An elementary approach to the Daugavet equation

Dirk Werner

Abstract. Let $T\colon C(S) \to C(S)$ be a bounded linear operator. We present a necessary and sufficient condition for the so-called Daugavet equation

$$\|Id + T\| = 1 + \|T\|$$

to hold, and we apply it to weakly compact operators and to operators factoring through c_0. Thus we obtain very simple proofs of results by Foiaş, Singer, Pełczyński, Holub and others.

If E is a real Banach space, let us say that an operator $T\colon E \to E$ satisfies the *Daugavet equation* if

$$\|Id + T\| = 1 + \|T\|.$$

Daugavet [5] proved that every compact operator $T\colon C[0,1] \to C[0,1]$ satisfies this equation, and Foiaş and Singer [7] extended his result to weakly compact operators. Later, these theorems were rediscovered by Kamowitz [11] and Holub [9]. Pełczyński observed that the Foiaş-Singer argument can be used to prove the Daugavet equation for weakly compact operators on a $C(S)$-space provided S has no isolated points, cf. [7, p. 446]. (This restriction on S is easily seen to be necessary.) Holub also showed, with no assumption on T, that T or $-T$ fulfills the Daugavet equation for which Abramovich [1] gave another proof valid for general S rather than the unit interval. Yet another argument was suggested in [8, p. 343]. Dually, a number of authors have investigated the Daugavet equation for operators on an L^1-space; for more precise references we refer to [2] and [12]. In the latter paper Schmidt proved the Daugavet equation for weakly compact operators on an atomless L^1-space using Banach lattice techniques. (Actually, his result is more general than that.) A different class of operators was considered by Holub [10] who showed the Daugavet equation for the ideal of operators on $C[0,1]$ factoring through c_0; Ansari [4] generalised this result to such operators on $C(S)$-spaces, where S has no isolated points.

In this paper we suggest a unified and elementary approach to all the results just mentioned. Our basic idea is to represent an operator $T\colon C(S) \to C(S)$ by its stochastic kernel, that is the family of measures $(\mu_s)_{s\in S}$ defined by $\mu_s = T^*\delta_s$; i.e.,

$$\int_S f\,d\mu_s = \langle f, \mu_s\rangle = \langle Tf, \delta_s\rangle = (Tf)(s).$$

We then have $\|T\| = \sup_s \|\mu_s\|$, and the function $s \mapsto \mu_s$ is continuous for the weak* topology of $M(S) \cong C(S)^*$. The operator T is weakly compact if and only if $s \mapsto \mu_s$ is continuous for the weak topology of $M(S)$ (meaning the $\sigma(M(S), M(S)^*)$-topology), and T is compact if and only if $s \mapsto \mu_s$ is norm continuous; see [6, p. 490]. Note that the identity operator is represented by the family of Dirac measures $(\delta_s)_{s\in S}$.

A different approach to the Daugavet equation for weakly compact operators was taken by Abramovich, Aliprantis, and Burkinshaw [3] who used ideas from Banach lattice theory, and Ansari [4] was able to incorporate Holub's result on c_0-factorable operators into their scheme. However, these arguments seem to be less elementary than the very simple calculations presented here.

We finally mention the recent papers [2] and [13] whose results are not covered by this note.

Acknowledgement. I would like to thank the referee for pointing out Ansari's paper to me.

Proposition 1 *If S is a compact Hausdorff space and $T\colon C(S) \to C(S)$ is a bounded linear operator, then*

$$\max\{\|Id + T\|, \|Id - T\|\} = 1 + \|T\|.$$

Proof. Let $(\mu_s)_{s\in S}$ be the representing kernel of T. Then

$$\begin{aligned}
\max_{\pm} \|Id \pm T\| &= \max_{\pm} \sup_{s\in S} \|\delta_s \pm \mu_s\| \\
&= \sup_{s\in S} \max_{\pm} (|\delta_s \pm \mu_s|(\{s\}) + |\delta_s \pm \mu_s|(S\backslash\{s\})) \\
&= \sup_{s\in S} \max_{\pm} (|1 \pm \mu_s(\{s\})| + |\mu_s|(S\backslash\{s\})) \\
&= \sup_{s\in S} (1 + |\mu_s(\{s\})| + |\mu_s|(S\backslash\{s\})) \\
&= \sup_{s\in S} (1 + \|\mu_s\|) \;=\; 1 + \|T\|.
\end{aligned}$$

□

Corollary 2 *If E is an (AL)-space or an (AM)-space and $T\colon E \to E$ is a bounded linear operator, then*

$$\max\{\|Id + T\|, \|Id - T\|\} = 1 + \|T\|.$$

Proof. An (AL)-space E is representable as $L^1(\mu)$ for some localisable measure μ, hence E^* is representable as $L^\infty(\mu) \cong C(S)$. So the assertion follows from Proposition 1 by passing to T^*. If E is an (AM)-space, then E^* is an (AL)-space, and again we obtain the assertion by considering the adjoint operator. □

We now formulate a technical condition that will allow us to prove the Daugavet equation for weakly compact operators and for c_0-factorable operators.

Lemma 3 *Let S be a compact Hausdorff space and $T\colon C(S) \to C(S)$ a bounded linear operator with representing kernel $(\mu_s)_{s\in S}$. If the kernel satisfies*

$$\sup_{s\in U} \mu_s(\{s\}) \geq 0 \textit{ for all nonvoid open sets } U \subset S, \tag{$*$}$$

then

$$\|Id + T\| = 1 + \|T\|.$$

In fact, a necessary and sufficient condition for this to hold is

$$\sup_{\{s:\ \|\mu_s\| > \|T\| - \varepsilon\}} \mu_s(\{s\}) \geq 0 \qquad \forall \varepsilon > 0. \tag{$**$}$$

Proof. We have

$$\|Id + T\| = \sup_{s\in S} \|\delta_s + \mu_s\| = \sup_{s\in S}(|1 + \mu_s(\{s\})| + |\mu_s|(S\backslash\{s\}))$$

and

$$1 + \|T\| = \sup_{s\in S}(1 + \|\mu_s\|) = \sup_{s\in S}(1 + |\mu_s(\{s\})| + |\mu_s|(S\backslash\{s\}));$$

so problems with showing the Daugavet equation can only arise in case some of the $\mu_s(\{s\})$ are negative.

Given $\varepsilon > 0$, we now apply $(*)$ to the open set $U = \{s \in S\colon \|\mu_s\| > \|T\| - \varepsilon\}$ (that is, we apply $(**)$) and obtain

$$\begin{aligned} \|Id + T\| &\geq \sup_{s\in U} \|\delta_s + \mu_s\| \\ &= \sup_{s\in U}(|1 + \mu_s(\{s\})| + |\mu_s|(S\backslash\{s\})) \\ &\geq \sup_{\substack{s\in U \\ \mu_s(\{s\})\geq -\varepsilon}} (1 + \|\mu_s\| + \mu_s(\{s\}) - |\mu_s(\{s\})|) \\ &\geq 1 + \|T\| - \varepsilon + \sup_{\substack{s\in U \\ \mu_s(\{s\})\geq -\varepsilon}} (\mu_s(\{s\}) - |\mu_s(\{s\})|) \\ &\geq 1 + \|T\| - 3\varepsilon; \end{aligned}$$

hence T satisfies the Daugavet equation.

A similar calculation shows that $(**)$ is not only sufficient, but also necessary. $\Box$

Next, we deal with weakly compact operators.

Lemma 4 *If S is a compact Hausdorff space without isolated points and $T\colon C(S) \to C(S)$ is weakly compact, then T fulfills $(*)$ of Lemma 3.*

Proof. To prove this lemma we argue by contradiction. Suppose there is a nonvoid open set $U \subset S$ and some $\beta > 0$ such that

$$\mu_s(\{s\}) < -2\beta \qquad \forall s \in U.$$

At this stage we note that, for each $t \in S$, the function $s \mapsto \mu_s(\{t\})$ is continuous, since T is weakly compact. For $\mu \mapsto \mu(\{t\})$ is in $M(S)^*$ and, as noted in the introduction, $s \mapsto \mu_s$ is weakly continuous.

Returning to our argument we pick some $s_0 \in U$ and consider the set

$$U_1 = \{s \in U\colon |\mu_s(\{s_0\}) - \mu_{s_0}(\{s_0\})| < \beta\}$$

which—as we have just observed—is an open neighbourhood of s_0. Since s_0 is not isolated, there is some $s_1 \in U_1$, $s_1 \neq s_0$. We thus have

$$\mu_{s_1}(\{s_1\}) < -2\beta,$$

because $s_1 \in U$, and

$$\mu_{s_1}(\{s_0\}) < \mu_{s_0}(\{s_0\}) + \beta < -2\beta + \beta = -\beta.$$

In the next step we let

$$U_2 = \{s \in U_1\colon |\mu_s(\{s_1\}) - \mu_{s_1}(\{s_1\})| < \beta\} \ (\subset U).$$

Likewise, this is an open neighbourhood of s_1, hence there is some $s_2 \in U_2$, $s_2 \neq s_1$, $s_2 \neq s_0$. We conclude, using that $s_2 \in U$, $s_2 \in U_2$ and $s_2 \in U_1$,

$$\begin{array}{rcl} \mu_{s_2}(\{s_2\}) & < & -2\beta, \\ \mu_{s_2}(\{s_1\}) & < & -\beta, \\ \mu_{s_2}(\{s_0\}) & < & -\beta. \end{array}$$

Thus we inductively define a descending sequence of open sets $U_n \subset U$ and distinct points $s_n \in U$ by

$$\begin{array}{rcl} U_{n+1} & = & \{s \in U_n\colon |\mu_s(\{s_n\}) - \mu_{s_n}(\{s_n\})| < \beta\}, \\ s_{n+1} & \in & U_{n+1} \backslash \{s_0, \dots, s_n\} \end{array}$$

yielding

$$\mu_{s_n}(\{s_j\}) < -\beta \qquad \forall j = 0, \dots, n-1.$$

Consequently,

$$\|T\| \geq \|\mu_{s_n}\| \geq |\mu_{s_n}|(\{s_0, \dots, s_{n-1}\}) \geq n\beta \qquad \forall n \in \mathbf{N},$$

which furnishes a contradiction. $\square$

Lemmas 3 and 4 immediately yield the first main result of this note.

Theorem 5 *Suppose S is a compact Hausdorff space without isolated points. If $T\colon C(S) \to C(S)$ is weakly compact, then*

$$\|Id + T\| = 1 + \|T\|.$$

Corollary 6 *If μ is an atomless measure and $T\colon L^1(\mu) \to L^1(\mu)$ is weakly compact, then*

$$\|Id + T\| = 1 + \|T\|.$$

Proof. By changing measures if necessary we may assume that $L^1(\mu)^* \cong L^\infty(\mu)$ canonically. (If $L^1(\mu) \cong L^1(\nu)$ and μ is atomless, then so is ν, since atomless measure spaces are characterised by the fact that the unit balls of the corresponding L^1-spaces fail to possess extreme points.) Now $L^\infty(\mu)$ is isometric to some $C(S)$-space, where S does not contain any isolated point.

It remains to observe that T^* is weakly compact as well [6, p. 485] and to apply Theorem 5. $\square$

Remarks. (1) If T is compact, the proof of Lemma 4 can considerably be simplified. In fact, if $\mu_s(\{s\}) < -2\beta < 0$ on an open nonvoid set U, let us pick some $s \in U$ and consider the set

$$U_1 = \{t \in U\colon \|\mu_s - \mu_t\| < \beta\}.$$

Since T is compact, this is an open neighbourhood of s, and for each $t \in U_1$ we deduce that

$$\mu_s(\{t\}) \leq \mu_t(\{t\}) + |\mu_t(\{t\}) - \mu_s(\{t\})| < -2\beta + \|\mu_t - \mu_s\| < -\beta.$$

Since s is not isolated, there are infinitely many distinct points $t_1, t_2, \ldots \in U_1$, and we obtain $|\mu_s|(\{t_1, t_2, \ldots\}) = \infty$, a contradiction.

(2) The proof of Theorem 5 shows that weakly compact operators on $C_0(S)$, S locally compact without isolated points, satisfy the Daugavet equation.

(3) We also see immediately that positive operators on $C(S)$-spaces (and likewise on (AL)- and (AM)-spaces) satisfy the Daugavet equation.

(4) For weakly compact operators T on $C(S)$, represented by $(\mu_s)_{s\in S}$, the functions $\varphi_A\colon s \mapsto \mu_s(A)$, $A \subset S$ a Borel set, are continuous; in fact, weakly compact operators are characterised by this property [6, p. 493]. In Lemma 4 it is even enough to assume that only the functions $\varphi_{\{t\}}$, $t \in S$, are continuous, provided S has no isolated points. Hence also such operators satisfy the Daugavet equation. A special case of this situation (a trivial one, though) occurs if $\mu_s(\{t\}) = 0$ for all $s, t \in S$; see also the following remark.

(5) A particular class of operators for which $(*)$ of Lemma 3 is valid are those for which

$$\{t \in S\colon \mu_s(\{t\}) = 0 \ \forall s \in S\} \text{ is dense in } S. \qquad (***)$$

Since this class is seen to contain the *almost diffuse operators* of Foiaş and Singer, we have obtained their result that almost diffuse operators satisfy the Daugavet equation.

This last remark easily leads to Ansari's extension of Holub's theorem that operators on $C[0,1]$ factoring through c_0 satisfy the Daugavet equation.

Theorem 7 *If S is a compact Hausdorff space without isolated points and $T: C(S) \to C(S)$ factors through c_0, then*

$$\|Id + T\| = 1 + \|T\|.$$

Proof. Let $(\mu_s)_{s \in S}$ be the representing kernel of T. By remark (5) it is enough to show that

$$S' := \{t \in S: \mu_s(\{t\}) = 0 \ \forall s \in S\}$$

is dense in S. Let us write $T = T_2 T_1$ with bounded linear operators T_1: $C(S) \to c_0$, T_2: $c_0 \to C(S)$. We have

$$\begin{aligned}(T_1 f)(n) &= \int_S f\, d\rho_n \qquad \forall n \in \mathbf{N},\\ (T_2(a_n))(s) &= \sum_{n=1}^{\infty} \nu_s(n) a_n \qquad \forall s \in S\end{aligned}$$

for a sequence of measures ρ_n and a family $(\nu_s(n))_n$ of sequences in ℓ_1. Consequently,

$$\mu_s = \sum_{n=1}^{\infty} \nu_s(n) \rho_n.$$

Now $S' \supset \bigcap_n \{t \in S: \rho_n(\{t\}) = 0\}$, which is a set whose complement is at most countable. Since no point in S is isolated, countable sets are of the first category, and Baire's theorem implies that S' is dense. □

More remarks. (6) The same proof applies to operators that factor through a $C(K)$-space where K is a countable compact space, since on such spaces all regular Borel measures are discrete. We recall that there are countable compact spaces K such that $C(K)$ is not isomorphic to c_0.

(7) The Baire argument in Theorem 7 implies a very simple proof of Theorem 5 if in addition S is supposed to be separable. In fact, let us show that then $(***)$ of Remark (5) holds. The complement of the set spelt out there is $\{t \in S: \exists s \in S \ \mu_s(\{t\}) \neq 0\}$. Since $s \mapsto \mu_s(\{t\})$ is continuous, this is, with $\{s_1, s_2, \ldots\}$ denoting a countable dense subset of S, $\bigcup_n \{t \in S: \mu_{s_n}(\{t\}) \neq 0\}$ and hence a countable union of countable sets, i.e., of the first category. Again, $\{t \in S: \mu_s(\{t\}) = 0 \ \forall s \in S\}$ must be dense.

(8) We finally wish to comment on the case of complex scalars. All the results and proofs in this paper remain valid—mutatis mutandis—in the

setting of complex Banach spaces. In Proposition 1 the proper formulation of the conclusion is

$$\max_{|\lambda|=1} \|Id + \lambda T\| = 1 + \|T\|,$$

and $(*)$ in Lemma 3 should be replaced by

$$\sup_{s\in U}(|1+\mu_s(\{s\})| - (1+|\mu_s(\{s\})|)) \geq 0 \textit{ for all nonvoid open sets } U \subset S.$$

References

[1] Y. Abramovich. *A generalization of a theorem of J. Holub.* Proc. Amer. Math. Soc. **108** (1990) 937–939.

[2] Y. Abramovich. *New classes of spaces on which compact operators satisfy the Daugavet equation.* J. Operator Theory **25** (1991) 331–345.

[3] Y. Abramovich, C. D. Aliprantis, and O. Burkinshaw. *The Daugavet equation in uniformly convex Banach spaces.* J. Funct. Anal. **97** (1991) 215–230.

[4] S. I. Ansari. *Essential disjointness and the Daugavet equation.* Houston J. Math. **19** (1993) 587–601.

[5] I. K. Daugavet. *On a property of completely continuous operators in the space C.* Uspekhi Mat. Nauk **18.5** (1963) 157–158. (Russian).

[6] N. Dunford and J. T. Schwartz. *Linear Operators. Part 1: General Theory.* Interscience Publishers, New York, 1958.

[7] C. Foiaş and I. Singer. *Points of diffusion of linear operators and almost diffuse operators in spaces of continuous functions.* Math. Z. **87** (1965) 434–450.

[8] P. Harmand, D. Werner, and W. Werner. *M-Ideals in Banach Spaces and Banach Algebras.* Lecture Notes in Math. 1547. Springer, Berlin-Heidelberg-New York, 1993.

[9] J. R. Holub. *A property of weakly compact operators on $C[0,1]$.* Proc. Amer. Math. Soc. **97** (1986) 396–398.

[10] J. R. Holub. *Daugavet's equation and ideals of operators on $C[0,1]$.* Math. Nachrichten **141** (1989) 177–181.

[11] H. Kamowitz. *A property of compact operators.* Proc. Amer. Math. Soc. **91** (1984) 231–236.

[12] K. D. Schmidt. *Daugavet's equation and orthomorphisms.* Proc. Amer. Math. Soc. **108** (1990) 905–911.

[13] P. Wojtaszczyk. *Some remarks on the Daugavet equation.* Proc. Amer. Math. Soc. **115** (1992) 1047–1052.

I. Mathematisches Institut
Freie Universität Berlin
Arnimallee 3
D-14 195 Berlin
Federal Republic of Germany
e-mail: werner@math.fu-berlin.de