The grade of an M-ideal

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Abstract. We assign two parameters to an M-ideal J in a Banach space X, called the lower and upper grade, which are defined in terms of the size of balls contained in the set of best approximants from J. These quantities serve to measure how far J resembles an M-summand, and they enter into geometric descriptions in various ways. For instance they allow estimates of the interior of the metric complement of J and of lower bounds for Lipschitz projections onto M-ideals.

1. Introduction

In this paper we consider the notion of a grade of an M-ideal which serves to distinguish between M-ideals and M-summands in a quantitative way. Recall that a subspace J of a Banach space X is called an *M-summand* if J is the range of an M-projection, i.e., a linear projection $Q : X \to X$ satisfying

$$\|x\| = \max\{\|Qx\|, \|x - Qx\|\} \qquad \forall x \in X.$$

J is an *M-ideal* if $J^\perp := \{x^* \in X^* \mid x^*|_J = 0\}$ is the range of an L-projection, i.e., a linear projection $P : X^* \to X^*$ satisfying

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\| \qquad \forall x^* \in X^*.$$

The kernel of this projection can be identified with J^* so that J^* is isometric with a subspace of X^*. More precisely, the 'orthogonal' complement of $J^\perp$ is

$$\ker P = J^\# := \{x^* \in X^* \mid \|x^*\| = \|x^*|_J\|\}. \tag{1.1}$$

Clearly, M-summands are M-ideals, but the converse does not hold; M-ideals which are not M-summands will be termed *proper*. For example, in

$X = C(K)$, K a compact Hausdorff space, the M-ideals are precisely the subspaces

$$J_D = \{f \in C(K) \mid f|_D = 0\}$$

for some closed subset $D \subset K$, and J_D is an M-summand if and only if D is clopen (i.e., closed and open). A detailed exposition of M-ideal theory can be found in [3].

In order to distinguish between M-ideals and M-summands, we attach a number $g \in [0, 1]$ to an M-ideal $J \subset X$. Actually, we propose a 'lower grade' g_* and an 'upper grade' g^* for that purpose, with $g^* = 0$ characterising M-summands. We present here, in a coherent form, results related to these notions, some of which can also be found in [3]. (In fact, we take the chance to correct an invalid statement from [3, p. 54], cf. (2.2) below.) In addition, we discuss pertinent examples.

We use standard notation such as B_X for the closed unit ball of X, $\operatorname{ex} C$ for the set of extreme points of C and $L(X, Y)$ (resp. $K(X, Y)$) for the space of bounded (resp. compact) linear operators from X into Y. The symbol $B_X(x, r)$ denotes the closed ball in X with centre x and radius r.

2. The lower grade

Let us first recall some notions from approximation theory. A subspace J of a Banach space X is called *proximinal* if

$$\forall x \in X \quad \exists y \in J \quad \|x - y\| = d(x, J) := \inf\{\|x - \xi\| \mid \xi \in J\}.$$

The set of all such y is called the set of best approximants and denoted by $P_J(x)$. Thus J is proximinal if and only if $P_J(x) \neq \emptyset$ for all $x \in X$. The set-valued map P_J is called the *metric projection.* The *metric complement* J^θ is defined as

$$J^\theta = \{x \in X \mid \|x\| = d(x, J)\} = \{x \in X \mid 0 \in P_J(x)\}.$$

It is a basic fact in M-ideal theory that M-ideals are proximinal [3, Proposition II.1.1].

The following definition comes from [2].

Definition 2.1 A closed convex bounded subset B of a Banach space Y is called a *pseudoball of radius* r if its diameter is $2r > 0$ and if for each finite collection $y_1, \dots, y_n$ of points with $\|y_i\| < r$ there is $y \in B$ such that

$$y + y_i \in B \text{ for } i = 1, \dots, n.$$

If $\rho = \sup\{s \geq 0 \mid B \text{ contains a ball of radius } s\}$, then the *grade* of B is defined to be

$$g(B) = 1 - \frac{\rho}{r}.$$

A pseudoball which is not a ball will be called *proper*. Singletons are considered as pseudoballs with radius 0 and grade 0.

Equivalently B is a pseudoball of radius $r > 0$ if and only if

$$\bigcap_i (y_i + B) \neq \emptyset$$

for each finite family $y_1, \dots, y_n$ satisfying $\|y_i\| < r$.

Note that B is a closed ball of radius $r > 0$ if and only if

$$\bigcap_{\|y\|<r} (y + B) \neq \emptyset,$$

in which case the intersection consists of the centre of the ball. In a pseudoball there is only a 'centre' for any finite set of directions. Also, note that $g(B) = 0$ if and only if B is a ball. Thus, the pseudoball B_2 should be considered as 'more proper' than B_1 if $g(B_1) \leq g(B_2)$.

To see a simple example, let $0 \leq s \leq 1$. It is easy to check that

$$B_s = \{(s_n) \in c_0 \mid |s_n - s| \leq 1 \ \forall n \in \mathbb{N}\} \tag{2.1}$$

is a pseudoball in c_0 with radius 1 and $g(B_s) = s$.

The next two results explain our interest in pseudoballs here.

Proposition 2.2

(a) *Let J be an M-ideal in X, and let $x \in X$. Then $P_J(x)$ is a pseudoball in J with radius $d(x, J)$.*

(b) *If $P_J(x)$ is a pseudoball of radius $d(x, J)$ for all $x \in X$, then J is an M-ideal.*

Proof. See [5] or [3, Prop. II.1.3]. □

Conversely, every pseudoball arises as in Proposition 2.2(a).

Theorem 2.3 *Let B be a pseudoball of radius 1 in Y. Then there are a Banach space X containing Y with $\dim X/Y = 1$ and $x \in X$ with $d(x, Y) = 1$ such that Y is an M-ideal in X and $P_Y(x) = B$.*

Proof. See [2] or [3, Th. II.3.10]. □

The following result has first appeared in [2].

Theorem 2.4 *For a closed convex bounded subset $B \subset Y$ and $r \geq 0$, equivalence between* (i) *and* (ii) *holds:*

(i) *B is a pseudoball of radius r.*

(ii) *The weak* closure of B in Y^{**} is a ball with radius r (and centre y_B^{**}, say).*

In this case

$$r \cdot g(B) = d(y_B^{**}, Y).$$

Proof. See [2] or [3, Th. II.1.6]. □

We shall now identify the centre y_B^{**}.

Corollary 2.5 *Let J be an M-ideal in X, and let P be the associated M-projection from X^{**} onto $J^{\perp\perp}$. Consider the pseudoball $B = P_J(x)$ for some $x \in X$. Then $Px = y_B^{**}$, the centre of $\overline{B}^{w*}$. More precisely: the canonical isometry i_X^{**} from J^{**} onto $J^{\perp\perp}$ maps y_B^{**} onto Px. Moreover* (*if* $d(x, J) = 1$)

$$g(P_J(x)) = d(Px, J) = \inf_{y \in J} \sup_{y^* \in B_{J^*}} |\langle y^*, x - y\rangle|. \tag{2.2}$$

Proof. We consider J^* via unique Hahn-Banach extensions as a subspace of X^*, cf. (1.1). We have for $y^* \in J^*$

$$y^*(B_{J^{**}}(y_B^{**}, 1)) = B_{\mathbb{K}}(\langle y_B^{**}, y^*\rangle, \|y^*\|).$$

On the other hand we know from Theorem 2.4 (assuming without loss of generality $d(x, J) = 1$)

$$B_{\mathbb{K}}(\langle y_B^{**}, y^*\rangle, \|y^*\|) = \overline{y^*(B)} \subset \overline{y^*(B_X(x, 1))} = B_{\mathbb{K}}(\langle y^*, x\rangle, \|y^*\|).$$

Hence $\langle y_B^{**}, y^*\rangle = \langle x, y^*\rangle$ for all $y^* \in J^*$, which gives $y_B^{**} = Px$.

That $g(P_J(x)) = d(Px, J)$ now follows from Theorem 2.4. Also, the M-projection P is the adjoint of the L-projection P_* on X^* with range J^*. Therefore

$$\sup_{y^* \in B_{J^*}} |\langle y^*, x\rangle| = \sup_{x^* \in B_{X^*}} |\langle P_* x^*, x\rangle| = \sup_{x^* \in B_{X^*}} |\langle x^*, Px\rangle| = \|Px\| \tag{2.3}$$

and

$$d(Px, J) = \inf_{y \in J} \|Px - y\| = \inf_{y \in J} \|P(x - y)\| = \inf_{y \in J} \sup_{y^* \in B_{J^*}} |\langle y^*, x - y \rangle|. \quad \square$$

The grade of the pseudoballs $P_J(x)$ is the appropriate tool for investigating the (possibly empty) interior of the metric complement J^θ of an M-ideal J. The following proposition answers a question raised in [4] where it is conjectured that an M-ideal is an M-summand as soon as its metric complement has an interior point. Although this conjecture could have been refuted on the basis of the familiar M-ideals J_D in $C(K)$ (cf. also Example 2.9), we now have a precise estimate of the interior of J^θ by this proposition.

Proposition 2.6 *Let J be an M-ideal in X, and let $D = \{x \in X \mid \|x\| = d(x, J) = 1\}$.*

(a) *If $B_J(y_0, 2r) \subset P_J(x_0)$ for some $x_0 \in D$, then $B_X(x_0 - y_0, r) \subset J^\theta$.*

(b) *If $B_X(x_0, r) \subset J^\theta$ for some $x_0 \in D$, then $P_J(x_0)$ contains a ball of radius $2r - \varepsilon$ for whatever $\varepsilon > 0$.*

Therefore, J^θ has empty interior if and only if

$$g_*(J, X) := \inf\{g(P_J(x)) \mid d(x, J) = 1\} = 1.$$

Proof. (a) We may assume that $y_0 = 0$. Define

$$|x| = \sup\{|\langle x, y^* \rangle| \mid y^* \in B_{J^*}\}. \tag{2.4}$$

This is a seminorm, and we have

$$\|x\| = \max\{|x|, d(x, J)\}. \tag{2.5}$$

In fact, we may write $X^* = J^\perp \oplus_1 J^*$ from which we easily conclude that $\operatorname{ex} B_{X^*} = \operatorname{ex} B_{J^\perp} \cup \operatorname{ex} B_{J^*}$. It is left to observe that $\|x\| = \sup\{|\langle x, x^* \rangle| \mid x^* \in \operatorname{ex} B_{X^*}\}$, $|x| = \sup\{|\langle x, y^* \rangle| \mid y^* \in \operatorname{ex} B_{J^*}\}$, and $d(x, J) = \sup\{|\langle x, x^* \rangle| \mid x^* \in \operatorname{ex} B_{J^\perp}\}$.

Now let $z \in X$, $\|z - x_0\| < r$. We wish to show that $|z| < d(z, J)$ (so that $z \in J^\theta$), then (a) will follow from the closedness of J^θ. First note that $|z - x_0| < r$ and $d(z - x_0, J) < r$ by (2.5). Secondly, we have

$$|x_0| \le 1 - 2r. \tag{2.6}$$

To see this, choose, given $\varepsilon > 0$, $y^* \in S_{Y^*}$ such that

$$|x_0| \le \langle y^*, x_0 \rangle + \varepsilon.$$

Next choose $y \in B_J$ such that

$$\langle y^*, y \rangle \le -1 + \varepsilon.$$

Note $2ry \in P_J(x_0)$, i.e., $\|x_0 - 2ry\| = 1$. The estimate

$$|x_0| \le \langle y^*, x_0 \rangle + \varepsilon = \langle y^*, x_0 - 2ry \rangle + 2r\langle y^*, y \rangle + \varepsilon \le 1 + 2r(-1 + \varepsilon) + \varepsilon$$

now proves (2.6).

Altogether one obtains

$$|z| < |x_0| + r \le 1 - r = d(x_0, J) - r < d(z, J),$$

as desired.

(b) Let P be the M-projection from X^{**} onto $J^{\perp\perp} \cong J^{**}$. In view of Theorem 2.4 and Corollary 2.5 we have to show that $d(Px_0, J) \le 1 - 2r$. This is implied by (cf. (2.2))

$$|\langle y^*, x_0 \rangle| \le 1 - 2r \qquad \forall y^* \in B_{J^*}.$$

To prove this for some given $y^* \in B_{J^*}$, fix $\varepsilon > 0$ and choose $y \in \operatorname{int} B_J$ such that $\langle y^*, y \rangle$ is real and so close to 1 that

$$|\langle y^*, y \rangle \pm \alpha| \le 1 \text{ only if } |\alpha| \le \varepsilon.$$

In particular $\langle y^*, y \rangle \ge 1 - \varepsilon$.

Since $P_J(x_0)$ is a pseudoball (Proposition 2.2), we may find $\overline{y} \in P_J(x_0)$ with $\overline{y} \pm y \in P_J(x_0)$. Let us first observe

$$|\langle y^*, \overline{y} \rangle - \langle y^*, x_0 \rangle| \le \varepsilon. \tag{2.7}$$

In fact,

$$\langle y^*, \overline{y} \rangle \pm \langle y^*, y \rangle \in y^*(P_J(x_0)) \subset B_{\mathbb{K}}(\langle y^*, x_0 \rangle, 1)$$

(cf. Theorem 2.4) so that (2.7) follows. On the other hand, for $z = (1 - r)x_0 + r\overline{y}$ we have

$$y^*(z) = y^*(z + ry) - ry^*(y) \le (1 - r) - r(1 - \varepsilon) \tag{2.8}$$

where we used $\|x_0 - (z + ry)\| \le r$ since $\overline{y} + y \in P_J(x_0))$, consequently

$$\|z + ry\| = d(z + ry, J) = d((1 - r)x_0, J) = 1 - r.$$

(2.7) and (2.8) together give our claim. $\square$

The previous result suggests the following definition.

Definition 2.7 If J is an M-ideal in X, we call the number

$$g_*(J, X) = \inf\{g(P_J(x)) \mid d(x, J) = 1\}$$

the *lower grade* of J in X.

We shall see in Example 3.2 that the infimum in the above definition need not be attained.

In order to compute $g_*(J, X)$ the following lemma is useful. Recall that the Dixmier characteristic of a subspace $V \subset X^*$ is defined to be

$$r(V, X^*) = \sup\{\rho \mid \rho B_{X^*} \subset \overline{B_V}^{w*}\}.$$

Lemma 2.8 *Let J be an M-ideal in X with codimension 1. Then*

$$g_*(J, X) = r(J^*, X^*).$$

Proof. We introduce the canonical operator

$$I_{X,J^*} : X \to J^{**}, \quad \langle I_{X,J^*}(x), y^* \rangle = \langle y^*, x \rangle.$$

The definition of the characteristic implies that I_{X,J^*} is an (into-) isomorphism with $\|I_{X,J^*}^{-1}\| = r(J^*, X^*)^{-1}$ if the characteristic is positive. Therefore,

$$r(J^*, X^*) = \inf_{\substack{x \in X \\ \|x\|=1}} \sup_{\substack{y^* \in J^* \\ \|y^*\| \le 1}} |\langle y^*, x \rangle|, \tag{2.9}$$

and this formula also holds in case $r(J^*, X^*) = 0$. In the notation of (2.4) we thus have $r(J^*, X^*) = \inf_{\|x\|=1} |x|$, and (2.5) and (2.3) imply that in fact

$$r(J^*, X^*) = \inf_{\substack{\|x\|=1 \\ d(x,J)=1}} |x| = \inf_{\substack{\|x\|=1 \\ d(x,J)=1}} \|Px\|.$$

On the other hand, $g(P_J(x))$ is independent of $x \in X \backslash J$, provided $d(x,J) = 1$, since $\dim X \backslash J = 1$. Therefore, if $\|x\| = d(x,J) = 1$, then

$$g_*(J,X) = g(P_J(x)) = d(Px,J) \le \|Px\|$$

by (2.3) and so $g_*(J,X) \le r(J^*,X^*)$. To prove the converse inequality, let x as above and $y \in J$. Then $\|x-y\| \ge 1$ and for $\xi = (x-y)/\|x-y\|$ we have

$$\|P\xi\| = \frac{\|Px-y\|}{\|x-y\|} \le \|Px-y\|,$$

which proves that $r(J^*,X^*) \le g_*(J,X)$. □

Example 2.9 For the M-ideals $J_D \subset C(K)$ we have

$$\begin{aligned} g_*(J_D, C(K)) &= 0 \quad \text{if } D \text{ has nonempty interior,} \\ g_*(J_D, C(K)) &= 1 \quad \text{otherwise.} \end{aligned}$$

If D has nonempty interior, we may pick a continuous function $f \neq 0$ with $\operatorname{supp} f \subset D$. Hence J_D is an M-summand in $Y := J_D \oplus \mathbb{K}\{f\}$ so that $r(J_D^*, Y^*) = 0$. On the other hand, if $\operatorname{int} D = \emptyset$, then $K \backslash D$ is dense, and for every $f \in C(K) \backslash J_D$, with Y as above, the canonical operator I_{Y,J_D^*} is an isometry which implies that $r(J_D^*, Y^*) = 1$. An appeal to Lemma 2.8 yields the desired result.

Example 2.10 We will show next that for function algebras the same dichotomy as for $C(K)$-spaces arises, namely the lower grade of an M-ideal in a function algebra $A \subset C(K)$ is either 0 or 1. Let us recall that precisely the subspaces $J_D \cap A = \{f \in A \mid f|_D = 0\}$, with D a p-set, are the M-ideals of A; see [3, p. 15 and Th. V.4.2]. We may and shall assume that K is the Shilov boundary of A so that the set of p-points is dense in K. Then we have

$$\begin{aligned} g_*(J_D \cap A, A) &= 0 \quad \text{if } D \text{ has nonempty interior,} \\ g_*(J_D \cap A, A) &= 1 \quad \text{otherwise.} \end{aligned}$$

Let $J = J_D \cap A$. Suppose first that D has nonempty interior. Let $U \subset D$ be open and $t \in U$ be a p-point. Then there are functions $f_n \in A$ such that $f_n(t) = 1$, $\|f_n\| \le 2$ and $|f_n| \le 1/n$ off U, in particular off D (see [3, p. 15]). If $\mu \in J^*$, $\|\mu\| = 1$, then μ can be represented by a measure of norm 1 that is supported on $K \backslash D$. Therefore

$$\|I_{A,J^*}(f_n)\| = \sup_{\mu \in B_{J^*}} |\langle \mu, f_n \rangle| \to 0,$$

and the canonical operator I_{A,J^*} has no bounded inverse. Lemma 2.8 and (2.9) imply that $g_*(J,A) = 0$.

If $\operatorname{int} D = \emptyset$, then by the same token $g_*(J,A) = 1$, since the functionals $\mu_k = \delta_k|_A$, $k \in K\backslash D$, are in J^* and have norm 1 and since $K\backslash D$ is dense:

$$\|f\| = \sup_{k\notin D} |f(k)| = \sup_{k\notin D} |\langle \mu_k, f\rangle| \leq \sup_{\mu\in B_{J^*}} |\langle \mu, f\rangle| = \|I_{A,J^*}(f)\|.$$

Example 2.11 If J is an M-ideal in its bidual (see Chapter III in [3] for examples), then $g_*(J, J^{**}) = 1$; this follows from Lemma 2.8 and Goldstine's theorem.

Example 2.12 If $K(X,Y)$ is an M-ideal in $L(X,Y)$ (see Chapter VI in [3]) and $K(X,Y) \neq L(X,Y)$, then $g_*(K(X,Y), L(X,Y)) = 1$. To show this it is enough to prove that the unit ball of $K(X,Y)^*$ is weak* dense in the unit ball of $L(X,Y)^*$, by Lemma 2.8. If this were not the case, then there would be some operator $T \in L(X,Y)$, $\psi \in B_{L(X,Y)^*}$ and $\alpha \geq 0$ such that

$$\langle \psi, T\rangle = 1 > \alpha \geq \langle \varphi, T\rangle \qquad \forall \varphi \in B_{K(X,Y)^*}.$$

In particular, we have $\|T\| \geq 1$. But checking the above inequality for all functionals $\varphi : S \mapsto \langle Sx, y^*\rangle$, $x \in B_X$, $y^* \in B_{Y^*}$, yields $\|T\| \leq \alpha < 1$: a contradiction.

We now turn to geometric descriptions of M-ideals, especially of those of codimension 1. If J is an M-ideal in X and $\dim X/J = 1$, then from what was observed in the proof of Proposition 2.6 it follows that $J = \ker p$ with $p \in \operatorname{ex} B_{X^*}$ and $d(x,J) = |p(x)|$. Moreover p attains its norm since J is proximinal.

The first result extends the idea of a cylindrically shaped unit ball from M-summands to M-ideals: the lid L of the cylinder is the translate of a pseudoball in J.

Proposition 2.13 *Let J be an M-ideal in X, $\dim X/J = 1$, and*

$$L := \{x \in X \mid \|x\| = 1 = p(x)\}$$

with $X^* = J^* \oplus_1 \mathbb{K}p$. Then

$$\operatorname{int} B_X \subset \operatorname{co} \bigcup_{|\theta|=1} \theta L \subset B_X. \tag{2.10}$$

Proof. The second inclusion is obvious. To prove the first for real scalars take x with $\|x\| < 1$ and put $\alpha := p(x)$. Then $|\alpha| < 1$ and $\alpha = \lambda \cdot 1 + (1-\lambda)\cdot(-1)$ for a suitable $\lambda \in (0,1)$. We will get x as a convex combination of elements from L and $-L$ by an application of the strict 2-ball property of M-ideals; see [3, Th. I.2.2]. To use this argument we have to translate x into J.

Choose $x_0 \in X$ with $\|x_0\| = 1$ and $p(x_0) = -1$. Then $y := x + \alpha x_0 \in J = \ker p$. Since $d(\lambda x_0, J) = |p(\lambda x_0)| = \lambda$ and $d(y + (1-\lambda)x_0, J) = |p(y + (1-\lambda)x_0)| = 1 - \lambda$ each of the two balls

$$B_X(\lambda x_0, \lambda) \quad \text{and} \quad B_X(y + (1-\lambda)x_0, 1-\lambda) \tag{2.11}$$

meets J. The inequality

$$\|\lambda x_0 - [y + (1-\lambda)x_0]\| = \|\alpha x_0 - y\| < 1$$

shows that the interiors of the two balls have nonempty intersection. Hence, by the strict 2-ball property

$$B_X(\lambda x_0, \lambda) \cap B_X(y + (1-\lambda)x_0, 1-\lambda) \cap J \neq \emptyset. \tag{2.12}$$

Putting $P := P_J(x_0) = B_X(x_0, 1) \cap J$ we see that

$$\lambda P \cap (y + (1-\lambda)P) \neq \emptyset.$$

So there are $x_1, x_2 \in P$ with $\lambda x_1 = y + (1-\lambda)x_2$, i.e., $y = \lambda x_1 + (1-\lambda)(-x_2)$. With the definition of y and λ this yields the desired representation

$$x = \lambda(x_1 - x_0) + (1-\lambda)(-x_2 + x_0) \in \lambda L + (1-\lambda)(-L).$$

For $\mathbb{K} = \mathbb{C}$ choose $|\theta| = 1$ such that $p(\theta x)$ is real. The argument for real scalars then shows that $\theta x \in \mathrm{co}(L \cup -L)$. □

It is clear from the above proof that we get equality in the right-hand inclusion of (2.10) if J even has the so-called strong 2-ball property, which means that to conclude (2.12) it is enough to assume that the two balls in (2.11) meet instead of requiring that the interiors of these balls have nonempty intersection.

Let us record two consequences of Proposition 2.13. We get a statement analogous to (2.10) for M-ideals of arbitrary codimensions if we replace L by the set D from Proposition 2.6.

Corollary 2.14 *Let J be an M-ideal in X, and $D = \{x \in X \mid \|x\| = 1 = d(x, J)\}$. Then*

$$\operatorname{int} B_X \subset \operatorname{co} D \subset B_X.$$

Proof. This follows from Proposition 2.13 by observing that every point in $\operatorname{int} B_X$ is in a space $Y = J \oplus \mathbb{K}\{x_0\}$, J is an M-ideal in Y, and $\theta L \subset D$. □

The sets θL serve as substitutes for B_J also in another way.

Corollary 2.15 *With the assumption and notation from Proposition 2.13 we have for $f \in J^*$ and $|\theta| = 1$*

$$\|f\| = \sup\{\operatorname{Re} f(x) \mid x \in \theta L\}.$$

Proof. From Proposition 2.13 we obtain for $g \in X^*$

$$\|g\| = \sup\{\operatorname{Re} g(x) \mid x \in \gamma L,\ |\gamma| = 1\}.$$

Using this and $\|f + \theta^{-1} p\| = \|f\| + 1$, we find, given $\varepsilon > 0$, an $x = \gamma y \in \gamma L$ such that

$$\operatorname{Re}(f + \theta^{-1} p)(x) = \operatorname{Re}[f(x) + \theta^{-1}\gamma] > \|f\| + 1 - \varepsilon.$$

Since $\operatorname{Re} f(x) \le \|f\|$, this yields $\operatorname{Re} \theta^{-1}\gamma \ge 1 - \varepsilon$, hence $|\theta - \gamma| < \sqrt{2\varepsilon}$. So

$$\operatorname{Re} f(\theta y) = \operatorname{Re} f((\theta - \gamma)y) + \operatorname{Re} f(x) > -\sqrt{2\varepsilon}\|f\| + \|f\| - \varepsilon. \qquad \square$$

The next result shows the limitations of the idea of a cylindrical unit ball: for M-ideals with lower grade 1 there is nothing above the lid and nothing below the bottom. The reader is advised to draw a two-dimensional picture to understand the reformulation of this claim in the second statement of the following proposition.

Proposition 2.16 *Let J be an M-ideal in X. Then $g_*(J, X) = 1$ if and only if for all $x \in X$, $\|x\| > 1$, there is $y \in \operatorname{co}(B_X \cup \{x\}) \cap J$ with $\|y\| > 1$.*

Proof. "$\Rightarrow$": By the definition of g_* and Lemma 2.8, J is an M-ideal in $Y = Y_x = J \oplus \mathbb{K}\{x\}$ and B_{J^*} is $\sigma(Y^*, Y)$-dense in B_{Y^*}. Writing $Y^* = J^* \oplus_1 \mathbb{K}\{p\}$, we may assume that $p(x) =: \alpha > 0$; the claim is trivial for $x \in J$ ($= \ker p$).

Choose $g \in S_{Y^*}$ with $g(x) = \|x\| > 1$. Because of $\overline{B_{J^*}}^{w*} = B_{Y^*}$ and the w^*-lower semicontinuity of the norm, there is

$$f \in S_{J^*} \quad \text{with} \quad f(x) =: 1 + \eta > 1.$$

Applying Corollary 2.15 with $\theta = -1$ to f we find

$$x_0 \in S_Y \quad \text{with} \quad p(x_0) = -1 \text{ and } \operatorname{Re} f(x_0) > 1 - \frac{\eta}{\alpha}.$$

For $y = \dfrac{1}{\alpha+1}\, x + \dfrac{\alpha}{\alpha+1}\, x_0 \in \operatorname{co}(B_Y \cup \{x\}) \subset \operatorname{co}(B_X \cup \{x\})$ we obtain

$$p(y) = \frac{1}{\alpha+1}\, \alpha + \frac{\alpha}{\alpha+1}(-1) = 0, \text{ i.e., } y \in J$$

and

$$\operatorname{Re} f(y) > \frac{1}{\alpha+1}(1+\eta) + \frac{\alpha}{\alpha+1}\Bigl(1 - \frac{\eta}{\alpha}\Bigr) = 1, \text{ i.e., } \|y\| > 1.$$

"$\Leftarrow$": If $g_*(J,X) < 1$, then, again by the definition of g_* and Lemma 2.8, there is a $z \in X$ such that, with $Y = J \oplus \mathbb{K}\{z\}$, B_{J^*} is not $\sigma(Y^*,Y)$-dense in B_{Y^*}. Separating $\overline{B_{J^*}}^{w*}$ from p we find for a suitable $x \in Y$ $(\subset X)$

$$\sup_{f \in B_{J^*}} |f(x)| = 1 < p(x) \le \|x\|.$$

For $x_0 \in B_X$ and $f \in B_{J^*}$ this yields

$$|f(\lambda x + (1-\lambda)x_0)| \le 1,$$

therefore, if $\lambda x + (1-\lambda)x_0 \in J$, then $\|\lambda x + (1-\lambda)x_0\| \le 1$. □

3. The upper grade

We will now attach another number to an M-ideal J in X.

Definition 3.1 If J is an M-ideal in X, we call the number

$$g^*(J,X) = \sup\{g(P_J(x)) \mid d(x,J) = 1\}$$

the *upper grade* of J in X.

As a result of Theorem 2.4 and Corollary 2.5, the condition $g^*(J,X)=0$ characterises M-summands among M-ideals.

We also note in the case $\dim X/J = 1$ that $g_*(J,X) = g^*(J,X) = g(P_J(x))$ whenever $d(x,J)=1$ and that $g_*(Y,X)=g^*(Y,X)=g(B)$ in the context of Theorem 2.3.

If J is an M-ideal in X, then a fortiori it is an M-ideal in $Y_x = J \oplus \mathbb{K}\{x\}$ for each $x \in X \backslash J$. If $g(P_J(x)) = 0$, then J is an M-summand in Y_x, and if $g(P_J(x)) = 1$, then J is far from being an M-summand. Thus, if $g_*(J,X)=1$, this indicates that J behaves like a 'very' proper M-ideal in every direction x, whereas $g^*(J,X)=1$ only yields the existence of one such direction.

Note that $g_*(J,X) \le g^*(J,X) \le 1$ so that $g^*(J,J^{**})=1$ if J is an M-ideal in J^{**}, by Example 2.11, and $g^*(K(X,Y),L(X,Y))=1$ in the setting of Example 2.12. We also have

$$\begin{aligned} g^*(J_D, C(K)) &= 0 \quad \text{if } D \text{ is clopen,} \\ g^*(J_D, C(K)) &= 1 \quad \text{otherwise.} \end{aligned}$$

If D is clopen, then J_D is an M-summand, and the upper grade is 0. If D is not clopen, we will argue that the canonical operator I_{Y,J_D^*}, for $Y = J_D \oplus \mathbb{K}\{\mathbf{1}\}$, is an isometry. In fact, for $f = g + \lambda \mathbf{1} \in Y$ we have, since $\overline{K\backslash D} \cap D \neq \emptyset$, that $\sup_{t\in K\backslash D} |f(t)| \ge |\lambda|$ so that $\|f\|_\infty = \sup_{t \in K\backslash D} |f(t)|$ which is enough to prove our claim.

The following example shows that neither the infimum in the definition of the lower grade nor the supremum in the definition of the upper grade need be attained.

Example 3.2 If $0 < s \le 1$ let

$$X_s = (c, \|\ \|_s) \qquad \text{where} \qquad \|(\xi_n)\|_s = \max\{\|(\xi_n)\|_\infty, (1/s)|\xi_\infty|\}$$

and $\xi_\infty = \lim \xi_n$. Define for a sequence (s_n) in $(0,1)$

$$X = \Big(\oplus \sum X_{s_n}\Big)_{c_0} \qquad \text{and} \qquad J = \Big(\oplus \sum c_0\Big)_{c_0}.$$

Then J is an M-ideal in X and

(a) if $\inf s_n = 0$ then $g_*(J,X) = 0$, but $g_*(J,Y) > 0$ for all Y with $J \subset Y \subset X$ and $\dim Y/J = 1$,

(b) if $\sup s_n = 1$ then $g^*(J,X) = 1$, but $g^*(J,Y) < 1$ for all Y with $J \subset Y \subset X$ and $\dim Y/J = 1$.

Proof. Note first that $X_s^* \cong \ell^1 \oplus_1 \mathbb{K}\{p_s\}$ with $p_s((\xi_n)) = (1/s)\xi_\infty$. Hence c_0 is an M-ideal in X_s, and one deduces easily using the restricted 3-ball property [3, Th. I.2.2] that J is an M-ideal in X. Moreover, we have $g_*(c_0, X_s) = g^*(c_0, X_s) = s$.

(a) Using sequences supported only in the n-th coordinate space one can check that $g_*(J, X) = 0$, since $\inf s_n = 0$. Assume that there is an $x \in X \setminus J$ such that J is an M-summand in $Y = J \oplus \mathbb{K}\{x\}$, i.e., x is M-orthogonal to J. The projection P_n from X onto the n-th coordinate space X_{s_n} is an M-projection, so by [1, Cor. 3] $x_n = P_n x$ is M-orthogonal to J, hence in particular M-orthogonal to $J_n = c_0$. This gives $x_n = 0$, since c_0 is not an M-summand in X_{s_n}, and thus $x = 0$: a contradiction.

(b) As in (a) it is easy to see that $g^*(J, X) = 1$. We will show that for all $x = (x_n) \in X$ with $d(x, J) = 1$ the pseudoball $P_J(x) = B_X(x, 1) \cap J$ has non-empty interior. Without restriction we may additionally assume that $\|x\| = 1$. By the definition of the space X there are a finite set $E \subset \mathbb{N}$ and some $\varepsilon_0 > 0$ such that $\|x_n\|_{s_n} = 1$ for $n \in E$ and $\|x_n\|_{s_n} \le 1 - \varepsilon_0$ for $n \notin E$. This gives, for $n \notin E$, the inclusion $\varepsilon_0 B_{J_n} \subset B_{X_{s_n}}(x_n, 1) \cap J_n$ where again J_n is the c_0-space in the n-th coordinate. For $n \in E$ we have $d(x_n, J_n) \le 1$. Using $g^*(J_n, X_{s_n}) = s_n$ one obtains from this for some suitable $\varepsilon_n > 0$ and some $j_n^0 \in J_n$ that $B_{J_n}(j_n^0, \varepsilon_n) \subset B_{X_{s_n}}(x_n, 1) \cap J$; note that $d(x_n, J_n) < 1$ or even $d(x_n, J_n) = 0$ does not spoil this. Defining $j^0 = (j_n^0) \in J$ with $j_n^0 = 0$ for $n \notin E$ and $\varepsilon = \min\{\varepsilon_n \mid n \in E \text{ or } n = 0\}$ one finds that $B_J(j^0, \varepsilon)$ is contained in $B_X(x, 1) \cap J$. $\square$

The upper grade of an M-ideal tells us something about the existence of M-orthogonal directions for finite dimensional subspaces.

Proposition 3.3 *Let J be an M-ideal in X and put $\alpha := g^*(J, X)$. Then, given $\varepsilon > 0$ and a finite dimensional subspace $E \subset J$, one may find $y_0 \in J$ satisfying*

$$\frac{\alpha}{1+\varepsilon} \max\{\|e\|, |\lambda|\} \ \le\ \|e + \lambda y_0\| \ \le\ (1+\varepsilon) \max\{\|e\|, |\lambda|\} \tag{3.1}$$

for all $e \in E$, $\lambda \in \mathbb{K}$.

Proof. If $\alpha = 0$ then the statement is trivially satisfied with $y_0 = 0$. For $\alpha > 0$ we may assume, because we have an $\varepsilon > 0$ at our disposal in (3.1), that there is an $x \in X$ such that $d(x, J) = 1$ and $\alpha = g(P_J(x))$. Then J is an M-ideal in $Y = Y_x = J \oplus \mathbb{K}\{x\}$ and $\alpha = g_*(J, Y) = g^*(J, Y)$.

As in the proof of Lemma 2.8 the canonical operator

$$I_{Y,J^*} : Y \to J^{**},\ \langle I_{Y,J^*}(y), x^* \rangle = \langle x^*, y \rangle$$

is an (into-) isomorphism whose inverse has norm α^{-1}. To get started, consider the decomposition

$$Y^{**} = J^{\perp\perp} \oplus_\infty \operatorname{lin}\{x_0^{**}\}$$

in which we assume $\|x_0^{**}\| = 1$. Let $\delta > 0$ and choose with the help of the principle of local reflexivity an injective operator

$$T : \operatorname{lin}(E \cup \{x_0^{**}\}) \to Y$$

such that $\|T\| \cdot \|T^{-1}\| \le 1 + \delta$ and $Te = e$ for all $e \in E$. For $x_0 := Tx_0^{**}$ we obtain

$$\frac{1}{1+\delta} \max\{\|e\|, |\lambda|\} \ \le\ \|e + \lambda x_0\| \ \le\ (1+\delta) \max\{\|e\|, |\lambda|\}$$

for all $e \in E,\ \lambda \in \mathbb{K}$.

We have to 'push' x_0 into J. As a first step, we map it into J^{**} by means of $y_0^{**} := I_{Y,J^*}(x_0)$. It follows easily that

$$\frac{\alpha}{1+\delta} \max\{\|e\|, |\lambda|\} \le \|e + \lambda y_0^{**}\| \le (1+\delta) \max\{\|e\|, |\lambda|\}$$

for all $e \in E,\ \lambda \in \mathbb{K}$. As a second step, we again apply the principle of local reflexivity to obtain an injection

$$S : \operatorname{lin}(E \cup \{y_0^{**}\}) \to J$$

with $\|S\| \cdot \|S^{-1}\| \le 1 + \delta$ which extends the identity on E.

With an appropriate choice of δ and $y_0 := S(y_0^{**})$ we finally achieve the desired result. □

If the upper grade of an M-ideal J in X equals 1, then it follows from the above proposition and an obvious induction process that J contains a sequence equivalent to the standard basis of c_0. The same conclusion holds if only $g^*(J, X) > 0$. This can be deduced from Proposition 3.3 and the classical Bessaga-Pełczyński theorem, which characterises spaces containing c_0 in terms of weakly unconditionally Cauchy series. However, in M-ideal theory the containment of c_0 in proper M-ideals is usually – and more naturally – proved using the so-called intersection property; see [3, Th. II.4.7].

We have seen in (3.1) that the grade of an M-ideal J in X says something about the space J itself, irrespective of its position in X. Conversely, the geometry of the space J – more precisely, the set of pseudoballs in J – determines which grades are possible if J is an M-ideal in some superspace X. If J contains no proper pseudoball (e.g., if J is reflexive or, by what was remarked above, if J fails to contain a copy of c_0), then J is already an M-summand if it is an M-ideal; i.e., only $g^*(J,X)=0$ is possible.

If a space J has a proper pseudoball, one asks if it contains even pseudoballs B with empty interior, i.e., with $g(B)=1$. This is the case for $J=c_0$, which is the only space so far, where we have exhibited in (2.1) a pseudoball of grade different from 0 and 1. The following result will help to decide this question.

Proposition 3.4 *If a Banach space Y contains a pseudoball with empty interior, then*

$$0 \in \overline{\mathrm{ex}}^{w*} B_{Y^*}.$$

Proof. If B is a pseudoball with radius 1 and grade 1, then by Theorem 2.3 there is a Banach space X containing Y such that Y is an M-ideal in X and $g^*(Y,X)=1$.

Let $y_1,\dots,y_n \in Y$, $\|y_i\|=1$, and $\delta>0$. We have to produce some $p\in \mathrm{ex}\, B_{Y^*}$ such that

$$|p(y_i)| \le \delta \qquad \forall i=1,\dots,n.$$

To this end consider $E := \mathrm{lin}\{y_1,\dots,y_n\}$ and $\varepsilon=\delta/3>0$. Choose y_0 according to Proposition 3.3. In particular (since $\alpha=1$ here) $\|y_0\|\ge 1/(1+\varepsilon)$ so that there exists $p\in \mathrm{ex}\, B_{Y^*}$ with $p(y_0)\in\mathbb{R}$ and $p(y_0)>1/(1+\varepsilon)-\varepsilon$. Hence for $i=1,\dots,n$ and suitable scalars θ_i with modulus 1

$$|p(y_i)| = p(\theta_i y_i + y_0) - p(y_0) \le \|\theta_i y_i + y_0\| - p(y_0) \le 1+\varepsilon - \Big(\frac{1}{1+\varepsilon} - \varepsilon\Big) \le \delta.$$

$\square$

Example 3.5 The real ℓ^1-predual space $Y=\{y\in c \mid y_1 + 2y_\infty = 0\}$ (where $y_\infty := \lim y_n$) contains proper pseudoballs, but no pseudoballs with empty interior.

Proof. All the claims will follow once we identify the dual of

$$X = (c, \| \ \|) \qquad \text{where} \quad \|x\| = \max\{\|x\|_\infty, |x_1 + 2x_\infty|\}.$$

To this end put $e_k^*(x) = x_k$ and $f_\infty^*(x) = x_1 + 2x_\infty$. Then for $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$, $k = 1, \dots, n$ or $k = \infty$,

$$\left\| \sum_{k=1}^n a_k e_k^* + a_\infty f_\infty^* \right\|_{X^*} = \sum_{k=1}^n |a_k| + |a_\infty|. \tag{3.2}$$

The inequality $\le$ in (3.2) is clear by the definition of the norm $\| \ \|$. For the converse and $\operatorname{sgn} a_1 = \operatorname{sgn} a_\infty$ consider $x = (\operatorname{sgn} a_1, \dots, \operatorname{sgn} a_n, 0, \dots)$. Then $\|x\| = 1$ and

$$\left(\sum_{k=1}^n a_k e_k^* + a_\infty f_\infty^* \right)(x) = \sum_{k=1}^n |a_k| + |a_\infty|. \tag{3.3}$$

If $\operatorname{sgn} a_1 \ne \operatorname{sgn} a_\infty$ and $a_1, a_\infty \ne 0$ use $x = (\operatorname{sgn} a_1, \dots, \operatorname{sgn} a_n, \operatorname{sgn} a_\infty, \dots)$. Note that in this case $\operatorname{sgn} a_1 + 2 \operatorname{sgn} a_\infty = \operatorname{sgn} a_\infty$, so that $\|x\| = 1$, and again (3.3) holds. Also the remaining case, $a_1 = 0$ or $a_\infty = 0$, is easily handled in a similar way. By (3.2) X^* is isometrically isomorphic to ℓ^1; note that $\| \ \|$ is an equivalent norm on c. Also $\operatorname{ex} B_{X^*} = \{\pm e_k^* \mid k \in \mathbb{N}\} \cup \{\pm f_\infty^*\}$.

Now, $Y = \ker f_\infty^*$ is an M-ideal in X, $Y^* \cong \overline{\operatorname{lin}}\{e_k^* \mid k \in \mathbb{N}\} \cong \ell^1$, and $\operatorname{ex} B_{Y^*} = \{\pm e_k^* \mid k \in \mathbb{N}\}$. Since Y^* is $\sigma(X^*, X)$-dense in X^*, Y is a proper M-ideal in X, hence Y contains a proper pseudoball. However, the $\sigma(Y^*, Y)$-closure of $\operatorname{ex} B_{Y^*}$ equals $\operatorname{ex} B_{Y^*} \cup \{\pm 1/2 e_1^*\}$, so that Y can't contain a pseudoball with empty interior by Proposition 3.4. $\square$

With the notation of the above proof define $x \in X$ by $x_1 = -1$ and $x_k = 1$ for $k \ge 2$. Then $\|x\| = 1$ and $d(x, Y) = |f_\infty^*(x)| = 1$. The pseudoball

$$B = P_Y(x) = \{y \in Y \mid |y_1 - (-1)| \le 1 \text{ and } |y_k - 1| \le 1 \text{ for } k \ge 2\}$$

has grade $1/3$: $B_Y(y, 2/3) \subset B$ for y with $y_1 = -4/3$ and $y_k = 2/3$ for $k \ge 2$; and, as one easily shows, B contains no ball of Y with radius larger than $2/3$. Hence $g_*(Y, X) = g^*(Y, X) = 1/3$.

We don't know if Y contains pseudoballs with grades larger than $1/3$. We also don't know if there is a quantitative version of Proposition 3.4 relating the supremum of the grades of pseudoballs in Y with the minimum of the norms of elements in $\overline{\operatorname{ex}}^{w*} B_Y$ (which is $1/2$ in the above example).

With the help of the notion of an upper grade we will now prove:

Proposition 3.6 *Suppose J is an M-ideal in X. If there exists a Lipschitz projection π from X onto J, then its Lipschitz constant L is at least $2 \cdot g^*(J, X)$.*

Proof. Let $x \in X$, $d(x, J) = 1$, and consider $B = P_J(x) = B_X(x, 1) \cap J$. Then

$$B = \pi(B) \subset \pi(B_X(x, 1)) \subset B_J(\pi(x), L).$$

Now we take $\sigma(J^{**}, J^*)$-closures and use Proposition 2.2, Theorem 2.4, and Corollary 2.5 to obtain

$$B_{J^{**}}(Px, 1) \subset B_{J^{**}}(\pi(x), L)$$

where P is the M-projection from X^{**} onto $J^{\perp\perp} \cong J^{**}$. Hence

$$2\,Px - y \in B_{J^{**}}(\pi(x), L) \qquad \forall y \in B$$

so that by Theorem 2.4 and Corollary 2.5

$$L \geq 2 \cdot \|Px - (y + \pi(x))/2\| \geq 2 \cdot d(Px, J) = 2 \cdot g(B). \qquad \square$$

Proposition 3.6 in particular yields lower bounds for projection constants. For example, it is clear that c_0 is the range of a norm-2 linear projection on c, but since $g^*(c_0, c) = 1$ there is no projection onto c_0 with smaller norm. (This is no doubt a well-known fact.) More interesting is the case of $c_0 \subset \ell^\infty$. Here, no linear continuous projections exist, but Lindenstrauss [6] has produced a nonlinear 2-Lipschitz projection from $\ell^\infty_{\mathbb{R}}$ onto c_0. Again we conclude that the Lipschitz constant 2 is optimal.

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