalence class in X/J generated by x.

Lemma 5.17 *Suppose that J is an M-ideal in the Banach space X and that X/J does not have any smooth points. Then an element x in the unit sphere of X is smooth if and only if*

(1) $\|x + J\| < 1$,
(2) *there is precisely one $j^* \in B_{J^*}$ with $j^*(x) = 1$.*

PROOF: Suppose that x is smooth, has norm one, and that $p(x) = 1$ for one (and only one) $p \in \operatorname{ex} B_{X^*}$. By Lemma I.1.5, there are two possible cases: If $p \in \operatorname{ex} B_{J^\perp}$, then $x + J$ would have norm one and would consequently be a smooth point of the unit ball of X/J. Since this is not possible, we anyway end up with the case where $p \in \operatorname{ex} B_{J^*}$. This proves (2), and the above argument also shows that $|p(x)| < 1$ for all $p \in \operatorname{ex} B_{J^\perp}$, and hence $\|x + J\| < 1$.
Conversely, suppose that the above conditions are satisfied for x. Put $F = \{\varphi \in X^* \mid \varphi(x) = \|\varphi\| = 1\}$. Then X is a weak* closed face of B_{X^*} and, again by Lemma I.1.5,

$$\operatorname{ex} F = (F \cap \operatorname{ex} B_{J^\perp}) \cup (F \cap \operatorname{ex} B_{J^*}).$$

Since $F \cap \operatorname{ex} B_{J^\perp} = \emptyset$ as a result of $\|x + J\| < 1$, we must have $\operatorname{ex} F = \{j^*\}$, and the lemma is proven. □

Lemma 5.18 *Suppose that X and Y are separable and belong to classes (M_p) and (M_q) for some $1 < p, q \leq \infty$. Then $L(X,Y)/K(X,Y)$ has no smooth point.*

PROOF: By Corollary 5.14, $K(X,Y)$ is an M-ideal in $L(X,Y)$. As a consequence, $K(X,Y)$ is proximinal in $L(X,Y)$ (Proposition II.1.1), and we may suppose that for any fixed $T \in L(X,Y)$ with $\|T\|_e = 1$ we have $\|T\| = 1$ as well. (Note that the essential norm of T is the norm of the equivalence class $T + K(X,Y)$.) Furthermore, by Proposition 4.7 and separability, we find sequences (x_n) in S_X or (y_n^*) in S_{Y^*} converging to zero in the weak or, respectively, weak* topology, such that $\lim_n \|Tx_n\| = 1$ or $\lim_n \|T^* y_n^*\| = 1$. Since Y^* either belongs to the class (M_{p^*}), where, as usual $1/p + 1/p^* = 1$, or is the dual of a space belonging to (M_∞), the following argument works in either of the above cases, and we may suppose that $\lim_n \|Tx_n\| = 1$. Clearly, (x_n) cannot converge in norm and therefore, by Theorem 5.10, we may think of (x_n) as being equivalent to the standard basis of ℓ^p or c_0 and such that there is a projection P_0 from X onto $\overline{\text{lin}}\,\{x_n \mid n \in \mathbb{N}\}$. Let $P_1 : \overline{\text{lin}}\,\{x_n \mid n \in \mathbb{N}\} \to \overline{\text{lin}}\,\{x_{2n} \mid n \in \mathbb{N}\}$ be the canonical projection and put $P = P_1 P_0$. Then,

$$d(TP, \text{lin}\,\{T(Id - P)\} \cup K(X,Y)) \leq \|TP + T(Id - P)\| = 1.$$

On the other hand, for each scalar λ and any $K \in K(X,Y)$,

$$\|TP - (\lambda T(Id - P) + K)\| \geq \limsup_n \|Tx_{2n} - Kx_{2n}\| = 1$$

and therefore, $d(TP, \text{lin}\,\{T(Id-P)\} \cup K(X,Y)) = 1$. Similarly, we find that $d(T(Id-P), \text{lin}\,\{TP\} \cup K(X,Y)) = 1$. To conclude the proof, choose functionals ψ_1, ψ_2 of unit norm with

$$1 = \psi_1(TP) = \psi_2(T(Id - P))$$

and

$$0 = \psi_2|_{\text{lin}\,\{TP\} \cup K(X,Y)} = \psi_1|_{\text{lin}\,\{T(Id-P)\} \cup K(X,Y)}.$$

Then $\psi_1(T) = \psi_2(T) = 1$ and $\psi_1 \neq \psi_2$ whence $T + K(X,Y)$ cannot be smooth. $\square$

PROOF OF THEOREM 5.15:
By Lemmas 5.17 and 5.18, an operator T is smooth if and only if $\|T\|_e < \|T\|$ and there is exactly one functional ψ in $B_{K(X,Y)^*}$ norming T. Necessarily, ψ is an extreme point of the dual unit ball of $K(X,Y)$. The result then follows from the fact that, by Theorem 1.3, ψ is of the form $x^{**} \otimes y^*$, where $x^{**} \in \text{ex}\, B_{X^{**}}$ and $y^* \in \text{ex}\, B_{Y^*}$. $\square$

We finally use the results of this section to disprove the M-ideal property of $K(X)$ in some instances.

Proposition 5.19

(a) *Let $1 \leq p \leq \infty$. Then $K(X)$ is not an M-ideal in $L(X)$ if $p \neq 2$ and*
  - (1) *$X = L^p[0,1]$,*
  - (2) *$X = c_p$, the Schatten class,*
  - (3) *$X = H^p$, the p-th Hardy space.*

(b) *$K(X)$ is an M-ideal in $L(X)$ for an L^1-predual space X if and only if $X = c_0(I)$ for some index set I.*

PROOF: (a) The borderline cases $p = 1$ and $p = \infty$ are handled by Proposition 4.4 (except for c_∞). So let $1 < p < \infty$. To prove (1) it is enough to rule out that X have (M_p), since $X \cong X \oplus_p X$. In fact, X has a hilbertian subspace (as a consequence of the Khintchin inequalities, see e.g. [421, Th. 2.b.3]) so that the conclusion of Corollary 5.11 fails.

Unfortunately, in (2) $X \cong X \oplus_p X$ is false. However, c_p has a 1-complemented subspace Y isometric to $\ell^2 \oplus_p \ell^2$ for $1 < p \le \infty$ [31]. If $K(X)$ were an M-ideal in $L(X)$ then $K(Y)$ would be an M-ideal in $L(Y)$ (Proposition 4.2), and Corollary 5.11 furnishes the contradiction that ℓ^2 should contain a copy of ℓ^p.

The proof of part (3) does not depend on the theory of (M_p)-spaces, but uses Corollary 4.23. Let us suppose that $K(H^p)$ is an M-ideal in $L(H^p)$ for some p. Let $\varepsilon > 0$. Using Corollary 4.23 embed H^p isometrically into a Banach space X with a $(1+\varepsilon)$-unconditional basis (e_n). Denote by (e_n^*) the corresponding coefficient functionals and pick $N \in \mathbb{N}$ such that $\left\|\sum_{k>N} \langle 1, e_k^*\rangle e_k\right\| \le \varepsilon$. Now consider the functions $f_n(z) = z^n + z^{2n}$ and note that $f_n \to 0$ weakly by the Riemann-Lebesgue lemma. Hence there is $n_0 \in \mathbb{N}$ such that $\left\|\sum_{k=1}^N \langle f_n, e_k^*\rangle e_k\right\| \le \varepsilon$ for $n \ge n_0$. In the following calculation we use the symbol $a \approx b$ to indicate that a and b differ by a term which tends to 0 as $\varepsilon \to 0$. We have for $n \ge n_0$

$$
\begin{aligned}
\|1 + f_n\| &\approx \left\|\sum_{k=1}^N \langle 1, e_k^*\rangle e_k + \sum_{k>N} \langle f_n, e_k^*\rangle e_k\right\| \\
&\approx \left\|\sum_{k=1}^N \langle 1, e_k^*\rangle e_k - \sum_{k>N} \langle f_n, e_k^*\rangle e_k\right\| \\
&\approx \|1 - f_n\|,
\end{aligned}
$$

where we used that (e_n) is $(1+\varepsilon)$-unconditional. Since $f_n(z) = f_1(z^n)$, the f_n are identically distributed and thus $\|1 \pm f_n\|_p = \|1 \pm f_1\|_p$ for all n. Consequently $\|1 + f_1\|_p = \|1 - f_1\|_p$ which enforces $p = 2$.

(b) This is an immediate consequence of Proposition 4.4, Proposition III.2.7 and of course Corollary 5.4. □

A modification of the above argument for H^p shows that $K(C(\mathbb{T})/A)$ is not an M-ideal in $L(C(\mathbb{T})/A)$. Indeed, otherwise, by a strengthening of Theorem 4.21 proved in [396], C/A is isometric to a $(1+\varepsilon)$-complemented subspace of a space with a $(1+\varepsilon)$-unconditional *shrinking* finite dimensional Schauder decomposition, and thus $H_0^1 = (C/A)^*$ is isometric to a subspace of a space with a $(1+\varepsilon)$-unconditional basis, for every $\varepsilon > 0$. A reasoning similar to the above shows that this is impossible. (Note that C/A and H^1 have unconditional bases [640].)

We shall show in the next section that L^p and c_p cannot even be renormed so that the compact operators become an M-ideal. Since L^p is isomorphic to H^p for $1 < p < \infty$ [422, Prop. 2.c.17], this result will contain (3) of part (a) as a special case.

VI.6 Banach spaces for which $K(X)$ forms an M-ideal: ℓ^p-subspaces and renormings

We now return to the investigation of the general case of Banach spaces X for which $K(X)$ is an M-ideal in $L(X)$. The main result of the first part of this section will be that those spaces X necessarily contain good copies of ℓ^p for some $p > 1$ or of c_0 (Theorem 6.4). In the second part we will treat the problem of renorming a Banach space X so that $K(X)$ becomes an M-ideal. For all this the equivalences proved in Theorem 4.17 and in particular the property (M) (Definition 4.12) are of crucial importance.
In this section we will exclusively deal with separable Banach spaces, in which case it suffices to work with sequences in Definition 4.12 and Theorem 4.17 as an inspection of its proof shows.
We first present a new class of Banach spaces for which the compact operators form an M-ideal. These spaces will be useful throughout this section and will eventually be identified as isomorphs of Orlicz sequence spaces (Proposition 6.11). Let N be a norm on $\mathbb{K}^2$ such that

$$N(1,0) = 1 \qquad \text{and} \qquad N(\alpha,\beta) = N(|\alpha|,|\beta|) \quad \forall \alpha,\beta \in \mathbb{K};$$

i.e., N is an absolute norm. Define inductively norms on $\mathbb{K}^d$ by

$$N(\xi_0,\dots,\xi_d) = N(N(\xi_0,\dots,\xi_{d-1}),\xi_d)$$

and denote by $\widetilde{\Lambda}(N)$ the vector space of all sequences $\xi = (\xi_0,\xi_1,\dots)$ such that

$$\|\xi\|_N = \sup_d N(\xi_0,\dots,\xi_d) < \infty.$$

Equipped with the norm $\|\,.\,\|_N$, $\widetilde{\Lambda}(N)$ is a Banach space. We denote by $\Lambda(N)$ the closed linear span of the unit vectors $e_0, e_1, e_2, \dots$.

Proposition 6.1

(a) *The unit vectors form a* 1*-unconditional basis of* $\Lambda(N)$*.*
(b) *If* (e_{n_k}) *is a subsequence of* (e_n)*, then* $\overline{\text{lin}}\,\{e_{n_k} \mid k \in \mathbb{N}\}$ *is canonically isometric with* $\Lambda(N)$*.*
(c) $\Lambda(N)$ *has property* (M)*.*
(d) $\Lambda(N)$ *is an* M*-ideal in* $\widetilde{\Lambda}(N)$*.*
(e) *If* $\Lambda(N)^*$ *is separable, then* $K(\Lambda(N))$ *is an* M*-ideal in* $L(\Lambda(N))$*.*

PROOF: (a), (b) and (c) follow directly from the definition of the norm in $\Lambda(N)$; note that the finitely supported sequences are dense. (d) is an immediate consequence of the 3-ball property (Theorem I.2.2). Finally, (e) is implied by Theorem 4.17(vi): The coordinate projections P_n fulfill $\|Id - 2P_n\| = 1$ by (a), and $P_n^* \to Id_{\Lambda(N)^*}$ strongly (i.e., the unit vector basis is shrinking), since $\Lambda(N)^*$ is separable, see [421, Th. 1.c.9]. □

We now address the problem of ℓ^p-subspaces of Banach spaces for which $K(X)$ is an M-ideal in $L(X)$. Let us introduce some terminology. A *type* on a Banach space X is a function of the form $\tau(x) = \lim_{n\to\infty} \|x + x_n\|$ for some bounded sequence (x_n). We say that (x_n) generates the type τ. If $x_n \to 0$ weakly, τ is called a *weakly null type*, and τ is called nontrivial if (x_n) does not converge strongly. A diagonal argument shows that every bounded sequence in a separable Banach space contains a subsequence generating a type.

In a Banach space with property (M) a weakly null type is a function of $\|x\|$. Consequently, if (x_n) generates a weakly null type τ on such a space and $\|x\| = 1$, then $N(\alpha, \beta) = \lim_{n\to\infty} \|\alpha x + \beta x_n\|$ is a well-defined absolute seminorm on $\mathbb{K}^2$ which does not depend on the particular choice of $x \in S_X$; and it is a norm if τ is nontrivial.

Lemma 6.2 *Suppose X is a separable Banach space with property (M) and (x_n) is a weakly null sequence generating a nontrivial type. Let $N(\alpha, \beta) = \lim_{n\to\infty} \|\alpha x + \beta x_n\|$ for some $x \in S_X$. Then, whenever $\|u\| = 1$ and $\varepsilon > 0$, there is a subsequence (y_n) of (x_n) such that*

$$(1+\varepsilon)^{-1} \left\| \xi_0 u + \sum_{i=1}^{\infty} \xi_i y_i \right\| \le \|\xi\|_N \le (1+\varepsilon) \left\| \xi_0 u + \sum_{i=1}^{\infty} \xi_i y_i \right\|$$

for all finitely nonzero sequences $\xi = (\xi_0, \xi_1, \ldots)$. In particular, there is an absolute norm N such that X contains, for every $\varepsilon > 0$, a subspace which is $(1+\varepsilon)$-isomorphic to $\Lambda(N)$.

PROOF: We just sketch the construction of the y_n which proceeds inductively. It is clearly enough to gain the desired inequalities for $|\xi_i| \le 1$. In the first step we pick an index n_1 such that $\|\xi_0 u + \xi_1 x_{n_1}\|$ is close to $N(\xi_0, \xi_1)$ for all $|\xi_0|, |\xi_1| \le 1$. This is possible by definition of N. (The Arzelà-Ascoli theorem takes care of the uniform, not only pointwise approximation.) Let $y_1 = x_{n_1}$. Then pick an index $n_2 > n_1$ such that $\|\xi_0 u + \xi_1 y_1 + \xi_2 x_{n_2}\|$ is close to $N(N(\xi_0, \xi_1), \xi_2) = N(\xi_0, \xi_1, \xi_2)$. Let $y_2 = x_{n_2}$, etc. The strategy how to prove this lemma should now be clear. □

Lemma 6.3 *Let N be an absolute norm on $\mathbb{K}^2$ with $N(1, 0) = 1$. Then there is some p, $1 \le p \le \infty$, such that, for every $\varepsilon > 0$, $\Lambda(N)$ contains a subspace $(1 + \varepsilon)$-isomorphic to ℓ^p if $p < \infty$ and to c_0 if $p = \infty$.*

PROOF: The proof of this result relies on a difficult theorem due to Krivine ([382, Th. 0.1] and [391]) which asserts in our case that, given $\delta > 0$ and $n \in \mathbb{N}$, some normalised blocks $b_1, \ldots, b_n$ of the unit vector basis of $\Lambda(N)$ with increasing supports are $(1+\delta)$-equivalent to the standard basis in $\ell^p(n)$. It follows that there is a normalised block basic sequence (u_n) satisfying

$$\lim_{n\to\infty} \|\alpha u_{n-1} + \beta u_n\| = \|(\alpha, \beta)\|_p \qquad \forall \alpha, \beta \in \mathbb{K}.$$

But since $\Lambda(N)$ has property (M) by Proposition 6.1, any finitely supported ξ lies eventually "left of" u_{2n}; hence by definition of the norm in $\Lambda(N)$

$$\lim_{n\to\infty} \|\alpha \xi + \beta u_n\| = \|(\alpha, \beta)\|_p \qquad \forall \alpha, \beta \in \mathbb{K}$$

whenever $\|\xi\| = 1$. It follows from Lemma 6.2 that some subsequence of (u_{2n}) spans a $(1 + \varepsilon)$-copy of ℓ^p resp. c_0. □

We now come to the first major result of this section.

Theorem 6.4 *Let X be a separable Banach space with property (M) and Y a closed infinite dimensional subspace of X. Then there is some p, $1 \leq p \leq \infty$, such that, for every $\varepsilon > 0$, Y contains a subspace $(1+\varepsilon)$-isomorphic to ℓ^p if $p < \infty$ and to c_0 if $p = \infty$. In particular, this conclusion holds if $K(X)$ is an M-ideal in $L(X)$ and Y is an infinite dimensional subspace of a quotient of X, and then necessarily $1 < p \leq \infty$.*

PROOF: Y has property (M), too. If every weakly null sequence in Y converges strongly, i.e., Y has the Schur property, then Y contains a $(1+\varepsilon)$-copy of ℓ^1 by Rosenthal's ℓ^1-theorem and James' distortion theorem [421, Th. 2.e.5 and Prop. 2.e.3]. Otherwise, there is a normalised weakly null sequence in Y, and the desired conclusion follows from Lemma 6.2 and Lemma 6.3.
If $K(X)$ is an M-ideal, then every subspace of a quotient of X has (M), see the proof of Theorem 4.19, and $p = 1$ is ruled out by Corollary 4.5. □

We will later need the following corollary.

Corollary 6.5 *Let X be a separable Banach space with property (M) not containing a copy of ℓ^1. Then there exists p, $1 < p \leq \infty$, and a weakly null sequence (x_n) such that whenever $\|u\| = 1$,*

$$\lim_{n\to\infty} \|\alpha u + \beta x_n\| = \|(\alpha, \beta)\|_p \qquad \forall \alpha, \beta \in \mathbb{K}.$$

PROOF: Choose p according to Theorem 6.4, and let E be a subspace of X isomorphic to ℓ^p (resp. c_0). Applying Theorem 6.4 again we obtain that for each n, E contains a further subspace E_n which is $(1+\varepsilon_n)$-isomorphic to ℓ^p (resp. c_0), where $\varepsilon_n \to 0$ is arbitrary. Consequently each E_n contains a weakly null sequence $(x_{nk})_k$ which is $(1+\varepsilon_n)$-equivalent to the unit vector basis of ℓ^p (resp. c_0) and generates a nontrivial weakly null type. Using the assumption that X has (M) we deduce from this that for every $u \in X$ with $\|u\| = 1$,

$$(1+\varepsilon_n)^{-1} \|(\alpha, \beta)\|_p \leq \lim_{k\to\infty} \|\alpha u + \beta x_{nk}\| = (1+\varepsilon_n) \|(\alpha, \beta)\|_p$$

for all $\alpha, \beta \in \mathbb{K}$. We then obtain (x_n) by passing to a suitable diagonal subsequence (x_{n,k_n}). □

We are now in a position to single out the (M_p)-spaces, studied in the previous section, among those Banach spaces for which $K(X)$ is an M-ideal in $L(X)$.
In [383] the class of separable stable Banach spaces was introduced, see also [242]. A separable Banach space X is called stable if, whenever (x_m) and (y_n) are bounded sequences generating types, then the iterated limits $\lim_n \lim_m \|x_m + y_n\|$ and $\lim_m \lim_n \|x_m + y_n\|$ both exist and coincide. Among the stable Banach spaces are the spaces ℓ^r and L^r for $1 \leq r < \infty$ and their subspaces while c_0 is not stable [383]. Also, the quotient spaces of ℓ^r are stable for $1 < r < \infty$ [519]. We remind the reader that the stability of a Banach space is an isometric notion that will generally be spoilt by passing to an equivalent norm. It is a celebrated result due to Krivine and Maurey [383] that stable Banach spaces contain arbitrarily good copies of ℓ^p for some $p \in [1, \infty)$.

Theorem 6.6 *Let X be a separable Banach space such that $K(X)$ is an M-ideal in $L(X)$. Then the following are equivalent:*

(i) *X has property (M_p) for some $1 < p < \infty$.*
(ii) *X is stable.*
(iii) *There exists p, $1 < p < \infty$, such that for every normalised weakly null sequence (x_n) generating a weakly null type and every $u \in X$ with $\|u\| = 1$,*
$$\lim_{n\to\infty} \|\alpha u + \beta x_n\| = (|\alpha|^p + |\beta|^p)^{1/p}.$$
(iv) *There exists p, $1 < p < \infty$, such that for every weakly null sequence (x_n) and every $u \in X$,*
$$\limsup \|u + x_n\|^p = \|u\|^p + \limsup \|x_n\|^p.$$

Proof: (i) ⇒ (ii): To begin with, we make the following observation. Let (K_i) be as in Theorem 5.3. (By the way, since X is separable, this net may be chosen to be a sequence.) We fix an index i for which (2a) and (2b) from that theorem are fulfilled. Now suppose x_0 and y_0 are given such that
$$\begin{aligned} \|K_i x_0 - x_0\| &\leq \varepsilon, \\ \|K_i y_0\| &\leq \varepsilon. \end{aligned}$$
Then
$$\big|\ \|K_i x_0 + (Id - K_i) y_0\| - (\|x_0\|^p + \|y_0\|^p)^{1/p}\ \big| \leq f(\varepsilon) \tag{1}$$
where $f(\varepsilon)$ is a quantity which tends to 0 as ε tends to 0. ($f(\varepsilon)$ depends on M if M is an upper bound for $\|x_0\|$ and $\|y_0\|$.)
To prove this claim, we note for $z_i = K_i x_0 + (Id - K_i) y_0$ from (2b) that
$$\|z_i\| \geq \frac{1}{1+\varepsilon} (\|K_i z_i\|^p + \|(Id - K_i) z_i\|^p)^{1/p}$$
and that
$$\begin{aligned} \|K_i z_i - x_0\| &\leq 4\varepsilon \\ \|(Id - K_i) z_i - y_0\| &\leq 4\varepsilon \end{aligned}$$
by the triangle inequality so that (1) follows as a consequence of this and of (2a).
We now enter the main part of the proof. Thus, let (x_m) and (y_n) be given as above. Since X is reflexive, we may assume that (x_m) and (y_n) are weakly convergent (cf. [383, p. 276]), say
$$x_m \xrightarrow{w} \xi \qquad \text{and} \qquad y_n \xrightarrow{w} \eta.$$
Let $m \in \mathbb{N}$. Given $\varepsilon > 0$, fix i such that (2a) and (2b) are fulfilled for i and such that
$$\|K_i(x_m + \eta) - (x_m + \eta)\| \leq \varepsilon$$
holds. Then find n_0 with
$$\|K_i(y_n - \eta)\| \leq \varepsilon \qquad \text{for all } n \geq n_0.$$
It follows for these n
$$\big|\ \|x_m + y_n\| - \|K_i(x_m + \eta) + (Id - K_i)(y_n - \eta)\|\ \big| \leq 2\varepsilon,$$

hence by (1)

$$\lim_n \|x_m + y_n\| \;=\; (\|x_m + \eta\|^p \;+\; \beta^p)^{1/p}$$

where $\beta \;=\; \lim_n \|y_n - \eta\|$. Likewise,

$$\lim_m \|x_m + \eta\| \;=\; (\alpha^p \;+\; \|\xi + \eta\|^p)^{1/p}$$

(with $\alpha \;=\; \lim_m \|x_m - \xi\|$) so that by symmetry

$$\begin{aligned} \lim_m \lim_n \|x_m + y_n\| &= (\alpha^p \;+\; \|\xi + \eta\|^p \;+\; \beta^p)^{1/p} \\ &= \lim_n \lim_m \|x_m + y_n\|. \end{aligned}$$

(ii) $\Rightarrow$ (iii): Since X does not contain a copy of ℓ^1, we first mention that X must be reflexive; otherwise, X contains arbitrarily good copies of c_0 (Corollary III.3.7), and the conclusion that c_0 is stable would follow. Corollary 6.5 provides us with a normalised weakly null sequence (y_n) such that, for some $1 < p < \infty$ and all $\|u\| = 1$,

$$\lim_{n\to\infty} \|\alpha u + \beta y_n\| = \|(\alpha, \beta)\|_p \qquad \forall \alpha, \beta \in \mathbb{K}.$$

Now let (x_n) be a normalised weakly null sequence generating a nontrivial type. Since X has (M) by assumption, we get for $\|u\| = 1$

$$\begin{aligned} \lim_{m\to\infty} \|\alpha u + \beta x_m\| &= \lim_{n\to\infty} \lim_{m\to\infty} \|\alpha y_n + \beta x_m\| \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \|\alpha y_n + \beta x_m\| \\ &= (|\alpha|^p + |\beta|^p)^{1/p}. \end{aligned}$$

(iii) $\Leftrightarrow$ (iv): This follows by a simple subsequence argument.

(iii) $\Rightarrow$ (i): We shall verify condition (vi) of Theorem 4.17 for $X \oplus_p X$. If (x_n) and (y_n) generate nontrivial weakly null types on X, then

$$\begin{aligned} \lim_{n\to\infty} \|(x, y) + (x_n, y_n)\|^p &= \lim_{n\to\infty} (\|(x + x_n\|^p + \|y + y_n\|^p) \\ &= \|(x, y)\|^p + \lim_{n\to\infty} \|x_n + y_n\|^p \end{aligned}$$

by assumption. Hence $X \oplus_p X$ has (M) by Lemma 4.13(ii). Further, if (K_n) is a shrinking compact approximation of the identity on X with $\lim \|Id - 2K_n\| = 1$, then $K_n \oplus K_n$ serves the same purpose on $X \oplus_p X$. $\square$

To cover the case $p = \infty$ one has to consider the class of weakly stable Banach spaces [32], defined by the requirement that

$$\lim_n \lim_m \|x_m + y_n\| = \lim_m \lim_n \|x_m + y_n\|$$

whenever (x_n) and (y_n) generate types and are weakly convergent. The above proof then carries over to include the case of (M_∞)-spaces.

We now investigate renormings. We first present a sufficient condition for $K(X, Y)$ to be an M-ideal in $L(X, Y)$.

Lemma 6.7 *Let X and Y be separable Banach spaces and let $1 < q \le p < \infty$. Suppose X and Y admit shrinking compact approximations of the identity (K_n) resp. (L_n) such that*

$$\limsup(\|K_n x\|^q + \|(Id - K_n)x\|^q)^{1/q} \le \|x\|, \tag{1}$$
$$\limsup \|L_n y_1 + (Id - L_n)y_2\| \le (\|y_1\|^p + \|y_2\|^p)^{1/p}, \tag{2}$$

where the lim sup *conditions are supposed to hold uniformly on bounded sets. Then $K(X,Y)$ is an M-ideal in $L(X,Y)$.*

PROOF: The proof is the same as in (the easy half of) Theorem 5.3. □

The lemma extends to $p = \infty$ in the obvious way.
To discuss the scope of this lemma, we introduce the temporary notation that a space fulfilling the assumptions made on X resp. Y above has the *lower q-* resp. *upper p-property.* We first observe that a reflexive space X has the lower q-property if and only if X^* has the upper q^*-property where $1/q + 1/q^* = 1$. This follows since

$$(\|K_n x\|^q + \|x - K_n x\|^q)^{1/q} \le (1+\varepsilon)\|x\| \qquad \forall x \in X \tag{3}$$

is equivalent to

$$\|K_n^* x^* + (Id - K_n^*)y^*\| \le (1+\varepsilon)(\|x^*\|^{q^*} + \|y^*\|^{q^*})^{1/q^*} \qquad \forall x^* \in X^*. \tag{4}$$

In fact, (3) means that the operator

$$x \ \mapsto \ (K_n x, (Id - K_n)x)$$

from X to $X \oplus_q X$ has norm $\le 1+\varepsilon$, and (4) expresses the fact that its adjoint has norm $\le 1+\varepsilon$.
Thus, in order to find examples where Lemma 6.7 applies we need only worry about the upper p-property; examples with the lower q-property can then be constructed by duality. One can also show, using the convex combinations technique, that the upper p-property is inherited by subspaces and quotients with the compact approximation property. Moreover, a computation reveals that the ℓ^r-sum of spaces $Y_1, Y_2, \ldots$ with the upper $p_1, p_2, \ldots$-properties enjoys the upper p-property provided $p \le \inf\{r, p_1, p_2, \ldots\}$.
An effective way to produce upper p-spaces is to look for reflexive sequence spaces where the unit vectors form a Schauder basis and the inequality

$$\|x_1 + x_2\| \le (\|x_1\|^p + \|x_2\|^p)^{1/p}$$

holds for disjointly supported sequences. It is clear that under this hypothesis the sequence of coordinate projections is an approximation of the identity with the required features. Examples include besides the ℓ^p-spaces the Lorentz spaces $d(w,p)$ and, more generally, the p-convexification of a sequence space whose unit vector basis is 1-unconditional; in particular, the p-convexified Tsirelson space $T^{(p)}$ [115, p. 116] can be considered. (Of course, finite dimensional decompositions can be considered instead of bases.)
It follows for instance that $K(\ell^2, T^{(2)})$ is an M-ideal in $L(\ell^2, T^{(2)})$. On the other hand, the compact operators on $T^{(2)}$ do not form an M-ideal in $L(T^{(2)})$ since $T^{(2)}$ does not contain

any ℓ^p. In this respect the operators on ℓ^2 behave very differently from the operators on the "weak Hilbert space" [501] $T^{(2)}$. We refer to [115] for exhaustive information on Tsirelson-like spaces and for a summary of [501].

Lemma 6.7 also enables us to provide an answer to the question if the L^p-spaces can be renormed so that $K(L^p, L^q)$ becomes an M-ideal in $L(L^p, L^q)$. (Recall from Proposition 5.19 that $K(L^p)$ is not an M-ideal in $L(L^p)$ for the natural norm of $L^p = L^p[0,1]$, and see also Corollary 6.10 below.) We will be able to answer this affirmatively if $1 < p \le 2 \le q < \infty$.

Proposition 6.8 *If $1 < p \le 2 \le q < \infty$, then for some renormings X of $L^p[0,1]$ and Y of $L^q[0,1]$, $K(X,Y)$ is an M-ideal in $L(X,Y)$.*

Proof: Indeed, under these circumstances the Haar system is an unconditional basis, and we first renorm L^p and L^q so that the Haar basis (h_i) becomes 1-unconditional. Call these renormings X_1 and Y_1. Letting

$$\sum_{i=1}^{\infty} a_i h_i \succeq 0 \qquad \text{if and only if} \qquad a_i \ge 0 \ \ \forall i$$

induces the structure of a Banach lattice on each of X_1 and Y_1, and since Y_1 is isomorphic to L^q and $2 \le q < \infty$, Y_1 has type 2. Hence, the Banach lattice Y_1 can further be renormed to satisfy

$$\left\| \sum_{i=1}^{\infty} a_i h_i \right\| \le \left(\left\| \sum_{i=1}^{n} a_i h_i \right\|^2 + \left\| \sum_{i=n+1}^{\infty} a_i h_i \right\|^2 \right)^{1/2} \qquad \forall n \in \mathbb{N}$$

(cf. [422, p. 100 and Lemma 1.f.11]). Likewise, since L^p has cotype 2 for $p \le 2$ there is a renorming such that

$$\left(\left\| \sum_{i=1}^{n} a_i h_i \right\|^2 + \left\| \sum_{i=n+1}^{\infty} a_i h_i \right\|^2 \right)^{1/2} \le \left\| \sum_{i=1}^{\infty} a_i h_i \right\| \qquad \forall n \in \mathbb{N}.$$

If these renormings are called X resp. Y, then $K(X,Y)$ is an M-ideal in $L(X,Y)$ by Lemma 6.7. ☐

We refrain from formulating the abstract lemma behind Proposition 6.8 which involves reflexive Banach spaces with unconditional bases and type 2 resp. cotype 2.

We next wish to prove that unless $p = 2$ there is no renorming X of $L^p[0,1]$ for which $K(X)$ is an M-ideal in $L(X)$. This will turn out to be a corollary to the following result.

Proposition 6.9 *The space $\ell^p(\ell^r)$ for $1 < p, r < \infty$ has a renorming X so that $K(X)$ is an M-ideal in $L(X)$ if and only if $p = r$. In fact, $\ell^p(\ell^r)$ can be renormed to have (M) if and only if $p = r$.*

Proof: The "if"-part follows from Example 4.1. Now suppose $\ell^p(\ell^r)$ has an equivalent renorming $X = (\ell^p(\ell^r), \|\,.\,\|_M)$ for which $K(X)$ is an M-ideal in $L(X)$. A fortiori, X

has (M) in this case. Let X_k be the space of all sequences $(x_n) \in X$ such that $x_n = 0$ if $n \neq k$. Then $(X_k, \|\,.\,\|_M)$ is isomorphic to ℓ^r and, since $(X, \|\,.\,\|_M)$ has (M), we find by Corollary 6.5 normalised weakly null sequences $(u_{kn})_n \subset X_k$ such that

$$\lim_{n\to\infty} \|x + \beta u_{kn}\|_M = (\|x\|^r + |\beta|^r)^{1/r} \qquad \forall x \in X,\ \beta \in \mathbb{K}.$$

Thus for all $\alpha_1, \ldots, \alpha_k$,

$$\lim_{n_1\to\infty} \cdots \lim_{n_k\to\infty} \left\| \sum_{i=1}^k \alpha_i u_{in_i} \right\|_M = \left(\sum_{i=1}^k |\alpha_i|^r \right)^{1/r}.$$

But the left hand side is equivalent to $\left\|\sum_{i=1}^k \alpha_i u_{in_i}\right\|_{\ell^p(\ell^r)} = \left(\sum_{i=1}^k |\alpha_i|^p\right)^{1/p}$; and we get a contradiction unless $p = r$. □

The following corollary is the isomorphic version of Proposition 5.19. Recall that c_p denotes the Schatten class of operators on Hilbert space of index p.

Corollary 6.10 *Let $1 \leq p \leq \infty$.*

(a) *$L^p = L^p[0,1]$ has a renorming X such that $K(X)$ is an M-ideal in $L(X)$ if and only if $p = 2$.*

(b) *c_p has a renorming X such that $K(X)$ is an M-ideal in $L(X)$ if and only if $p = 2$.*

Proof: Both L^2 and c_2 are Hilbert spaces so that the compact operators form an M-ideal. Suppose now that $1 < p < \infty$ and that L^p has a renorming as stated. Since there are isomorphic embeddings (the first one results from the Khintchin inequality)

$$\ell^p(\ell^2) \hookrightarrow \ell^p(L^p) \hookrightarrow L^p(L^p) \cong L^p,$$

we deduce from Theorem 4.17 that $\ell^p(\ell^2)$ has a renorming satisfying (M). Now Proposition 6.9 yields $p = 2$. The argument for c_p is the same, since $\ell^p(\ell^2) \hookrightarrow c_p$ [31].
For c_∞ we observe that $c_0(\ell^2)$ embeds into c_∞, and if c_∞ had a renorming for which the compact operators are an M-ideal, then $c_0(\ell^2)$ could be renormed to have (M). An argument similar to the one in Proposition 6.9 shows that this is impossible.
For L^1, L^∞ and c_1 the same arguments as in Proposition 5.19 apply. □

We finally come to a positive result and resume the discussion of the spaces $\Lambda(N)$ introduced at the beginning of this section. Recall from Section III.1 the definition of the Orlicz sequence spaces h_M and ℓ_M.

Proposition 6.11

(a) *Every space $\widetilde{\Lambda}(N)$ (resp. $\Lambda(N)$) is isomorphic to an Orlicz sequence space ℓ_M (resp. h_M).*

(b) *Every Orlicz sequence space ℓ_M (resp. h_M) is isomorphic to a space $\widetilde{\Lambda}(N)$ (resp. $\Lambda(N)$) for some absolute norm N on $\mathbb{K}^2$.*

PROOF: (a) Let N be an absolute norm on $\mathbb{K}^2$ such that $N(1,0)=1$. Define $M(t) = N(1,t)-1$ for $t \geq 0$. Clearly, this is a continuous, increasing, convex function; however, it might be degenerate, meaning $M(t)=0$ for some $t>0$. We claim that $\widetilde{\Lambda}(N) = \ell_M$ as sets. A standard closed graph argument then shows that the two spaces are actually (canonically) isomorphic, and since $\Lambda(N)$, resp. h_M, is the closed linear span of the unit vectors, these spaces are canonically isomorphic, too.

We start with proving that $\widetilde{\Lambda}(N) \subset \ell_M$. Suppose that $\|\xi\|_N \leq 1$. Then $N(\xi_0,\ldots,\xi_k) \leq 1$ for all k; note that this expression increases with k. For simplicity of notation assume $\xi_0 \neq 0$ (otherwise switch to the smallest index k_0 where $\xi_{k_0} \neq 0$) so that $0 < N(\xi_0,\ldots,\xi_k) \leq 1$ for all k. For convenience we put $N(\xi_0) = |\xi_0|$. Then we obtain for all $k \geq 1$

$$\begin{aligned} N(\xi_0,\ldots,\xi_k) &= N(N(\xi_0,\ldots,\xi_{k-1}),\xi_k) \\ &= N(\xi_0,\ldots,\xi_{k-1})N\Big(1,\frac{\xi_k}{N(\xi_0,\ldots,\xi_{k-1})}\Big) \\ &\geq N(\xi_0,\ldots,\xi_{k-1})N(1,\xi_k) \\ &= N(\xi_0,\ldots,\xi_{k-1})(1+M(|\xi_k|)). \end{aligned}$$

Consequently, $N(\xi_0,\ldots,\xi_k) \geq \prod_{j=1}^k (1+M(|\xi_j|))|\xi_0|$ for each k, and $\prod_{j=1}^\infty (1+M(|\xi_j|)$ converges. Hence $\sum_{j=1}^\infty M(|\xi_j|)$ converges, and $\xi \in \ell_M$.

Let us now suppose for contradiction that $\widetilde{\Lambda}(N) \neq \ell_M$. Then there is a sequence $\xi = (\xi_j)$ with $\sum M(|\xi_j|) < \infty$, yet $\sup_k N(\xi_0,\ldots,\xi_k) = \infty$.

CLAIM: $$N\Big(1,\frac{\xi_k}{N(\xi_0,\ldots,\xi_{k-1})}\Big) > 1+M(|\xi_k|) \quad \text{infinitely often.}$$

Indeed, otherwise we would have "$\leq$" for $k \geq K$ instead. It would follow for those k, as in the first part of the proof, that

$$\sup_{k\geq K} N(\xi_0,\ldots,\xi_k) \leq \prod_{j=K+1}^{\infty} (1+M(|\xi_j|))N(\xi_0,\ldots,\xi_K) < \infty,$$

contradicting the choice of ξ. Thus, the claim is proved.

But the claim leads to a contradiction, too, since we have infinitely often

$$N\Big(1,\frac{\xi_k}{N(\xi_0,\ldots,\xi_{k-1})}\Big) > 1+M(|\xi_k|) = N(1,\xi_k) \geq N\Big(1,\frac{\xi_k}{N(\xi_0,\ldots,\xi_{k-1})}\Big).$$

This completes the proof of part (a).

(b) Given ℓ_M we wish to find an Orlicz function F such that $\ell_M = \ell_F$ and $F(t) = N(1,t)-1$, $t \geq 0$, for some absolute norm N. Since $\ell_F = \widetilde{\Lambda}(N)$ by (a), this will yield our assertion.

Without loss of generality we assume that $M(1)=1$. We define F by $F(t)=M(t)$ for $0 \leq t \leq \frac{1}{2}$ and $F(t) = M(\frac{1}{2}) + 2t - 1$ for $t > \frac{1}{2}$. By convexity of M the right-hand derivative of M at $\frac{1}{2}$ is ≤ 2; therefore F is continuous, increasing and convex. Clearly $\ell_F = \ell_M$. Let us define $N(s,t) = |s|(1+F(|t/s|))$ for $s \neq 0$ and $N(0,t) = 2|t|$. Note that N is continuous on $\mathbb{K}^2$. To complete the proof of the proposition it is enough to prove the triangle inequality for N. Let us first suppose that $s_1, s_2 > 0$. Then we have, using the convexity of F,

$$\begin{aligned} N(s_1+s_2, t_1+t_2) &= s_1+s_2+(s_1+s_2)F\Big(\frac{t_1+t_2}{s_1+s_2}\Big) \\ &= s_1+s_2+(s_1+s_2)F\Big(\frac{s_1}{s_1+s_2}\frac{t_1}{s_1}+\frac{s_2}{s_1+s_2}\frac{t_2}{s_2}\Big) \\ &\le s_1+s_2+s_1F\Big(\frac{t_1}{s_1}\Big)+s_2F\Big(\frac{t_2}{s_2}\Big) \\ &= N(s_1,t_1)+N(s_2,t_2). \end{aligned}$$

This inequality extends to the case in which one or both s_i are zero by continuity. It is left to observe that in the general case

$$\begin{aligned} N(s_1+s_2, t_1+t_2) &= N(|s_1+s_2|, |t_1+t_2|) \ \le\ N(|s_1|+|s_2|, |t_1|+|t_2|) \\ &\le N(|s_1|,|t_1|)+N(|s_2|,|t_2|) \ =\ N(s_1,t_1)+N(s_2,t_2) \end{aligned}$$

due to the fact that $(s,t) \mapsto N(s,t)$ is increasing in each variable on $[0,\infty)\times[0,\infty)$, which is elementary to verify. ◻

Corollary 6.12

(a) *Every Orlicz sequence space h_M can be renormed to have property (M).*

(b) *An Orlicz sequence space h_M can be renormed to a space X for which $K(X)$ is an M-ideal in $L(X)$ if and only if $(h_M)^*$ is separable.*

Proof: This follows from Propositions 6.1 and 6.11 and Corollary 4.5. ◻

Note that $K(h_M)$ is not an M-ideal in $L(h_M)$ for the Luxemburg norm introduced in Section III.1 unless $h_M = \ell^p$, since the unit vector basis is 1-symmetric for this norm, by Proposition 4.24. Also, no renorming of $h_M \neq \ell^p$ such that the compact operators form an M-ideal can be stable; indeed, otherwise some subsequence of the unit vector basis would be equivalent to some ℓ^p-basis by Theorem 6.6 and Lemma 6.2.

Corollary 6.13 (Lindenstrauss-Tzafriri)
Every infinite dimensional subspace of an Orlicz sequence space h_M contains an isomorphic copy of some ℓ^p for $1 \le p < \infty$ or of c_0.

Proof: This follows from Theorem 6.4 and Corollary 6.12(a). ◻

Our final result can be thought of as an isomorphic version of Proposition 4.24. To formulate it we need the notion of a subsymmetric basis [421, p. 114]: A basis (e_n) of a Banach space X is called *subsymmetric* if it is unconditional and if for every subsequence (n_k) the basic sequence (e_{n_k}) is equivalent to (e_n). Proposition 6.1 implies that the unit vector basis of $\Lambda(N)$ is subsymmetric.

Theorem 6.14 *Let X be a Banach space with a subsymmetric basis (e_n). Then X can be renormed to a space Y for which $K(Y)$ is an M-ideal in $L(Y)$ if and only if X is isomorphic to an Orlicz sequence space h_M and X^* is separable.*

PROOF: One direction is contained in Corollary 6.12, and it remains to prove that a space which can be renormed as stated is in fact isomorphic to some h_M; X^* is then necessarily separable by Corollary 4.5.
The separability of X^* implies that (e_n) is weakly null, since (e_n) is an unconditional basis; this has already been observed in Proposition 4.24. By Theorem 4.17 the renormed space Y has (M). Lemma 6.2 shows that some subsequence of (e_n) is equivalent to the unit vector basis of some $\Lambda(N)$ and thus of some h_M (Proposition 6.11). Since (e_n) is supposed to be subsymmetric, (e_n) itself is equivalent to the unit vector basis of some h_M. Hence, the claim is proved. $\square$

The proof of this theorem shows a little bit more, namely if $K(X)$ is an M-ideal in $L(X)$, then every subsymmetric basic sequence is actually symmetric.

VI.7 Notes and remarks

GENERAL REMARKS. The main references for Sections VI.1 through VI.3 are [60], [619], [625] and [628]. Section VI.1 is largely based on [628], where Theorem 1.2 and its corollaries can be found. Special cases of these results were previously obtained in [625] ($X = Y = C_{\mathbb{C}}(K)$) and [60] ($Z(X^*)$ trivial, $Y = C(K)$); for the case of arbitrary X and $Y = C(K)$ see [61]. Lemma 1.1(c) is a simple observation which can be traced back as least as far as [400]. Theorem 1.3 is due to Ruess and Stegall [545] in the real and to Lima and Olsen [407] in the complex case. Our proof, building on the Lemmas 1.4 and 1.5 from [619], covers both cases simultaneously. A similar approach is in [603].
In [107] it is proved, by vector measure techniques, that K is hyperstonean if X is reflexive and $L(X, C(K))$ is isometric to a dual space. This can also be deduced from Corollary 1.11, which is easily seen to extend to spaces X such that $Z(X^*)$ is finite dimensional, in particular to reflexive X. The paper [107] contains the additional information that, for reflexive X, the predual of $L(X, C(K))$ is strongly unique. Let us mention that the paper [516] deals with the dual space $L(X, C(K))$ from an M-structure point of view, too.
The results in Section VI.2 are taken from [60]. The main aspect of Theorem 2.3 is that, under the assumptions made, the M-ideals of $L(X, C(K))$ are in one-to-one correspondence with the M-ideals in $C(K)$. A result of this type was first proved by Flinn and Smith [235] for the case in which X is a complex $C(K)$-space. Their arguments rely on numerical range techniques for the Banach algebra $L(C(K))$. The representation of the M-ideals in $L(C(K))$ they give is embellished in [625]. There the subspaces $J_{(D)}$ are introduced, and their relation to the subspaces of J_D-valued operators is investigated. There are also some results concerning the M-ideals in $L(X, C_0(L))$ for a locally compact space L, but here a lot of technical problems arise; see [60] and [235] for details.
Our source for the first half of Section VI.3 is [619], which contains 3.1–3.7 and Proposition 3.12. This paper also discusses M-ideals for the projective hull of the ε-tensor norm and dually L-summands in spaces of absolutely summing operators. Theorem 3.2 is extended to the case of a space X with finitely many M-ideals in [547]. Corollary 3.4 has first appeared in [51]; the M-ideals of $C(K, X)$ are completely described in [51, Prop. 10.1]. Corollary 3.7 has been considered for $X = Y = \ell^p_{\mathbb{C}}$ in [579], and for re-

flexive X and Y in [292]. Theorem 3.8 and its corollaries are taken from [628], but Corollary 3.11 was originally proved in [278] where measure theoretic arguments are employed. Proposition 3.13 is a result from [629]. It was previously proved in [547] that the structure topology is stable by taking products if one of the Banach spaces involved has only finitely many M-ideals. In connection with Section VI.3 we also mention the papers [400] and [518] which deal with intersection properties in tensor products.

The problem of investigating for which Banach spaces $K(X,Y)$ is an M-ideal in $L(X,Y)$ has attracted a number of authors from the beginning of M-ideal theory. The interest in this question originated initially both from the approximation theoretic properties of M-ideals and from the uniqueness conclusion in the Hahn-Banach theorem in the M-ideal setting. Hennefeld [302] was the first to prove that $K(\ell^p)$ is an M-ideal in $L(\ell^p)$, for the explicit purpose of obtaining unique Hahn-Banach extensions. (We recall that Dixmier [164] proved that $K(H)$ is an M-ideal in $L(H)$ for a Hilbert space H.) Saatkamp [550] showed that $K(\ell^p,\ell^q)$ is an M-ideal (meaning, an M-ideal in the corresponding space of bounded operators); these authors directly construct the L-projection in the dual of the bounded operators. Å. Lima was the first to tackle the problem of M-ideals of compact operators by means of the 3-ball property; it is his proof from [401] that we have presented in Example 4.1. Actually the same proof yields the assertion of Corollary 5.4. Several authors observed that $K(X,c_0)$ is an M-ideal [219], [433], [550]; examples in the spirit of Lemma 6.7, which is taken from [623], can be found in [459] and [552]. In the other direction, a number of papers give ad hoc constructions in some spaces of operators to show that the 2-ball property fails and thus that $K(X,Y)$ is not an M-ideal. In this regard we mention [219], [550], [577]. Today these results can be obtained in a more systematic manner. For instance, in order to show that $K(\ell^1,\ell^2)$ is not an M-ideal observe that $K(\ell^1,\ell^2)\cong K(\ell^2,\ell^\infty)\cong C(\beta\mathbb{N},\ell^2)$, that ℓ^2 has the IP (Definition II.4.1) and that $C(K,X)$ has the IP whenever X has (this is easy). Consequently, should $K(\ell^1,\ell^2)$ be an M-ideal in $L(\ell^1,\ell^2)$, it would even be an M-summand by Theorem II.4.4. This is impossible by Proposition 4.3. In [406] similar reasoning is employed to show that $K(X,C(K))$ is an M-ideal in $L(X,C(K))$ only in the trivial case where X or $C(K)$ is finite dimensional. This paper also contains the result that, for a separable Banach space Y with the MCAP, $K(\ell^1,Y)$ is an M-ideal if and only if Y is an (M_∞)-space.

Lima proved that $K(L^p)$, $p\neq 2$, fails to be an M-ideal in $L(L^p)$, by using properties of the Haar basis. A different proof is due to Li [396], wheras our argument in Proposition 5.19(a), parts (1) and (2), comes from [466]; we remark that part (3) of that proposition is unpublished. However, as was pointed out to us by L. Weis, $K(L^p)$ is an M-ideal in a certain nonunital subalgebra of $L(L^p)$, viz. the algebra consisting of those $T\in L(L^p)$ such that

$$\forall\varepsilon>0\ \exists A,B\subset[0,1],\ \lambda(A\cup B)<\varepsilon:\quad T-P_ATP_B \text{ is compact.}$$

(Here P_A stands for the operator $f\mapsto\chi_Af$.) This can easily be shown with the help of the 3-ball property.

As we have already pointed out in the Notes and Remarks section to Chapter I, Hennefeld, in [304], introduced the notion of an *HB*-subspace as a more flexible means to deal with uniqueness of Hahn-Banach extensions. Results on *HB*-subspaces of compact operators can be found in [304], [305], [459], [462], [463], [465], [466]; for example, $K(\ell^p,X)$ and $K(X,\ell^p)$ are *HB*-subspaces for $1<p<\infty$, but in general not M-ideals.

The systematic treatment of M-ideals of compact operators was begun in Hennefeld's paper [304], which contains Proposition 4.24. Clearly his proof differs from ours in details, since property (M) was not at his disposal. Instead, his argument is based on the following lemma: If X has a 1-unconditional basis (e_i) and $K(X)$ is an M-ideal in $L(X)$, then, for every $\varepsilon > 0$ and every finite rank operator S with $\|S\| = 1$, we have $\|S + (Id - P_n)\| \le 1 + \varepsilon$ infinitely often; here P_n denotes the n^{th} coordinate projection relative to the basis (e_i). The appearance of this lemma more than a decade before similar *necessary* conditions (cf. Theorem 4.17) appeared in [630] and [366] is quite remarkable. Lima [403] contributed Propositions 4.2, 4.4 and 4.5; Proposition 4.3 was observed in [552], however with a less obvious proof. The real breakthrough came with the paper [292] where the MCAP of X and X^* was shown to be necessary for the M-ideal property of $K(X)$, cf. Proposition 4.10. Simpler proofs can be found in [627] and [621], the latter one being presented in the text. Subsequently, it was investigated by several authors to what extent the converse is true. Cho and Johnson [118] obtained Corollary 4.20 (for $p < \infty$). The new technique they introduced into this context was the convex combination argument that allows one to pass from strong operator convergence via Banach space weak convergence to operator norm convergence. This device, which we have used a number of times, seems to be due to Feder [221]; it is successfully applied in [343] too. Cho's and Johnson's proof was simplified in [57] and [620]; the corresponding result for subspaces of c_0 was established in [621] and, independently and with a different proof, in [461].

The next step was performed in [630] where the equivalences of (i) through (iii) of Theorem 4.17 were proved. The remaining equivalences are due to Kalton [366] (see also [365]) who succeeded in singling out a geometrical property of the norm, viz. property (M) resp. property (M^*), which in conjunction with a suitable version of the approximation property characterises Banach spaces for which $K(X)$ forms an M-ideal. The achievement of Kalton's extremely important paper is to devise a characterisation in terms of the Banach space X itself rather than in terms of the operators defined on X. It is this characterisation that permitted him to provide answers to many of the questions in the theory of M-ideals of compact operators left open until then, for instance the general Cho-Johnson type Theorem 4.19. Corollary 4.18 was recently observed by E. Oja [465]. Actually, Kalton only deals with separable Banach spaces, but the extension to the general case presents no major difficulty (see also [465]). Our proofs of these results occasionally differ from his in that we work with the 3-ball property and use the material of Chapter V, whereas Kalton's approach is via the projection in the dual; he also presents an independent argument for the equivalence (i) $\iff$ (ii) along these lines.

It seems remarkable that a contractive projection on $L(X,Y)^*$ with kernel $K(X,Y)^\perp$ can be constructed more or less explicitly when Y has the MCAP. This was done by J. Johnson [351, Lemma 1] as follows. Pick a net (K_α) in the unit ball of $K(Y)$ such that $K_\alpha \to Id_Y$ strongly, and let $\psi \in K(Y)^{**}$ be a weak* cluster point of (K_α). Let us assume for simplicity that $\psi = w^*\text{-}\lim K_\alpha$. For $\ell \in L(X,Y)^*$ consider $\pi(\ell) : T \mapsto \lim\langle \ell, K_\alpha T\rangle = \langle \psi, \varphi\rangle$ where $\varphi \in K(Y)^*$ is the functional $\varphi(K) = \langle \ell, KT\rangle$. Then π is a contractive projection with kernel $K(X,Y)^\perp$ and whose range is isometric to $K(X,Y)^*$. By Proposition I.1.2(b) π must be the L-projection if $K(X,Y)$ is an M-ideal in $L(X,Y)$, thus, to check that $K(X,Y)$ is an M-ideal it is enough to check that this particular projection is an L-projection. This remark applies even in the setting of semi

L-projections defined in the Notes and Remarks to Chapter I. Therefore, if one could prove that X has the MCAP if $K(X)$ is a semi M-ideal in $L(X)$, one would obtain that $K(X)$ is a semi M-ideal in $L(X)$ if and only if it is an M-ideal; as yet, this is open. A similar procedure as above, with $\pi(\ell) : T \mapsto \lim\langle \ell, TK_\alpha\rangle$ works if X^* has the MCAP with adjoint operators. Likewise, one obtains analogous results for the space $A(X,Y)$ of norm limits of finite rank operators if one employs the MAP. But here a slight technical improvement is possible. For, an application of the principle of local reflexivity shows that $\{S^* \mid S \in F(X),\ \|F\| \le 1\}$ is strongly dense in $B_{F(X^*)}$. Therefore X^* has the MAP with adjoint operators provided it has the MAP at all. By contrast, there are counterexamples to this last assertion for the MCAP [279]; in fact, only recently was the first example of a Banach space with the MCAP lacking the MAP (even the AP) constructed by Willis [636]. It appears to be still open whether or not such spaces exist among the subspaces of ℓ^p. We also refer to [269], [405] and [470] for related information.
Theorem 4.21, Lemma 4.22 and Corollary 4.23 suggest that the assumption that $K(X)$ is an M-ideal in $L(X)$ is related to unconditionality properties of X; these results are due to Li [396], cf. also [268] and [563]. (Concerning Lemma 4.22, see also the subsection on u-ideals of compact operators below.) The condition $\lim\|Id - 2K_\alpha\| = 1$ from Theorem 4.17 is an indication of this, too. Corollary 6.11(b) supports this point of view from another direction since c_p fails to have local unconditional structure (see [500, Cor. 8.20]). In a similar vein, one can understand the assumption that $K(X,Y)$ is an M-ideal as a richness property; for example it follows as in the proof of Proposition 4.10 that $B_{K(X,Y)}$ is strongly dense in $B_{L(X,Y)}$ which yields the existence of compact operators with some desirable properties. On the other hand, once $K(X,Y) \neq L(X,Y)$ is an M-ideal in $L(X,Y)$ it is necessarily a proper M-ideal (Definition II.3.1) and hence contains c_0 (Theorem II.4.7). But then it follows that $K(X,Y)$ is an uncomplemented subspace of $L(X,Y)$ (see [364], [222], [203] and [344]) which indicates a rich supply of nontrivial (= noncompact) operators. (This can also be derived from Theorem I.2.10 with an appeal to [222].) By the way, up to now *no* example of a complemented subspace $K(X,Y) \neq L(X,Y)$ is known.
Condition (3) in Theorem 5.3 was known to experts as an easy-to-check sufficient condition for $K(Y,X)$ to be an M-ideal in $L(Y,X)$ for all Banach spaces Y, cf. [621]. That (3) is also necessary was proved in [480]. In this paper the idea to consider $X \oplus X$ in order to derive information about X emerged, and it allowed the authors of [480] to provide a proof of Theorem 5.3(b) and Proposition 5.6. The formal definition of the class of (M_p)-spaces was given in [466]. In particular it was shown there that the (M_p)-spaces are stable for $p < \infty$, a fact that might be considered as a first step towards property (M). Our sources for Section VI.5 are, apart from [466] and [480], [631] and [632]; the simple proof that (M_1)-spaces are finite dimensional was pointed out to us by T. S. S. Rao. Theorem 5.7 comes from [631].
Theorem 5.15 and the preparatory Propositions 4.7 and 4.8 are proved in [632]. Let us mention that $K(X,Y)^\theta$ is always nowhere dense (see [552] or [632]); we used a particular case of this in the proof of Proposition 4.8. The quantity $w(T)$ from Proposition 4.7 was introduced in the context of Hilbert space operators in [319] where $w(T) = \|T\|_e$ is proved for $T \in L(H)$. Proposition 4.8 was observed for Hilbert spaces in [320]. The method of proof of Theorem 5.15 is adapted from [378]; this paper, again, deals with Hilbert space operators and uses in addition spectral theoretic arguments. Characterisations of

smooth points in $K(X,Y)$ rather than $L(X,Y)$ can be found in [297] and [546]; these papers also investigate Fréchet smoothness in operator spaces. In another direction one might ask, starting from the main result in [378], i.e. Theorem 5.15 for $X = Y =$ Hilbert space, for characterisations of smooth points in subalgebras of $L(H)$. Such results are proved in [598] and [599].

Section VI.6 is almost entirely based on Kalton's paper [366] and contains the most important consequences of the results presented so far; only the easy implication (i) $\Rightarrow$ (ii) in Theorem 6.6 [466] and Lemma 6.7 together with Proposition 6.8 [623] are not found there. Actually, Kalton's results are more general than the versions stated; for example he shows that a separable order continuous nonatomic Banach lattice can be renormed to have property (M) if and only if it is lattice isomorphic to L^2. Also, instead of the spaces $\Lambda(N)$ one can construct a more general class of spaces enjoying (M) resp. such that the compact operators form an M-ideal. One considers a sequence $\mathcal{N} = (N_1, N_2, \ldots)$ of absolute norms on $\mathbb{K}^2$ with $N_k(1,0) = 1$ and defines inductively $\mathcal{N}(\xi_0, \ldots, \xi_d) = N_d(\mathcal{N}(\xi_0, \ldots, \xi_{d-1}), \xi_d)$. The resulting space, $\Lambda(\mathcal{N})$, fulfills the conclusion of Proposition 6.1 (apart from part (b)). As in Proposition 6.11 one can show that $\Lambda(\mathcal{N})$ is isomorphic to a so-called modular sequence space (see [421, Sect. 4.d], these spaces are also known as Musielak-Orlicz spaces) and vice versa. As a consequence, one obtains that every modular sequence space contains a copy of some ℓ^p. Originally this result is due to Woo [643] who reduced it to the Lindenstrauss-Tzafriri theorem, Corollary 6.13. Lindenstrauss and Tzafriri proved their theorem with the help of a fixed point argument [421, Th. 4.a.9]; this argument even yields $(1+\varepsilon)$-copies of ℓ^p for the Luxemburg norm introduced in Section III.1. This does not imply automatically that one also gets $(1+\varepsilon)$-copies of ℓ^p for the $\Lambda(N)$-norm. Still, this is true as proved in Section VI.6; the nontrivial ingredient of the proof this time being an appeal to Krivine's theorem in Lemma 6.3. Note that isomorphs of ℓ^p not containing $(1+\varepsilon)$-copies of ℓ^p were recently constructed by Odell and Schlumprecht [458]. Yet another approach, via the Krivine-Maurey theorem [383], can be found in [242, p. 167] where Garling proved that h_M is stable (for the Luxemburg norm).

As for Lemma 6.7, we would like to mention that under the assumptions made there, for subspaces $E \subset X$ and $F \subset Y$ such that $K(E,F)$ is strongly dense in $L(E,F)$, the space $K(E,F)$ is an M-ideal in $L(E,F)$ as well. This is shown in [623]. Further, let us record the result from [624] that $K(X,Y)$ is an M-ideal in $L(X,Y)$ if and only if for all $T \in L(X,Y)$, $\|T\| = 1$, there is a net (K_α) in $K(X,Y)$ with $K_\alpha^* y^* \to T^* y^*$ for all $y^* \in Y^*$ satisfying $\limsup \|S + (T - K_\alpha)\| \le 1$ whenever $S \in K(X,Y)$, $\|S\| = 1$.

M-ideals of compact operators are also considered in some papers quoted in the Notes and Remarks to Section V.6 where additional comments can be found. For the general topic of M-ideals in operator spaces we also mention [34], [231], [463], [464], [517].

More on (M_∞)-spaces. One of the main problems left open in connection with these spaces is to decide whether the class of (M_p)-spaces coincides with the class of subspaces of quotients of $\ell^p(X_\alpha)$, $\dim X_\alpha < \infty$, resp. of $c_0(\Gamma)$ with the MCAP. (Note that a quotient of c_0 is almost isometric to a subspace of c_0, by [14] or [335].) Theorem 5.7 gives a contribution to this question. Another subclass of (separable) (M_∞)-spaces which are known to embed almost isometrically into c_0 are those where the K_α appearing in condition (3) of Theorem 5.3 are projections; then the K_α clearly have finite rank.

Let us sketch the simple argument. Our assumption is that there exists a sequence (F_n) of finite rank projections on X such that

$$F_n x \to x \qquad \forall x \in X,$$

$$\limsup_n \sup_{\|x\|,\|y\|\le 1} \|F_n x + (Id - F_n)y\| \le 1.$$

Let us now observe the following lemma.

Lemma A. *For all $\varepsilon > 0$ there is some $\delta > 0$ with the following property: If E is a Banach space and the operators S, $T \in L(E)$ satisfy*

$$\left.\begin{array}{rcll} \|Sx + (Id - S)y\| & \le & 1 + \delta & \forall x, y \in B_E \\ \|Tx + (Id - T)y\| & \le & 1 + \delta & \forall x, y \in B_E \end{array}\right\} \tag{1}$$

then

$$\|ST - TS\| \le \varepsilon. \tag{2}$$

Assuming the lemma to be false we try to achieve a contradiction. So let us suppose that there are some $\varepsilon_0 > 0$, some Banach spaces E_n and operators S_n, $T_n \in L(E_n)$ such that

$$\begin{array}{rcll} \|S_n x_n + (Id - S_n)y_n\| & \le & 1 + \dfrac{1}{n} & \forall x_n, y_n \in B_{E_n}, \\ \|T_n x_n + (Id - T_n)y_n\| & \le & 1 + \dfrac{1}{n} & \forall x_n, y_n \in B_{E_n}, \end{array}$$

but

$$\|S_n T_n - T_n S_n\| > \varepsilon_0.$$

We choose a free ultrafilter $\mathcal{U}$ over the integers and consider the ultraproduct $E := \prod_{\mathcal{U}} E_n$ and the operators $S, T : E \to E$ defined by $[(x_n)] \mapsto [(S_n x_n)]$, and $[(x_n)] \mapsto [(T_n x_n)]$. Then S and T satisfy (1) with $\delta = 0$. On the other hand, there are $\xi_n \in E_n$, $\|\xi_n\| = 1$, such that $\|(S_n T_n - T_n S_n)\xi_n\| > \varepsilon_0$. It follows for $\xi = [(\xi_n)] \in E$ that

$$\|(ST - TS)\xi\| = \lim_{\mathcal{U}} \|(S_n T_n - T_n S_n)\xi_n\| \ge \varepsilon_0,$$

hence $ST \ne TS$, and (2) fails with $\varepsilon = 0$. This contradicts Corollary I.3.9, since the centralizer of E is commutative.
With a similar technique one can show:

Lemma B. *For all $\varepsilon > 0$ there is some $\delta > 0$ with the following property: If E is a Banach space and $T \in L(E)$ is a projection satisfying*

$$\|Tx + (Id - T)y\| \le 1 + \delta \qquad \forall x, y \in B_E,$$

then

$$\|x\| \le (1 + \varepsilon) \max\{\|Tx\|, \|x - Tx\|\} \qquad \forall x \in E.$$

Thus our projections are almost M-projections (cf. $(*)$ on p. 2), and by Lemma A they almost commute. Let now $0 < \varepsilon < 1/2$. Pick a decreasing sequence of positive numbers (ε_n) such that $\prod_{n=1}^{\infty}(1+\varepsilon_n) \le 1+\varepsilon$, and choose $0 < \delta_n \le \varepsilon_n$ according to Lemmas A and B. After possibly discarding a number of the F_n, we have from our assumption

$$F_n x \to x \qquad \forall x \in X,$$

$$\|F_n x + (Id - F_n)y\| \le 1 + \delta_n \qquad \forall x, y \in B_X.$$

Denote $Y_n := \operatorname{ran}(F_n)$, $Z := (Y_1 \oplus Y_2 \oplus \ldots)_{\ell^\infty}$ and let $I : X \to Z$ be defined by

$$I(x) = (F_1 x, F_2(Id - F_1)x, F_3(Id - F_2)(Id - F_1)x, \ldots).$$

It is clear that I is well-defined and that $\|I\| \le 1 + \varepsilon$. We wish to prove that $\operatorname{ran}(I) \subset Y := (Y_1 \oplus Y_2 \oplus \ldots)_{c_0}$ and that

$$\frac{1}{1+\varepsilon}\|x\| \le \|I(x)\| \le (1+\varepsilon)\|x\| \qquad \forall x \in X. \tag{3}$$

This will show our assertion since Y is almost isometric to a subspace of c_0 and ε is arbitrarily small.

For convenience we let $G_k := Id - F_k$. To prove the former we have to show that $G_n G_{n-1} \cdots G_1 x \to 0$ for each $x \in X$. The idea is to let G_n wander from the left to the right until we end up with something small; the error terms that arise can be estimated by Lemma A. We skip the details. We now turn to the proof of the left hand inequality in (3). From Lemma B we have

$$\|\xi\| \le (1+\varepsilon_n)\max\{\|F_n\xi\|, \|\xi - F_n\xi\|\} \qquad \forall \xi \in X,\ n \in \mathbb{N}. \tag{4}$$

Let $x \in X$. We first apply (4) to $\xi = x$ with $n = 1$. If the left hand term in the maximum in (4) is the bigger one, we get

$$\|x\| \le (1+\varepsilon_1)\|F_1 x\| \le (1+\varepsilon)\|I(x)\|.$$

Otherwise, we have $\|x\| \le (1+\varepsilon_1)\|G_1 x\|$. Then we apply (4) to $\xi = G_1 x$ with $n = 2$. If now the maximum in (4) is attained at the first item, we get

$$\|x\| \le (1+\varepsilon_1)\|G_1 x\| \le (1+\varepsilon_1)(1+\varepsilon_2)\|F_2 G_1 x\| \le (1+\varepsilon)\|I(x)\|.$$

Otherwise, we have $\|G_1 x\| \le (1+\varepsilon_2)\|G_2 G_1 x\|$, and we continue exploiting (4) with $\xi = G_2 G_1 x$, $n = 3$, etc. If, for some n, the first item in the maximum in (4) is the bigger one, we deduce $\|x\| \le (1+\varepsilon)\|I(x)\|$, which is the desired inequality. But should for no n the first item exceed the second, we get for all $n \in \mathbb{N}$

$$\|x\| \le \prod_{k=1}^{n}(1+\varepsilon_k)\|G_n \cdots G_1 x\| \le (1+\varepsilon)\|G_n \cdots G_1 x\|,$$

which, however, tends to 0, as was noticed above. This completes the proof of the inequality $\|x\| \le (1+\varepsilon)\|I(x)\|$ and thus of our claim.

A direct proof of the almost commutativity of almost M-projections is given in [110]; the usage of ultraproduct techniques in the above proof is inspired by [340]. Another piece of information on (M_∞)-spaces is the result from [406] already mentioned that a separable space Y with the MCAP has (M_∞) if and only if $K(\ell^1, Y)$ is an M-ideal in $L(\ell^1, Y)$.

The centralizer of an injective tensor product. The centralizers of injective tensor products are discussed in various contexts (C^*-algebras, $A(K)$-spaces) for example in [126], [296], [376] and [615]. The most far-reaching results in a general framework seem to have been obtained in [629]. In this paper the centralizer of an injective tensor product is represented as a function space on a suitable product space. More precisely, the following can be shown. We have seen in Example I.3.4(g) that $Z(X)$ can be regarded as a subalgebra of $C^b(Z_X)$, where $Z_X = \overline{\mathrm{ex}}^{w^*} B_{X^*} \setminus \{0\}$, and it follows from Theorem I.3.6 that $Z(X)$ can in fact be identified with the space of bounded continuous functions on the quotient space $\Theta_X = Z_X/\!\sim$ derived from the equivalence relation

$$p \sim q \iff a_T(p) = a_T(q) \quad \forall T \in Z(X).$$

Let us now address the question how $\Theta_{X \widehat{\otimes}_\varepsilon Y}$ relates to the product of Θ_X and Θ_Y. The answer involves the so-called k-product of topological spaces [581]. Let S_1 and S_2 be Hausdorff spaces. Then $S_1 \times_k S_2$ denotes the space $S_1 \times S_2$ endowed with the finest topology which agrees with the product topology on the compact subsets of $S_1 \times S_2$. Here is the main theorem from [629].

Theorem. *$Z(X \widehat{\otimes}_\varepsilon Y)$ can be identified with $C^b(\Theta_X \times_k \Theta_Y)$.*

The example at the end of Section VI.3 shows that one cannot replace the k-product by the ordinary product here.
Prior to [629] it was known that the algebraic tensor product $Z(X) \otimes Z(Y)$ is dense in $Z(X \widehat{\otimes}_\varepsilon Y)$ for the strong operator topology [634], another proof of this fact was given in [50]. (That $Z(X) \otimes Z(Y) \subset Z(X \widehat{\otimes}_\varepsilon Y)$ follows from the easy part of Theorem 1.3.) The latter paper also contains sufficient conditions for $Z(X) \otimes Z(Y)$ to be norm dense, i.e. $Z(X \widehat{\otimes}_\varepsilon Y) = Z(X) \widehat{\otimes}_\varepsilon Z(Y)$. For example, norm density holds if X and Y are dual spaces; in view of the above theorem this follows once the compactness of Θ_X and Θ_Y is shown, which is done in [629].
A special case of these results is that $Z(C(K, X)) \cong C(K)$ if $Z(X)$ is trivial. For a description of the centralizer of $C(K, X)$ in the general case see [51, Prop. 10.3].

Best approximation by compact operators and the "basic inequality". The concept of an M-ideal has proved particularly useful for obtaining best approximation results for compact operators, because M-ideals are proximinal (Proposition II.1.1). Thus, M-ideal techniques provide a systematic approach to proximity questions. The close connection between M-ideal theory and best approximation by compact operators was first noticed in [320].
The proximinality of $K(H)$ in $L(H)$, H a Hilbert space, is a result originally due to Gohberg and Krein [275, p. 62]. To obtain the same result now, one just has to take into account that $K(H)$ is an M-ideal in $L(H)$. Since $K(\ell^p)$ is an M-ideal in $L(\ell^p)$ provided $1 < p < \infty$, $K(\ell^p)$ is seen to be proximinal in $L(\ell^p)$ for these p. This corollary extends

to $p = 1$ although $K(\ell^1)$ is not an M-ideal, but not to $p = \infty$ [220]. We remark that a "constructive" proof of the proximinality of $K(\ell^p)$ can be found in [433]. The natural question whether this result extends to L^p was answered in the negative by Benyamini and Lin [74]. Of course, this is stronger than the fact, proved in Proposition 5.19, that $K(L^p)$ is not an M-ideal in $L(L^p)$. Incidentally Benyamini and Lin use the concept of U-proximinality, which we mentioned in the Notes and Remarks to Chapter II, as well: They point out that $K(L^p)$ would even be U-proximinal ("satisfy the successive approximation scheme" in their terminology) if it were proximinal at all, and they show that it is not.

Another approach to proximinality questions was suggested by Axler, Berg, Jewell and Shields [38], [39]; see also [75]. They prove proximinality of $K(X)$ for Banach spaces X such that X^* enjoys the bounded approximation property and X satisfies the so-called "basic inequality", which means the following:

For all $S \in L(X)$, for all bounded nets $(A_\alpha) \subset L(X)$ such that $A_\alpha \to 0$ and $A_\alpha^* \to 0$ strongly and for all $\varepsilon > 0$ there is some index α_0 such that

$$\|S + A_{\alpha_0}\| \le \max\{\|S\|, \|S\|_e + \|A_{\alpha_0}\|\} + \varepsilon. \tag{1}$$

(Here $\|S\|_e$ denotes the essential norm of S, i.e. the norm of the equivalence class $S+K(X)$ in the quotient space $L(X)/K(X)$.) They go on to show that ℓ^p satisfies the basic inequality for $1 < p < \infty$ as does c_0, whereas ℓ^1, ℓ^∞ and the L^p-spaces (for $p \ne 2$) fail the basic inequality. Therefore, they get that $K(\ell^p)$ is proximinal in $L(\ell^p)$ for $1 < p < \infty$. Another result of [38] is the proximinality of $H^\infty + C$ in L^∞, which can also be derived with the help of M-ideal techniques; see Corollary III.1.5. Davidson and Power also obtain theorems on best approximation both by M-ideal methods and by basic inequality techniques [148]; and there are proximinality results on nest algebras (which we discussed in the Notes and Remarks to Chapter V) that can be obtained either using M-ideals or a variant of the basic inequality ([224] and [225]).

These similarities suggest that there might be a close relation between the two methods. This relation is elucidated in [624] where it is proved that, although neither does the basic inequality imply that $K(X)$ is an M-ideal in $L(X)$ nor does the converse hold, the two techniques are basically equivalent. More precisely, one can show:

THEOREM. *For a Banach space X, the following assertions are equivalent:*

(i) *$K(X)$ is an M-ideal in $L(X)$.*
(ii) *For all $T \in L(X)$ there is a net (K_α) in $K(X)$ such that $K_\alpha^* \to T^*$ strongly and, for all $S \in L(X)$,*

$$\limsup \|S + T - K_\alpha\| \le \max\{\|S\|, \|S\|_e + \|T\|\} \tag{2}$$

Here, condition (ii) can be regarded as a "revised basic inequality"; and an inspection of the proofs in [38] and [39] shows that the arguments in these papers are based on (2) rather than the basic inequality (1). Thus it seems all the more surprising that one can actually prove (1) for the ℓ^p-spaces. Let us point out in addition that one can take the coordinate projections in (ii) above for $X = \ell^p$; this contrasts the situation in Theorem 4.17(ii) where they don't work.

For other papers on the problem of best approximation by compact operators we refer e.g. to [22], [41], [73], [217], [218], [220], [319], [387], [617], [618], [653]. Papers dealing with the impact of M-ideals on approximation theory – apart from those already mentioned – include [129], [130], [219], [232], [233], [306], [552], [616].

Banach-Stone theorems. The classical Banach-Stone theorem asserts that two compact Hausdorff spaces K and L are homeomorphic provided that the sup-normed spaces $C(K)$ and $C(L)$ are isometrically isomorphic [178, p. 442]. A number of authors have investigated the validity of this theorem for spaces of vector-valued continuous functions. An in-depth analysis of this problem is presented in part II of E. Behrends' monograph [51], where the relevance of M-structure concepts for proving vector-valued Banach-Stone theorems was pointed out for the first time. We wish to sketch the main ideas and some of their recent extensions.

It is clear that some geometric condition has to be imposed on a Banach space X in order to render the Banach-Stone theorem for X-valued function spaces valid; the simplest example to demonstrate this is the isometric isomorphism $C(\{1\}, c_0) \cong c_0 \cong c_0 \oplus_\infty c_0 \cong C(\{1,2\}, c_0)$. The example suggests that the richness of the M-structure of c_0 is responsible for the failure of the Banach-Stone theorem in this case. Hence one might expect positive results for Banach spaces whose M-structure is scarce or even lacking. This is indeed so and can be seen as follows. Suppose $\Phi : C(K,X) \to C(L,X)$ is an isometric isomorphism. Then Φ maps M-ideals onto M-ideals. Let us now suppose that X has no nontrivial M-ideals. Then, by Corollary 3.4, $\Phi(J_{\{k\}} \widehat{\otimes}_\varepsilon X) = J_D \widehat{\otimes}_\varepsilon X$ for some closed subset D of L. Next one shows, by a maximality argument or otherwise, that D must be a singleton, say $D = \{l\}$, so that a map $\varphi : K \to L$, $k \mapsto l$ is defined. Finally one concludes by routine arguments that φ is actually a homeomorphism from K onto L. In fact, one can even obtain more information by this method. Namely it turns out that Φ must have the special form $(\Phi f)(l) = \Phi_l(f(\varphi^{-1}(l)))$, where $(\Phi_l)_{l\in L}$ is a strongly continuous family of isometrical isomorphisms on X. Also, if $\Phi : C(K,X) \to C(L,Y)$ is an isometrical isomorphism, with X and Y a priori distinct, then the same arguments apply, and X and Y must be isometric.

A second method to obtain a vector-valued Banach-Stone theorem consists in analysing the centralizer of $C(K,X)$. If $Z(X)$ is trivial, then one knows that $Z(C(K,X)) \cong C(K)$; this was mentioned above. Plainly, if $C(K,X)$ and $C(L,X)$ are isometrically isomorphic, so are their centralizers; hence K and L must be homeomorphic by the scalar Banach-Stone theorem. (Clearly, this is the mechanism on which the proof of Corollary 1.14 relies.) We remark that also in this case deeper consequences can be revealed, see [51, Th. 8.10]. We have just scratched the surface of the topic of $C(K,X)$-isometries; for full treatment we again refer to [51].

In another direction, the original Banach-Stone theorem has been extended by Amir [15] and Cambern [104] to include small-bound isomorphisms; see also [172]. Their result is that K and L are homeomorphic if $d(C(K), C(L)) < 2$ (d denotes the Banach-Mazur distance). The constant 2 is known to be optimal here [135]. (At this stage the Milutin theorem is worth recalling, which states that $C(K)$ and $C(L)$ are isomorphic whenever K and L are uncountable compact metric spaces; see [420, p. 174].) Let us agree to say that a Banach space X has the isomorphic Banach-Stone property if there exists some $\delta > 0$ such that, whenever $d(C(K,X), C(L,X)) < 1 + \delta$, K and L are homeomorphic.

The isomorphic Banach-Stone property is tackled in [337] and [105], and more recently in [59] and [65] where the above-sketched techniques are adapted to the nonisometric setting. For instance, it is shown in [65] that an isomorphism $\Phi : C(K,X) \to C(L,X)$ maps the M-ideal $J_{\{k\}}\widehat{\otimes}_\varepsilon X$ onto a subspace close enough to an M-ideal $J_{\{l\}}\widehat{\otimes}_\varepsilon X$ to recover l uniquely from k, provided $\|\Phi\|\,\|\Phi^{-1}\|$ is sufficiently small and X has in some sense no isomorphic M-structure. More precisely, in that paper it is required that the two-dimensional space $\ell^1(2)$ is not finitely representable in X^*, i.e.,

$$\lambda(X^*) := \inf\{d(E,\ell^1(2)) \mid E \subset X^*,\ \dim E = 2\} > 1.$$

In the real case this condition means precisely that X is uniformly nonsquare in the sense of [334]. Clearly, uniformly convex spaces X fulfill $\lambda(X^*) > 1$ and hence have the isomorphic Banach-Stone property. This was originally proved in [105].

Another approach to the problem, suggested in [59] and further elaborated in [109], is to investigate the so-called ε-mulipliers, which are defined by the requirement that the condition of Theorem I.3.6(ii) be fulfilled up to ε, and the class $\mathfrak{C}$ of Banach spaces on which ε-multipliers are close to multiples of the identity operator. It can be shown that any Banach space in $\mathfrak{C}$ enjoys the isomorphic Banach-Stone property, and a sufficient condition to belong to $\mathfrak{C}$ is that some ultrapower of X has a trivial multiplier algebra. In particular, L^1-spaces have the isomorphic Banach-Stone property; note that $\lambda(L^\infty) = 1$ so that the previous method does not apply.

Another recent direction of research inquires into Banach-Stone theorems for spaces of weak* continuous functions $C(K,(X^*,w^*))$. In this case the operator space $L(X,C(K))$ can be represented as $C(K,(X^*,w^*))$ [178, p. 490], and so the Corollaries 1.14 and 1.15 contribute to this circle of ideas. Particular results, apart from these corollaries, can be found in the papers [61], [106], [108], [109] and their references.

We also refer to [338] and [341] for information on Banach-Stone theorems, and to [110], [277] and [278] for similar results on Bochner L^p-spaces, where the L^p-structure rather than the M-structure is essential.

Daugavet's equation. We have pointed out in Proposition 4.3 that, for an infinite dimensional Banach space X, $K(X)$ is not an L^p- or M-summand in any subspace of $L(X)$ properly containing it, for $p > 1$. On the other hand, Theorem 4.17 asserts that $K(X)$ is an M-ideal in $L(X)$ once it is an M-ideal in $K(X) \oplus \mathbb{K}\{Id\}$. Therefore the following result, due to Daugavet [145], is surprising:

$K(C[0,1])$ *is an* L*-summand in* $K(C[0,1]) \oplus \mathbb{K}\{Id\}$.

In a plain formula this result can be rephrased as

$$\|Id + T\| = 1 + \|T\| \qquad \forall T \in K(C[0,1]).$$

The above equation is nowadays referred to as Daugavet's equation. Today it is known that every weakly compact operator on X satisfies Daugavet's equation if $X = C(K)$, K without isolated points, or $X = L^1(\mu)$, μ without atoms. This theorem appears in the literature several times; instead of giving a complete account of the relevant references, we mention the papers [3] and [560] where this is done. Other papers on this subject are [1], [322] and [642].

Another result along these lines, proved, rediscovered and reproved in [177], [321], [2] and [560], states that, if X is a real $C(K)$- or L^1-space, then

$$\max \|Id \pm T\| = 1 + \|T\| \qquad \forall T \in L(X); \tag{$*$}$$

in other words, T or $-T$ satisfies Daugavet's equation.
We would like to offer one more argument for this which attempts to make the geometric background clear. Let E be a real Banach space. For $x \in E$ we let $I(x)$ be the intersection of all closed balls containing both x and $-x$. It is proved in [52, Ex. 2.3(e)] that $p(I(x)) \subset [-p(x), p(x)]$ for every extreme functional $p \in \operatorname{ex} B_{E^*}$. Hence, if $\|x\| = 1$ and $I(x) = B_E$, then $|p(x)| = 1$, from which we deduce

$$1 + p(y) = \max p(\pm x + y) \le \max \|x \pm y\|.$$

Thus $\max \|x \pm y\| = 1 + \|y\|$ for all $y \in E$. It is left to observe that $I(x) = B_E$ holds for $E = L(C(K))$ and $x = Id$, whence ($*$) for operators on $C(K)$-spaces. The case of L^1-spaces follows by duality. Likewise it follows for complex scalars that $\max\{\|Id + \theta T\| \mid |\theta| = 1\} = 1 + \|T\|$.

u-IDEALS OF COMPACT OPERATORS. In the Notes and Remarks to Chapter IV we presented a number of results on u-ideals from the important paper [263]. Now we resume the discussion of this notion and turn our interest to u-ideals of compact operators. It was in this connection that u-ideals were first introduced by Casazza and Kalton in [114]. Recall that a u-ideal is a closed subspace J of a Banach space E such that there is a projection P on E^* with kernel $J^\perp$ satisfying $\|Id - 2P\| = 1$. In case the scalars are complex and P satisfies $\|Id - (1+\lambda)P\| = 1$ for all $|\lambda| = 1$ (which amounts to saying that P is a hermitian projection), then J is called an h-ideal.
The study of u-ideals of compact operators is intimately related to the investigation of an unconditional version of the metric compact approximation property. Let us say that a separable Banach space X has the unconditional MCAP if there exists a sequence of compact operators (K_n) converging strongly to Id_X such that $\lim \|Id_X - 2K_n\| = 1$. (Note that this really implies the MCAP.) In the case of complex scalars we define the $\mathbb{C}$-unconditional MCAP by requiring that $\lim \|Id_X - (1+\lambda)K_n\| = 1$ for all $|\lambda| = 1$. Likewise, the unconditional MAP and the $\mathbb{C}$-unconditional MAP are introduced. These properties can be characterised in terms of unconditional expansions. By [263, Prop. 8.2] (see also [114, Th. 3.8]) X has the unconditional MCAP if and only if for all $\varepsilon > 0$ there is a sequence (A_n) of compact operators such that $\sum_{n=1}^\infty A_n x = x$ for all $x \in X$ and $\|\sum_{n=1}^m \varepsilon_n A_n\| \le 1 + \varepsilon$ for all $m \in \mathbb{N}$ and $|\varepsilon_n| = 1$. Let us observe that $\sum A_n x$ is unconditionally convergent, which explains why this version of the MCAP is called unconditional. Similar results hold for the remaining unconditional approximation properties. Thus we see that the conclusion of Theorem 4.21 holds under much weaker assumptions than those made there; note that X has the unconditional MCAP if $K(X)$ is an M-ideal in $L(X)$ by (a variant of) Lemma 4.22. It is clear that a Banach space with a 1-unconditional basis enjoys the unconditional MAP; on the other hand, the argument in [396] shows that $L^p[0,1]$, $1 < p < \infty$, $p \ne 2$, fails the unconditional MAP for its canonical norm, but clearly can be renormed to have it, because the Haar basis is unconditional. Unconditional versions of the AP have been employed by Kalton [364] and Feder [221],

[222], among others, to show that the compact operators between certain Banach spaces form an uncomplemented subspace in the space of all bounded linear operators.

Suppose now that a separable Banach space X has the unconditional MCAP (resp., if the scalars are complex, the $\mathbb{C}$-unconditional MCAP). Then one can show that X is a u-ideal in X^{**} and that $K(X)$ is a u-ideal in $L(X)$ (resp., they are h-ideals). The converse holds for reflexive X or, in the complex case which is better understood, if X^* is separable [263, Th. 8.3]. Note that $K(X)^{**} \cong L(X)$ for reflexive X with the CAP so that the preceding result actually deals with the embedding of $K(X)$ in its bidual in this case. The general problem of when $K(X)$ is a u- or h-ideal in $K(X)^{**}$ is more subtle. Here one can show for complex spaces with separable duals satisfying the $\mathbb{C}$-unconditional MCAP that $K(X)$ has property (u), actually $\varkappa_h(K(X)) = 1$ (see p. 213 for this notation), however $K(X)$ need not be an h-ideal in $K(X)^{**}$. The definite criterion to check this involves M-ideals in a decisive manner, since [263, Th. 8.6] asserts that, for a complex Banach space with a separable dual, $K(X)$ is an h-ideal in $K(X)^{**}$ if and only if X has the $\mathbb{C}$-unconditional MCAP and X is an M-ideal in X^{**}. Let us mention that the Orlicz sequence spaces and the preduals of the Lorentz sequence spaces discussed in Example III.1.4 provide examples where these conditions are fulfilled.