THE PERFEKT THEORY OF M-IDEALS

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Dedicated to Ehrhard Behrends, teacher, mentor and friend, on the occasion of his 75th birthday.

Abstract. We revisit some ideas of K.-M. Perfekt who has provided an elegant framework to detect the biduality between function or sequence spaces defined in terms of some o- resp. O-condition. We present new proofs under somewhat weaker assumptions than before and apply the result to Lipschitz spaces.

1. Introduction

It has long been known that a number of pairs (E_0, E) of function or sequence spaces defined in terms of a "little o"- resp. "big O"-condition provide examples of spaces in biduality, i.e., $E_0^{**} = E$ with the identical inclusion $E_0 \hookrightarrow E$ corresponding to the canonical embedding β_{E_0} of E_0 into its bidual. In addition it turns out that often E_0 is an M-ideal in E (the definition will be recalled shortly); see for example [11]. The best known and simplest example of this kind is the pair (c_0, ℓ_∞); another example is the pair (B_0, B) consisting of the "little" Bloch space and the usual Bloch space.

In [7] and [8], K.-M. Perfekt provided an elegant general framework to accommodate many examples of this phenomenon including the pair (VMO, BMO). Other examples and applications can be found in [6]. In this note we shall revisit his construction, detailed in Section 3, and will give new proofs under somewhat less restrictive assumptions. We then apply this framework to Lipschitz spaces over general compact pointed metric spaces, which was left out in [7] where only compact subsets of $\mathbb{R}^n$ were considered.

Let us finish this section by recalling the notion of an M-ideal introduced by E.M. Alfsen and E.G. Effros in their seminal paper [1]. Let E be a Banach space and E_0 a closed subspace. Then E_0 is called an M-ideal if there is a projection $P\colon E^* \to E^*$ with $\ker P = E_0^\perp$, the annihilator of E_0 in E^*, such that

$$\|x^*\| = \|Px^*\| + \|x^* - Px^*\| \qquad \text{for all } x^* \in E^*. \tag{1.1}$$

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Detailed information on M-ideals can be found in E. Behrends's monograph [2] and in [4].

A very important special situation is when $E = E_0^{**}$; in this case E_0 is called M-embedded; see Chapter III in [4]. One can show that E_0 is M-embedded if and only if it is the restriction map P: $x^{***} \mapsto x^{***}|_{E_0}$ that satisfies (1.1).

The key result of part I of the Alfsen-Effros paper is a characterisation of the M-ideal property in purely geometric terms, by means of an intersection property of balls. We shall employ the following version, originally due to Å. Lima; cf. [4, Th. I.2.2]. The closed unit ball of a Banach space E is denoted by B_E.

Theorem 1.1. *E_0 is an M-ideal in E if and only if the following 3-ball property holds: For all $x \in B_E$, $y_1, y_2, y_3 \in B_{E_0}$ and $\varepsilon > 0$ there is some $y \in E_0$ such that*

$$\|x + y_i - y\| \le 1 + \varepsilon \qquad (i = 1, 2, 3). \tag{1.2}$$

This is a very versatile tool to prove the M-ideal property since no prior information on the dual space is involved.

2. M-ideals in subspaces of $C^b(L, Y)$.

Perfekt's approach naturally leads to subspaces of $C^b(L, Y)$, the space of bounded continuous functions on a locally compact Hausdorff space L with values in a Banach space Y.

We have the following result; $C_0(L, Y)$ stands for the space of continuous functions vanishing at infinity, i.e., a continuous function f is in $C_0(L, Y)$ if and only if $\{t \in L\colon \|f(t)\|_Y \ge \varepsilon\}$ is compact for each $\varepsilon > 0$.

Theorem 2.1. *Let L be a locally compact Hausdorff space, Y be a Banach space, let $E \subset C^b(L, Y)$ be a closed subspace and $E_0 = E \cap C_0(L, Y)$. Assume that B_{E_0} is dense in B_E for the topology of uniform convergence on compact subsets of L. Then E_0 is an M-ideal in E.*

Proof. We shall verify the 3-ball property from Theorem 1.1. So let $\varepsilon > 0$, $f \in B_E$ and $g_1, g_2, g_3 \in B_{E_0}$. In the following we shall use the notation $\|h\|_S = \sup_{t\in S} \|h(t)\|_Y$ for a function on L and $S^c = L\setminus S$ for the complement of S.

In the first step take a compact subset $K_0 \subset L$ such that all $\|g_i\|_{K_0^c} \le \varepsilon$. By the density assumption there is some $h_1 \in B_{E_0}$ such that $\|f - h_1\|_{K_0} \le \varepsilon$. Pick a compact set $K_1 \supset K_0$ such that $\|h_1\|_{K_1^c} \le \varepsilon$; obviously $\|g_i\|_{K_1^c} \le \varepsilon$ as well.

In the nest step pick some function $h_2 \in B_{E_0}$ such that $\|f - h_2\|_{K_1} \le \varepsilon$ and a compact set $K_2 \supset K_1$ such that $\|h_2\|_{K_2^c} \le \varepsilon$; clearly, still $\|h_1\|_{K_2^c} \le \varepsilon$. Inductively, one can find functions $h_j \in B_{E_0}$ and compact sets $K_0 \subset K_1 \subset K_2 \subset \cdots \subset L$ such that

$$\|f - h_j\|_{K_u} \le \varepsilon \qquad \text{for } j > u \tag{2.1}$$

and

$$\|h_j\|_{K_u^c} \le \varepsilon \qquad \text{for } j \le u. \tag{2.2}$$

Let $r > 1/\varepsilon$ and

$$g = \frac{1}{r}\sum_{j=1}^{r} h_j \in B_{E_0}.$$

We shall verify that

$$\|f + g_i - g\| \le 1 + 3\varepsilon \qquad (i = 1, 2, 3), \tag{2.3}$$

which implies the 3-ball property.

Indeed, if $t \in K_0$, then $\|f(t) - h_j(t)\|_Y \le \varepsilon$ for all j and $\|g_i(t)\|_Y \le 1$ for all i, hence

$$\|f(t) + g_i(t) - g(t)\|_Y \le \|g_i(t)\|_Y + \|f(t) - g(t)\|_Y \le 1 + \varepsilon.$$

Next, suppose $t \in K_u \setminus K_{u-1}$ for some $u \in \{1, \dots, r-1\}$. Then we have $\|f(t) - h_j(t)\|_Y \le 1 + \varepsilon$ for $j < u$ by (2.2) and the triangle inequality and $\|f(t) - h_j(t)\|_Y \le \varepsilon$ for $j > u$ by (2.1), and trivially $\|f(t) - h_u(t)\|_Y \le 2$. This shows for such t

$$\begin{aligned}\|f(t) + g_i(t) - g(t)\|_Y &\le \|f(t) - g(t)\|_Y + \|g_i(t)\|_Y \\ &\le \frac{1}{r}\big((u-1)(1+\varepsilon) + 2 + (r-u)\varepsilon\big) + \varepsilon \\ &= \frac{u+1}{r} + \Big(\frac{r-1}{r} + 1\Big)\varepsilon \le 1 + 2\varepsilon\end{aligned}$$

Finally, if $t \notin K_{r-1}$, then $\|h_j(t)\|_Y \le \varepsilon$ for $j \le r-1$ by (2.2) and $\|h_r(t)\|_Y \le 1$ so that

$$\begin{aligned}\|f(t) + g_i(t) - g(t)\|_Y &\le \|f(t)\|_Y + \|g_i(t)\|_Y + \|g(t)\|_Y \\ &\le 1 + \varepsilon + \frac{1}{r}\big((r-1)\varepsilon + 1\big) \le 1 + 3\varepsilon.\end{aligned}$$

Altogether we have proved (2.3). □

3. The Perfekt construction

We now recall the setup of Perfekt's approach; actually, we have removed some unnecessary restrictions. Let X be a reflexive space and Y be any Banach space. Consider a subset $\mathscr{L} \subset L(X, Y)$ (the space of bounded linear operators from X into Y) and equip it with a locally compact Hausdorff topology τ that is finer than the strong operator topology sot. Hence, for each $x \in X$, the mapping $\hat{x}\colon \mathscr{L} \to Y$, $\hat{x}(T) = Tx$, is continuous on $(\mathscr{L}, \tau)$. Now define the vector subspace

$$E = \Big\{x \in X\colon \sup_{T \in \mathscr{L}} \|Tx\|_Y < \infty\Big\}$$

of X. By definition, $\hat{x} \in C^b(\mathscr{L}, Y)$ for $x \in E$. We further assume that $\mathscr{L}$ is rich enough to make $x \mapsto \hat{x}$ injective and consequently $x \mapsto \|\hat{x}\|_\infty = \sup_{T \in \mathscr{L}} \|Tx\|_Y$ a norm on E. In this situation $\|x\|_\infty := \|\hat{x}\|_\infty$ is a norm on

E which makes E isometric to a subspace of $C^b(\mathscr{L},Y)$; henceforth we shall consider $E\subset C^b(\mathscr{L},Y)$ in a canonical way. We also assume that E is closed in $C^b(\mathscr{L},Y)$ so that both E and $E_0 := E\cap C_0(\mathscr{L},Y)$ are Banach spaces. Then the canonical inclusion mapping $x\mapsto x$ from $(E,\|\,.\,\|_\infty)$ to $(X,\|\,.\,\|_X)$ is continuous by the closed graph theorem. These assumptions will be in place throughout the whole section.

We finally consider the following crucial density assumption.

(A) B_{E_0} is dense in B_E for the topology generated by the norm $\|\,.\,\|_X$ of X.

Under these assumptions we will now give a new proof of Perfekt's biduality theorem [7, Th. 2.2]. At several points, the argument below is the same as in [7], but it differs at a decisive juncture so that we can dispense with some assumptions in [7]. Instead of using Singer's theorem representing the dual of $C_0(\mathscr{L},Y)$ by vector measures our argument relies on the $C_0(\mathscr{L})$-module structure of that space.

Theorem 3.1. *The space E is canonically isometric to E_0^{**} provided assumption* (A) *is valid.*

Before proceeding to the proof let us explain which isomorphism is meant in the theorem and what makes it canonical. Let us introduce, as in [7], the operators $i_0\colon E_0\to X$, $i_0(x)=x$; $J\colon X^*\to E_0^*$, $J=i_0^*$ (i.e., $Jx^*=x^*|_{E_0}$); $I\colon E_0^{**}\to X$, $I=J^*$. Then the claim is that $\operatorname{ran} I=E$ and I is an isometry for $\|\,.\,\|_\infty$. Note that $I\beta_{E_0}=\mathrm{Id}_{E_0\to E}$ (with β_{E_0} the canonical map from E_0 into its bidual), which makes I canonical.

Proof. In the first step we prove that $\operatorname{ran} J$ is norm dense in E_0^*. Let $\ell^0\in E_0^*$ and consider a Hahn-Banach extension $\ell\in C_0(\mathscr{L},Y)^*$. Let $\varphi\in C_0(\mathscr{L})$ with compact support $\mathscr{K}$ and $0\le\varphi\le 1$; then the functional

$$\ell_\varphi\in C_0(\mathscr{L},Y)^*,\quad \ell_\varphi(f)=\ell(\varphi f)$$

is well defined. If $x\in E_0$ we have

$$|\ell_\varphi(x)|=|\ell(\varphi x)|\le\|\ell\|\cdot\|\varphi x\|_\infty$$

and

$$\sup_{T\in\mathscr{L}}\|\varphi(T)Tx\|_Y=\sup_{T\in\mathscr{K}}\|\varphi(T)Tx\|_Y\le\sup_{T\in\mathscr{K}}\|T\|\cdot\|x\|_X.$$

Now $\mathscr{K}$ is τ-compact and hence sot-compact; as a result it is pointwise bounded and so $\sup_{T\in\mathscr{K}}\|T\|<\infty$ by the uniform boundedness principle. This shows that ℓ_φ is continuous on E_0 for the norm of X and therefore the restriction of some $x^*\in X^*$ to E_0; consequently $\ell_\varphi|_{E_0}\in\operatorname{ran} J$.

Let us now prove:

- *For every $\varepsilon>0$ there exists $\varphi\in C_0(\mathscr{L})$ with compact support, $0\le\varphi\le 1$, such that $\|\ell-\ell_\varphi\|_{C_0(\mathscr{L},Y)^*}\le\varepsilon$.*

Indeed, assume that for some $\varepsilon_0>0$ there is no such φ. Then for all such φ

$$\|\ell_{1-\varphi}\|_{C_0(\mathscr{L},Y)^*}=\|\ell-\ell_\varphi\|_{C_0(\mathscr{L},Y)^*}>\varepsilon_0.$$

Let us start with $\varphi_1 = 0$; so there is some $f_1 \in C_0(\mathscr{L}, Y)$, $\|f_1\|_\infty = 1$, with $|\ell((1-\varphi_1)f_1)| > \varepsilon_0$; upon replacing f_1 with $-f_1$ if necessary we even have

$$\ell((1-\varphi_1)f_1) > \varepsilon_0.$$

Since the functions of compact support are dense in $C_0(\mathscr{L}, Y)$ we may as well assume that $\operatorname{supp} f_1$ is compact. Next consider a function $\varphi_2 \in C_0(\mathscr{L})$ with compact support, $0 \le \varphi_2 \le 1$, such that $\varphi_2(T) = 1$ on $\operatorname{supp} f_1$. By our assumption there is some $f_2 \in C_0(\mathscr{L}, Y)$ of compact support, $\|f_2\|_\infty = 1$, such that

$$\ell((1-\varphi_2)f_2) > \varepsilon_0.$$

Inductively we find $\varphi_j \in C_0(\mathscr{L})$ between 0 and 1 and $f_j \in C_0(\mathscr{L}, Y)$ of norm 1, both with compact support, such that $\varphi_j(T) = 1$ on $\operatorname{supp} f_1 \cup \dots \cup \operatorname{supp} f_{j-1}$ and

$$\ell((1-\varphi_j)f_j) > \varepsilon_0.$$

By construction, if $(1-\varphi_j)(T)f_j(T) \neq 0$, then $(1-\varphi_i)(T)f_i(T) = 0$ for all $i < j$; consequently for all $r \in \mathbb{N}$

$$\Big\|\sum_{j=1}^{r}(1-\varphi_j)f_j\Big\|_\infty \le 1,$$

but

$$\ell\Big(\sum_{j=1}^{r}(1-\varphi_j)f_j\Big) > r\varepsilon_0;$$

this is a contradiction if $r \ge \|\ell\|/\varepsilon_0$. Thus, the above claim is proved.

Since the estimate in the above claim is a fortiori true for the restrictions of the functionals to E_0, we get that, given $\varepsilon > 0$,

$$\|\ell^0 - \ell_\varphi|_{E_0}\| \le \varepsilon.$$

Together with the first part of the proof this implies that the range of J is dense in E_0^*.

Now that we know that J has dense range, it is clear that I is injective. Let us show that $\operatorname{ran} I \subset E$. Let $e_0^{**} \in E_0^{**}$; we wish to prove that the element $Ie_0^{**} \in X$ satisfies

$$\sup_{T\in\mathscr{L}} \|T(Ie_0^{**})\|_Y < \infty.$$

Without loss of generality we can assume $\|e_0^{**}\| = 1$. Then there is a net (e_α) in B_{E_0} such that $e_\alpha \to e_0^{**}$ for the topology $\sigma(E_0^{**}, E_0^*)$. Since $I = J^*$ is weak* continuous, it follows $Ie_\alpha \to Ie_0^{**}$ for the topology $\sigma(X, X^*)$. Since $I\beta_{e_0}e_\alpha = e_\alpha$, we can conclude for each $T \in \mathscr{L}$ and $y^* \in Y^*$

$$(T^*y^*)e_\alpha \to (T^*y^*)(Ie_0^{**}),$$

that is

$$y^*(Te_\alpha) \to y^*(TIe_0^{**}).$$

Now

$$|y^*(Te_\alpha)| \le \|y^*\|\,\|Te_\alpha\|_Y \le \|y^*\|\,\|e_\alpha\|_\infty \le \|y^*\|$$

and therefore

$$|y^*(TIe_0^{**})| \le \|y^*\|$$

which shows

$$\|T(Ie_0^{**})\|_Y = \sup_{\|y^*\|\le 1} |y^*(TIe_0^{**})| \le 1$$

and thus $\|Ie_0^{**}\|_\infty \le 1$. This proves that $Ie_0^{**} \in E$ and that I is a contraction as an operator from E_0^{**} to $(E, \|\,.\,\|_\infty)$.

Finally consider the linear mapping Λ: $E \to E_0^{**}$, $(\Lambda e)(x^*|_{E_0}) = x^*(e)$, which is well defined since $\operatorname{ran} J$ is dense in E_0^*. One has, given $e \in E$,

$$x^*(e) = (\Lambda e)(Jx^*) = (I\Lambda e)(x^*) \qquad \text{for all } x^* \in X^*,$$

hence $e = I\Lambda e$; therefore I is surjective and

$$\|e\| = \|I\Lambda e\| \le \|\Lambda e\|.$$

To complete the proof of the theorem it remains to show that Λ is contractive, that is

$$x^* \in X^*,\ |x^*(e_0)| \le 1 \text{ for all } e_0 \in B_{E_0} \quad \Rightarrow \quad |x^*(e)| \le 1 \text{ for all } e \in B_E.$$

It is here that the assumption (A) enters. Let $x^* \in X^*$ as above and $\|e\|_\infty = 1$; pick a sequence (e_n) in B_{E_0} satisfying $\|e_n - e\|_X \to 0$, in particular $x^*(e_n) \to x^*(e)$. Consequently $|x^*(e)| \le 1$, as requested. □

Corollary 3.2. *Under the assumptions of Theorem 3.1, E_0 is an M-ideal in E and hence an M-embedded space.*

Proof. We just have to verify the density condition of Theorem 2.1, that is, B_{E_0} is dense in B_E for the topology of uniform convergence on compact subsets of $\mathscr{L}$.

Let $e \in E$, $\|e\|_\infty = 1$. By assumption (A) there are (e_n) in B_{E_0} such that $\|e_n - e\|_X \to 0$; as a result $\|Te_n - Te\|_Y \le \|T\|\,\|e_n - e\|_X \to 0$ for each $T \in \mathscr{L}$. This says that the associated functions $\hat{e}_n$ on $\mathscr{L}$ converge pointwise on $\mathscr{L}$. But we have already observed in the previous proof that a compact subset of $\mathscr{L}$ is bounded for the norm of $L(X,Y)$ by the uniform boundedness principle. Hence the convergence is uniform on compact subsets of $\mathscr{L}$. □

There is a limit to this method of detecting M-embedded spaces since the simplest nonseparable M-embedded space, $c_0(\Gamma)$ for some uncountable Γ, cannot be injected into a reflexive space. (This is a known result; here is a sketch of the proof: Suppose j_0: $c_0(\Gamma) \to X$ is a continuous injection into a reflexive space. Then j_0^* is weakly compact with pointwise dense range in $\ell_1(\Gamma)$, which has the Schur property; thus j_0^* is compact and hence has norm-separable range. This means that $\operatorname{ran} j_0^* \subset \ell_1(\Gamma_0)$ for some countable $\Gamma_0 \subset \Gamma$, a contradiction.)

4. Lipschitz spaces

Let (M,d) be a metric space with a distinguished point $p \in M$ (a "pointed" metric space). We denote by $\mathrm{Lip}_0(M)$ the Banach space of all real-valued Lipschitz functions on M vanishing at p with the Lipschitz constant as its norm:

$$\|F\|_{\mathrm{Lip}} = \sup_{s\neq t} \frac{|F(s)-F(t)|}{d(s,t)}.$$

We also consider the "little" Lipschitz space

$$\mathrm{lip}_0(M) = \Big\{ f \in \mathrm{Lip}_0(M)\colon \lim_{d(s,t)\to 0} \frac{|f(s)-f(t)|}{d(s,t)} = 0 \Big\}.$$

This is a closed subspace of $\mathrm{Lip}_0(M)$ which might however reduce to $\{0\}$, e.g., for $M=[0,1]$ with the Euclidean metric. For a Hölder metric on $[0,1]$, $d_\alpha(s,t) = |s-t|^\alpha$ where $0<\alpha<1$, the subspace $\mathrm{lip}_0([0,1],d_\alpha)$ is nontrivial, indeed $\mathrm{Lip}_0([0,1],d_\beta) \subset \mathrm{lip}_0([0,1],d_\alpha)$ for $\alpha<\beta\leq 1$. The authoritative source about Lipschitz spaces is N. Weaver's monograph [10].

We shall now consider compact (pointed) metric spaces with a countable dense subset P having the following property:

(B) $B_{\mathrm{lip}_0(M)}$ is dense in $B_{\mathrm{Lip}_0(M)}$ for the topology of pointwise convergence on P.

Since $B_{\mathrm{Lip}_0(M)}$ is equicontinuous this is the same as saying:

(B′) $B_{\mathrm{lip}_0(M)}$ is dense in $B_{\mathrm{Lip}_0(M)}$ for the topology of uniform convergence on M.

Proposition 4.1. *If M is a compact pointed metric space with* (B)*, then $\mathrm{lip}_0(M)$ is an M-ideal in* $\mathrm{Lip}_0(M)$.

Proof. Let $\Delta_M = \{(t,t)\colon t\in M\}$ be the diagonal in $M\times M$ and $L = (M\times M)\setminus \Delta_M$; this is a locally compact space. We further associate to a function F on M a new function ΦF on L defined by

$$(\Phi F)(s,t) = \frac{|F(s)-F(t)|}{d(s,t)};$$

this is the approach of K. de Leeuw's classical paper [3]. Note that Φ is a linear isometry from $\mathrm{Lip}_0(M)$ into $C^b(L)$ that takes $\mathrm{lip}_0(M)$ into $C_0(L)$. Let $\Lambda = \Phi(\mathrm{Lip}_0(M))$ and $\lambda = \Phi(\mathrm{lip}_0(M))$; then $\lambda = \Lambda \cap C_0(L)$.

To prove that λ is an M-ideal in Λ (and thus $\mathrm{lip}_0(M)$ is an M-ideal in $\mathrm{Lip}_0(M)$) it is enough, by Theorem 2.1, to show that B_λ is dense in B_Λ for the topology of uniform convergence on compact subsets of L. If $K\subset L$ is compact, then $\inf\{d(s,t)\colon (s,t)\in K\} =: \delta > 0$. Let $F\in B_{\mathrm{Lip}_0(M)}$; by (B) (or rather (B′)) there are $f_n \in B_{\mathrm{lip}_0(M)}$ such that $\|f_n - F\|_\infty \to 0$; hence for

$(s,t) \in K$

$$\begin{aligned} |(\Phi f_n)(s,t) - (\Phi F)(s,t)| &\le \frac{1}{\delta} |(f_n(s) - f_n(t)) - (F(s) - F(t))| \\ &\le \frac{1}{\delta} (|f_n(s) - F(s)| + |f_n(t) - F(t)|) \\ &\le \frac{2}{\delta} \|f_n - F\|_\infty \to 0 \end{aligned}$$

so that $\Phi f_n \to \Phi F$ uniformly on K.

This completes the proof of the proposition. □

We now come to the biduality theorem, originally due to N. Weaver in [9] who was the first to handle lip_0-spaces and even covered certain non-compact spaces. In [7] Perfekt considered the case of a Hölder metric on a compact subset $M \subset \mathbb{R}^n$ and showed that $\mathrm{lip}_0(M)^{**} \cong \mathrm{Lip}_0(M)$ by means of his method using a Besov space as the reflexive space X. He asked for a proof along these lines for a general compact metric space; this will be accomplished in the proof of the next theorem.

Theorem 4.2. *Let M be a compact pointed metric space satisfying* (B) *above. Then* $\mathrm{lip}_0(M)^{**} \cong \mathrm{Lip}_0(M)$.

Proof. We shall set up the Perfekt scenario as follows. Let X be the weighted ℓ_2-space

$$X = \Big\{ (x_n) \colon \sum_{k=1}^{\infty} |x_k|^2 2^{-k} < \infty \Big\}$$

with its canonical norm. Pick a dense sequence (p_n) in M and consider the functionals $\ell_{n,m} \in X^*$ (so $Y = \mathbb{R}$) defined by

$$\ell_{n,m}(x) = \frac{x_n - x_m}{d(p_n, p_m)} \qquad (n \neq m)$$

and equip $\mathscr{L} = \{\ell_{n,m} \colon n, m \in \mathbb{N},\ n \neq m\}$ with the discrete topology. For $F \in \mathrm{Lip}_0(M)$ let $x_F = (F(p_n))$, then $x_F \in X$ since F is bounded. Further define $E = \{x_F \colon F \in \mathrm{Lip}_0(M)\}$. Note that

$$\sup_{n \neq m} |\ell_{n,m}(x_F)| = \sup_{n \neq m} \Big| \frac{F(p_n) - F(p_m)}{d(p_n, p_m)} \Big| = \|F\|_{\mathrm{Lip}};$$

therefore $(E, \|\,.\,\|_\infty)$ is isometric to $\mathrm{Lip}_0(M)$, and E is a closed subspace of $C^b(\mathscr{L})$.

Let $E_0 = E \cap C_0(\mathscr{L})$. We shall argue that $x_F \in E_0$ if and only if $F \in \mathrm{lip}_0(M)$. Indeed, write $\alpha\mathscr{L} = \mathscr{L} \cup \{\infty\}$ for the Alexandrov compactification of $\mathscr{L}$. Suppose $\lim_{\ell_{n,m} \to \infty} \ell_{n,m}(x_F) = 0$. Then, given $\varepsilon > 0$, there is a finite set $K \subset (\mathbb{N} \times \mathbb{N}) \setminus \Delta_{\mathbb{N}}$ such that $|\ell_{n,m}(x_F)| < \varepsilon$ for $(n,m) \notin K$. Let $\delta = \inf\{d(p_n, p_m) \colon (n,m) \in K\} > 0$. Therefore, if $d(p_n, p_m) < \delta$, then $(n,m) \notin K$ and as a result

$$\Big| \frac{F(p_n) - F(p_m)}{d(p_n, p_m)} \Big| < \varepsilon$$

for these (n,m), which implies $F \in \mathrm{lip}_0(M)$ by density of $\{p_1, p_2, \dots\}$. Conversely, if $F \in \mathrm{lip}_0(M)$, then, if $\varepsilon > 0$ and $\ell_{(n,m)}(F) \geq \varepsilon$, we have $d(p_n, p_m) \geq \delta$ for some $\delta > 0$. In a compact space there can only be finitely many δ-separated points; this proves that $\lim_{\ell_{n,m}\to\infty} \ell_{n,m}(x_F) = 0$.

To conclude the proof of the theorem it is only left to verify condition (A) from Section 3 and to apply Theorem 3.1. By (B′) there are, given $F \in \mathrm{Lip}_0(M)$ with $\|F\|_{\mathrm{Lip}} = 1$, functions $f_n \in B_{\mathrm{lip}_0(M)}$ such that $\|f_n - F\|_\infty \to 0$. Then

$$\begin{aligned}\|x_{f_n} - x_F\|_X^2 &= \sum_{k=1}^\infty |f_n(p_k) - F(p_k)|^2 2^{-k} \\ &\leq \sum_{k=1}^\infty \|f_n - F\|_\infty^2 2^{-k} = \|f_n - F\|_\infty^2 \to 0,\end{aligned}$$

as required. □

Let (M,d) be a metric space. For $0 < \alpha < 1$, d^α is a metric as well; we write M^α to indicate that M is equipped with the metric d^α. (Sometimes M^α is called a snow-flaked version of M.) The corresponding Lipschitz spaces are also called Lipschitz-Hölder spaces, in accordance with the classical notation in $\mathbb{R}^n$. We shall prove that the compact metric spaces M^α satisfy the condition (B).

Proposition 4.3. *Let (M,d) be a compact pointed metric space and $0 < \alpha < 1$. Then M^α satisfies condition* (B).

Proof. Let $P = \{p_1, p_2, \dots\}$ be a countable dense subset of M with $p_1 = p$, the base point of M, and let $P_n = \{p_1, \dots, p_n\}$. Let $F \in \mathrm{Lip}_0(M^\alpha)$ with $\|F\|_{\mathrm{Lip}_0(M^\alpha)} = 1$. Define a function g_n on P_n by $g_n(p_j) = F(p_j)$. Clearly $\|g_n\|_{\mathrm{Lip}_0(P_n^\alpha)} \leq 1$. Choose $\beta_n \in (\alpha, 1)$ such that $\|g_n\|_{\mathrm{Lip}_0(P_n^{\beta_n})} \leq 1 + \frac{1}{n}$ and $(\operatorname{diam} M)^{\beta_n - \alpha} \leq 1 + \frac{1}{n}$; the former is possible since P_n is finite. Now apply the McShane extension theorem [10, Th. 1.33] and extend g_n to a function $G_n\colon M \to \mathbb{R}$ having the same Lipschitz constant as g_n for the metric d^{β_n}. Finally let $f_n = G_n/(1+\frac{1}{n})^2$. Then $f_n \in \mathrm{Lip}_0(M^{\beta_n})$ and thus $f_n \in \mathrm{lip}_0(M^\alpha)$. Moreover, $f_n \to F$ pointwise on P by construction and

$$\begin{aligned}\|f_n\|_{\mathrm{Lip}_0(M^\alpha)} &\leq \|f_n\|_{\mathrm{Lip}_0(M^{\beta_n})} (\operatorname{diam} M)^{\beta_n - \alpha} \\ &\leq \frac{\|G_n\|_{\mathrm{Lip}_0(M^{\beta_n})}}{(1+\frac{1}{n})^2} \Big(1 + \frac{1}{n}\Big) \leq 1.\end{aligned}$$

This proves condition (B) for d^α. □

Corollary 4.4. *Let (M,d) be a compact pointed metric space and $0 < \alpha < 1$. Then $(\mathrm{lip}_0(M^\alpha))^{**}$ is canonically isometric to $\mathrm{Lip}_0(M^\alpha)$, and $\mathrm{lip}_0(M^\alpha)$ is an M-ideal in its bidual $\mathrm{Lip}_0(M^\alpha)$.*

Proof. This follows from Proposition 4.1, Theorem 4.2 and Proposition 4.3. □

It is a remarkable fact, proved by N. Kalton [5, Th. 6.6], that, for a compact metric space, $\mathrm{lip}_0(M)$ is always M-embedded since it embeds almost isometrically into c_0. I do not know whether $\mathrm{lip}_0(M)$ is always an M-ideal in $\mathrm{Lip}_0(M)$; this trivially holds if $\mathrm{lip}_0(M)$ is trivial.

The biduality theorem of Corollary 4.4 is originally due to N. Weaver [9]; in fact he gave a number of conditions that are equivalent to the validity of the biduality theorem on a compact metric space. Let us add that condition (B) is actually equivalent to $\mathrm{Lip}_0(M)$ being the bidual of $\mathrm{lip}_0(M)$ under the canonical duality. Indeed, if $\mathrm{lip}_0(M)$ and $\mathrm{Lip}_0(M)$ are in canonical biduality, then by Goldstine's theorem $B_{\mathrm{lip}_0(M)}$ is weak* dense in $B_{\mathrm{Lip}_0(M)}$, and checking this on point evaluations $F \mapsto F(t)$, which span the predual $\mathscr{F}(M)$ of $\mathrm{Lip}_0(M)$ (known as the Lipschitz free space, Arens-Eells space or transportation cost space), shows that (B) holds.

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